On the arithmetic sum of regular Cantor sets

by

J. PALIS

IMPA, Rio de Janeiro, Brazil.
E-mail: jpalis@impa.br

and

J. C. YOCCOZ

Dep. Math., Université Paris-Sud, Orsay, France.

ABSTRACT. – We prove here that for any pair of regular Cantor sets in the line, either its arithmetic sum has Lebesgue measure zero or the pair can be approximated (through the image of a smooth diffeomorphism) by another one whose arithmetic sum contains an interval. The latter occurs when the Hausdorff dimension of the product of the sets is bigger than one.

RÉSUMÉ. – Nous démontrons que pour toute paire d'ensembles de Cantor réguliers sur la droite, ou bien leur somme arithmétique est de mesure de Lebesgue nulle, ou bien la paire peut être approchée (par l'image d'un difféomorphisme) par une autre dont la somme arithmétique contient un intervalle. Le dernier cas se produit quand la dimension de Hausdorff du produit des ensembles est plus grande que 1.

1. INTRODUCTION

Besides its perhaps more classical role in number theory and related topics, as it can be seen in the reference to Hall in [PT2], the arithmetic sum (difference) of Cantor sets plays a central role in dynamic bifurcations,

especially homoclinic bifurcations. These bifurcations in turn, can be a key to better understand the dark realm of dynamics: we conjecture that bifurcating homoclinic systems are dense in the complement of the hyperbolic or stable ones [PT2, Ch. 7].

Roughly speaking, the arithmetic sum of Cantor sets appear naturally in dynamics, when we analyse the set of intersections (often the set of tangencies) of invariant foliations, say one-dimensional stable and unstable foliations, and how this set varies with a parameter. If such foliations are associated to the same invariant hyperbolic set and their leaves are tangent along a transversal line, we are lead to analyse the sum of the two Cantor sets thus obtained, to study the bifurcations that appear when varying the diffeomorphism. They include, in the dissipative case, infinitely many simultaneous sinks, strange attractors, and cascades of period doubling bifurcations (see [PT2]). The only known way to produce infinitely many sinks, due to Newhouse, is to show that the sum of the Cantor sets contains intervals. That is what we prove here for a dense subset of pairs of regular Cantor sets whose Cartesian product has Hausdorff dimension bigger than one. The precise statement is presented in Section 2. Notice that if the Hausdorff dimension of the product of the Cantor sets is smaller than one, then its sum has Lebesgue measure zero [PT2]. In this last situation, in the context of homoclinic bifurcations, we have a total prevalence of hyperbolicity for the nearby diffeomorphisms [PT1], whereas in the latter it is shown with techniques quite different from the ones exhibited here, that this is definitively not the case [PY].

Finally, we have been conjecturing for more than a decade that the result in the present paper should be true for generic, or even an open and dense subset of pairs of regular Cantor sets and also for affine Cantor sets in general [P]. The conjecture may be of central interest in bifurcations of homoclinic tangencies.

2. DEFINITIONS AND RESULTS

We consider the following setting:

– a compact non trivial interval I;

– $C^{1+\alpha}$ diffeomorphisms $\varphi_1, \ldots, \varphi_p$ of I onto disjoint subintervals of I, which satisfy $|D\varphi_i(x)| < 1$, $1 \leq i \leq p$, $x \in I$.

– a matrix $A = (a_{ij})$, $1 \leq i, j \leq p$, with $a_{ij} \in \{-1, 1\}$.

We assume that the subshift of finite type Σ_A of the full shift on p elements is *infinite* and *topologically mixing*. For $1 \leq i_o \leq p$, let:

$$K(i_o) = \bigcup \bigcap_{k \geq 1} \varphi_{i_1} \ldots \varphi_{i_k}(I)$$

the union being taken over all sequences $(i_o, i_1 \ldots)$ such that $a_{i_k i_{k+1}} = 1$ for $k \geq 0$. This is a Cantor subset of I; a Cantor subset of $\mathbf{R}$ obtained in this way is called *regular*.

The $K(i_o)$, for $1 \leq i_o \leq p$, are called *twin* regular Cantor sets.

The diffeomorphic image of a regular Cantor set is regular: conjugate the φ_i by the diffeomorphism.

For $K = K(i_o)$ as above, we define:

$$\Phi(K) = \{\varphi = \varphi_{i_1} \ldots \varphi_{i_k} \mid k \geq 1,\ a_{i_\ell i_{\ell+1}} = 1 \text{ for } 0 \leq \ell < k\}.$$

For $\varphi = \varphi_{i_1} \ldots \varphi_{i_k} \in \Phi(K)$, we have:

$$(1) \qquad\qquad \varphi(I) \cap K(i_o) = \varphi(K(i_k)).$$

This shows that the *cylinder* $\varphi(I) \cap K(i_o)$ is *regular*; $\varphi(I)$ is the *box* of $\varphi(I) \cap K(i_o)$. Two cylinders (or boxes) are either disjoint, or one is contained in the other.

We recall that the arithmetic sum of two subsets A' and A'' of the line is defined as

$$A' + A'' = \{t \in \mathbf{R}; \exists\, a' \in A', a'' \in A'' \text{ such that } a' + a'' = t\}$$

We can now state our result.

THEOREM. – *Let K', K'' be two regular Cantor sets, of respective Hausdorff dimensions d', d''. Assume that $d' + d'' > 1$. Then, given $\delta > 0$, one can find C^∞ diffeomorphisms h of $\mathbf{R}$, arbitrarily near the identity, such that any point in $h(K') + K''$ is at a distance less than δ from some nontrivial interval in $h(K') + K''$.*

It is actually sufficient to find h such that $h(K') + K''$ contains *one* nontrivial interval. Indeed, let $J'_1, \ldots, J'_k$ and $J''_1, \ldots, J''_k$ be disjoint cylinders of K' and K'', respectively, such that any point in $K' + K''$ contains some $J'_i + J''_i$ in its $\delta/2$ neighbourhood. Gluing together diffeomorphisms $h_1 \ldots h_k$ near the identity such that $h_i(J'_i) + J''_i$ contains one nontrivial interval, we get the conclusion of the theorem (observe that J'_i, resp. J''_i, is regular with the same Hausdorff dimension as K', respectively K'').

3. BASIC ESTIMATES

We give here the basic estimates for the regular set K', with constants $c_1', c_2' \ldots$. Similar estimates hold for K'', with constants $c_1'', c_2'' \ldots$. Let I', $\varphi_1', \ldots, \varphi_p'$ be as in 2., defining K'.

There exist c_1', c_2', c_3', c_4' such that:

$$(2) \qquad 0 < c_1'^{-1} < |D\varphi_i'(x)| < c_2' < 1, \quad 1 \le i \le p, x \in I'$$

$$(3) \qquad |\varphi_i'(x) - \varphi_j'(y)| > c_3'^{-1} > 0, \quad 1 \le i, j \le p, i \ne j, x, y \in I'$$

$$(4) \quad |\log|D\varphi_i'(x)| - \log|D\varphi_i'(y)|| \le c_4'|x - y|^\alpha, \quad 1 \le i \le p, \ x, y \in I'.$$

From (2) and (4), we get:

Distortion estimate. – For $\varphi \in \Phi(K')$, $x, y \in I'$.

$$(5) \qquad |\log|D\varphi(x)| - \log|D\varphi(y)|| \le c_5' = c_4'(1 - c_2'^\alpha)^{-1}|I'|^\alpha.$$

Let $Q' = \varphi(I')$, $\varphi \in \Phi(K')$, be a box of K', with $Q' \ne I'$. Writing $\varphi = \varphi'\varphi_i'$, we have $|Q'| \le c_2'|I'|\max|D\varphi'|$ and the distance from Q' to $K' - Q'$ is at least $c_3'^{-1}\min|D\varphi'|$. Hence we get from (5):

$$(6) \qquad d(Q', K' - Q') \ge c_6'^{-1}|Q'|, \quad \text{with} \quad c_6' = c_3' \, e^{c_5'} \, c_2'|I'|.$$

Let m' be the d'-dimensional Hausdorff measure. Because of the topologically mixing property, each twin of K' contains a diffeomorphic image of any other, so they all have Hausdorff dimension d'. Moreover, regularity implies that there are constants c_7', c_8' such that:

$$(7) \qquad 0 < c_7'^{-1} < m'(K_1') < c_8'$$

for any K_1' twin of K' (including itself). See [F], [PT2].

LEMMA 1. – *Let J' be a cylinder of K', with box Q'; one has:*

$$c_9'^{-1}|Q'|^{d'} \le m'(J') \le c_{10}'|Q'|^{d'},$$

with $c_9' = c_7' \exp(c_5'd')|I'|^{d'}$, $c_{10}' = c_8' \exp(c_5'd')|I'|^{-d'}$.

Proof. – One has $J' = \varphi(K_1')$, $Q' = \varphi(I')$ for some $\varphi \in \Phi(K')$ and some twin K_1' of K'. From this, we get:

$$|Q'| = |D\varphi(x_o)|\,|I'|, \quad \text{for some} \quad x_o \in I'$$

$$m'(J') = \int_{K_1'} |D\varphi(x)|^{d'}\, dm'(x) = m'(K_1')|D\varphi(x_1)|^{d'}$$

for some $x_1 \in I'$. The lemma then follows from (5) and (7). $\quad\square$

Let $\beta > 0$. A β-*decomposition* of K' is a partition of K' in (disjoint) cylinders whose boxes have length between β and $c_1'^{-1}\beta$. Such a decomposition exists and is non trivial if $\beta < |I'|$ (see (2)). Let π' be a β-decomposition of K'.

LEMMA 2. – *We have:*

$$m'(J_1') \le c_{11}'\, m'(J_2'), \quad \text{for} \quad J_1', J_2' \in \pi'$$

with $c_{11}' = c_9'c_{10}'c_1'^{d'}$.

Proof. – Trivial from Lemma 1 and the definition of a β-decomposition. $\quad\square$

LEMMA 3. – *Let Q_0' of length greater than β. The number N of elements of π' contained in Q_0' satisfies:*

$$c_{12}'^{-1}\big(\beta^{-1}|Q_0'|\big)^{d'} \le N \le c_{13}'\big(\beta^{-1}|Q_0'|\big)^{d'}$$

with $c_{12}' = c_9'c_{10}' \quad c_{13}' = c_9'c_{10}'c_1'^{d'}$.

Proof. – One has, if J_0' is the cylinder of Q_0':

$$c_9'^{-1}|Q_0'|^{d'} \le m'(J_0') \le c_{10}'|Q_0'|^{d'}, \quad \text{by Lemma 1;}$$

$$m'(J_0') = \sum_{\substack{J' \in \pi' \\ J' \subset Q_0'}} m'(J'),$$

and $c_9'^{-1}c_1'^{-d'}\beta^{d'} \le m'(J') \le c_{10}'\beta^{d'}$ for $J' \in \pi'$.

The lemma follows. $\quad\square$

4. THE ENERGY ESTIMATES

We now consider K', K'' as in the theorem. We let $K = K' \times K'' \subset \mathbf{R}^2$, $D = d' + d'' > 1$, $m_D = m' \times m''$ the D-dimensional Hausdorff measure on K. The Euclidean metrics on $\mathbf{R}^2$ is noted $\| \quad \|$.

A β-*decomposition* of K is a partition of K which is product of two β-decompositions of K', K''. In a similar way, we define cylinders and boxes for K.

For simplicity, we assume that:

$$(8) \qquad\qquad |I'| \leq 1, \quad |I''| \leq 1, \quad |I'| + |I''| \geq 1,$$

which can be obtained by an homothety of the *same* ratio on K', K''. We also assume that c_1', c_1'' have been taken so that:

$$(9) \qquad\qquad |I'| \geq c_1'^{-1}, \quad |I''| \geq c_1''^{-1},$$

in such that a way that $\pi(0) = \{K\}$ is a 1-decomposition of K.

Let π be a β-decomposition of K.

LEMMA 4. – *For* $J_1, J_2 \in \pi$, *one has:*

$$m_D(J_1) \leq c_{11}\, m_D(J_2), \quad with \quad c_{11} = c_{11}'\, c_{11}''$$

Proof. – Direct consequence of Lemma 2.

LEMMA 5. – *Let* $\beta_1 < (c_1' + c_1'')^{-1} \beta$ *and* π_1 *a* β_1-*decomposition of* K. *For* $J \in \pi_0$, *the number* N *of elements of* π_1 *contained in* J *satisfies:*

$$c_{14}^{-1} \big(\beta\beta_1^{-1}\big)^D \leq N \leq c_{14}\big(\beta\beta_1^{-1}\big)^D$$

with $c_{14} = c_9'\, c_9''\, c_{10}'\, c_{10}''\, c_1'^{d'}\, c_1''^{d''}$.

Proof. – Use Lemma 3, knowing that if $Q' \times Q''$ is the box of J, one has $c_1'^{-1} \beta \leq |Q'| \leq \beta$, $c_1''^{-1} \beta \leq |Q''| \leq \beta$. $\square$

DEFINITION. – The *energy* of a cylinder J of K is the value of the integral:

$$\mathcal{I}(J) = \iint_{J \times J} \frac{dm_D(z)\, dm_D(z')}{\|z - z'\|}.$$

LEMMA 6. – *One has*

$$\mathcal{I}(K) \leq c_{15} = 2^D \big(1 - 2^{1-D}\big)^{-1} c_3'^{d'}\, c_3''^{d''}\, (c_8'\, c_8'')^2 \exp(c_5' d' + c_5'' d'').$$

Proof. – Let $z = (x', x'') \in K$, $r > 0$, and $B_r(z)$ the Euclidean ball of center z, radius r. Let $Q = Q' \times Q''$ be the smallest box

of K containing $B_r(z) \cap K$; write $Q' = \varphi'(I')$, $Q'' = \varphi''(I'')$ with $\varphi' \in \Phi(K')$, $\varphi'' \in \phi(K'')$. Let i (resp. j) be such that $x' \in \varphi'\varphi'_i(I')$ (resp. $x'' \in \varphi''\varphi''_j(I'')$). By the minimality of Q, the interval of center x' (resp. x'') and radius r must intersect $\varphi'\varphi'_k(I')$ (resp. $\varphi''\varphi''_\ell(I'')$) for some $k \neq i$ (resp. $\ell \neq j$). Then, we get from (3) that:

$$r \geq (\inf |D\varphi'|)c_3'^{-1}$$
$$r \geq (\inf |D\varphi''|)c_3''^{-1}.$$

If J', J'' are the cylinders associated to Q', Q'', we obtain (using (5)):

$$m'(J') = \int_{K_1'} |D\varphi'(y)|^{d'} \, dm'(y) \leq (c_3'r)^{d'} c_8' \exp(c_5'd')$$
$$m''(J'') = \int_{K_1'} |D\varphi''(y)|^{d''} \, dm''(y) \leq (c_3''r)^{d''} c_8'' \exp(c_5''d''),$$

where K_1', K_1'' are twins of K', K''. This gives:

$$m_D(B_r(z) \cap K) \leq c_{16} \, r^D, \text{ with } c_{16} = c_8'c_8''c_3'^{d'} c_3''^{d''} \exp(c_5'd' + c_5''d'').$$

One has $K \subset B_2(z)$ by (8), so we can write

$$K - \{z\} = \bigcup_{k=0}^{\infty} \left((B_{2^{1-k}}(z) - B_{2^{-k}}(z)) \cap K\right),$$

to get

$$\int_K \frac{dm_D(z')}{||z - z'||} \leq \sum_{k=0}^{\infty} 2^k c_{16}\left(2^{1-k}\right)^D = 2^D c_{16}\left(1 - 2^{1-D}\right)^{-1}.$$

Using $m_D(K) \leq c_8'c_8''$, we get the lemma. $\square$

LEMMA 7. – *Let J be an element of a β-decomposition of K. One has:*

$$\mathcal{I}(J) \leq c_{17}\beta^{2D-1} \text{ with } c_{17} = c_{15}c_1'^{2d'} c_1''^{2d''} e^{2c_5'd' + 2c_5''d''} Max\left(c_1'e^{c_5'}, c_1''e^{c_5''}\right).$$

Proof. – Write $J = J' \times J''$, Q', Q'' for the boxes of J', J''; let $\varphi' \in \Phi(K)$, $\varphi'' \in \Phi(K'')$ be such that $Q' = \varphi'(I')$, $Q'' = \varphi''(I'')$, $J' = \varphi'(K_1')$, $J'' = \varphi''(K_1'')$, for some twins K_1', K_1'' of K', K''. The estimate of Lemma 6 holds with $K_1 = K_1' \times K_1''$ instead of K. One has:

$$\inf |D\varphi'| \geq e^{-c_5'}|Q'| |I'|^{-1} \geq e^{-c_5'} c_1'^{-1}\beta \text{ (see (8))}$$
$$\inf |D\varphi''| \geq e^{-c_5''} c_1''^{-1}\beta$$

and, for $z = (x', x'')$, $z' = (y', y'') \in K_1$:

$$\|\varphi' \times \varphi''(z) - \varphi' \times \varphi''(z')\| \geq \min(\inf|D\varphi'|, \inf|D\varphi''|)\|z - z'\|.$$

We then get:

$$\iint_{J \times J} \frac{dm_D(z)dm_D(z')}{\|z - z'\|}$$

$$\leq \iint_{K_1 \times K_1} \frac{dm_D(z)dm_D(z')|D\varphi'(x')|^{d'}|D\varphi'(y')|^{d'}|D\varphi''(x'')|^{d''}|D\varphi''(y'')|^{d''}}{\|z - z'\| \min(\inf|D\varphi'|, \inf|D\varphi''|)}$$

$$\leq \beta^{-1} c_{15} \max(c_1' e^{c_5'}, c_1'' e^{c_5''}) \exp(2c_5'd' + 2c_5''d'') \left(\frac{|Q'|}{|I'|}\right)^{2d'} \left(\frac{|Q''|}{|I''|}\right)^{2d''}$$

which gives the lemma, using (9).   □

5. MEASURE OF PROJECTIONS

For $\lambda > 0$, let π_λ the linear map from $\mathbf{R}^2$ to $\mathbf{R}$ defined by:

$$\pi_\lambda(x, y) = \lambda x + y.$$

Let J be a cylinder of K. We write m_J for the restriction of m_D to J and $m_{J,\lambda}$ for the image of m_J by π_λ.

LEMMA. – *Let J be an element of a β-decomposition π of K. Assume that for some $\lambda \in [1/2, 1]$, $\gamma > 0$, we have:*

$$(10) \qquad \int_{-\infty}^{+\infty} |\hat{m}_{J,\lambda}(\zeta)|^2 \, d\zeta \leq \gamma^{-1} \beta^{2D-1}.$$

Let J_o be a measurable subset of J satisfying $m_D(J_o) \geq k\, m_D(J)$, with $0 < k \leq 1$. Then, one has:

$$m(\pi_\lambda(J_o)) \geq c_{18}^{-1} k^2 \gamma \beta, \text{ with } m \text{ the Lebesgue measure on } \mathbf{R}$$

and $c_{18} = \left(c_9' c_9'' c_1'^{d'} c_1''^{d''}\right)^2.$

Proof. – Let m_{J_o} be the restriction of m_J to J_o, and $m_{J_o,\lambda}$ its image by π_λ. The hypothesis implies that $m_{J,\lambda}$ (resp. $m_{J_o,\lambda}$) is absolutely continuous with respect to m and its density g (resp. g_o) satisfy:

$$(11) \qquad \int_{-\infty}^{+\infty} (g_o(u))^2 \, du \leq \int_{-\infty}^{+\infty} (g(u))^2 \, du < \gamma^{-1} \beta^{2D-1}.$$

By Cauchy-Schwarz's inequality, one has:

$$(12) \qquad m(\pi_\lambda(J_o)) \int_{-\infty}^{+\infty} (g_o(u))^2 \, du \geq \left[\int_{-\infty}^{+\infty} g_o(u) \, du \right]^2.$$

On the other hand, one has, with

$$(13) \qquad \int_{-\infty}^{+\infty} g_o(u) \, du = m_{J_o,\lambda}(\mathbf{R}) = m_D(J_o)$$

$$\geq k \, m_D(J) \geq k \, c_9'^{-1} c_9''^{-1} c_1'^{-d'} c_1''^{-d''} \beta^D$$

by Lemma 1-Lemma 8 follows from (10), (12), (13). $\qquad \square$

LEMMA 9. – *Let* $\beta_1 < (c_1' + c_1'')^{-1} \beta$, π_1 *a* β_1-*decomposition of* K *and* $J \in \pi$. *Let* J_o *be the union of at least one eight of all elements of* π_1 *which are contained in* J. *If* λ, γ *are as in Lemma 8 and* (10) *holds, then we have:*

$$m(\pi_\lambda(J_o)) \geq c_{19}^{-1} \gamma \beta,$$

with $c_{19} = 64 c_{18} c_{11}^2$.

Proof. – By Lemma 4, we can take $k = \frac{1}{8} c_{11}^{-1}$ in Lemma 8. $\qquad \square$

LEMMA 10. – *There is a universal constant* c *such that, for any cylinder* J *of* K, *we have:*

$$\int_{1/2}^1 \left(\int_{-\infty}^{+\infty} |\hat{m}_{J,\lambda}(\zeta)|^2 \, d\zeta \right) d\lambda \leq c \mathcal{I}(J).$$

Proof. – This is a trivial consequence of a deep inequality of Kaufman; see the proof of Marstrand's theorem in [F]. $\qquad \square$

6. TWO COUNTING LEMMAS

LEMMA 11. – *Let* $E = E_1 \cup \ldots \cup E_N$ *be a partition of a finite set* E. *Then at least half of the elements of* E *belong to a* E_j *such that* $\#E_j \geq \frac{1}{2N} \#E$.

Proof. – Immediate. $\qquad \square$

Let $\beta_1 < (c_1' + c_1'')^{-1} \beta$, and π_1 (resp. π) a β_1 (resp. β) decomposition of K. We consider a subset $\tilde{\pi}_1$ of π_1 such that:

$$(14) \qquad \#\tilde{\pi}_1 \geq (1 - \rho_1) \#\pi_1,$$

with some $\rho_1 \in [0; 1/2]$. We then define, for $J \in \pi$:

$$\pi_1(J) = \{ J_1 \in \pi_1, J_1 \subset J \}$$
$$\tilde{\pi}_1(J) = \pi_1(J) \cap \tilde{\pi}_1,$$

and

$$\pi_2 = \left\{ J \in \pi \mid \#\tilde{\pi}_1(J) \geq \frac{1}{2}(1 - \rho_1)\#\pi_1(J) \right\}$$

LEMMA. – *We have*

$$\#\pi_2 \geq (1 - \rho)\#\pi, \text{ with } \rho = 4c_{14}^2 \rho_1.$$

Proof. – By definition of π_2, one has:

$$(1 - \rho_1)\#\pi_1 \leq \#\tilde{\pi}_1 \leq \sum_{J \in \tilde{\pi}} \#\pi_1(J) + \frac{1}{2}(1 - \rho_1) \sum_{J \in \pi - \pi_2} \#\pi_1(J)$$

which implies:

$$\frac{1}{2}(1 - \rho_1) \sum_{J \in \pi - \pi_2} \#\pi_1(J) \leq \rho_1 \sum_{J \in \pi_2} \#\pi_1(J).$$

Using Lemma 5, we get (as $\rho_1 \leq 1/2$):

$$\#(\pi - \pi_2) \leq 4c_{14}^2 \rho_1 \#\pi_2. \qquad \square$$

7. STARTING THE CONSTRUCTION

The construction we are going to make depends on a certain number of parameters, which are:

– an integer $N \geq 1$;

– a positive number $\varepsilon \in (0, 1/2]$

– a decreasing sequence $1 = \beta_o > \beta_1 > \cdots > \beta_{2N} > 0$

These parameters will be specified later.

We define $\pi(0) = \{K\}$ and choose, for $1 \leq i \leq 2N$, a β_i-decomposition $\pi(i) = \pi'(i) \times \pi''(i)$ of K.

We assume that

(15) $\beta_i > (c_1' + c_1'')\beta_{i+1}$

so that each element of $\pi(i+1)$ is contained in one element of $\pi(i)$. From Lemmas 7 and 10, we have:

$$(16) \qquad \int_{1-\varepsilon}^{1} \left(\int_{-\infty}^{+\infty} |\hat{m}_{J,\lambda}(\zeta)|^2 \, d\zeta \right) d\lambda \leq cc_{17} \beta_i^{2D-1} \quad \text{if } J \in \pi(i).$$

From this we deduce:

$$(17)$$
$$\int_{1-\varepsilon}^{1} \left[\sum_{i=0}^{2N} 2^{-i} \beta_i^{1-2D} [\#\pi(i)]^{-1} \sum_{J \in \pi(i)} \int_{-\infty}^{+\infty} |\hat{m}_{J,\lambda}(\zeta)|^2 \, d\zeta \right] d\lambda \leq 2cc_{17}.$$

We can therefore find $\lambda \in [1-\varepsilon, 1]$, *fixed in the following* such that we have, for $0 \leq i \leq 2N$:

$$(18) \qquad [\#\pi(i)]^{-1} \sum_{J \in \pi(i)} \int_{-\infty}^{+\infty} |\hat{m}_{J,\lambda}(\zeta)|^2 \, d\zeta \leq 2^{i+1} \varepsilon^{-1} cc_{17} \beta_i^{2D-1}.$$

For $1 \leq i \leq 2N$, let $\pi_1(i)$ be the subset of $\pi(i)$ formed by the J which satisfy:

$$(19) \qquad \int_{-\infty}^{+\infty} |\hat{m}_{J,\lambda}(\zeta)|^2 \, d\zeta \leq c_{20} \left(16c_{14}^2\right)^{i-1} \varepsilon^{-1} \beta_i^{2D-1}$$

with $c_{20} = 16cc_{17}$.

From (18), (19), we get:

$$(20) \qquad \#\pi_1(i) \geq \left(1 - \frac{1}{4}\left(8c_{14}^2\right)^{1-i}\right) \#\pi(i).$$

We now define inductively subsets $\tilde{\pi}(i)$ of $\pi(i)$. Define $\rho_i = \frac{1}{2}(8c_{14}^2)^{1-i}$, for $1 \leq i \leq 2N$. Let:

$$\tilde{\pi}(2N) = \pi_1(2N).$$

For $i < 2N$, if $\tilde{\pi}(i+1)$ has already been defined, let

$$(21) \quad \pi_2(i) = \left\{ J \in \pi(i) \mid \#\tilde{\pi}(i+1, J) \geq \frac{1}{2}(1 - \rho_{i+1}) \#\pi(i+1, J) \right\}$$

where the notations $\pi(i+1, J)$, $\tilde{\pi}(i+1, J)$ is as before Lemma 12: the sets of elements of $\pi(i+1)$, $\tilde{\pi}(i+1)$ which are contained in J. Then let

$$(22) \qquad \tilde{\pi}(i) = \pi_1(i) \cap \pi_2(i).$$

We show inductively that

(23) $$\#\tilde{\pi}(i) \geq (1 - \rho_i)\#\pi(i).$$

This is true for $i = 2N$ by (20). Assuming it true for $i + 1$, one can apply Lemma 12 to get:

$$\#\pi_2(i) \geq \left(1 - \frac{1}{2}\rho_i\right)\#\pi(i),$$

which, taking (20), (22) into account, gives (23). Observe that $\rho_1 = \frac{1}{2}$, and $\rho_i \in [0, 1/2]$ for $1 \leq i \leq 2N$. Finally, we define:

$$\tilde{\pi}(0) = \pi(0) = \{K\}.$$

8. THE MAIN ESTIMATES

8.1. For $1 \leq i \leq 2N$, $k \in \mathbf{Z}$, let:

$$M(i, k) = [(2k - 2)\beta_i, (2k + 2)\beta_i].$$

For $J \in \pi(i)$, the diameter of $\pi_\lambda(J)$ is less than $2\beta_i$, so we have $\pi_\lambda(J) \subset M(i, k)$ for at least one (and at most 3) $k \in \mathbf{Z}$.

We choose a *partition* of $\pi(i)$ in subsets $\pi(i, k)$ such that $J \in \pi(i, k)$ implies $\pi_\lambda(J) \subset M(i, k)$. We use the notation:

$$\pi(i + 1, J, k) = \pi(i + 1, k) \cap \pi(i + 1, J)$$
$$\tilde{\pi}(i + 1, J, k) = \pi(i + 1, k) \cap \tilde{\pi}(i + 1, J), \quad J \in \pi(i).$$

8.2. Let $0 \leq i \leq 2N - 1$, $J \in \tilde{\pi}(i)$.

By (21), (22) and Lemma 5, one has:

(24) $$\#\tilde{\pi}(i + 1, J) \geq \frac{1}{4}\#\pi(i + 1, J) \geq \frac{1}{4}c_{14}^{-1}\left(\beta_i\beta_{i+1}^{-1}\right)^D.$$

On the other hand, the number of $k \in \mathbf{Z}$ such that $M(i + 1, k)$ intersects $\pi_\lambda(J)$ (whose diameter is $\leq 2\beta_i$) is at most $3 + \beta_i\beta_{i+1}^{-1}$, therefore assuming

(25) $$\beta_i > 3\beta_{i+1}$$

we get

(26) $$\#\{k \in \mathbf{Z}, \tilde{\pi}(i + 1, J, k) \neq \phi\} \leq 2\beta_i\beta_{i+1}^{-1}.$$

Let $A(i+1, J)$ be the set of $k \in \mathbf{Z}$ such that:

$$(27) \qquad \#\tilde{\pi}(i+1, J, k) \geq (16c_{14})^{-1} \left(\beta_i \beta_{i+1}^{-1}\right)^{D-1}.$$

From Lemma 11, we have:

$$(28) \qquad \sum_{k \in A(i+1,J)} \#\tilde{\pi}(i+1, J, k) \geq \frac{1}{2} \#\tilde{\pi}(i+1, J) \geq \frac{1}{8} \#\pi(i+1, J).$$

If we define:

$$J_o = \bigcup_{\substack{k \in A(i+1,J) \\ J_1 \in \tilde{\pi}(i+1,J,k)}} J_1,$$

we can apply Lemma 9, using (19) and (28) to conclude that:

$$(29) \qquad m(\pi_\lambda(J_o)) \geq c_{21}^{-1} c_{22}^{1-i} \varepsilon \beta_i, \text{ with } c_{21} = c_{19}c_{20}, c_{22} = 16c_{14}^2.$$

As $\pi_\lambda(J_o)$ is included in the union of $M(i+1, k)$ for $k \in A(i+1, J)$, one deduces from (29) that:

$$(30) \qquad \#A(i+1, J) \geq (4c_{21})^{-1} c_{22}^{1-i} \varepsilon \beta_i \beta_{i+1}^{-1}.$$

8.3. Keeping the same notations as above, we assume $i \leq 2N - 2$. For $\ell \in \mathbf{Z}$, $J_1 \in \tilde{\pi}(i+1, J)$, we define:

$$\chi(\ell, J_1) = 1 \quad \text{if} \quad \tilde{\pi}(i+2, J_1, \ell) \neq \phi$$
$$= 0 \quad \text{if} \quad \tilde{\pi}(i+2, J_1, \ell) = \phi.$$

From (30), we have

$$(31) \qquad \sum_\ell \chi(\ell, J_1) \geq (4c_{21})^{-1} c_{22}^{1-i} \varepsilon \beta_{i+1} \beta_{i+2}^{-1}.$$

Let $k \in A(i+1, J)$, and let $\chi(\ell, k) = \sum_{J_1 \in \tilde{\pi}(i+1,J,k)} \chi(\ell, J_1)$. If $\chi(\ell, k) \neq 0$, $M(i+2, \ell)$ must intersect $M(i+1, k)$, so there are at most $2\beta_{i+1}\beta_{i+2}^{-1}$ such ℓ. On the other hand, by (31):

$$(32) \qquad \sum_\ell \chi(\ell, k) \geq (4c_{21})^{-1} c_{22}^{-i} \varepsilon \beta_{i+1} \beta_{i+2}^{-1} \#\tilde{\pi}(i+1, J, k).$$

Let $B(i+2, J, k)$ be the set of $\ell \in \mathbf{Z}$ such that:

$$(33) \qquad \chi(\ell, k) \geq (16c_{21})^{-1} c_{22}^{1-i} \varepsilon \#\tilde{\pi}(i+1, J, k).$$

Then we have by (32) and the remark preceding it:

$$(34) \qquad \sum_{\ell \in B(i+2,J,k)} \chi(\ell, k) \geq (8c_{21})^{-1} c_{22}^{1-i} \varepsilon \beta_{i+1} \beta_{i+2}^{-1} \#\tilde{\pi}(i+1, J, k).$$

As trivially $\chi(\ell, k) \leq \#\tilde{\pi}(i+1, J, k)$, we conclude that:

$$(35) \qquad \#B(i+2, J, k) \geq (8c_{21})^{-1} c_{22}^{-1} \varepsilon \beta_{i+1} \beta_{i+2}^{-1}.$$

9. PERTURBATIONS

We choose once and for all a C^∞ function η on $\mathbf{R}$ which satisfies:

$$\eta(x) = 0, \text{ for } x \leq -2, \ \eta(x) = 1 \text{ for } x \geq -1, \ 0 \leq \eta(x) \leq 1, \ \forall x.$$

Let Q' be a box of $\pi'(i)$, $1 \leq i \leq 2N$; by (6) we have:

$$(36) \qquad d(Q', K' - Q') > c_{23}^{-1} \beta_i, \text{ with } c_{23} = c_6' c_1'.$$

Define $\eta_{Q'}$ by (if $Q' = [a, b]$):

$$(37) \qquad \eta_{Q'}(x) = \eta(6c_{23}\beta_i^{-1}(x - a)) + \eta(6c_{23}\beta_i^{-1}(b - x)) - 1.$$

The following properties hold:

$$(38) \qquad \eta_{Q'}(x) = 1 \quad \text{if} \quad d(x, Q') \leq (6c_{23})^{-1} \beta_i;$$

$$(39) \qquad \eta_{Q'}(x) = 0 \quad \text{if} \quad d(x, Q') \geq (3c_{23})^{-1} \beta_i$$

$$(40) \qquad 0 \leq \eta_{Q'}(x) \leq 1, \quad \forall x$$

$$(41) \qquad ||D^k \eta_{Q'}||_{C^o} \leq (6c_{23}\beta_i^{-1})^k ||D^k\eta||_{C^o}, \text{ for } k \geq 0.$$

For $t \in [-1/2, +1/2]$, define a map $h_{Q',t}$ by:

$$(42) \qquad h_{Q',t}(x) = x + 16\beta_{i+1} t \, \eta_{Q'}(x).$$

We require that

$$(43) \qquad 48c_{23}||D\eta||_{C^o} \beta_{i+1} < \beta_i$$

so that, by (41), $h_{Q',t}$ is a C^∞ diffeomorphism.

Let $T = (t(Q'))_{Q' \in \pi'(i)}$ be a family with $t(Q') \in [-1/2, 1/2]$ for all Q'. Then $h_{Q_1', t(Q_1')} - \text{Id}$ and $h_{Q_2', t(Q_2')} - \text{Id}$, for distinct $Q_1', Q_2' \in \pi'(i)$, have disjoint support by (39), (36); hence the various $h_{Q',t(Q')}$ glue in a diffeomorphism h_T which satisfy, by (41):

$$(44) \qquad ||D^k(h_T - \text{id}_{\mathbf{R}})||_{C^o} \leq 8\beta_{i+1}(6c_{23}\beta_i^{-1})^k ||D^k\eta||_{C^o}.$$

We denote by $C(i)$ the cube $\left[-\frac{1}{2}, \frac{1}{2}\right]^{\pi'(i)}$, equipped the Lebesgue measure m.

If h is a diffeomorphism of the form $h = h_{T_i} \circ h_{T_{i+1}} \circ \ldots \circ h_{T_{2N-1}}$, with $T_j \in C(j)$ for $i \leq j < 2N$, we get from (40), (42) that:

$$(45) \qquad |h(x) - x| \leq 8 \sum_{j > i} \beta_j \leq 16\beta_{i+1} \quad \text{by (25)}.$$

10. INDUCTION

10.1. We begin with some easy consequences of the estimates in 8. For $0 \leq i \leq 2N - 2$, $J \in \tilde{\pi}(i)$, let:

$$B(i + 2, J) = \bigcup_{k \in A(i+1, J)} B(i + 2, J, k).$$

If $\ell \in B(i + 2, J, k)$, $M(i + 2, \ell)$ must intersect $M(i + 1, k)$, so ℓ can belong to at most 3 of the $B(i + 2, J, k)$.

Then using (30) and (35), we obtain:

$$(46) \qquad \#B(i + 2, J) \geq c_{24}^{-1} c_{25}^{-i} \, \varepsilon^2 \beta_i \beta_{i+2}^{-1},$$

with $c_{24} = 96 c_{21}^2 c_{22}^{-2}$, $c_{25} = c_{22}^2$.

Let $\ell \in B(i + 2, J)$, and $N(\ell, J)$ be the number of $J_1 \in \tilde{\pi}(i + 1, J)$ such that $\tilde{\pi}(i + 2, J_1, \ell) \neq \phi$. By (33) and (27), one has:

$$(47) \qquad N(\ell, J) \geq c_{26}^{-1} c_{22}^{-i} \, \varepsilon \left(\beta_i \beta_{i+1}^{-1}\right)^{D-1}, \quad c_{26} = 256 c_{14} c_{21} c_{22}^{-1}.$$

10.2. Let $J_1, J_2 \in \tilde{\pi}(i + 1, J)$, such that $\tilde{\pi}(i + 2, J_p, \ell) \neq \phi$ for $p = 1, 2$, and let a_1, a_2 be the smallest points of $\pi_\lambda(J_1)$, $\pi_\lambda(J_2)$. If we assume that J_1, J_2 have the same component in $\pi'(i + 1)$, one must have, like in (36), $|a_1 - a_2| \geq \frac{1}{2} \tilde{c}_{23}^{-1} \beta_{i+1}$, where $\tilde{c}_{23} = c_6'' c_9''$. On the other hand, $\pi_\lambda(J_p)$ intersects $M(i + 2, \ell)$ and has diameter less than $2\beta_{i+1}$ for $p = 1, 2, \ldots$, hence $|a_1 - a_2| \leq 2\beta_{i+1} + 4\beta_{i+2} \leq 4\beta_{i+1}$. We conclude that at most $(8\tilde{c}_{23} + 1)$ elements $\tilde{J}$ of $\tilde{\pi}(i + 1, J)$ with the *same* component in $\pi'(i + 1)$ can verify $\tilde{\pi}(i + 2, \tilde{J}, \ell) \neq \phi$.

Using (47), we see that for $\ell \in B(i + 2, J)$, we can find w_i elements $J_1 \in \tilde{\pi}(i + 1, J)$ with *distinct* components in $\pi'(i + 1)$ such that $\tilde{\pi}(i + 2, J_1, \ell) \neq \phi$, where:

$$(48) \qquad w_i = c_{27}^{-1} c_{22}^{-i} \, \varepsilon \left(\beta_i \beta_{i+1}^{-1}\right)^{D-1}, \quad c_{27} = c_{26}(8\tilde{c}_{23} + 1).$$

10.3. We now construct a diffeomorphism of the form:

$$h = h_{T_1} \circ h_{T_3} \circ \ldots \circ h_{T_{2N-1}}, \quad \text{with} \quad T_{2j+1} \in C(2j + 1).$$

Denote $h_j = h_{T_{2j+1}} \circ \ldots \circ h_{T_{2N-1}}$ so that $h_N = \text{id}$, $h_j = h_{T_{2j+1}} \circ h_{j+1}$, $h = h_o$. We will determine the T_{2j+1} inductively, starting from $T_{2N-1} = 0$, in order to have the following property:

$$P(i): \quad \text{For any } J \in \tilde{\pi}(2i), \text{ and any } \ell \in B(2i + 2, J),$$
$$\text{any point in } M(2i + 2, \ell) \text{ is at a distance less than } 4\beta_{2N}$$
$$\text{from } \pi_\lambda(h_i(J') \times J''), \text{ where } J = J' \times J''.$$

We take $T_{2N-1} = 0$, so that $h_{N-1} = \text{id}$. Each interval $M(2N, \ell)$ has diameter $4\beta_{2N}$, so $P(N-1)$ is a consequence of the definition of $B(2N, J)$.

We now assume that $T_{2i+1}, \ldots, T_{2N-3}$ have been determined in order to have $P(i)$, and we will choose T_{2i-1} in order to satisfy $P(i-1)$.

Let $J \in \tilde{\pi}(2i-2)$, $\ell \in B(2i, J)$, and $x \in M(2i, \ell)$; by 10.2 one can find $J_1, \ldots, J_{w_{2i-2}} \in \tilde{\pi}(2i-1, J)$, having disjoint projections $J'_1, \ldots, J'_{w_{2i-2}}$ on K', such that:

$$\tilde{\pi}(2i, J_k, \ell) \neq \phi, \qquad 1 \leq k \leq w_{2i-2}.$$

Let $1 \leq k \leq w_{2i-2}$, and $\widetilde{J}_k \in \tilde{\pi}(2i, J_k, \ell)$. Let

$$M_k = \bigcup_{n \in B(2i+2, \widetilde{J}_k)} \widetilde{M}(2i+2, n),$$

where $\widetilde{M}(2i+2, n)$ is the interval with same centre and half length than $M(2i+2, n)$. The $\widetilde{M}(2i+2, n)$ have disjoint interiors so, by (46):

$$(49) \qquad\qquad m(M_k) \geq 2c_{24}^{-1} c_{25}^{-2i} \varepsilon^2 \beta_{2i}.$$

As each $\widetilde{M}(2i+2, n)$ must intersect $M(2i, \ell)$, M_k is contained in the interval of center x, radius $5\beta_{2i}$. On the other hand, let

$$E(J, x, k) = \left\{ t \in \left[-\frac{1}{2}, \frac{1}{2}\right], x - 16\beta_{2i}\lambda t \notin M_k \right\}.$$

We assume $\varepsilon < \frac{1}{4}$, hence $\lambda \in \left[\frac{3}{4}, 1\right]$, and we get, from (49):

$$(50) \qquad\qquad m(E(J, x, k)) \leq 1 - (8c_{24})^{-1} c_{25}^{-2i} \varepsilon^2.$$

Consider now the subset $E(J, x)$ of $C(2i-1)$ formed by the T_{2i-1} such that, for $1 \leq k \leq w_{2i-2}$, the coordinate t_k of T_{2i-1} corresponding to J'_k belongs to $E(J, x, k)$. As these coordinates are distinct, we obtain:

$$(51) \qquad\qquad m(E(J, x)) \leq \left(1 - (8c_{24})^{-1} c_{25}^{-2i} \varepsilon^2\right)^{w_{2i-2}}.$$

We now take the union E of the sets $E(J, x)$, where J runs over $\tilde{\pi}(2i-2)$, and x is any point of the form $p\beta_{2i+2}$, $p \in \mathbf{Z}$ in $M(2i, \ell)$ for some $\ell \in B(2i, J)$, some $J \in \tilde{\pi}(2i-2)$. The number of these sets is at most $2\beta_{2i+2}^{-1} c_{14}\beta_{2i-2}^{-D}$ (Lemma 5, and (8)). We assume:

$$(52) \qquad\qquad 2\beta_{2i+2}^{-1} c_{14}\beta_{2i-2}^{-D} \left(1 - (8c_{24})^{-1} c_{25}^{-2i} \varepsilon^2\right)^{w_{2i-2}} < 1,$$

which implies $m(E) < 1$, and we choose $T_{2i-1} \in C(2i-1) - E$.

10.4. We now check that $P(i-1)$ is satisfied by this choice of T_{2i-1}. Let $J, \ell, x \ldots$ be as above. Let $x_o = p\beta_{2i+2}$, $p \in \mathbf{Z}$, be such that $x_o \in M(2i, \ell)$ and $|x - x_o| < \beta_{2i+2}$. As $T_{2i-1} \notin E(J, x_o)$, there exists $1 \le k \le w_{2i-2}$ such that $x_o - 16\beta_{2i}\lambda t_k \in M_k$ (where t_k is the coordinate of T_{2i-1} corresponding to J'_k). This means that $x_o - 16\beta_{2i}\lambda t_k \in \widetilde{M}(2i+2, n)$ for some $n \in B(2i+2, \tilde{J}_k)$; as $|x - x_o| < \beta_{2i+2}$, $x - 16\beta_{2i}\lambda t_k \in M(2i+2, n)$. By the induction hypothesis $(P(i))$, there exists $z = (z', z'') \in \tilde{J}_k$ such that:

$$|x - 16\beta_{2i}\lambda t_k - \pi_\lambda(h_i(z'), z'')| \le 4\beta_{2N},$$

which can be rewritten as:

$$|x - \pi_\lambda(h_i(z') + 16\beta_{2i}t_k, z'')| \le 4\beta_{2N}.$$

Finally, from (38), (42), (45) we have $h_i(z') + 16\beta_{2i}t_k = h_{T_{2i-1}} \circ h_i(z')$ if we assume that

$$(53) \qquad\qquad 96c_{23}\,\beta_{2i+1} < \beta_{2i}.$$

This proves $P(i-1)$.

11. CONCLUSION

11.1. We have made on the β_i's the assumptions (15), (25), (43), (53) and (52) the first four reduce to a single assumption:

$$(54) \qquad\qquad \beta_i > c_{28}\beta_{i+1} \quad \text{for} \quad 0 \le i < 2N.$$

On the other hand, as $e^x < (1-x)^{-1}$ for $0 < x < 1$, and using formula (48) for w_{2i-2}, (52) is implied by:

$$(55) \qquad \beta_{2i+2}^{-1} < (2c_{14})^{-1}\,\beta_{2i-2}^{D}\,\exp\left[c_{29}^{-1}c_{30}^{-i}\,\varepsilon^3\left(\beta_{2i-2}\beta_{2i-1}^{-1}\right)^{D-1}\right]$$

with $c_{29} = 8c_{24}c_{27}c_{22}^{-2}$, $c_{30} = c_{25}^2 c_{22}^2$, for $1 \le i \le N-1$.

11.2. We now choose the *infinite* sequence $(\beta_i)_{i \ge 0}$ as:

$$\beta_i = \varepsilon^{n_i}, \qquad n_i = a^{2^i} - a,$$

for some $a > 1$. It is then easy to check that $\beta_o = 1$, and (54), (55) are satisfied if we have $a > c_{31}$ (where c_{31} depends only on c_{28}, c_{29}, c_{30}, c_{14} and D, but *not* on ε).

We now fix a closed neighbourhood $\mathcal{U}$ of the identity in $\mathrm{Diff}^\infty(\mathbf{R})$. It is not difficult (but fastidious) to check that one can choose ε and $a > c_{31}$ such that the following hold:

If we fix $N \geq 1$ and construct from the finite sequence $\beta_o = 1, \ldots, \beta_{2N}$ a number $\lambda \in [1 - \varepsilon, 1]$ as in 7. and a diffeomorphism h as in 10., then, for the diffeomorphism $H_N = \lambda h$, we have:

(a) $H_N \in \mathcal{U}$, for all $N \geq 1$,

(b) the sequence $H_N - \mathrm{Id}$ is bounded in the C^r topology, for all $r \geq 1$.

Then we can extract from H_N a subsequence which converges in the C^∞-topology to a C^∞ diffeomorphism H in $\mathcal{U}$. By $P(0)$ for H_{N_k}, each diffeomorphism H_{N_k} in this subsequence has at least one interval I_k of length $4\beta_2$ such that any point in I_k is at distance less than $4\beta_{2N_k}$ from $H_{N_k}(K') + K''$. There exists an interval I length $2\beta_2$ which is contained in infinitely many I_k. Passing to the limit, I must be contained in $H(K') + K''$.

This proves the theorem.   □

REFERENCES

[F]   K. J. FALCONER, The geometry of fractal sets, *Cambridge Univ. Press*, 1985.
[PT1]   J. PALIS and F. TAKENS, Cycles and measure of bifurcation sets for two-dimensional diffeomorphisms, *Inventiones Math.*, Vol. **82**, 1985, pp. 397-422.
[P]   J. PALIS, Homoclinic orbits, hyperbolic dynamics and dimension of Cantor sets, *Contemporary Math.*, Vol. **58**, 1987, pp. 203-215.
[PT2]   J. PALIS and F. TAKENS, Hyperbolicity and sensitive-chaotic dynamics at homoclinic bifurcations, *Cambridge Univ. Press*, 1993, 2nd edition 1994.
[PY]   J. PALIS and J-C.YOCCOZ, Homoclinic tangencies for hyperbolic sets of large Hausdorff dimension, *Acta Math.*, Vol. **172**, 1994, pp. 91-136.