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Homoclinics: Poincaré-Melnikov type results via a variational approach

by

Antonio AMBROSETTI and Marino BADIALE

Scuola Normale Superiore, Pisa, 56100 Italy.

ABSTRACT. – We introduce a variational approach to obtain some Poincaré-Melnikov type results on the existence and multiplicity of homoclinics.

1. INTRODUCTION

This paper deals with the existence of homoclinics, namely doubly-asymptotic solutions, for a broad class of perturbed differential equations, variational in nature.

The existence of homoclinics has been faced both from the local and from the global point of view. The existence for perturbed time periodic systems with one degree of freedom was first proved by Poincaré [15], see also [10]. The results by Poincaré have been the starting point for a great deal of work. In particular, Melnikov [14] has proved by analytical methods the existence of homoclinics for non conservative perturbations, leading to chaos, see for example [11]. A common feature of these results is the use of an integral function, the Poincaré function or – roughly – its derivative, the Melnikov function. The non degenerate zeros of the latter give rise to homoclinics.

On the other side, more recently Critical Point Theory has been used to prove the existence of homoclinics for a class of Hamiltonian systems like

$$\ddot{u} - u + \nabla U(t, u) = 0,$$

when the potential $U \simeq |u|^{p+1}$ with $p > 1$ and depends periodically (or almost periodically) on t, see [7, 8, 9, 16, 17] and references therein.

Although these approaches are apparently considered different in nature, we will show that they are connected, in the sense that an appropriate use of Critical Point Theory permits also to find the classical perturbation results. More precisely, we discuss an approach, variational in nature, which furnishes a general frame to deal with several different kinds of perturbed differential equations. We not only find Poincaré-Melnikov like results (both for systems with several degrees of freedom and for autonomous systems) without any non degeneracy assumption, but also handle Partial Differential Equations.

Moreover, specializing the recent variational works cited before to our setting, they cannot provide results in the same generality: we localize the solutions and find multiplicity results; in addition, we can also handle potentials with a more general dependence on t, not only periodic or almost periodic.

In order to have an idea of our setting, let us consider the second order Hamiltonian system with N degrees of freedom

$$\ddot{u} - u + \nabla V(u) = \epsilon \nabla_u W(t, u), \tag{1}$$

where $V(0) = 0$, $\nabla V(0) = 0$, $D^2V(0) = 0$, and roughly $W(t, 0) = 0$, $\nabla_u W(t, 0) = 0$. Homoclinics of (1) correspond to stationary points $u \in W^{1,2}(\mathbb{R})$ of the Lagrangian functional

$$\int_{\mathbb{R}} [L_0 + \epsilon W], \quad L_0(x, p) = \frac{1}{2}\left[|p|^2 + |x|^2\right] + V(x),$$

whose Euler equation is (1). Suppose that the unperturbed equation

$$\ddot{u} - u + \nabla V(u) = 0$$

has a non trivial homoclinic $u_0(t)$. Connected with u_0, the unperturbed functional $\int L_0$ possesses a manifold of critical points $Z = \{u_0(t + \theta) : \theta \in \mathbb{R}\}$ and we are lead to search homoclinics near one of these translates $u_0(\cdot + \theta)$ by looking for critical points of $\int L_0 + \epsilon \int W$ nearby Z. It turns out that these critical points exist provided that

$$\Gamma(\theta) = \int_{\mathbb{R}} W(t, u_0(t + \theta))dt,$$

has a (possibly degenerate) critical point. Such a Γ is the Poincaré function and its derivative is the Melnikov function.

More in general, the abstract set up, discussed in Section 2, deals with the existence of stationary points for a class of functionals like

$$f_\epsilon(u) = f_0(u) + \epsilon G(u),$$

and is related to some previous work [4, 2], see also [1]. It is assumed that the unperturbed functional f_0 possesses a manifold of critical points Z such that $T_z Z = Ker[f_0''(z)]$ for all $z \in Z$. For example, in the case of (1), it suffices to require that the solutions of the linearized equation

$$\ddot{\phi} - \phi + D^2 V(z)\phi = 0$$

form a one dimensional space. Then it is shown that, roughly, the (possibly degenerate) critical points of G on Z give rise to stationary points of f_ϵ. The main ingredient is a kind of finite dimensional reduction which permits to search the critical points by studying f_ϵ constrained on a manifold Z_ϵ, locally diffeomorphic to Z. In the specific applications it turns out that this $G_{|Z}$ is nothing but a Poincaré-Melnikov type function.

These abstract results are applied in Section 3 to (1) as well as to perturbed radial systems like

$$\ddot{u} - u + |u|^{p-1}u = \epsilon \nabla_u W(t, u), \quad (p > 1) \tag{2}$$

In this latter case $Z = \{\xi r(t + \theta)\} \simeq S^{N-1} \times \mathbb{R}$ where $\xi \in S^{N-1}$ and $r > 0$ satisfies

$$\ddot{r} - r + r^p = 0, \quad \lim_{t \to \pm\infty} r(t) = 0.$$

One shows that the condition $T_z Z = Ker[f_0''(z)]$ is still satisfied and hence the abstract approach yields the existence of homoclinics in connection with the critical points of

$$\Gamma(\xi, \theta) = \int_{\mathbb{R}} W(t, \xi r(t + \theta))dt, \quad (\xi, \theta) \in S^{N-1} \times \mathbb{R}.$$

As for the perturbation W, we can also consider the case that $W(t, u) = g(t) \cdot u$, when (1) becomes a forced system. In particular, the classical case of systems with a periodic or quasi periodic forcing term can be handled, see section 4.

Our functional approach also applies when W is independent of time and (2) becomes an autonomous system, see Section 5. In such a case we can show the existence of two distinct homoclinics, a multiplicity result which improves the one of [18].

The generality of our abstract setting allows us to handle Partial Differential Equations, too. Applications to the existence of semiclassical states of a class of Schrödinger equations with potential have been discussed in [2]. Here, see Section 6, we prove the existence of two solutions of forced Schrödinger equations like

$$\begin{cases} -\Delta u + u = |u|^{p-1}u - \epsilon g(x), & x \in \mathbb{R}^N, \\ u(x) \to 0 \quad \text{as } |x| \to \infty. \end{cases}$$

provided $1 < p < (N+2)/(N-2)$ and $g \in L^2$, improving a recent result of [12].

2. ABSTRACT SETTING

Let E be a real Hilbert space with inner product $(\cdot|\cdot)$ and norm $\|\cdot\|$. We consider a family of functionals f_ϵ satisfying

(h_0) $f_\epsilon \in C^2(E, \mathbb{R})$ and has the form

$$f_\epsilon(u) = \frac{1}{2}\|u\|^2 - F(u) + \epsilon G(u).$$

Motivated by the applications, we will further assume that the unperturbed functional $f_0(u) = \frac{1}{2}\|u\|^2 - F(u)$ inherits some simmetries in such a way that it possesses at a certain level b a *non degenerate manifold* Z of critical points. Precisely we will assume:

(h_1) $f_0 \in C^2(E, \mathbb{R})$ has a d-dimensional C^2 manifold Z of critical points;

(h_2) For all $z \in Z$ the linear operator $F''(z)$ is compact;

(h_3) For all $z \in Z$ one has: $T_z Z = Ker f_0''(z)$.

Above, $T_z Z$ denotes the tangent space to Z at z.

Remark 1. – Since, obviously, $T_z Z \subseteq Ker[f_0''(z)]$, then (h_3) is a non degeneracy condition that amounts to require that, for any $\phi \in E$, $\phi - F''(z)\phi = 0$ implies $\phi \in T_z Z$. ■

Our first goal is to show that, locally near any $z \in Z$, there exists a manifold Z_ϵ diffeomorphic to Z which is a natural constraint for f_ϵ. By this we mean that $u \in Z_\epsilon$ and $f'_{\epsilon|Z_\epsilon}$ implies $f'_\epsilon(u) = 0$. In this way the search of critical points of f_ϵ on E (near Z) is reduced to the search of critical points of $f'_{\epsilon|Z_\epsilon}$. Such a procedure, carried out in Lemmas 2 and

4 below, is essentially known, see for example [4], and is reported here for the reader convenience.

We will henceforth assume that $Z = \zeta(\mathbb{R}^d)$ whith $\zeta \in C^2(\mathbb{R}^d, E)$. Let $B_R = \{\theta \in \mathbb{R}^d : \|\theta\| < R\}$ and $Z^R = \zeta(B_R)$. However all the results hold true in the general case giving to Z^R the meaning of a relatively compact subset of Z.

LEMMA 2. – *Given $R > 0$, there exists $\epsilon_0 > 0$ and a smooth function*

$$w = w(z, \epsilon) : M = Z^R \times (-\epsilon_0, \epsilon_0) \to E \tag{3}$$

such that

(i) $w(z, 0) = 0 \quad \forall z \in Z^R$;

(ii) $f'_\epsilon(z + w(z, \epsilon)) \in T_z Z \quad \forall (z, \epsilon) \in M$;

(iii) $w(z, \epsilon)$ is orthogonal to $T_z Z \quad \forall (z, \epsilon) \in M$.

Proof. – Let $q_i = q_i(z)$, $1 \le i \le d$ denote an orthogonal basis for $T_z Z$. We will find $w(z, \epsilon)$ by means of the Local Inversion Theorem applied to the map

$$H : M \times E \times \mathbb{R}^d \to E \times \mathbb{R}^d$$

with components $H_1 \in E$ and $H_2 \in \mathbb{R}^d$ given by

$$H_1(z, \epsilon, w, \alpha) = z + w - F'(z + w) + \epsilon G'(z + w) - \sum_i \alpha_i q_i,$$

$$H_2(z, \epsilon, w, \alpha) = (\,(w|q_1), \dots, (w|q_d)\,)$$

Let us remark that $H_1 = 0$ means that $f'_\epsilon(z + w) \in T_z Z$, namely that (ii) holds, while $H_2 = 0$ means that w is orthogonal to $T_z Z$, namely that (iii) holds.

Plainly, there results $H_1(z, 0, 0, 0) = z - F'(z) = 0$ and $H_2(z, 0, 0, 0) = 0$. Furthermore, fixed $z^* \in Z$, we consider the derivatives of H evaluated on $(z^*, 0, 0, 0)$:

$$L(z^*) = \frac{\partial H}{\partial(\alpha, w)} = \left(\frac{\partial H_1}{\partial(\alpha, w)}, \frac{\partial H_2}{\partial(\alpha, w)} \right).$$

One easily finds, for $(v, \beta) \in E \times \mathbb{R}^d$,

$$\frac{\partial H_1}{\partial(\alpha, w)}[v, \beta] = v - F''(z^*)v - \sum_i \beta_i q_i,$$

$$\frac{\partial H_2}{\partial(\alpha, w)}[v, \beta] = (\,(v|q_1), \dots, (v|q_d)\,).$$

In order to prove that $L(z^*)$ is invertible we notice that (h_2) implies that L is of the form *"Identity - Compact"*, so it is enough to prove that it is injective. Then let us assume that $L(z^*)[\beta, v] = (0,0)$. From

$$v - F''(z^*)v = \sum_i \beta_i q_i, \tag{4}$$

taking the inner product with q_i, we infer

$$-(F''(z^*)v|q_i) = \beta_i \|q_i\|^2.$$

Using (h_3) one has that $q_i \in Ker f_0''(z^*)$, namely that $F''(z^*)q_i = q_i$, and hence

$$(F''(z^*)v|q_i) = (F''(z^*)q_i|v) = (v|q_i) = 0.$$

Then it follows that $\beta_i = 0$ and (4) becomes

$$v - F''(z^*)v = 0.$$

Using again (h_3), we deduce that $v \in T_{z^*}Z$. On the other side, $\frac{\partial H_1}{\partial(v,\beta)} = 0$ implies that v is orthogonal to $T_{z^*}Z$ and thus $v = 0$. This shows that $L(z^*)$ is invertible and an application of the Implicit Function Theorem yields the existence of smooth, unique functions w and α, defined in a neighbourhood U of z^* (relative to Z) and for ϵ small, satisfying

$$H_i(z, \epsilon, w(z,\epsilon), \alpha(z,\epsilon)) = 0, \quad 1 \le i \le d, \ \forall z \in U.$$

Since Z is finite dimensional one can extend by compactness w on all M and the proof is completed. ■

Remark 3. – The function w is smooth and $w(z,0) = 0 \ \forall z \in Z^R$. In particular it follows that $w(z,\epsilon) = O(\epsilon)$ as $\epsilon \to 0$, uniformly in $z \in Z^R$. ■

Let

$$Z_\epsilon = \{z + w(z,\epsilon) \,:\, (z,\epsilon) \in M\}. \tag{5}$$

LEMMA 4. – *Z_ϵ is a natural constraint for f'_ϵ, namely: if $u \in Z_\epsilon$ and $f'_{\epsilon|Z_\epsilon}(u) = 0$ then $f'_\epsilon(u) = 0$.*

Proof. – Suppose that $f'_{\epsilon|Z_\epsilon}(u) = 0$ for some $u = z + w(z,\epsilon) \in Z_\epsilon$. Then $f'_\epsilon(u)$ is orthogonal to $T_u Z_\epsilon$. On the other hand $f_\epsilon(u) \in T_z Z$ (by the definition of $w(z,\epsilon)$) and $T_u Z_\epsilon$ is near $T_z Z$ provided ϵ is small enough.Thus $f'_\epsilon(u) = 0$. ■

Remark 5. – Let u_ϵ be possible critical points of f_ϵ on Z_ϵ. One can show that if $u_\epsilon \to z^* \in Z$ as $\epsilon \to 0$ then z^* is a critical points of G on Z. See, for example, Section 4 of [4] or Section 7 of [2] for more details. This localization of the critical points of f_ϵ can be useful to distinguish these critical points from other ones and can permit to obtain multiplicity results, see for example Remark 17-(iv) and Theorem 22 below. ■

The following theorems provide the existence of critical points of f_ϵ. Although the first two can be actually deduced from the latter, we prefer to state and prove them separately to make our arguments more transparent. We begin with the case in which G has a local minimum or maximum on Z.

THEOREM 6. – *Let* $(h_0 - h_1 - h_2 - h_3)$ *hold and assume*
(h_4) *there exists an open bounded set* $A \subseteq Z$ *and* $z_0 \in A$ *such that*

$$G(z_0) < \inf_{\partial A} G. \quad (\textit{resp. } G(z_0) > \sup_{\partial A} G)$$

Then f_ϵ *has at least a critical point* $u_\epsilon \in Z_\epsilon$ *provided* $\epsilon > 0$ *is small enough.*

Proof. – Let $R > 0$ be such that $\overline{A} \subseteq Z^R$. By Lemma 2 we know that for $\epsilon > 0$ small there exists a diffeomorphism $\Phi_\epsilon : Z^R \to Z_\epsilon$, such that $\Phi_\epsilon(z) = z + w(z, \epsilon)$. Moreover there results

$$\Phi_\epsilon(\partial A) = \partial(\Phi_\epsilon(A)), \tag{6}$$

where the boundary on the left-hand side is relative to Z_ϵ. According to Remark 3, for any $u = \Phi_\epsilon(z)$ we have $u = z + \epsilon w_0 + o(\epsilon)$, where $w_0 = w_0(z)$ is bounded provided $z \in Z^R$. Using this fact, we infer that

$$||z + w||^2 = ||z||^2 + 2\epsilon(z|w_0) + o(\epsilon); \tag{7}$$

$$F(z + w) = F(z) + \epsilon(F'(z)|w_0) + o(\epsilon); \tag{8}$$

$$G(z + w) = G(z) + \epsilon(G'(z)|w_0) + o(\epsilon). \tag{9}$$

Putting together (7), (8) and (9) one finds

$$\begin{aligned} f_\epsilon(u) &= \frac{1}{2}||z + w||^2 - F(z + w) + \epsilon G(z + w) \\ &= \frac{1}{2}||z||^2 + \epsilon(z|w_0) - F(z) - \epsilon(F'(z)|w_0) + \epsilon G(z) + o(\epsilon) \\ &= f_0(z) + \epsilon[(z|w_0) - (F'(z)|w_0) + G(z)] + o(\epsilon). \end{aligned}$$

As $f_0'(z) = 0$ for all $z \in Z$, we get

$$(z|w_0) = (F'(z)|w_0)$$

and hence

$$f_\epsilon(z + w(z, \epsilon)) = b + \epsilon G(z) + o(\epsilon), \quad \text{where } b = f_0(z). \tag{10}$$

As $G(z_0) < \inf_{\partial A} G$, it is easy to get, for ϵ small,

$$f_\epsilon(\Phi_\epsilon(z_0)) < \inf_{z \in \partial A} f_\epsilon(\Phi_\epsilon(z)). \tag{11}$$

Recalling that Z_ϵ is finite dimensional, (11) implies that f_ϵ has a point of minimum in the open (relative to Z_ϵ) set $\Phi_\epsilon(A)$. By Lemma 4, this minimum (constrained on Z_ϵ) is a critical point of f_ϵ. ■

The preceding Theorem, together with Remark 5, immediately yields:

THEOREM 7. – *Let* $(h_0 - h_1 - h_2 - h_3)$ *hold and assume*

(h_5) *G has a proper local minimum (or maximum) at some* $z_0 \in Z$.

Then f_ϵ *has at least a critical point* u_ϵ *such that* $u_\epsilon \to z_0$ *as* $\epsilon \to 0$.

Our next result deal with a more general critical point of G. Given $c \in \mathbb{R}$ we set

$$A^c = \{z \in Z : G(z) < c\}.$$

We say that $z_0 \in Z$ is an *essential* critical point of $G_{|Z}$ if, letting $c_0 = G(z_0)$, the sublevel $A^{c_0+\delta}$ cannot be deformed into $A^{c_0-\delta}$, for any $\delta > 0$ small. For example, Morse non degenerate critical points are essential. A critical level c_0 is called an *essential critical level* if it carries an essential critical point.

THEOREM 8. – *Let* $(h_0 - h_1 - h_2 - h_3)$ *hold and suppose:*

(h_6) G is coercive on Z and has k essential critical levels.

Then, f_ϵ *possesses at least* k *critical points* $u_\epsilon \in Z_\epsilon$.

Proof. – Let $c_0 := G(z_0)$ be an essential critical level and take $c > c_0$. Since Γ is coercive, there exists $R > 0$ such that $A^c \subset Z^R$. Then, as in the proof of Theorem 6, we find (see Equation (10))

$$f_\epsilon(u) = b + \epsilon G(z) + o(\epsilon),$$

for all $u \in A^c_\epsilon = \Phi_\epsilon(A^c)$ (hereafter we use the notation introduced in Theorem 6). From the above equation we infer

$$f_\epsilon(u) = b + \epsilon c + o(\epsilon) \quad \forall\, u \in \partial A^c_\epsilon$$

and hence, for ϵ small,

$$\sigma_\epsilon := \inf_{\partial A^c_\epsilon} > b + \epsilon c_0. \tag{12}$$

Define

$$\{f_\epsilon \leq a\} = \{u \in A_\epsilon^c : f_\epsilon(u) \leq a\}.$$

According to (12), let $\delta > 0$ be such that

$$\{f_\epsilon \leq b + \epsilon(c_0 + 2\delta)\} \subset \{f_\epsilon \leq \sigma_\epsilon\}.$$

Taking ϵ possibly smaller, one also finds

$$A_\epsilon^{c_0+\delta} \subset \{f_\epsilon \leq b + \epsilon(c_0 + 2\delta)\},$$

Indeed, if $u \in A_\epsilon^{c_0+\delta}$ then $u = \Phi_\epsilon(z)$ with $G(z) \leq c_0+\delta$. Therefore one has

$$f_\epsilon(u) = b + \epsilon G(z) + o(\epsilon) \leq b + \epsilon(c_0 + \delta) + o(\epsilon) < b + \epsilon(c_0 + 2\delta)$$

for ϵ small enough. Similarly there results

$$\{f_\epsilon \leq b + \epsilon(c_0 - 2\delta)\} \subset A_\epsilon^{c_0-\delta}.$$

After these preliminaries, we can use standard arguments (note that they apply in a straight way because all the deformations take place in the sublevel $\{f_\epsilon \leq \sigma_\epsilon\}$, and this is compact) to conclude. Indeed, if f_ϵ has no critical points in the strip $\{c_0-2\delta \leq f_\epsilon \leq c_0+2\delta\}$, then $\{f_\epsilon \leq b+\epsilon(c_0+2\delta)\}$ could be deformed into $\{f_\epsilon \leq b + \epsilon(c_0 - 2\delta)\}$. According to the preceding inclusions, this would induce a deformation of $A_\epsilon^{c_0+\delta}$ into $A_\epsilon^{c_0-\delta}$ and finally a deformation of $A^{c_0-\delta}$ into $A^{c_0-\delta}$, a contradiction because z_0 is essential. Thus f_ε has a critical level near each essential critical level of G, proving the Theorem. ■

We end this section with another multiplicty result.

THEOREM 9. – *Let* $(h_0 - h_1 - h_2 - h_3)$ *hold. Furthermore, suppose there exist* $c \in \mathbb{R}$ *and* $\Sigma \subset Z$, $\Sigma \neq \emptyset$, *such that:*

(h_7) A^c *is bounded;*

(h_8) $\Sigma \subsetneq A^c$ *and* $\sigma := \sup_\Sigma f_0 < c$.

Then f_ϵ *has at least* $cat(\Sigma, A^c)$ *critical points in* Z_ϵ *(cat denotes the usual Lusternik-Schnirelman category, see for example [1], section 2).*

Proof. – As above, we can consider the map $\Phi_\epsilon : Z^R \to Z_\epsilon$. We set $\Sigma_\epsilon = \Phi_\epsilon(\Sigma)$ and $A_\epsilon^c = \Phi_\epsilon(A^c)$. We also have, for $u = \Phi_\epsilon(z)$,

$$f_\epsilon(u) = b + \epsilon G(z) + o(\epsilon).$$

Then the hypothesis $\sigma < c$ readily implies, for ϵ small,

$$\sup_{\Sigma_\epsilon} f_\varepsilon < \inf_{\partial A_\varepsilon^c} f_\varepsilon.$$

Take $\beta \in \mathbb{R}$ such that $\sup_{\Sigma_\epsilon} f_\varepsilon < \beta < \inf_{\partial A_\varepsilon^c} f_\varepsilon$. Then we have

$$\Sigma_\varepsilon \subseteq f_\varepsilon^\beta \subseteq A_\varepsilon^c,$$

and hence

$$cat\big(f_\varepsilon^\beta, f_\varepsilon^\beta\big) \geq cat(\Sigma_\varepsilon, f_\varepsilon^\beta) \geq cat(\Sigma_\varepsilon, A_\varepsilon^c) = cat(\Sigma, A^c).$$

Now the result follows by standard critical point theory for differentiable functions on manifolds. ■

Remark 10. – The same result holds if $A^c = \{z \in Z \ : \ G(z) > c\}$ and $\sigma > c$. ■

Remark 11. – Dealing with a more general perturbation term $G(\epsilon, u)$, the above arguments may require some modification. This is the case, for example, in [2] where we refer for more details. ■

3. APPLICATIONS TO TIME DEPENDENT SYSTEMS

The preceding abstract setting will be used to study the existence of homoclinics of perturbed systems of differential equations like

$$\ddot{u} - u + \nabla V(u) = \epsilon \nabla_u W(t, u). \tag{13}$$

Here we will take $u \in \mathbb{R}^N$ with $N \geq 1$ and we assume

(V_1) $V \in C^2(\mathbb{R}^N, \mathbb{R})$, $V(0) = 0$, $\nabla V(0) = 0$, $D^2V(0) = 0$.

(W_1) $W \in C^2(\mathbb{R} \times \mathbb{R}^N)$ and $\exists W^* \in C^2(\mathbb{R}^N)$, with $W^*(0) = 0$, $\nabla W^*(0) = 0$, such that $|W(t, u)| \leq |W^*(u)|$, $|\nabla_u W(t, u)| \leq |\nabla W^*(u)|$ and $|W_{uu}(t, u) - W_{uu}(t, v)| \leq |D^2W^*(u) - D^2W^*(v)|$.

Let E denote the Sobolev space $W^{1,2}(\mathbb{R}, \mathbb{R}^N)$ endowed with scalar product $(u|v) = \int_{\mathbb{R}}(\dot{u} \cdot \dot{v} + u \cdot v)dt$ and norm $||u||^2 = (u|u)$. We define the linear continuous operator $K : L^2(\mathbb{R}, \mathbb{R}^N) \to E$ by setting

$$Kv = u \quad \Longleftrightarrow \quad -\ddot{u} + u = v.$$

For $u \in E$ we set

$$F(u) = \int_{\mathbb{R}} V(u)\, dt, \quad G(u) = \int_{\mathbb{R}} W(t, u)\, dt.$$

Homoclinic solutions of (13) are the critical points $u \in E$ of the functional

$$f_\epsilon(u) = \frac{1}{2}\|u\|^2 - F(u) + \epsilon G(u).$$

To use the abstract results, some assumptions on the unperturbed equation

$$\ddot{u} - u + \nabla V(u) = 0 \tag{14}$$

are in order. Precisely we will either assume (V_1) and

(V_2) $\exists\, u_0, \phi_0 \in E$ such that u_0 solves (14) and $Ker f''(u_o) = span\{\phi_0\}$;

or

(V_3) $V(u) = \frac{1}{p+1}|u|^{p+1}$, with $p > 1$.

In the former case equation (14) has a homoclinic $u_0 \in E$ such that the solutions $\phi \in E$ of the linearized equation

$$\ddot{\phi} - \phi + D^2 V(u_0(t))\phi = 0$$

form a one dimensional space. Letting, for any $\theta \in \mathbb{R}$, $u_\theta(t) = u(t + \theta)$ one has that $Z = \{u_\theta \,:\, \theta \in \mathbb{R}\}$ and

$$G(z) = \Gamma(\theta) = \int W(t, u_0(t + \theta))dt.$$

If (V_3) holds, let $r = r(t)$ be the unique solution of

$$\begin{cases} -\ddot{r} + r = r^p, \ r > 0 \\ \dot{r}(0) = 0, \\ \lim_{|t| \to \infty} r = 0. \end{cases} \tag{15}$$

and let r_θ denote the translated function $r_\theta(t) = r(t + \theta)$, $\theta \in \mathbb{R}$. Then, for $\xi \in S^{N-1}$, $z = \xi\, r_\theta$ are homoclinic solutions of (14) giving rise to a critical manifold Z of f_0. For future reference we notice that $Z \simeq S^{N-1} \times \mathbb{R}$ and that $G_{|Z}$ has the form

$$\Gamma(\xi, \theta) = \int_{\mathbb{R}} W(t, \xi r(t + \theta))\, dt.$$

In the sequel we will denote by ω either $\theta \in \mathbb{R}$ or $(\xi, \theta) \in S^{N-1} \times \mathbb{R}$ according that V satisfies $(V_1 - V_2)$ or (V_3). We also use y_0, respectively

y_θ, to indicate either u_0 or ξr_0, resp. u_θ or ξr_θ, and so we have $\Gamma(\omega) = \int_{\mathbb{R}} W(t, y_0(t+\theta))\, dt$.

LEMMA 12. – *Assumptions* $(h_0 - h_1 - h_2 - h_3)$ *hold.*

Proof. – (h_0) and (h_1) are obvious. As for (h_2), we have to show that $F''(z)v_n \to 0$ strongly in E, whenever v_n converges weakly to 0. There results

$$F''(y_\theta)v_n = K\big[D^2V(y_\theta)v_n\big].$$

Since $y_\theta \in E$, by (V_1) it readily follows that $D^2V(y_\theta)v_n \to 0$ in $L^2(\mathbb{R}, \mathbb{R}^N)$ and by the continuity of K we deduce $F''(y_\theta)v_n \to 0$.

As to (h_3), we have to treat separately the cases that $(V_1 - V_2)$ or (V_3) hold. In the former, it immediately follows from $T_zZ \subseteq Ker f_0''$ (see Remark 1) and (V_2).

In the latter, let us first notice that

$$T_zZ = \{\eta r_\theta + \alpha \dot{r}_\theta \,:\, \alpha \in \mathbb{R},\ \xi \in \mathbb{R}^N,\ \eta \cdot \xi = 0\}$$

as well as that $\phi \in Ker[f_0''(z)]$ if and only if

$$-\ddot{\phi} + \phi = (p-1)r_\theta^{p-1}(\xi \cdot \phi)\xi + \dot{r}_\theta^{p-1}\phi \tag{16}$$

Furthermore, according to Remark 1, we have to show that for any ϕ satisfying (16) there results

$$\phi = \eta r_\theta + \alpha \xi \dot{r}_\theta$$

for some $\alpha \in \mathbb{R}$, $\eta \in \mathbb{R}^N$ with $\eta \cdot \xi = 0$. To prove this claim, let us set $\psi(t) = \xi \cdot \phi(t)$. The real valued function ψ satisfies the equation

$$-\ddot{\psi} + \psi = (p-1)r_\theta^{p-1}\psi + r_\theta^{p-1}\psi = pr_\theta^{p-1}\psi. \tag{17}$$

Let us remark that $\dot{r}_\theta$ is a solution of (17). We can then apply Theorem 3.3 of [5], which implies that all non diverging solution of (17) are given by $\alpha\dot{r}_\theta$, $\alpha \in \mathbb{R}$. In other words, $\exists \alpha \in \mathbb{R}$ such that $\xi \cdot \phi(t) = \alpha \dot{r}_\theta(t)$.

We now define

$$\sigma = \phi - (\xi \cdot \phi)\xi = \phi - \alpha\dot{r}_\theta\xi$$

and notice that $\xi \cdot \sigma(t) = 0$ for all t. Substituting $\phi = \sigma + \alpha\dot{r}_\theta\xi$ in (16) we find

$$\begin{aligned} &-\ddot{\sigma} + \alpha \dddot{r}_\theta\xi + \sigma + \alpha\dot{r}_\theta\xi = \\ &= (p-1)r_\theta^{p-1}(\xi \cdot \sigma + \alpha\dot{r}_\theta) + \xi + r_\theta^{p-1}\sigma + \alpha\dot{r}_\theta r_\theta^{p-1}\xi = \\ &= r_\theta^{p-1}\sigma + p\alpha\dot{r}_\theta r_\theta^{p-1}\xi. \end{aligned} \tag{18}$$

There results $\dddot{r}_\theta + \dot{r}_\theta = p\dot{r}_\theta r_\theta^{p-1}$ and hence from (18) we deduce

$$-\ddot{\sigma} + \sigma = r_\theta^{p-1}\sigma \tag{19}$$

System (19) is equivalent to the N de-coupled equations

$$-\ddot{\sigma}_i + \sigma_i = r_\theta^{p-1}\sigma_i \tag{20}$$

Obviously r_θ is a solution of (20) and another application of Theorem 3.3 of [5] yields that $\exists \eta_i \in \mathbb{R}$ such that $\sigma_i = \eta_i r_\theta$, and there results

$$\sigma = \eta r_\theta, \quad \eta = (\eta_1, ..., \eta_N).$$

Then one has $\xi \cdot \eta = 0$ and

$$\phi = \eta r_\theta + \alpha\xi\dot{r}_\theta,$$

and the proof is completed. ■

We are now in position to state the results concerning the existence of homoclinics. The first one is an immediate application of Theorems 7, 8 and Lemma 12.

THEOREM 13. – *Let $(V_1 - V_2)$, respectively (V_3), and (W_1) hold.*

(i) *if Γ has a proper local minimum (or maximum) at some θ_0, resp. (ξ_0, θ_0), then for $|\epsilon|$ small, (13) has a homoclinic solution near $u_0(t + \theta_0)$, resp. $\xi_0 r(t + \theta_0)$;*

(ii) *if Γ is coercive and has k essential critical levels then, for $|\epsilon|$ small, (13) has at least k homoclinic solutions.*

The above Theorem is a general result that applies in a large variety of specific cases. In the sequel we list some of them.

THÉOREM 14. – *Let $(V_1 - V_2)$, respectively (V_3), and (W_1) hold and assume that $W(t + T, u) = W(t, u)$. Then, for $|\epsilon|$ small enough, (13) has a pair of homoclinic solutions.*

Proof. – Instead applying the general theorems 6 and 9 we can use Lemmas 2 and 4, directly. By local uniqueness of the function w (see Lemma 2), one gets easily, for all $t, \theta \in \mathbb{R}$,

$$w(y_{\theta+T}, \epsilon)(t) = w(y_\theta, \epsilon)(t + T) \tag{21}$$

Recalling that $Z_\epsilon = \{y_\theta + w(y_\theta, \epsilon)\}$, we obtain that f_ϵ is periodic as a function of θ namely

$$f_\epsilon(y_{\theta+T} + w(y_{\theta+T}, \epsilon)) = f_\epsilon(y_\theta + w(y_\theta, \epsilon)) \tag{22}$$

By compactness, (22) implies that, when θ varies in a closed bounded interval larger than T, f_ϵ will assume maximum and minimum values. The points of maximum and minimum give rise to two critical points of f_ϵ (by Lemma 4), and then to two homoclinic solutions of (13). ■

Let us now consider, for the sake of simplicity,
a systems of the form

$$\ddot{u} - u + \nabla V(u) = \epsilon\, g(t) \nabla W(u), \tag{23}$$

with $g(t)$ not constant. The case of a constant g will be discussed separately in Section 5 below.

THEOREM 15. – *Let (W_1) hold and suppose that $g \in L^\infty$ satisfies*

(g_1) $\lim_{|t|\to\infty} g(t) = \gamma$;

(g_2) Γ *is not identically constant.*

Then for $|\epsilon| > 0$ small enough, (23) has a homoclinic prvided that either $(V_1 - V_2)$ are satisfied or (V_3) holds and $\gamma = 0$.

Proof. – Suppose $(V_1 - V_2)$ hold. In order to apply Theorem 6, we first remark that from (g_1) it follows

$$\lim_{|\theta|\to\infty} \Gamma(\theta) = \lim_{|\theta|\to\infty} \int_{\mathbb{R}} g(t-\theta) W(u_0)\, dt = \gamma \int_{\mathbb{R}} W(u_0) dt \equiv \gamma'. \tag{24}$$

As Γ is not constant on Z, it has a global minimum or maximum value $\neq \gamma'$. Assume the former (in the other case the proof is the same, by using Remark 10) and let θ_0 be such a minimum. Then there exists $\theta^* > 0$ such that

$$\Gamma(\theta_0) < \Gamma(\pm\theta^*).$$

Letting $A = \{u_\theta : |\theta| < \theta^*\}$, we can apply Theorem 6, jointly with Lemma 12, yielding the existence of a homoclinic solution to (23).

If (V_3) holds and $\gamma = 0$, the preceding arguments imply Γ achieves an extremum at some (ξ_0, θ_0). ■

In this last result the presence of symmetries allows us to get not only existence but also multiplicity results.

THEOREM 16. – *Let $N > 1$ and (V_3) hold. Assume (W_1), $g \in L^\infty$ and (g_1) with $\gamma = 0$. Furthermore, let us suppose that both g and W have constant sign.*
Then, for $|\epsilon|$ small enough, (23) has a pair of homoclinic solutions.

Proof. – Arguing as above we find that $\Gamma(\xi,\theta) \to 0$ as $|\theta| \to \infty$. Moreover Γ has constant sign, say $\Gamma(\xi,\theta) < 0$ for all (ξ,θ). Then by compactness it follows

$$\sup_{|\xi|=1} \Gamma(\xi, 0) < 0. \tag{25}$$

Let $c \in \mathbb{R}$ be such that $\sup_{|\xi|=1} \Gamma(\xi,0) < c < 0$ and define

$$A^c = \{\xi r_\theta \in Z : \Gamma(\xi,\theta) < c\}, \quad \Sigma = \{\xi r : |\xi| = 1\}.$$

It is then easy to see that $\Gamma \to 0$ and (25) imply that the hypotheses of Theorem 4 hold. So we obtain at least $cat(\Sigma, A^c) = 2$ homoclinic solutions of (23). ■

Remarks 17. – (i) A specific one dimensional problem like (13) has been studied in [6] by variational methods, obtaining results related to Theorem 13-(i).

(ii) The case of a periodic time dependence of W is extensively studied in a global setting, see for example [8, 9]. Roughly, the potential is usually assumed to behave like $|u|^{p+1}$. In particular, the case that $(V_1 - V_2)$ hold is not covered in those papers.

(iii) A result like Theorem 14 in the case $(V_1 - V_2)$ (but not (V_3)) is also proved by analytical methods in [13].

(iv) As $\epsilon \to 0$ the solutions we have found converge to non trivial homoclinics of the unperturbed system (14), according to Remark 5. In particular, they are not small in the L^∞ norm. As a consequence, an additional homoclinic exists, to be found by using the Local Inversion Theorem near to $u = 0$.

(v) In all the above results we can take a perturbed potential like $W(t, u, \dot{u})$. See also Remark 21. ■

4. FORCED SYSTEMS

In this section we deal with the forced system

$$\ddot{u} - u + \nabla V(u) = \epsilon g(t). \tag{26}$$

We will always assume that either $(V_1 - V_2)$ or (V_3) hold, so that assumptions $(h_0 - h_3)$ are satisfied. It is convenient to distinguish between the cases that $g \in L^2$ or that $g \in L^\infty$. In the former we can proceed in a straight way defining $G \in C^\infty(E, \mathbb{R})$ by

$$G(u) = \int_{\mathbb{R}} g(t) \cdot u(t)\, dt$$

and carrying out the preceding arguments. The Poincaré-Melnikov type function Γ has the form $\Gamma = \int_{\mathbb{R}} g(t) \cdot u_0(t+\theta)dt$, where it is understood that $u_0 = \xi\, r$ when $V(u) = |u|^{p+1}/(p+1)$. In any case one finds

$$\lim_{|\theta|\to\infty} \Gamma = \lim_{|\theta|\to\infty} \int_{\mathbb{R}} g(t-\theta) \cdot u_0(t) = 0$$

and hence Γ has a minimum (or a maximum) at some θ_0, whenever it is not identically $= 0$. Then an application of Theorem 6 yields

THEOREM 18. – *Let $(V_1 - V_2)$, respectively (V_3), hold and suppose that $g \in L^2(\mathbb{R}, \mathbb{R}^N)$ and that $\Gamma \not\equiv 0$. Then for $|\epsilon|$ small enough, (26) has a homoclinic solution near to $u_0(t+\theta_0)$ for some $\theta_0 \in \mathbb{R}$.*

Let us now consider the case that $g \in L^\infty$. Here we cannot proceed as before, because G is not well defined on E. To overcome this problem we make a standard change of variable. Using the Implicit Function Theorem, the system

$$\ddot{u} - u + \nabla V(u) = \epsilon g(t)$$

has a solution $\gamma_\varepsilon \in L^\infty$ near $u = 0$. Letting $u = v + \gamma_\epsilon$, we are lead to search a $v \in E$ satisfing

$$-\ddot{v} + v = \nabla V(v + \gamma_\varepsilon) - \nabla V(\gamma_\varepsilon).$$

If we put $W(\epsilon, t, v) = V(v) - V(v + \gamma_\varepsilon) + \nabla V(\gamma_\varepsilon)v + V(\gamma_\varepsilon)$, this equation becomes

$$-\ddot{v} + v = \nabla V(v) - \nabla_v W(\epsilon, t, v), \tag{27}$$

which is in a form suited to our abstract approach (see also Remark 11). Precisely, one has that $\gamma_\varepsilon(t) = \epsilon\gamma_0(t) + o(\epsilon)$ where γ_0 solves $\ddot{\gamma}_0 - \gamma_0 = g(t)$ and hence $W(\epsilon, t, v) = -\epsilon\nabla V(v)\gamma_0(t) + o(\epsilon)$. It follows that (27) has a homoclinic (to 0) provided

$$\Gamma^*(\theta) := -\int_{\mathbb{R}} \nabla V(u_0(t+\theta) \cdot \gamma_0(t)dt = -\int_{\mathbb{R}} \nabla V(u_0(t)) \cdot \gamma_0(t-\theta)dt$$

has a (possibly degenerate) essential critical point.

Since u_0 is a solution of $-\ddot{v}+v=\nabla V(v)$ one finds

$$\Gamma^*(\theta)=-\int_{\mathbb{R}}[-\ddot{u}_0(t)+u_0(t)]\cdot\gamma_0(t-\theta)dt.$$

Integrating by parts one immediately finds that

$$\Gamma^*(\theta)=\int_{\mathbb{R}}u_0(t)\cdot g(t-\theta)dt=\Gamma(\theta),$$

the usual Poincaré-Melnikov function. In conclusion we have

THEOREM 19. – *Let (V_1-V_2), respectively (V_3), hold and suppose that $g\in L^\infty$. Then (26) has a solution which is doubly asymptotic to γ_ε, provided Γ has a proper local minimum (or maximum). Moreover, if Γ is coercive and has k essential critical levels, then (26) has at least k homoclinics to γ_ε.*

It is worth mentioning that, the specific cases that g is periodic, or quasi periodic, or almost periodic, are covered by the preceding Theorem. Indeed, when g is periodic, the result would follow from Theorem 14.

5. MULTIPLE HOMOCLINICS FOR AUTONOMOUS SYSTEMS

In this section we will discuss the existence of multiple homoclinics for autonomous systems like

$$\ddot{u}-u+|u|^{p-1}u=\epsilon\nabla_u W(u). \tag{28}$$

Rather than using Theorems 6-9, we will take advantage of the fact that W is independent of time and use directly Lemmas 2 and 4.

THEOREM 20. – *Suppose that (W_1) hold. Then for $|\epsilon|$ small enough, (28) has at least 2 geometrically distinct homoclinics.*

Proof. – Let $N>1$, otherwise the result is trivial. As in Section 2, we define $G\in C^1(E,\mathbb{R})$ by setting

$$G(u)=\int_{\mathbb{R}}W(u)\,dt$$

and take $\epsilon>0$. Since W does not depend on time, the perturbed functional f_ϵ is, likewise f_0, time translation invariant. As a consequence, the function w, see Lemma 2, satisfies

$$w(\xi\,r_{\theta+\tau},\epsilon)=w(\xi\,r_\theta,\epsilon),\quad\forall\,\theta,\tau\in\mathbb{R}.$$

and the perturbed manifold Z_ϵ has the form $Z_\epsilon = \Sigma_\epsilon \times \mathbb{R}$, where Σ_ϵ is diffeomorphic to S^{N-1}. We remark that f_ϵ does not depend on θ but only on ξ: $f_\epsilon(\xi) = f_0(\xi r) - \epsilon G(\xi r + w(\xi r, \epsilon))$. Obviously, f_ϵ has at least two critical points $\xi_1(\epsilon)$, $\xi_2(\epsilon)$ on $\Sigma_\epsilon \times \{0\}$ which give rise to two (geometrically distinct) homoclinics $u_{i,\epsilon}(t) = \xi_i(\epsilon) r(t) + w(\xi_i(\epsilon) r(t), \epsilon)$, $i = 1, 2$. ■

Remarks 21. – (i) The same arguments can be used to handle potentials $W = W(u, \dot u)$. In such a case we assume $W \in C^2(\mathbb{R}^N \times \mathbb{R}^N, \mathbb{R})$, $W(0,0) = 0$ and $\nabla_u W(0,0) = \nabla_{\dot u} W(0,0) = 0$, so that the critical points of f_ε are classical solutions of

$$\ddot u - u + |u|^{p-1} u = \epsilon\left(\nabla_x W(u, \dot u) + \frac{d}{dt}\nabla_y W(u, \dot u)\right),$$

(ii) Taking also into account the preceding Remark, Theorem 20 improves [18]. In particular, it is worth pointing out that the approach used in [18] only works for reversible systems and cannot be used to handle potential depending on $\dot u$. See also [3] for other results concerning the existence of multiple homoclinics for autonomous systems.

(iii) As pointed out in Remark 17-(iv), also here the solutions $u_{i,\epsilon}$ converge to $z_i(t) = \xi_i r(t)$ as $\epsilon \to 0$, where $\xi_i \in S^{N-1}$ are such that

$$D_\xi\left(\int_{\mathbb{R}} W(\xi r(t), \xi \dot r(t))\, dt\right) = 0. \quad ■$$

6. SCHRÖDINGER EQUATIONS

Our last result is concerned with the nonlinear Schrödinger equation

$$\begin{cases} -\Delta u + u = |u|^{p-1} u - \epsilon g(x), & x \in \mathbb{R}^N, \\ u(x) \to 0 \quad \text{as } |x| \to \infty. \end{cases} \tag{29}$$

with $g \in L^2(\mathbb{R}^N)$ and $1 < p < 2^* - 1$, where $2^* = 2N/(N-2)$ if $N > 2$, and $= +\infty$ otherwise. We will use the preceding notation, with $E = W^{1,2}(\mathbb{R}^N, \mathbb{R})$. It is well known that the unperturbed equation

$$-\Delta u + u = |u|^{p-1} u \tag{30}$$

has a unique positive solution $z_0 \in E$ such that $\nabla z_0(0) = 0$. Setting $z_\theta(x) = z_0(x + \theta)$ and $Z := \{z_\theta : \theta \in \mathbb{R}^N\}$, it turns out that Z is a critical manifold, diffeomorphic to $\mathbb{R}^N$, of the unperturbed functional

$$f_0(u) = \frac{1}{2}\|u\|^2 - \frac{1}{p+1}\int_{\mathbb{R}^N} |u|^{p+1} dx.$$

Moreover, assumptions $(h_1 - h_2 - h_3)$ hold, see for example [2] and references therein. Letting

$$G(u) = \int_{\mathbb{R}^N} g(x)u(x)\,dx$$

one has that $G \in C^1(E, \mathbb{R})$ and (h_0) holds true, too. For $\theta \in \mathbb{R}$ we define

$$\Gamma(\theta) = G(z_\theta) = \int_{\mathbb{R}^N} g(x)z_0(x+\theta)\,dx$$

and hence

$$\lim_{|\theta|\to\infty} \Gamma(\theta) = \lim_{|\theta|\to\infty} \int_{\mathbb{R}^N} g(x-\theta)z_0(x)\,dx = 0.$$

Furthemore, let us assume that

(g_3) $\exists\ \overline{\theta} \in \mathbb{R}$ such that $\Gamma(\overline{\theta}) \neq 0$.

Then Γ has either a minimum or a maximum on Z and Theorem 6 applies yielding a solution $u_\epsilon \in E$ of (29) such that $u_\epsilon \to z_0(t+\theta_0)$, for some $\theta_0 \in \mathbb{R}^N$. A second solution can be found near the manifold $Z_1 = \{-z_0(x+\theta)\}$. Finally to find a third solution of (29), we can use the Local Inversion Theorem near $u = 0$. Since, obviously, the linearized equation $-\Delta v + v = 0$, $v \in E$, has the trivial solution, only, we find a solution $\widetilde{u}_\epsilon \in E$ of (29) such that $\widetilde{u}_\epsilon \to 0$ as $\epsilon \to 0$. In particular, $u_\epsilon \neq \widetilde{u}_\epsilon$ for ϵ sufficiently small, proving the following result:

THEOREM 22. – *Let $g \in L^2(\mathbb{R}^N)$ and $1 < p < 2^* - 1$. If (g_3) holds then (29) has 3 distinct solutions provided ϵ is sufficiently small.*

Remarks 23. – (i) Obviously, condition (g_3) is satisfied if $g(x)$ has constant sign on all of $\mathbb{R}^N$. Hence our Theorem 22 improves some results of a recent paper [12].

(ii) The nonlinearity $|u|^{p-1}u$ could be substituted by a more general function $\psi(u)$ satisfying suitable conditions in such a way that (g_3) still holds. A class of admissible ψ is discussed, for example, in [2], Section 6. Similar remark applies to all the problems discussed above. ■

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REFERENCES

[1] A. AMBROSETTI, Critical points and nonlinear variational problems. *Supplément au Bull. Soc. Math. de France*, Vol. **120**, 1992.
[2] A. AMBROSETTI, M. BADIALE and S. CINGOLANI, Semiclassical states of Nonlinear Schrödinger equations, *Archive Rat. Mech. Analysis*, to appear.
[3] A. AMBROSETTI and V. COTI ZELATI, Multiple homoclinic orbits for a class of conservative systems, *Rend. Sem. Mat. Univ. Padova*, Vol. **89**, 1993, pp. 177-194, and *Note C.R.A.S.*, Vol. **314**, 1992, pp. 601-604.
[4] A. AMBROSETTI, V. COTI ZELATI and I. EKELAND, Symmetry breaking in Hamiltonian systems, *Jour. Diff. Equat.*, Vol. **67**, 1987, pp. 165-184.
[5] F. A. BEREZIN and M. A. SHUBIN, *The Schrödinger Equation*. Kluwer Acad. Publ., Dordrecht, 1991.
[6] U. BESSI, A variational proof of a Sitnikov-like theorem, *Nonlin. Anal. TMA*, Vol. **20**, 1993, pp. 1303-1318.
[7] S. V. BOLOTIN, Homoclinic orbits to invariant tori of Hamiltonian systems, *A.M.S. Transl.*, Vol. **168**, 1995, pp. 21-90.
[8] V. COTI ZELATI, I. EKELAND and E. SÉRÉ, A variational approach to homoclinic orbits in Hamiltonian systems. *Math. Ann.*, Vol. **288**, 1990, pp. 133-160.
[9] V. COTI ZELATI and P. H. RABINOWITZ, Homoclinic orbits for a second order Hamiltonian systems possessing superquadratic potentials. *Jour. Am. Math. Soc.*, Vol. **4**, 1991, pp. 693-727.
[10] V.V. KOZLOV, Integrability and non-integrability in Hamiltonian Mechanics. *Russian Math. Surveys*, Vol. **38**, 1983, pp. 1-76.
[11] U. KIRCHGRABER and D. STOFFER, Chaotic behaviour in simple dynamical systems, *SIAM Review*, Vol. **32**, 1990, pp. 424-452.
[12] L. JEANJEAN, *Two positive solutions for a class of nonhomogeneous elliptic equations*, preprint.
[13] S. MATHLOUTI, Bifurcation d'horbites homoclines pour les systèmes hamiltoniens. *Ann. Fac. Sciences Toulouse*, Vol. **1**, 1992, pp. 211-235.
[14] V. K. MELNIKOV, On the stability of the center for time periodic perturbations, *Trans. Moscow Math. Soc.*, Vol. **12**, 1963, pp. 3-52.
[15] H. POINCARÉ, *Les Méthodes nouvelles de la méchanique céleste.*, 1892.
[16] E. SÉRÉ, Existence of infinitely many homoclinic orbits in Hamiltonian systems, *Math. Z.*, Vol. **209**, 1992, pp. 27-42.
[17] E. SÉRÉ, Looking for the Bernoulli Shift, *Ann. Inst. H. Poincaré, Anal. nonlin.*, Vol. **10**, 1993, pp. 561-590.
[18] K. TANAKA, A note on the existence of multiple homoclinics orbits for a perturbed radial potential, *NoDEA*, Vol. **1**, 1994, pp. 149-162.

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