On the correction to Einstein's formula for the effective viscosity

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Abstract. This paper is a follow-up to Gérard-Varet and Hillairet (2020) on the derivation of accurate effective models for viscous dilute suspensions. The goal is to identify an effective Stokes equation providing an $o(\lambda^2)$ approximation of the exact fluid–particle system, with λ the solid volume fraction of the particles. This means that we look for an improvement of Einstein's formula for the effective viscosity in the form $\mu_{\text{eff}}(x) = \mu + \frac{5}{2}\mu\rho(x)\lambda + \mu_2(x)\lambda^2$. Under a separation assumption on the particles, we proved in the article above that *if an $o(\lambda^2)$ Stokes effective approximation exists*, the correction μ_2 is necessarily given by a mean field limit, which can then be studied and computed under further assumptions on the particle configurations. Roughly, we go here from the conditional result of the article above to an unconditional result: we show that such an $o(\lambda^2)$ Stokes approximation indeed exists, as soon as the mean field limit exists. This includes the case of periodic and random stationary particle configurations.

1. Introduction

We consider a suspension of spherical particles, modelled by a collection of balls $B_i = B(x_i, r_n)$, $1 \le i \le n$, all included in a fixed compact of $\mathbb{R}^3$. The centres x_i of the balls may (and will) depend on n, but we omit it in the notation. We consider a regime where n is large, and $r_n \sim n^{-1/3}$. More precisely, we assume for simplicity that $\lambda := n\frac{4}{3}\pi r_n^3$ is independent of n. We also assume that the balls occupy a volume of size 1, in the sense that

$$\rho_n := \frac{1}{n}\sum_{i=1}^{n}\delta_{x_i} \to \rho(x)\,dx, \quad n \to +\infty, \tag{A0}$$

where ρ is a bounded density with support $\overline{\mathcal{O}}$ for a smooth bounded domain $\mathcal{O}$ such that $|\mathcal{O}| = 1$. In particular, λ can be interpreted as the solid volume fraction. The suspension is immersed in a viscous fluid. We consider particles light enough so that neglecting inertia of the fluid and the particles is reasonable. Setting $\Omega_n := \mathbb{R}^3 \setminus \bigcup_i B_i$,

$$\begin{cases} -\operatorname{div}(2\mu D(u_n) - \mathbb{I}p_n) = g_n & \text{on } \Omega_n, \\ \operatorname{div}(u_n) = 0 & \text{on } \Omega_n, \\ u_n = u_i + \omega_i \times (x - x_i) & \text{on } B_i, \end{cases} \tag{1}$$

where $g_n \in L^2(\mathbb{R}^3) \cap L^{6/5}(\mathbb{R}^3)$ models some forcing. The constant vectors u_i and ω_i are the translation velocity and rotation vector of ball B_i. They are unknowns, associated to the Newtonian dynamics of the particles: in the absence of inertia, relations are of the form

$$\int_{\partial B_i} \sigma(u_n, p_n)v = -\int_{B_i} g_n \, dx,$$

$$\int_{\partial B_i} [\sigma(u_n, p_n)v] \times (x - x_i) = -\int_{B_i} g_n \times (x - x_i) \, dx, \tag{2}$$

where $\sigma(u, p) = 2D(u) - pI$ is the Newtonian stress tensor, and v is the unit normal vector pointing outward. These relations correspond to prescribing the force and the torque on each particle. One further assumes decay of u_n at infinity, which will be encoded in the functional setting.

As $n \to +\infty$, one may expect that some averaging takes place. The hope is to replace the fluid–particle system above by a Stokes equation, with a viscosity coefficient $\mu_{\text{eff}} = \mu_{\text{eff}}(x)$, different from μ in the domain $\mathcal{O}$, reflecting there the rigidity of the particles. Convergence to such a Stokes equation can indeed be shown through homogenization techniques, if one further assumes periodicity or stationarity assumptions on the distributions of balls: see [4], or [13] for the scalar case. Note that such homogenization results are valid for any λ, but somehow abstract, as the expression of the effective viscosity involves a corrector equation which is not much simpler than the original system. They are, moreover, restricted to homogeneous distributions. In the present paper, we aim at more explicit formulas for the effective viscosity in the *dilute regime*, namely when λ is small (but not vanishing as n goes to infinity). We want to show that for n large, the solution u_n of (1) has for $o(\lambda^2)$ approximation the solution $\bar{u}$ of

$$\begin{cases} -\operatorname{div}(2[\mu + \mu_1\lambda + \mu_2\lambda^2]D(\bar{u}) - \mathbb{I}\bar{p}) = g & \text{on } \mathbb{R}^3, \\ \operatorname{div}(\bar{u}) = 0 & \text{on } \mathbb{R}^3, \end{cases} \tag{3}$$

for appropriate first- and second-order corrections $\mu_1 = \mu_1(x)$, $\mu_2 = \mu_2(x)$. Clearly, μ_1 and μ_2 should be nonzero only in the region $\mathcal{O}$ where the suspension is located. Note also that, if the distribution of the particles is anisotropic, μ_1 and μ_2 are not expected to be scalar functions. In full generality, we look for μ_1, μ_2 in the set

$$\operatorname{Sym}(\operatorname{Sym}_{3,\sigma}(\mathbb{R})) := \{M : \operatorname{Sym}_{3,\sigma}(\mathbb{R}) \to \operatorname{Sym}_{3,\sigma}(\mathbb{R}), \ M^t = M\}$$

of symmetric isomorphisms of the space of trace-free symmetric 3×3 matrices (denoted by $\operatorname{Sym}_{3,\sigma}(\mathbb{R})$). This space can be identified with the space of four-tensors satisfying

$$M = (M_{ijkl})_{1 \leq i,j,k,l \leq 3}, \quad M_{ijkl} = M_{jikl} = M_{jilk} = M_{lkji},$$

and the trace conditions

$$\sum_i M_{iikl} = 0 \text{ for } k \neq l, \quad \sum_i M_{ii11} = \sum_i M_{ii22} = \sum_i M_{ii33}.$$

The search for the effective viscosity has a long history, starting from the work of Einstein ([5]): he showed that if the suspension is homogeneous, and if the interaction between the particles can be neglected, an $o(\lambda)$ approximation is given by $\mu_{\text{eff}} = \mu + \frac{5}{2}\lambda\mu$. A rigorous derivation of this formula and of inhomogeneous extensions was later provided under more or less stringent separation assumptions on the particles. We refer to [10, 14, 16] for periodic distributions of balls, and to [12, 15] where the periodicity assumption is relaxed into a lower bound on the minimal distance:

$$d_n := \min_{i \neq j} |x_i - x_j| \geq cn^{-1/3}. \tag{A1}$$

See also the most recent paper [9], where the formula $\mu_1(x) = \frac{5}{2}\rho(x)\mu$ is established under mild requirements. Our concern in the present paper is related to an $o(\lambda^2)$ effective approximation that is beyond Einstein's formula. Such second-order effective viscosity has been discussed in several papers; see [1,2,19]. However, one can observe discrepancies between the results, and very different approaches depending on the types of suspensions considered. A more global analysis was initiated by the first author and M. Hillairet in the recent paper [8]. Roughly, this paper shows that under (A1), if $\limsup_n \|u_n - \bar{u}\| = o(\lambda^2)$, where $\|\ \|$ is a weak norm and where $\bar{u}$ is a solution of (3) with $\mu_1 = \frac{5}{2}\rho(x)\mu$, then necessarily,

$$v_2 := \int_{\mathcal{O}} \mu_2 = \frac{75}{16\pi} \lim_{n\to\infty} \left(\frac{1}{n^2} \sum_{j \neq i} \mathcal{M}(x_i - x_j) - \iint \mathcal{M}(x - y)\rho(x)\rho(y)\, dx\, dy \right) \tag{4}$$

where $\mathcal{M} = \mathcal{M}(x) \in \text{Sym}(\text{Sym}_{3,\sigma}(\mathbb{R}))$ is given by

$$\mathcal{M}(x)S : S' = -D\left(\frac{x \otimes x : S}{|x|^5}x\right) : S'$$
$$= -2\frac{Sx \cdot S'x}{|x|^5} + 5\frac{(S : x \otimes x)(S' : x \otimes x)}{|x|^7} \quad \forall S, S' \in \text{Sym}_{3,\sigma}(\mathbb{R}). \tag{5}$$

Hence, *if a second-order effective model exists*, the average v_2 of the second-order correction over the domain (which coincides with μ_2 if μ_2 does not depend on x) is given by the mean field limit (4)–(5). The second part of article [8] consists of an analysis of such a mean field limit, using ideas introduced by S. Serfaty and her coauthors in the context of Coulomb gases (see [17] and references therein). More explicit formulas are provided, notably in the periodic case.

The limitation of the results in [8] is that they hold *conditionally to the existence of a second-order effective model of type* (3). Ideally, one would like to prove the existence of an effective model as soon as the mean field limit in (4)–(5) does exist. This necessary condition is however not enough: indeed, v_2 corresponds to an average over the whole domain $\mathcal{O}$, so that it is unlikely to guarantee the existence of an effective local coefficient $\mu_2 = \mu_2(x)$. Nevertheless, as we will show, we can exhibit more local in nature mean field limits, whose existence ensures the existence of an effective model. Moreover, such

limits allow us to determine μ_2, and not only its average ν_2. We introduce

$$\mu_{2,n} := \frac{75\mu}{16\pi}\big(\mathcal{M}(x-y)1_{x\neq y}\rho_n(dx)\rho_n(dy) - \mathcal{M}(x-y)\rho(x)\,dx\,\rho(y)\,dy\big). \quad (6)$$

It can be seen as a compactly supported distribution on $\mathbb{R}_x^3 \times \mathbb{R}_y^3$, with values in the space $\mathrm{Sym}(\mathrm{Sym}_{3,\sigma}(\mathbb{R}))$: for $F = F(x,y) \in C^\infty(\mathbb{R}^3 \times \mathbb{R}^3)$ (even for $F \in C^1(\mathbb{R}^3 \times \mathbb{R}^3)$),

$$\langle \mu_{2,n}, F \rangle = \frac{75\mu}{16\pi}\Big(\frac{1}{n^2}\sum_{i\neq j}\mathcal{M}(x_i - x_j)F(x_i, x_j)$$

$$- \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\mathcal{M}(x-y)F(x,y)\rho(x)\rho(y)\,dy\,dx\Big). \quad (7)$$

We stress that $\mathcal{M}(x)$ is a Calderon–Zygmund kernel, hence not integrable. In particular, the last integral must be understood in a weak sense: it can be defined rigorously through the decomposition

$$\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\mathcal{M}(x-y)F(x,y)\rho(x)\rho(y)\,dx\,dy$$

$$:= \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\mathcal{M}(x-y)[F(x,y)-F(y,y)]\rho(x)\rho(y)\,dx\,dy$$

$$+ \int_{\mathbb{R}^3}(\mathcal{M}\star\rho)(y)F(y,y)\rho(y)\,dy,$$

where the first integral in the decomposition exists in the usual sense, while the second one is defined because $h \to \mathcal{M}\star h$ is continuous from $L^p(\mathbb{R}^3)$ to $L^p(\mathbb{R}^3)$ for any $1 < p < \infty$ by the Calderon–Zygmund theorem. Of course, when F is of the form $F(x,y) = f(x)g(y)$, one can write directly

$$\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\mathcal{M}(x-y)F(x,y)\rho(x)\rho(y)\,dx\,dy = \int_{\mathbb{R}^3}(\mathcal{M}\star(\rho f))\rho g,$$

which allows us to give a meaning to $\langle \mu_{2,n}, f \otimes g \rangle$ for much less regular f and g. Our theorem reads as follows.

Theorem 1.1. *Let* $\lambda > 0$, $g \in L^{3+\varepsilon}$, $\varepsilon > 0$, $\mu_2 \in L^\infty(\mathbb{R}^3, \mathrm{Sym}(\mathrm{Sym}_{3,\sigma}(\mathbb{R})))$. *For all* n, *let* r_n *be such that* $\lambda = \frac{4\pi}{3}nr_n^3$, $g_n \in L^{\frac{6}{5}}(\mathbb{R}^3)$. *Let* $u_{n,\lambda}$ *be the solution of* (1)–(2) *in* $\dot{H}^1(\mathbb{R}^3) \cap L^6(\mathbb{R}^3)$. *Assume* (A0)–(A1), *that* $g_n \to g$ *in* $L^{\frac{6}{5}}(\mathbb{R}^3)$, *and that*

$$\mu_{2,n} \to \mu_2(x)\delta_{x=y} \quad \text{in } \mathcal{D}'(\mathbb{R}^3 \times \mathbb{R}^3, \mathrm{Sym}(\mathrm{Sym}_{3,\sigma}(\mathbb{R}))), \quad (A2)$$

with $\mu_{2,n}$ *defined in* (6). *Then any accumulation point* u_λ *of* $u_{n,\lambda}$ *solves*

$$\begin{cases} -\mathrm{div}(2[\mu + \frac{5}{2}\mu\rho\lambda + \mu_2\lambda^2]D(u_\lambda) - \mathbb{I}p_\lambda) = g + R_\lambda & \text{in } \mathbb{R}^3, \\ \mathrm{div}(u_\lambda) = 0 & \text{in } \mathbb{R}^3, \end{cases} \quad (8)$$

where R_λ *satisfies, for all* $q \geq 3$,

$$|\langle R_\lambda, \phi \rangle| \leq C\lambda^{\frac{7}{3}}\|D\phi\|_q \quad \forall \phi \in \dot{H}^1(\mathbb{R}^3) \cap \dot{W}^{1,q}(\mathbb{R}^3). \quad (9)$$

A few remarks are in order.

Remark 1.1. One has directly from estimate (9) that for any $p \leq \frac{3}{2}$,

$$\|u - u_\lambda\|_{\dot{W}^{1,p}(\mathbb{R}^3)} \leq C\lambda^{\frac{7}{3}},$$

where u is the solution of the effective model

$$\begin{cases} -\operatorname{div}(2[\mu + \frac{5}{2}\mu\rho\lambda + \mu_2\lambda^2]D(u) - \mathbb{I}p) = g & \text{on } \mathbb{R}^3, \\ \operatorname{div}(u) = 0 & \text{on } \mathbb{R}^3. \end{cases} \tag{10}$$

It follows by Sobolev embedding that $\|u - u_\lambda\|_{L^r_{\text{loc}}} = O(\lambda^{\frac{7}{3}})$, for any $r = \frac{3p}{3-p} \leq 3$. Moreover, as will be seen below, $(u_{n,\lambda})_{n\in\mathbb{N}}$ is bounded in $\dot{H}^1 \cap L^6$. Combining the last estimate with Rellich's theorem, it follows easily that

$$\limsup_n \|u_{n,\lambda} - u\|_{L^r_{\text{loc}}} = O(\lambda^{\frac{7}{3}}) \quad \forall r \leq 3.$$

Remark 1.2. The main assumption of the theorem is the convergence of $\mu_{2,n}$ to $\mu_2(x)\delta_{x=y}$. With regard to the form of $\langle \mu_{2,n}, F \rangle$, cf. (7), this convergence corresponds to the local mean field limits alluded to above. In particular, the necessary condition (4) derived in [8] corresponds to the convergence of $\langle \mu_{2,n}, F \rangle$ for the special case $F(x, y) = 1$.

The outline of the paper is the following. After preliminary results on the Stokes system, gathered in Section 2, we address the proof of Theorem 1.1 in Section 3. Finally, we turn in Section 4 to the discussion of assumption (A2). Roughly, we show that it is fulfilled by both periodic and random stationary particle distributions that satisfy the separation assumption (A1), and we discuss the corresponding limit μ_2. There we closely follow the analysis in [8]. Notably, we show that when the distribution of particles is given by an isotropic process (plus technical conditions), then $\mu_2 S : S = \frac{5}{2}\mu|S|^2$, a result that was not given in [8].

2. Reminder on the Stokes problem

In this section we recall some properties regarding the Stokes equation on an exterior domain. We denote by $(\mathcal{U}, \mathcal{P})$ the Green function of the Stokes equation:

$$\mathcal{U}(x) := \frac{1}{8\pi}\left(\frac{I_3}{|x|} + \frac{x \otimes x}{|x|^3}\right), \quad \mathcal{P}(x) = \frac{1}{4\pi}\frac{x}{|x|^3}. \tag{11}$$

Let $A \in \operatorname{Sym}_{3,\sigma}(\mathbb{R})$. We denote by $(V[A], Q[A])$ the solution to the Dirichlet problem

$$\begin{cases} -\Delta u + \nabla p = 0 & \text{on } \mathbb{R}^3 \setminus B(0,1), \\ \operatorname{div}(u) = 0 & \text{on } \mathbb{R}^3 \setminus B(0,1), \\ u = -Ax & \text{on } B(0,1), \end{cases} \tag{12}$$

given by the explicit formulas

$$V[A](x) = -\frac{5}{2}\frac{A : x \otimes x}{|x|^5}x - \frac{1}{|x|^5}Ax + \frac{5}{2}\frac{(A : x \otimes x)}{|x|^7}x, \tag{13}$$

$$Q[A](x) = -5\frac{A : x \otimes x}{|x|^5}. \tag{14}$$

An important feature of this solution is that, as easily seen from symmetry considerations, it fulfills the extra conditions

$$\int_{\partial B(0,1)} \sigma(V[A], Q[A])\nu = 0, \quad \int_{\partial B(0,1)} x \times \sigma(V[A], Q[A])\nu = 0. \tag{15}$$

Moreover, we can link $V[A]$ to the Green function through the identity

$$V[A](x) = \frac{20\pi}{3}\nabla \mathcal{U}(x)A + R[A](x), \tag{16}$$

where $R[A]$ is homogeneous of degree -4. Here, $\nabla \mathcal{U}$ is a third-rank tensor defined using the Einstein summation convention by

$$\nabla \mathcal{U}A = (\partial x_k \mathcal{U}_{ij} A_{jk})_{1 \le i \le 3} = -\frac{3}{8\pi}\frac{A : x \otimes x}{|x|^5}x. \tag{17}$$

Moreover, a simple calculation yields the identity

$$D(\nabla \mathcal{U}A) : B = \frac{3}{8\pi}\mathcal{M}(x)A : B = D(\nabla \mathcal{U}B) : A, \tag{18}$$

where $\mathcal{M}$ was defined in (5), so that

$$D(V[A])(x) = \frac{5}{2}\mathcal{M}(x)A + D(R[A])(x). \tag{19}$$

We also introduce the extensions

$$\mathcal{V}[A](x) = \begin{cases} V[A] & \text{on } B(0,1)^c, \\ -Ax & \text{on } B(0,1), \end{cases} \quad \mathcal{Q}[A](x) = \begin{cases} Q[A] & \text{on } B(0,1)^c, \\ 0 & \text{on } B(0,1). \end{cases} \tag{20}$$

Direct computation shows that

$$-\operatorname{div}(\sigma(\mathcal{V}[A], \mathcal{Q}[A])) = 5Axs^1 \quad \text{in } \mathbb{R}^3, \tag{21}$$

where s^η is the surface measure on the sphere of radius η.

We finish this part by recalling a classical estimate for the Stokes equation. Let $w \in W^{1,2}(\bigcup B_i)$ be divergence-free. We consider the unique solution (v, q) satisfying

$$\begin{cases} -\operatorname{div}(2D(v) - \mathbb{I}q) = 0 & \text{on } \Omega_n, \\ \operatorname{div}(v) = 0 & \text{on } \Omega_n, \\ D(v) = D(w) & \text{on } \bigcup B_i, \end{cases} \tag{22}$$

with the following conditions:

$$\int_{\partial B_i} \sigma(v, q)n = 0, \quad \int_{\partial B_i} [\sigma(v, q)n] \times (x - x_i) = 0, \quad \forall 1 \leq i \leq n. \tag{23}$$

Using integration by parts, one can show that v is a minimizer of

$$\left\{\int_{\mathbb{R}^3} |D(u)|^2, \ u \in \dot{H}_\sigma^1(\mathbb{R}^3), \text{ such that } D(u) = D(w) \text{ on } \bigcup B_i\right\}.$$

Combining this minimizing property with [15, Lem. 4.4] we have the following result:

Proposition 2.1. *The unique solution v of* (22), (23) *satisfies*

$$\|\nabla v\|_{L^2(\mathbb{R}^3)}^2 = 2\|D(v)\|_{L^2(\mathbb{R}^3)}^2 \leq C\|D(w)\|_{L^2(\bigcup B_i)}^2.$$

Proof. The first equality is well known: it follows from the identity $\Delta v = 2 \operatorname{div}(D(v))$ and integration by parts. For the inequality, by the minimizing property of v, it is enough to construct a divergence-free velocity field u that matches the condition $D(u) = D(w)$ on $\bigcup B_i$ and satisfies the same inequality. Classical considerations about the Bogovskii operator ensure the existence of fields $u_i \in H_0^1((B(x_i, 2r_n)))$ such that

$$\operatorname{div}(u_i) = 0 \text{ on } B(x_i, 2r_n), \quad u_i = w - \fint_{B_i} w \text{ on } B_i,$$

where, using the Poincaré–Wirtinger inequality, we get for all i,

$$\|\nabla u_i\|_{H_0^1(B(x_i, 2r_n))} \leq C \left\| w - \fint w \right\|_{H^1(B_i)} \leq C' \|\nabla w\|_{L^2(B_i)},$$

with C, C' independent of n by scaling considerations. See [11, Lem. 18] for details. We then take $u = \sum_i u_i$. Since the balls $B(x_i, 2r_n)$ are disjoint by (A1) for λ small enough, u satisfies $D(u) = D(w)$ on $\bigcup B_i$ and we have

$$\|u\|_{\dot{H}^1(\mathbb{R}^3)}^2 = \sum_i \|\nabla u_i\|_{L^2(B(x_i, 2r_n))}^2 \leq C \sum_i \|\nabla w\|_{L^2(B_i)}^2.$$

Moreover, adding a proper rigid vector field to w on each B_i, which does not change $D(w)$ on each B_i, we can always assume that $\int_{\partial B_i} w = \int_{\partial B_i} w \times (x - x_i) = 0$. We conclude by applying [15, Lem. 4.4]. ∎

3. Proof of Theorem 1.1

By linearity of the Stokes equation, we can restrict to the case $\mu = 1$. Let $q \geq 3$. The goal is to show that any accumulation point u_λ of $u_{n,\lambda}$ satisfies a system of type (8) *with remainder R_λ satisfying*

$$\langle R_\lambda, \phi \rangle \leq C \lambda^{\frac{7}{3}} \|\phi\|_{W^{1,q}}$$

for all divergence-free $\phi \in \dot{H}^1 \cap W^{1,q}$. By density, it is enough to show such an inequality for all divergence-free $\phi \in C_c^\infty(\mathbb{R}^3)$. For any such ϕ, we consider ϕ_n satisfying

$$
\begin{cases}
-\operatorname{div}(2D(\phi_n) - \mathbb{I}q_n) = 2\operatorname{div}((\frac{5}{2}\rho\lambda + \lambda^2\mu_2)D(\phi)) & \text{on } \Omega_n, \\
\operatorname{div}(\phi_n) = 0 & \text{on } \Omega_n, \\
\phi_n = \phi + \text{translation} + \text{rotation} & \text{on } B_i,
\end{cases}
\tag{24}
$$

with the following conditions:

$$
\int_{\partial B_i} \sigma(\phi_n, q_n)v = -2\int_{\partial B_i}\left(\frac{5}{2}\rho\lambda + \mu_2\lambda^2\right)D(\phi)v\,dx,
$$
$$
\int_{\partial B_i}[\sigma(u_n, p_n)v] \times (x - x_i) = -2\int_{\partial B_i}\left[\left(\frac{5}{2}\rho\lambda + \mu_2\lambda^2\right)D(\phi)v\right] \times (x - x_i)\,dx.
\tag{25}
$$

Note that ϕ_n depends implicitly on λ. Similarly, we shall use u_n instead of $u_{n,\lambda}$ for short. Testing $\phi - \phi_n$ in equation (1) we get

$$
\int_{\Omega_n}(\phi - \phi_n) \cdot g_n = -\int_{\Omega_n}\operatorname{div}(2D(u_n) - \mathbb{I}p_n) \cdot (\phi - \phi_n)
$$
$$
= 2\int_{\Omega_n}D(u_n) : D(\phi - \phi_n) + \sum_i\int_{\partial B_i}\sigma(u_n, p_n)v \cdot (\phi - \phi_n)
$$
$$
= 2\int_{\Omega_n}D(u_n) : D(\phi) - 2\int_{\Omega_n}D(u_n) : D(\phi_n) - \sum_i\int_{B_i}g_n \cdot (\phi - \phi_n).
$$

We recall that in the second line of the above computations, the unit normal vector v is pointing outward the balls. Using equations (24) and (25) we have

$$
-2\int_{\Omega_n}D(u_n) : D(\phi_n)
$$
$$
= \int_{\Omega_n}\operatorname{div}(2D(\phi_n) - q_n\mathbb{I}) \cdot u_n + \sum_i\int_{\partial B_i}[\sigma(\phi_n, q_n)v] \cdot u_n
$$
$$
= -\int_{\Omega_n}2\operatorname{div}\left(\left(\frac{5}{2}\rho\lambda + \mu_2\lambda^2\right)D(\phi)\right) \cdot u_n + \sum_i\int_{\partial B_i}[\sigma(\phi_n, q_n)v] \cdot u_n
$$
$$
= 2\int_{\Omega_n}\left(\frac{5}{2}\rho\lambda + \mu_2\lambda^2\right)D(\phi) : D(u_n).
$$

We get the following relation for all ϕ using the fact that $D(u_n) = 0$ on B_i:

$$
2\int_{\mathbb{R}^3}D(u_n) : \left(1 + \frac{5}{2}\rho\lambda + \mu_2\lambda^2\right)D(\phi) = \int_{\mathbb{R}^3}\phi \cdot g_n - \int_{\mathbb{R}^3}\phi_n \cdot g_n.
\tag{26}
$$

By a simple energy estimate, u_n also satisfies

$$
\int_{\mathbb{R}^3}|\nabla u_n|^2 = 2\int_{\mathbb{R}^3}|D(u_n)|^2 = \int_{\mathbb{R}^3}u_n \cdot g_n \leq \|u_n\|_{L^6}\|g_n\|_{L^{\frac{6}{5}}} \leq C\|\nabla u_n\|_{L^2},
$$

where the last inequality comes from the Sobolev embedding and the boundedness of $(g_n)_{n\in\mathbb{N}}$ in $L^{\frac{6}{5}}$. Hence, $(u_n)_{n\in\mathbb{N}}$ is bounded in $\dot{H}^1 \cap L^6$. Denoting by $u_\lambda = \lim_k u_{n_k}$ an accumulation point, we deduce from (26) with $n = n_k$ that

$$2\int_{\mathbb{R}^3} D(u_\lambda) : \left(1 + \frac{5}{2}\rho\lambda + \mu_2\lambda^2\right)D(\phi) = \int_{\mathbb{R}^3} \phi \cdot g + \langle R_\lambda, \phi \rangle,$$

where $\langle R_\lambda, \phi \rangle = -\lim_k \int_{\mathbb{R}^3} \phi_{n_k} \cdot g_{n_k}$. Note that this limit exists because all other terms in (26) converge when $n = n_k, k \to +\infty$. Moreover, it is clearly linear in ϕ as ϕ_n is linear in ϕ. Furthermore, testing $\phi_n - \phi$ in (24), similar integrations by parts lead to

$$2\int_{\mathbb{R}^3} |D(\phi_n)|^2 = 2\int_{\mathbb{R}^3} D(\phi_n) : D(\phi) - 2\int_{\mathbb{R}^3} \left(\frac{5}{2}\lambda\rho + \mu_2\lambda^2\right)D(\phi) : (D(\phi_n) - D(\phi)).$$

The Cauchy–Schwarz inequality implies that ϕ_n is bounded uniformly in n in $\dot{H}^1$, hence in L^6. Eventually, as $g_n \to g$ strongly in $L^{\frac{6}{5}}$, we have $\langle R_\lambda, \phi \rangle = -\lim_k \int_{\mathbb{R}^3} \phi_{n_k} \cdot g$. Finally, to prove the theorem, it is enough to show that

$$\forall q \geq 3, \quad \limsup_n \left| \int_{\mathbb{R}^3} \phi_n \cdot g \right| \leq C\lambda^{\frac{7}{3}} \|\phi\|_{W^{1,q}}. \tag{27}$$

In order to obtain (27), we shall write $(\phi_n, q_n) = (\phi_n^1 + \phi_n^2, q_n^1 + q_n^2)$, where ϕ_n^1 is a (somehow natural) approximation of ϕ_n and where ϕ_n^2 is a remainder. Namely, we look for an approximation ϕ_n^1 of the form

$$\phi_n^1 = \nabla\mathcal{U} \star \left(2\left(\lambda\frac{5}{2}|\mathcal{O}|\rho + \lambda^2\mu_2\right)D(\phi)\right) - r_n \sum_i \mathcal{V}[A_i]\left(\frac{x-x_i}{r_n}\right),$$

where $\mathcal{U}, \mathcal{V}$ are defined in (11), (20). The rough idea behind this approximation is that the first term on the right-hand side should take care of the source term in (24), while the second term should take care of the boundary conditions at the balls B_i. In particular, the field $\phi_{\mathbb{R}^3} := \nabla\mathcal{U} * (2(\lambda\frac{5}{2}|\mathcal{O}|\rho + \lambda^2\mu_2)D(\phi))$ solves the Stokes equation

$$-\Delta\phi_{\mathbb{R}^3} + \nabla q_{\mathbb{R}^3} = \text{div}\left(2\left(\lambda\frac{5}{2}|\mathcal{O}|\rho + \lambda^2\mu_2\right)D(\phi)\right), \quad \text{div}\,\phi_{\mathbb{R}^3} = 0 \quad \text{in } \mathbb{R}^3,$$

while each term in the sum, that is, $\phi_{i,n} := -r_n\mathcal{V}[A_i](\frac{x-x_i}{r_n})$, solves

$$-\Delta\phi_{i,n} + \nabla q_{i,n} = 0, \quad \text{div}\,\phi_{i,n} = 0 \text{ in } \mathbb{R}^3 \setminus B_i, \quad \phi_{i,n}|_{B_i} = A_i(x - x_i).$$

By looking at (24), it is tempting to take $A_i = D\phi_i$, where

$$D\phi_i := D(\phi)(x_i),$$

as ϕ should be close to this value on the small ball B_i. However, this approximation is not accurate enough, and would only allow us to recover Einstein's formula. To go beyond, one must account for two extra contributions. The first one is the trace left at the balls

by $\phi_{\mathbb{R}^3}$. More precisely, it will be enough to correct the trace of the $O(\lambda)$ term in $\phi_{\mathbb{R}^3}$. The second contribution is the trace left by all $\phi_{j,n}$, $j \neq i$, on ball B_i, which corresponds to binary interactions between particles. Again, it will be enough to account for the least decaying term in $\phi_{j,n}$; cf. decomposition (16). This leads to the following definition:

$$\phi_n^1 = \nabla \mathcal{U} * \left(2\left(\lambda \frac{5}{2}\rho + \lambda^2 \mu_2\right) D(\phi)\right) - r_n \sum_i \mathcal{V}[D\phi_i + S_i^1 + S_i^2]\left(\frac{x - x_i}{r_n}\right), \quad (28)$$

where, using relation (18),

$$S_i^1 := -\fint_{B_i} D\big(\nabla \mathcal{U} * (\lambda 5|\mathcal{O}|\rho D(\phi))\big)(x)\, dx = -\frac{15}{8\pi}\lambda \fint_{B_i} \big(\mathcal{M} * (\rho D(\phi))\big)(x)\, dx$$

$$= -\frac{15}{8\pi}\lambda \fint_{B_i} \left(\int_{\mathbb{R}^3} \mathcal{M}(x - y) D(\phi)(y)\rho(y)\, dy\right) dx, \quad (29)$$

while, still using (18),

$$S_i^2 := \sum_{j \neq i} \fint_{B_i} \frac{20\pi}{3} D(\nabla \mathcal{U} D\phi_j)\left(\frac{x - x_j}{r_n}\right) dx = \frac{5}{2}r_n^3 \fint_{B_i} \sum_{j \neq i} \mathcal{M}(x - x_j) D\phi_j\, dx$$

$$= \frac{15}{8\pi}\frac{\lambda}{n} \sum_{j \neq i} \fint_{B_i} \mathcal{M}(x - x_j) D\phi_j\, dx. \quad (30)$$

Direct computations using formula (21) yield

$$-\operatorname{div}(\sigma(\phi_n^1, q_n^1))$$

$$= \operatorname{div}\left(2\left(\frac{5}{2}\rho\lambda + \mu_2\lambda^2\right)D(\phi)\right) - 5\sum_i (D\phi_i + S_i^1 + S_i^2)\nabla 1_{B_i}$$

$$= \operatorname{div}\left((5\rho\lambda + 2\mu_2\lambda^2)D(\phi) - \frac{5\lambda}{n}\sum_i \frac{1}{|B_i|}1_{B_i}(D\phi_i + S_i^1 + S_i^2)\right) \quad \text{in } \mathbb{R}^3.$$

Using the definitions of S_i^1 and S_i^2, (29) and (30), we find

$$-\operatorname{div}(\sigma(\phi_n^1, q_n^1))$$

$$= \operatorname{div}\Bigg((5\lambda\rho + 2\lambda^2\mu_2)D(\phi) - \frac{5\lambda}{n}\sum_i \frac{1}{|B_i|}1_{B_i} D\phi_i$$

$$- \frac{5\lambda}{n}\sum_i \frac{1}{|B_i|}1_{B_i}\left(-\frac{15\pi}{8}\lambda \fint_{B_i}\int_{\mathbb{R}^3}\mathcal{M}(x - y)D(\phi)(y)\rho(y)\, dy\, dx\right)$$

$$- \frac{5\lambda}{n}\sum_i \frac{1}{|B_i|}1_{B_i}\left(-\frac{15\pi}{8}\frac{\lambda}{n}\fint_{B_i}\sum_{j \neq i}\mathcal{M}(x - x_j)D\phi_j\, dx\right)\Bigg)$$

$$= 5\lambda \operatorname{div}\left(\rho D(\phi) - \frac{1}{n}\sum_i \frac{1}{|B_i|}1_{B_i} D\phi_i\right) \tag{31}$$

$$+ 2\lambda^2 \operatorname{div}\left(\mu_2 D(\phi) - \frac{75}{16\pi}\frac{1}{n^2}\sum_i \frac{1}{|B_i|}1_{B_i}\sum_{j\neq i}\fint_{B_i} \mathcal{M}(x-x_j)D\phi_j\,dx\right.$$

$$\left. + \frac{75}{16\pi}\frac{1}{n}\sum_i \frac{1}{|B_i|}1_{B_i}\fint_{B_i}\int_{\mathbb{R}^3}\mathcal{M}(x-y)D(\phi)(y)\rho(y)\,dy\,dx\right) \quad \text{in } \mathbb{R}^3.$$

Moreover, thanks to property (15), it is easily seen that

$$\int_{\partial B_i}\sigma(\phi_n^1, q_n^1)v = -2\int_{\partial B_i}\left(\frac{5}{2}\rho\lambda + \mu_2\lambda^2\right)D(\phi)v\,dx,$$

$$\int_{\partial B_i} x\times(\sigma(\phi_n^1, q_n^1)v) = -2\int_{\partial B_i}\left(\left(\frac{5}{2}\rho\lambda + \mu_2\lambda^2\right)D(\phi)v\right)\times(x-x_i)\,dx.$$

It follows that (ϕ_n^2, q_n^2) satisfies

$$\operatorname{div}(\sigma(\phi_n^2, q_n^2)) = 0 \text{ on } \Omega_n, \quad \int_{\partial B_i}\sigma(\phi_n^2, p_n^2)v = 0,$$

$$\int_{\partial B_i}\sigma(\phi_n^2, p_n^2)v\times(x-x_i) = 0 \tag{32}$$

with boundary condition

$$D(\phi_n^2) = D(\psi_n^2 + \tilde{\psi}_n^2) \quad \text{on } \bigcup B_i, \tag{33}$$

where ψ_n^2 and $\tilde{\psi}_n^2$ are defined by

$$\psi_n^2(x)|_{B_i} := \phi(x) - D\phi_i\cdot(x-x_i), \quad x\in B_i \tag{34}$$

and

$$\tilde{\psi}_n^2(x) = -(S_i^1 + S_i^2)\cdot(x-x_i) - \left(\nabla\mathcal{U} * \left(2\left(\frac{5}{2}\rho\lambda + \mu_2\lambda^2\right)D(\phi)\right)\right. \tag{35}$$

$$\left. - r_n\sum_{j\neq i}V[D\phi_j + S_j^1 + S_j^2]\left(\frac{x-x_j}{r_n}\right)\right), \quad x\in B_i.$$

We aim now to estimate both terms $\int g\phi_n^1$ and $\int g\phi_n^2$.

3.1. Estimate of ϕ_n^2

Most of this subsection is dedicated to the derivation of the following proposition:

Proposition 3.1. *For all $q\geq 2$,*

$$\limsup_n\|D(\phi_n^2)\|_{L^2(\mathbb{R}^3)} \leq C_q\left(\lambda^{\frac{5}{2}-\frac{2}{q}} + \lambda^{\frac{11}{6}}\right)\|D(\phi)\|_{L^q}.$$

Before we prove this proposition, we show that it implies the following result:

Proposition 3.2. *For all $q \geq 2$,*

$$\limsup_n \left| \int_{\mathbb{R}^3} g\phi_n^2 \right| \leq C_{g,q} \lambda^{\frac{1}{2}} \left(\lambda^{\frac{5}{2}-\frac{2}{q}} + \lambda^{\frac{11}{6}} \right) \|D(\phi)\|_{L^q}.$$

Note the extra factor $\lambda^{1/2}$ compared to Proposition 3.1, which is crucial to obtain an $o(\lambda^2)$ error. Note also that for $q \geq 3$, this term is bounded by $\lambda^{\frac{7}{3}} \|D(\phi)\|_{L^q}$.

Proof of Proposition 3.2. We introduce the solution u_g of the Stokes equation

$$- \Delta u_g + \nabla p_g = g, \quad \text{div } g = 0, \quad \text{in } \mathbb{R}^3. \tag{36}$$

As $g \in L^{3+\varepsilon}$, $u_g \in W^{2,3+\varepsilon}_{\text{loc}}$, so that $D(u_g)$ is continuous. Integrations by parts yield

$$\int_{\mathbb{R}^3} g\phi_n^2 = \int_{\mathbb{R}^3} (-\Delta u_g + \nabla p_g)\phi_n^2 = 2 \int_{\mathbb{R}^3} D(u_g) : D(\phi_n^2)$$

$$= 2 \int_{\bigcup B_i} D(u_g) : D(\phi_n^2) - \sum_i \int_{\partial B_i} u_g \cdot \sigma(\phi_n^2, q_n^2)\nu$$

$$= 2 \int_{\bigcup B_i} D(u_g) : D(\phi_n^2) - \sum_i \int_{\partial B_i} (u_g + u_g^i + \omega_g^i \times (x - x_i)) \cdot \sigma(\phi_n^2, q_n^2)\nu$$

for any constant vectors u_g^i, ω_g^i, $1 \leq i \leq n$, by the last two relations in (32). As $u_g + u_g^i + \omega_g^i \times (x - x_i)$ is divergence-free, one has

$$\int_{\partial B_i} (u_g + u_g^i + \omega_g^i \times (x - x_i)) \cdot \nu = 0.$$

We can apply classical considerations on the Bogovskii operator ([6]): for any $1 \leq i \leq n$, there exists $U_g^i \in H_0^1(B(x_i, 2r_n))$ such that

$$\text{div } U_g^i = 0 \text{ in } B(x_i, 2r_n), \quad U_g^i = u_g + u_g^i + \omega_g^i \times (x - x_i) \text{ in } B_i,$$

and with

$$\|\nabla U_g^i\|_{L^2} \leq C_{i,n} \|u_g + u_g^i + \omega_g^i \times (x - x_i)\|_{W^{1,2}(B_i)}.$$

Furthermore, by a proper choice of u_g^i and ω_g^i, we can ensure the Korn inequality:

$$\|u_g + u_g^i + \omega_g^i \times (x - x_i)\|_{W^{1,2}(B_i)} \leq c'_{i,n} \|D(u_g)\|_{L^2(B_i)},$$

resulting in

$$\|\nabla U_g^i\|_{L^2} \leq C \|D(u_g)\|_{L^2(B_i)},$$

where the constant C in the last inequality can be taken independent of i and n by translation and scaling arguments. Extending U_g^i by zero, and denoting $U_g = \sum U_g^i$, we have for $d_n > 4r_n$ (which is implied by (A1) for λ small enough):

$$\|\nabla U_g\|_{L^2} \leq C \|D(u_g)\|_{L^2(\bigcup B_i)}. \tag{37}$$

Back to our calculation, we find

$$\int_{\mathbb{R}^3} g\phi_n^2 = 2 \int_{\bigcup B_i} D(U_g) : D(\phi_n^2) - \sum_i \int_{\partial B_i} U_g \cdot \sigma(\phi_n^2, q_n^2)\nu$$

$$= 2 \int_{\mathbb{R}^3} D(U_g) : D(\phi_n^2).$$

By using (37) and the Cauchy–Schwarz inequality, we obtain

$$\left| \int_{\mathbb{R}^3} g\phi_n^2 \right| \le C \|D(u_g)\|_{L^2(\bigcup B_i)} \|D(\phi_n^2)\|_{L^2(\mathbb{R}^3)} \le C \|D(u_g)\|_{L^\infty} \lambda^{\frac{1}{2}} \|D(\phi_n^2)\|_{L^2(\mathbb{R}^3)},$$

so that combining with Proposition 3.1 yields the result. ∎

We now turn to the proof of Proposition 3.1. According to the definition of ϕ_n^2 in (32), (33), Proposition 2.1 implies

$$\|\nabla\phi_n^2\|_{L^2(\Omega_n)}^2 \le C(\|D(\psi_n^2)\|_{L^2(\bigcup B_i)}^2 + \|D(\tilde{\psi}_n^2)\|_{L^2(\bigcup B_i)}^2). \tag{38}$$

As regards ψ_n^2, defined in (34), we compute

$$\|D(\psi_n^2)\|_{L^2(\bigcup B_i)}^2 = \sum_i \int_{B_i} |D(\phi) - D\phi_i|^2 \, dx$$

$$\le \|\nabla^2 \phi\|_\infty^2 \sum_i \int_{B_i} r_n^2 \, dx \le \|\nabla^2 \phi\|_\infty^2 \lambda r_n^2 \xrightarrow[n\to\infty]{} 0. \tag{39}$$

As regards $\tilde{\psi}_n^2$, defined in (35), we use identities (18) and (19) to write for all $x \in B_i$,

$$D(\tilde{\psi}_n^2)(x) = -\frac{15}{8\pi} \lambda \mathcal{M} * (\rho D(\phi))(x) - S_i^1$$

$$+ \frac{15\lambda}{8\pi n} \sum_{j\neq i} \mathcal{M}(x - x_j) D\phi_j - S_i^2$$

$$+ r_n^2 \frac{3\lambda}{4\pi n} \sum_{j\neq i} D(R[D\phi_j])(x - x_j)$$

$$- \frac{3\pi}{4} \lambda^2 \mathcal{M} * (\mu_2 D(\phi))(x)$$

$$+ \sum_{j\neq i} D(V[S_j^1 + S_j^2])\left(\frac{x - x_j}{r_n}\right) = \sum_{i=1}^5 E_i(x).$$

For E_4 we have for all $q \in [2, \infty)$,

$$\|E_4\|_{L^2(\bigcup B_i)}^2 = \frac{9\pi^2}{16} \lambda^4 \|\mathcal{M} * (\mu_2 D(\phi))\|_{L^2(\bigcup B_i)}^2$$

$$\le \frac{9\pi^2}{16} \lambda^4 \|\mathcal{M} * (\mu_2 D(\phi))\|_{L^q}^2 \left| \bigcup B_i \right|^{1-\frac{2}{q}}$$

$$\le C\lambda^4 \|\mu_2\|_\infty^2 \lambda^{1-\frac{2}{q}} \|D(\phi)\|_{L^q}^2, \tag{40}$$

because $\mathcal{M}$ is a Calderon–Zygmund operator. The estimate of E_5 is more difficult. We shall rely on (A1), and notably make crucial use of the following lemma, taken from [8]:

Lemma 3.3 ([8, Lem. 2.4][1]). *Under assumption* (A1)*, for all* $q \in (1, \infty)$*, there exists* $C > 0$*, such that for all* $A_1, \ldots, A_n$ *in* $\mathrm{Sym}_{3,\sigma}(\mathbb{R})$,

$$\sum_i \left| \sum_{j \neq i} r_n^3 \mathcal{M}(x_i - x_j) A_j \right|^q \leq C \lambda^{q-1} \sum_i |A_i|^q,$$

as well as

$$\sum_i \left| \sum_{j \neq i} r_n^3 \fint \mathcal{M}(x - x_j)\, dx\, A_j \right|^q \leq C \lambda^{q-1} \sum_i |A_i|^q.$$

In particular, this lemma can be applied to matrices $A_j = S_j^2$; cf. (30). We find

$$\sum_i |S_i^2|^q \leq C \sum_i \left| \sum_{j \neq i} r_n^3 \fint_{B_i} \mathcal{M}(x - x_j)\, dx\, D\phi_j \right|^q$$

$$\leq C \lambda^{q-1} \sum_i |D\phi_j|^q,$$

where *in this computation and all computations below, the constant C may change from line to line.* Moreover, by (A0),

$$\frac{1}{n} \sum_i |D\phi_j|^q \xrightarrow[n \to +\infty]{} \int_{\mathbb{R}^3} |D(\phi)|^q(x)\rho(x)\, dx \tag{41}$$

so that

$$\limsup_n \frac{1}{n} \sum_i |S_i^2|^q \leq C \lambda^{q-1} \|D(\phi)\|_{L^q}^q. \tag{42}$$

Besides Lemma 3.3, we shall also make use of the following easy generalization of Young's inequality:

$$\forall q \geq 1, \quad \sum_i \left(\sum_j |a_{ij} b_j| \right)^q \leq \max\left(\sup_i \sum_j |a_{ij}|, \sup_j \sum_i |a_{ij}| \right)^q \sum_i |b_i|^q. \tag{43}$$

We now introduce $y_i := x_i n^{-1/3}$ so that $|y_i - y_j| \geq \frac{1}{2}(c + |y_i - y_j|)$ with c the constant appearing in (A1). Using decomposition (19) and the homogeneity of each term in this decomposition, we obtain, for all $q \geq 2$,

$$\|E_5\|_{L^2(\bigcup B_i)}^2 \leq \lambda^{1-\frac{2}{q}} \|E_5\|_{L^q(\bigcup B_i)}^2$$

$$\leq C \lambda^{1-\frac{2}{q}} \left(\sum_i \int_{B_i} \left| \sum_{j \neq i} D(V[S_j^1 + S_j^2]) \left(\frac{x - x_j}{r_n} \right) \right|^q dx \right)^{\frac{2}{q}}$$

---

[1]Only the first inequality is stated in [8, Lem. 2.4], but a look at the proof shows that it follows from the second one.

$$\leq C\lambda^{1-\frac{2}{q}}\left(\sum_i |B_1|\left|\sum_{j\neq i} r_n^3 \mathcal{M}(S_j^1 + S_j^2)(x_i - x_j)\right|^q dx\right)^{\frac{2}{q}}$$

$$+ C\lambda^{1-\frac{2}{q}} r_n^8 \left(\sum_i |B_1|\left|\sum_{j\neq i} \frac{|S_j^1 + S_j^2|}{|x_i - x_j|^4}\right|^q\right)^{\frac{2}{q}}$$

$$\leq C\lambda^{1-\frac{2}{q}}\left(\sum_i |B_1|\left|\sum_{j\neq i} r_n^3 \mathcal{M}(S_j^1 + S_j^2)(x_i - x_j)\right|^q dx\right)^{\frac{2}{q}}$$

$$+ C\lambda^{1-\frac{2}{q}}\lambda^{\frac{8}{3}}\left(\sum_i |B_1|\left|\sum_j \frac{|S_j^1 + S_j^2|}{c + |y_i - y_j|^4}\right|^q\right)^{\frac{2}{q}}.$$

We can then apply Lemma 3.3 to the first term, and apply (43) to the second term (together with the fact that $\sum_i \frac{1}{c+|y_i-y_j|^4}$ is uniformly bounded in j). We obtain

$$\|E_5\|_{L^2(\bigcup B_i)}^2 \leq C\lambda^{1-\frac{2}{q}}\left(|B_1|\lambda^{q-1}\sum_i |S_i^1 + S_i^2|^q\right)^{\frac{2}{q}}$$

$$+ C\lambda^{\frac{8}{3}}\lambda^{1-\frac{2}{q}}\left(\sum_i |B_1|\,|S_i^1 + S_i^2|^q\right)^{\frac{2}{q}}$$

$$\leq C\lambda^{3-\frac{2}{q}}\left(\frac{1}{n}\sum_i |S_i^1 + S_i^2|^q\right)^{2/q}$$

for any $q \geq 2$, where the last bound comes from Hölder's inequality. It remains to bound $\frac{1}{n}\sum_i |S_i^1 + S_i^2|^q$. By (29), we have

$$|S_i^1|^q \leq C\frac{\lambda^q}{|B_1|^q}\left(\int_{B_i} |M \star (\rho D(\phi))|\right)^q \leq C\frac{\lambda^q}{|B_1|}\int_{B_i} |M \star (\rho D(\phi))|^q,$$

so that

$$\frac{1}{n}\sum_i |S_i^1|^q \leq C\lambda^{q-1}\|M \star (\rho D(\phi))\|_{L^q}^q \leq C\lambda^{q-1}\|D(\phi)\|_{L^q}^q,$$

where we used the L^q continuity of the convolution with $\mathcal{M}$ again. Combining this inequality with (42), and injecting in the bound for E_5, we get that for all $q \geq 2$,

$$\limsup_n \|E_5\|_{L^2(\bigcup B_i)}^2 \leq C\lambda^{5-\frac{4}{q}}\|D(\phi)\|_{L^q}^2. \tag{44}$$

We now turn to E_3. We use the fact that $D(R[A])(x) = O(|A|\,|x|^{-5})$. We find

$$\|E_3\|_{L^2(\bigcup B_i)}^2 \leq C|B_1|\sum_i \left|\lambda^{\frac{5}{3}}\sum_{j\neq i}\frac{|D\phi_j|}{(c + |y_i - y_j|)^5}\right|^2$$

$$\leq C|B_1|\lambda^{\frac{10}{3}}\sum_i |D\phi_j|^2,$$

where the last line follows from (43). Using again (41), we get eventually

$$\limsup_n \|E_3\|_{L^2(\bigcup B_i)}^2 \leq C\lambda^{\frac{13}{3}} \|D(\phi)\rho^{\frac{1}{2}}\|_{L^2}^2 \leq C\lambda^{\frac{13}{3}} \|D(\phi)\|_{L^q}^2 \quad \forall q \geq 2. \tag{45}$$

We recall now the expressions for E_1 and E_2 on B_i:

$$E_1(x) = -\frac{15}{8\pi}\lambda\left(\int_{\mathbb{R}^3} \mathcal{M}(x-y)D(\phi)(y)\rho(y)\,dy\right.$$
$$\left. - \fint_{B_i}\int_{\mathbb{R}^3} \mathcal{M}(z-y)D(\phi)(y)\rho(y)\,dy\,dz\right),$$

$$E_2(x) = \frac{15}{8\pi}\lambda\left(\frac{1}{n}\sum_{j\neq i}\mathcal{M}(x-x_j)D\phi_j - \fint_{B_i}\frac{1}{n}\sum_{j\neq i}\mathcal{M}(z-x_j)D\phi_j\,dz\right).$$

We claim first that

$$\limsup_n \|E_1\|_{L^2(\bigcup B_i)} = 0. \tag{46}$$

To prove (46), the idea is to regularize $D(\phi)\rho$ using a convolution with a mollifier χ_η. Denoting by $\|\ \|_{0,\mu}$ the Hölder seminorm of exponent μ, we get

$$\|(D(\phi)\rho) \star \chi_\eta\|_{0,\mu} \leq C\|(D(\phi)\rho) \star \chi_\eta\|_{L^\infty}^{1-\mu}\|(D(\phi)\rho) \star \nabla\chi_\eta\|_{L^\infty}^{\mu}$$
$$\leq \frac{C}{\eta^{\mu+\frac{3}{q}}}\|D(\phi)\rho\|_{L^q}$$

for all $q \geq 1$. Hence we get, using the fact that $\mathcal{M}$ is a Zygmund–Calderon kernel, for all $q \geq 2$ and any $\mu \in (0,1)$,

$$\sum_i \int_{B_i} |E_1(x)|^2\,dx \leq C\lambda^2\sum_i\int_{B_i}|\mathcal{M}\star(D(\phi)\rho - D(\phi)\rho\star\chi_\eta)|^2$$
$$+ C\lambda^2 n|B_1|r_n^{2\mu}\|(D(\phi)\rho)*\chi_\eta\|_{0,\mu}^2$$
$$\leq \lambda^{3-\frac{2}{q}}\|D(\phi)\rho - (D(\phi)\rho)*\chi_\eta\|_q^2 + \lambda^3\frac{r_n^{2\mu}}{\eta^{2\mu+6/q}}\|D(\phi)\rho\|_{L^q}^2,$$

which vanishes when taking the lim sup in n for fixed η and then taking the limit $\eta \to 0$.

Finally, we bound E_2 by

$$\|E_2\|_{L^2(\bigcup B_i)}^2 \leq C\frac{\lambda^2}{n^2}\sum_i\int_{B_i}\left|\sum_{j\neq i}\left(\sup_{z\in B_i}|\nabla\mathcal{M}(z-x_j)|\right)r_n|D\phi_j|\right|^2$$
$$\leq C\frac{\lambda^3}{n^3}\sum_i\left|\sum_{j\neq i}\frac{r_n}{|x_i-x_j|^4}|D\phi_j|\right|^2$$
$$\leq C\lambda^{\frac{11}{3}}\frac{1}{n}\sum_i\left|\sum_j\frac{1}{(c+|y_i-y_j|)^4}|D\phi_j|\right|^2$$
$$\leq C\lambda^{\frac{11}{3}}\frac{1}{n}\sum_i|D\phi_i|^2,$$

where the last line comes from (43). We obtain

$$\limsup_n \|E_2\|_{L^2(\bigcup B_i)}^2 \le C\lambda^{\frac{11}{3}} \int_{\mathbb{R}^3} |D(\phi)|^2 \rho \le C\lambda^{\frac{11}{3}} \|D(\phi)\|_{L^q}^2 \quad \forall q \ge 2. \qquad (47)$$

By the combination of (46), (47), (45), (40), (44), we find

$$\limsup_n \|\nabla\widetilde{\psi}_n^2\|_{L^2(\bigcup B_i)} \le C_q \big(\lambda^{\frac{5}{2}-\frac{2}{q}} + \lambda^{\frac{11}{6}}\big)\|D(\phi)\|_{L^q} \quad \forall q \ge 2.$$

This inequality, together with (39) and (38), concludes the proof of Proposition 3.1.

3.2. Estimate of ϕ_n^1

The goal of this subsection is to show the following proposition:

Proposition 3.4. *For all $q \ge 1$,*

$$\limsup_n \left| \int_{\mathbb{R}^3} g\phi_n^1 \right| \le C\lambda^{\frac{7}{3}} \|D(\phi)\|_{L^q}.$$

The main estimate (27) is then an easy consequence of Propositions 3.2 and 3.4, which, as explained before (27), concludes the proof of Theorem 1.1.

The derivation of Proposition 3.4 will rely strongly on our assumption (A2). First, let us recall that ϕ_n is bounded uniformly in n in $\dot{H}^1$, and so is ϕ_n^2 by Proposition 3.1. It follows that ϕ_n^1 is bounded uniformly in n in $\dot{H}^1$. Hence, by a density argument, it is enough to establish the inequality in Proposition 3.4 for a C_c^∞ field g, as long as the constant C on the right-hand side only involves $\|g\|_{L^{3+\varepsilon}}$. From now on, we assume g to be smooth and compactly supported.

We introduce again the solution (u_g, p_g) of (36). As g is smooth, so is u_g. Moreover, by elliptic regularity and Sobolev embedding, we have for a compact K containing all the balls,

$$\|D(u_g)\|_{L^\infty(K)} \lesssim \|D(u_g)\|_{W^{1,3+\varepsilon}(K)} \lesssim \|g\|_{L^{3+\varepsilon}}. \qquad (48)$$

With (31), we compute

$$\int_{\mathbb{R}^3} \phi_n^1 \cdot g = \int_{\mathbb{R}^3} \phi_n^1 \cdot (-\Delta u_g + \nabla p_g) = \int_{\mathbb{R}^3} D(\phi_n^1) : D(u_g)$$

$$= -\int_{\mathbb{R}^3} \operatorname{div}(\sigma(\phi_n^1, q_n^1)) \cdot u_g$$

$$= -5\lambda \int_{\mathbb{R}^3} \left(\rho D(\phi) - \frac{1}{n}\sum_i \frac{1}{|B_i|} 1_{B_i} D\phi_i \right) : D(u_g)$$

$$- 2\lambda^2 \int_{\mathbb{R}^3} \left(\mu_2 D(\phi) - \frac{75}{16\pi}\frac{1}{n^2}\sum_i \frac{1}{|B_i|} 1_{B_i} \sum_{j\ne i} \fint_{B_i} \mathcal{M}(x - x_j) D\phi_j \, dx \right.$$

$$\left. + \frac{75}{16\pi}\frac{1}{n}\sum_i \frac{1}{|B_i|} 1_{B_i} \fint_{B_i} \int_{\mathbb{R}^3} \mathcal{M}(x - y) D(\phi)(y)\rho(y) \, dy \, dx \right) : D(u_g).$$

We set

$$\bar{\mu}_{2,n} := \frac{75}{16\pi}\left(\frac{1}{n^2}\sum_{i\neq j}\left(\fint_{B_i}\mathcal{M}(\cdot - x_j)\right)\frac{1}{|B_i|}1_{B_i}(x)\,dx\,\delta_{y_j}(dy)\right.$$

$$\left.-\sum_i\frac{1}{n}\left(\fint_{B_i}\mathcal{M}(\cdot - y)\right)\frac{1}{|B_i|}1_{B_i}(x)\,dx\,\rho(y)\,dy\right).$$

See (6) for comparison. We have

$$\int_{\mathbb{R}^3}\phi_n^1\cdot g = -5\lambda\int_{\mathbb{R}^3}\left(\rho D(\phi) - \frac{1}{n}\sum_i\frac{1}{|B_1|}1_{B_i}D\phi_i\right):D(u_g)$$

$$-2\lambda^2\big\langle(\mu_2\delta_{x=y} - \bar{\mu}_{2,n}), D(\phi)(y)\otimes D(u_g)(x)\big\rangle =: -5\lambda R_1 + 2\lambda^2 R_2,$$

with the notation that for M a distribution over $\mathbb{R}^3\times\mathbb{R}^3$ with values in $\mathrm{Sym}(\mathrm{Sym}_{3,\sigma}(\mathbb{R}))$,

$$\langle M, S\otimes S'\rangle := \langle M_{ijkl}, S_{ij}\otimes S'_{kl}\rangle.$$

From assumption (A0), it easily follows that

$$\lim_n R_1 = 0.$$

Then we write

$$R_2 = \big\langle(\mu_2\delta_{x=y} - \mu_{2,n}), D(\phi)(y)\otimes D(u_g)(x)\big\rangle + \tilde{R}_2,$$

with

$$\tilde{R}_2 = \big\langle(\mu_{2,n} - \bar{\mu}_{2,n}), D(\phi)(y)\otimes D(u_g)(x)\big\rangle.$$

The first term on the right-hand side goes to zero as $n\to +\infty$, by assumption (A2). For the remainder, we decompose it as

$$\tilde{R}_2 = \frac{75}{16\pi n^2}\sum_{i\neq j}\left(\left(\fint_{B_i}\mathcal{M}(\cdot - x_j)\right) - \mathcal{M}(x_i - x_j)\right)D\phi_j:\left(\fint_{B_i}D(u_g)\right)$$

$$+\frac{75}{16\pi n^2}\sum_{i\neq j}\mathcal{M}(x_i - x_j)D\phi_j:\left(\left(\fint_{B_i}D(u_g)\right) - D(u_g)(x_i)\right)$$

$$-\frac{75}{16\pi}\int_{\mathbb{R}^3}(\mathcal{M}\star(D(\phi)\rho))(x)\left(\frac{1}{n}\sum_i\frac{1}{|B_i|}\left(\fint_{B_i}D(u_g)\right)1_{B_i} - D(u_g)\rho\right)(x)\,dx$$

$$=: \tilde{R}_{2,1} + \tilde{R}_{2,2} + \tilde{R}_{2,3}.$$

The first term is bounded by

$$|\tilde{R}_{2,1}| \leq \frac{75}{16\pi n^2}\sum_{i\neq j}\sup_{z\in B_i}|\nabla\mathcal{M}(z - x_j)|\,r_n|D\phi_j|\,\|D(u_g)\|_{L^\infty(K)}$$

$$\leq \frac{C}{n^2}\sum_{i\neq j}\frac{r_n}{|x_i - x_j|^4}|D\phi_j| \leq \frac{C\lambda^{\frac{1}{3}}}{n}\sum_{i\neq j}\frac{1}{(c+|y_i - y_j|)^4}|D\phi_j|$$

$$\leq \frac{C\lambda^{\frac{1}{3}}}{n}\sum_i |D\phi_i|,$$

where the last line follows from (43). By (A0),

$$\limsup_n |\widetilde{R}_{2,1}| \leq C\lambda^{\frac{1}{3}}\int_{\mathbb{R}^3}|D(\phi)|\rho \leq C\lambda^{\frac{1}{3}}\|D(\phi)\|_{L^q} \quad \forall q \geq 1.$$

Note that the constant C may be chosen so that it depends on g only through $\|g\|_{L^{3+\varepsilon}}$, using (48). The second term is bounded by

$$|\widetilde{R}_{2,2}| \leq \frac{75}{16\pi n^2}\sum_i \left|\sum_{j\neq i}\mathcal{M}(x_i - x_j)D\phi_j\right|r_n\|\nabla^2 u_g\|_{L^\infty}$$

$$\leq \frac{C}{n^2 r_n^2}\sum_i \left|\sum_{j\neq i}r_n^3\mathcal{M}(x_i - x_j)D\phi_j\right| \leq \frac{C}{n^{\frac{3}{2}}r_n^2}\left(\sum_i \left|\sum_{j\neq i}r_n^3\mathcal{M}(x_i - x_j)D\phi_j\right|^2\right)^{1/2}$$

$$\leq \frac{C}{n r_n^2}\left(\frac{1}{n}\sum_i |D\phi_i|^2\right)^{1/2} \leq \frac{C\lambda}{n^{\frac{1}{3}}}\left(\int_{\mathbb{R}^3}|D(\phi)|^2\rho\right)^{1/2},$$

where the fourth (resp. fifth) inequality is a consequence of Lemma 3.3 (resp. (A0)). Hence,

$$\limsup_n |\widetilde{R}_{2,2}| = 0.$$

As regards the last term $\widetilde{R}_{2,3}$, we use the fact that, under (A0),

$$\frac{1}{n}\sum_i \frac{1}{|B_i|}\left(\fint_{B_i}D(u_g)\right)1_{B_i} \xrightarrow[n\to+\infty]{D} (u_g)\rho \quad \text{weakly* in } L^1.$$

See [9, Lem. 5] for a proof. Hence,

$$\limsup_n |\widetilde{R}_{2,3}| = 0.$$

Proposition 3.4 follows.

4. Discussion of the mean-field limit

In this final section we come back to the main assumption (A2): we discuss examples of particle distributions (x_i) for which $\mu_{2,n}$ converges as in (A2), and discuss how to compute the limit μ_2, that is, the $O(\lambda^2)$ correction to the effective viscosity. This discussion is closely related to the analysis performed in [8], and relies on results established there.

First, one notices that the sequence of compactly supported distributions $\mu_{2,n}$ defined in (6) is bounded with respect to n. More precisely, we have the following lemma:

Lemma 4.1. *There exists $C > 0$, such that for any smooth $F = F(x, y)$, for any n,*

$$|\langle \mu_{2,n}, F \rangle| \leq C \|F\|_{C^1(K \times K)},$$

where K is any fixed convex compact set containing $x_1, \ldots, x_n$ for all n.

Proof. We need to bound

$$\frac{1}{n^2} \sum_{i \neq j} \mathcal{M}(x_i - x_j) F(x_i, x_j) = \frac{1}{n^2} \sum_{i \neq j} \mathcal{M}(x_i - x_j) F(x_j, x_j)$$

$$+ \frac{1}{n^2} \sum_{i \neq j} \mathcal{M}(x_i - x_j)(F(x_i, x_j) - F(x_j, x_j))$$

$$=: I_{1,n} + I_{2,n}.$$

As $\mathcal{M}$ is homogeneous of degree -3 and $|F(x_i, x_j) - F(x_j, x_j)| \leq \|\nabla F\|_{L^\infty(K \times K)} \times |x_i - x_j|$,

$$|I_{2,n}| \leq \frac{C}{n^2} \|\nabla F\|_{L^\infty} \sum_{i \neq j} \frac{1}{|x_i - x_j|^2}.$$

We introduce $y_i := x_i n^{-1/3}$ again, so that $|y_i - y_j| \geq \frac{1}{2}(c + |y_i - y_j|)$ with c the constant appearing in (A1). We obtain

$$|I_{2,n}| \leq \frac{C}{n^2} n^{\frac{2}{3}} \|\nabla F\|_{L^\infty} \sum_{i,j} \frac{1}{(c + |y_i - y_j|)^2} \leq C \|\nabla F\|_{L^\infty},$$

using that for each i, under assumption (A1),

$$\sum_j \frac{1}{(c + |y_i - y_j|)^2} \leq C \int_{n^{1/3} K} \frac{1}{(c + |y_i - z|)^2} \, dz \leq C n^{\frac{1}{3}}.$$

To bound $I_{1,n}$, we introduce $s_n := c/(4n^{\frac{1}{3}})$, where c is again the constant appearing in (A1). We use the splitting

$$I_{1,n} = \frac{1}{n^2} \sum_{i \neq j} \left(\mathcal{M}(x_i - x_j) - \fint_{B(x_j, s_n)} \mathcal{M}(x_i - y) \, dy \right) F(x_j, x_j)$$

$$+ \frac{1}{n^2} \sum_{i \neq j} \left(\fint_{B(x_j, s_n)} \mathcal{M}(x_i - y) \, dy - \fint_{B(x_i, s_n)} \fint_{B(x_j, s_n)} \mathcal{M}(x - y) \, dx \, dy \right) F(x_j, x_j)$$

$$+ \frac{1}{n^2} \sum_{i \neq j} \fint_{B(x_i, s_n)} \fint_{B(x_j, s_n)} \mathcal{M}(x - y) \, dx \, dy \, F(x_j, x_j) =: J_{1,n} + J_{2,n} + J_{3,n}.$$

We find

$$|J_{1,n}| \leq \frac{C}{n^2} \sum_{i \neq j} \sup_{z \in B(x_j, s_n)} |\nabla \mathcal{M}(x_i - \cdot)| s_n \|F\|_{L^\infty}$$

$$\leq \frac{C s_n}{n^2} \sum_{i \neq j} \frac{1}{|x_i - x_j|^4} \|F\|_{L^\infty}$$

$$\leq \frac{C}{n} \sum_{i \neq j} \frac{1}{(c + |y_i - y_j|)^4} \|F\|_{L^\infty} \leq C \|F\|_{L^\infty}.$$

Similarly, we find $|J_{2,n}| \leq C$. Then, let χ be a truncation that is 0 on $B(0, 1)$ and 1 outside $B(0, 2)$. The last term can be written

$$J_{3,n} = \frac{1}{n^2} \int_{\mathbb{R}^3} \left(\left(\chi\left(\frac{\cdot}{s_n}\right) \mathcal{M} \right) \star \sum_i 1_{B(x_i, s_n)} \right) \sum_j F(x_j, x_j) 1_{B(x_j, s_n)}$$

so that

$$|J_{3,n}| \leq \frac{C}{n^2} \left\| \sum_i 1_{B(x_i, s_n)} \right\|_{L^2} \left\| \sum_j F(x_j, x_j) 1_{B(x_j, s_n)} \right\|_{L^2} \leq C \|F\|_{L^\infty}$$

using the Calderon–Zygmund theorem. This concludes the proof. ∎

Combining the previous lemma with the density of $\mathrm{span}(\{\varphi \otimes \tilde{\varphi}, \varphi, \tilde{\varphi} \in C^\infty(K)\})$ in $C^\infty(K \times K)$, we deduce that for $\mu_2 \in L^\infty(\mathbb{R}^3, \mathrm{Sym}(\mathrm{Sym}_{3,\sigma}(\mathbb{R})))$, assumption (A2) is satisfied if and only if for all smooth $\varphi, \tilde{\varphi}$,

$$\langle \mu_{2,n}, \varphi \otimes \tilde{\varphi} \rangle \to \int_{\mathbb{R}^3} \mu_2(x) \varphi(x) \tilde{\varphi}(x) \, dx \tag{49}$$

or, equivalently, if and only if for all $S \in \mathrm{Sym}_{3,\sigma}(\mathbb{R})$, for all smooth $\varphi, \tilde{\varphi}$,

$$\frac{75\mu}{16\pi} \left(\frac{1}{n^2} \sum_{i \neq j} g_S(x_i - x_j) \varphi(x_i) \tilde{\varphi}(x_j) - \int_{\mathbb{R}^3} g_S(x - y) \varphi(x) \tilde{\varphi}(y) \, dx \, dy \right)$$
$$\to \int_{\mathbb{R}^3} (\mu_2(x) S : S) \varphi(x) \tilde{\varphi}(x) \, dx,$$

where

$$g_S(x) = \mathcal{M}(x) S : S; \tag{50}$$

cf. (5). Since g_S is even, this amounts to the following: for all $S \in \mathrm{Sym}_{3,\sigma}(\mathbb{R})$, for all smooth φ,

$$\frac{75\mu}{16\pi} \left(\sum_{i \neq j} g_S(x_i - x_j) \varphi(x_i) \varphi(x_j) - \int_{\mathbb{R}^3} g_S(x - y) \varphi(x) \varphi(y) \, dx \, dy \right)$$
$$\to \int_{\mathbb{R}^3} (\mu_2(x) S : S) |\varphi(x)|^2 \, dx. \tag{51}$$

Of course, the existence of a nontrivial limit on the right-hand side of (51) is directly related to the singularity of g_s at zero. If g_s were smooth or not too singular, one could show that the limit on the right-hand side is zero.

Our first goal is to understand configurations of points (x_i) for which the limit as $n \to +\infty$ of $\frac{1}{n^2} \sum_{i \neq j} g_S(x_i - x_j)\varphi(x_i)\varphi(x_j)$ exists. This is closely related to the study in [8], in which the special case $\varphi = 1$ is investigated. This study relies on ideas from homogenization theory, together with the notion of renormalized energy, described in [17] in the context of Coulomb gases. We shall explain how the approach in [8] adapts to a general φ. One proceeds in several steps.

Step 1. Expression of the mean field as a (renormalized) energy

Proceeding as in [8, Lem. 3.1], one can restrict to the case where

$$x_i \in \mathcal{O} \quad \forall 1 \leq i \leq n.$$

Under this assumption, proceeding as in [8, Prop. 3.3], one shows that (51), and thus (A2), amounts to the following: for all smooth φ, for all $S \in \mathrm{Sym}_{3,\sigma}(\mathbb{R})$,

$$\lim_n . \mu \, \mathcal{W}_n[\varphi] = \int_{\mathbb{R}^3} (\mu_2(x)S : S)|\varphi(x)|^2 \, dx, \tag{52}$$

where

$$\mathcal{W}_n[\varphi] := \frac{75}{16\pi} \int_{\mathbb{R}^6 \setminus \mathrm{Diag}} g_S(x - y)\varphi(x)\big(\rho_n(dx) - \rho(x)\,dx\big)\varphi(y)\big(\rho_n(dy) - \rho(y)\,dy\big). \tag{53}$$

To investigate the convergence of $\mathcal{W}_n[\varphi]$, the main idea is then to associate to this quadratic quantity an energy. This idea is based on the following lemma:

Lemma 4.2 ([8, Lem. 2.2]). *For any $f \in L^2(\mathbb{R}^3)$,*

$$\int_{\mathbb{R}^6} g_S(x - y)f(x)f(y)\,dx\,dy = -\frac{16\pi}{3} \int_{\mathbb{R}^3} |D(u_S)|^2,$$

where u_S is the solution of the Stokes equation

$$-\Delta u_S + \nabla q_S = \mathrm{div}(Sf) = S\nabla f, \quad \mathrm{div}\, u_S = 0 \quad in \ \mathbb{R}^3.$$

Considering definition (53), it is tempting to replace f by $\varphi(\rho_n - \rho)$ in the lemma, which would yield the formal identity

$$``\int_{\mathbb{R}^6} g_S(x - y)\varphi(x)\big(\rho_n(dx) - \rho(x)\,dx\big)\varphi(y)\big(\rho_n(dy) - \rho(y)\,dy\big)$$
$$= -\frac{16\pi}{3n^2} \int_{\mathbb{R}^3} |D(h_n)|^2 \text{''}, \tag{54}$$

where h_n solves the equation

$$-\Delta h_n + \nabla p_n = n\,\mathrm{div}(S\varphi(\rho_n - \rho)), \quad \mathrm{div}\, h_n = 0 \quad in \ \mathbb{R}^3. \tag{55}$$

The solution of this equation decaying to zero is explicit, and given by

$$h_n := \sum \varphi(x_i)G_S(x - x_i) - n\,\mathcal{U} \star S\nabla(\varphi\rho), \tag{56}$$

where $G_S(x) = (S\nabla) \cdot \mathcal{U}(x) = -\frac{3}{8\pi}(Sx \cdot x)\frac{x}{|x|^5}$, while the corresponding pressure field is

$$p_n = \sum \varphi(x_i)p_S(x - x_i) - n\mathcal{P} \star S\nabla(\varphi\rho),$$

where $p_S(x) = (S\nabla) \cdot \mathcal{P}(x) = -\frac{3}{4\pi}\frac{(Sx \cdot x)}{|x|^5}$.

However, this formal identity is not justified, as both sides are infinite: the left-hand side is infinite due to the singularity of the kernel at the diagonal (which is excluded in definition (53)), and the right-hand side is infinite because the solution h_n of (55) is not in $\dot{H}^1(\mathbb{R}^3)$, as ρ_n is an atomic measure. Still, one can interpret $\mathcal{W}_n[\varphi]$ in terms of a so-called renormalized energy, intensively studied in the context of Coulomb gases and Ginzburg–Landau systems. See [17] and the references therein for more. One must first regularize the singular measure. More precisely, following [8], we consider for all $\eta > 0$, the field G_S^η satisfying

$$G_S^\eta = G_S, \quad |x| \geq \eta, \quad -\Delta G_S^\eta + \nabla p_S^\eta = 0, \quad \text{div } G_S^\eta = 0, \quad |x| < \eta,$$

Note that the condition $G_S^\eta = G_S$ at $|x| = \eta$ contained in the first condition can be seen as a Dirichlet condition for the Stokes problem satisfied by G_S^η inside $\{|x| < \eta\}$. Let us stress that G_S^η belongs to $H_{\text{loc}}^1(\mathbb{R}^3)$, and can be seen as a regularization of G_S. Accordingly, we set

$$h_n^\eta := \sum_i \varphi(x_i)G_S^\eta(x - x_i) - n\mathcal{U} \star S\nabla(\varphi\rho). \tag{57}$$

As shown in [8, Lem. 3.5], it solves the modified Stokes equation

$$-\Delta h_n^\eta + \nabla p_n^\eta = \sum_i \varphi(x_i)\text{div}(\Psi^\eta(\cdot - x_i)) - n\,\text{div}(S\varphi\rho), \quad \text{div } h_n^\eta = 0 \quad \text{in } \mathbb{R}^3, \tag{58}$$

where

$$\Psi^\eta := \begin{cases} \frac{3}{\pi\eta^5}\left(Sx \otimes x + x \otimes Sx - 5\frac{|x|^2}{2}S + \frac{5}{4}\eta^2 S\right) \\ \quad -2D(G_S^\eta)(x) + p_S^\eta(x)\text{Id}, & |x| < \eta, \\ 0, & |x| > \eta. \end{cases} \tag{59}$$

A main advantage of this regularization is that $h_n - h_n^\eta$ is supported in $\bigcup_i B(x_i, \eta)$. The main result, which is a straightforward adaptation of [8, Props. 3.7 and 3.8] (dedicated to the special case $\varphi = 1$) is given by the following proposition:

Proposition 4.3 (Renormalized energy formula). *We have*

$$\mathcal{W}_n[\varphi] = -\frac{25}{2}\lim_{\eta \to 0}\left(\frac{1}{n^2}\int_{\mathbb{R}^3}|\nabla h_n^\eta|^2 - \frac{1}{n^2\eta^3}\left(\int_{B(0,1)}|\nabla G_S^1|^2 + \frac{3}{10\pi}|S|^2\right)\sum_i|\varphi(x_i)|^2\right).$$

More precisely, for $\eta < \frac{c}{2}n^{-\frac{1}{3}}$, where c is the constant appearing in (A1)*, one has*

$$\left|\mathcal{W}_n[\varphi] + \frac{25}{2}\left(\frac{1}{n^2}\int_{\mathbb{R}^3}|\nabla h_n^\eta|^2 - \frac{1}{n^2\eta^3}\left(\int_{B(0,1)}|\nabla G_S^1|^2 + \frac{3}{10\pi}|S|^2\right)\sum_i|\varphi(x_i)|^2\right)\right| \leq C\eta.$$

The first formula in this proposition is the rigorous translation of the false identity (54): it connects the mean field $\mathcal{W}_n[\varphi]$ to an energy. However, two differences are apparent. First, the energy of h_n being infinite, it is replaced by the energy of the regularization h_n^η. Second, and more importantly, one must subtract the term

$$\frac{1}{\eta^3}\left(\int_{B(0,1)}|\nabla G_S^1|^2 + \frac{3}{10\pi}|S|^2\right)\sum_i|\varphi(x_i)|^2,$$

which corresponds to the energy of self-interaction, and blows up as $\eta \to 0$. It is only the difference of the two quantities that does not blow up as $\eta \to 0$ and gives $\mathcal{W}_n[\varphi]$. Note that the removal of the self-interaction is coherent with the fact that the expression for $\mathcal{W}_n[\varphi]$ excludes the diagonal; see (53).

Step 2. Existence of the mean-field limit for periodic and random particle configurations

Thanks to the second formula in Proposition 4.3, we deduce that

$$\lim_{n\to+\infty}\mathcal{W}_n[\varphi] = -\frac{25}{2}\lim_{n\to+\infty}\frac{1}{n^2}\int_{\mathbb{R}^3}|\nabla h_n^{\eta_n}|^2$$
$$-\frac{1}{n^2\eta_n^3}\left(\int_{B(0,1)}|\nabla G_S^1|^2 + \frac{3}{10\pi}|S|^2\right)\sum_i|\varphi(x_i)|^2,$$

when $\eta_n = \tilde\eta n^{-\frac{1}{3}}$, for any fixed $\tilde\eta < \frac{c}{2}$. Note that for such a scaling $\eta \sim n^{-\frac{1}{3}}$, contrary to the case where η goes to zero at fixed n, the second term has a finite limit: namely, by (A0),

$$\lim_n \frac{1}{n^2\eta^3}\left(\int_{B(0,1)}|\nabla G_S^1|^2 + \frac{3}{10\pi}|S|^2\right)\sum_i|\varphi(x_i)|^2$$
$$= \frac{1}{(\tilde\eta)^3}\left(\int_{B(0,1)}|\nabla G_S^1|^2 + \frac{3}{10\pi}|S|^2\right)\int_{\mathbb{R}^3}|\varphi|^2\rho.$$

Hence, the main point is to understand the limit of $\frac{1}{n^2}\int_{\mathbb{R}^3}|\nabla h_n^{\eta_n}|^2$. In the case $\varphi = 1$, this is analyzed in detail in [8], for two classes of point configurations: periodic and random stationary. Namely, in [8] one considers particles $x_1,\ldots,x_n$ given by

$$\{x_1,\ldots,x_n\} := \varepsilon\omega \cap \mathcal{O}, \quad \varepsilon \ll 1$$

for ω an infinite locally finite subset of $\mathbb{R}^3$, of two possible kinds:

(i) *Periodic patterns:* $\omega = \{z_1,\ldots,z_m\} + \mathbb{Z}^3$ for $z_1,\ldots,z_m$ distinct points of $(0,1)^3$, such that $|z - z'| > cm^{-\frac{1}{3}}$ for all $z \neq z' \in \omega$, where c is the constant in (A1).

(ii) *Stationary point process:* ω is the realization of an ergodic stationary point process, of mean intensity m, satisfying the hardcore condition $|z - z'| > cm^{-\frac{1}{3}}$ for all $z \neq z' \in \omega$.

Note that in the periodic case, the number n of particles in $\mathcal{O}$ is equivalent to $m\varepsilon^{-3}$ as $\varepsilon \to 0$. In the random case, this number n is random, but is almost surely still equivalent to $m\varepsilon^{-3}$ by the ergodic theorem (under an additional convexity assumption on $\mathcal{O}$). In particular, as $\varepsilon \to 0$, one has $n \to +\infty$ and one can consider the asymptotic behaviour of $\mathcal{W}_n[\varphi]$ or, as said before, of $\frac{1}{n^2} \int_{\mathbb{R}^3} |\nabla h_n^{\eta n}|^2$. Note that in both cases, assumption (A0) is satisfied with $\rho = 1_{\mathcal{O}}$, and for ε small enough (or n large enough), (A1) is also satisfied.

The limit of the renormalized energy can then be tackled through homogenization techniques. Indeed, we claim that equation (58) is close to systems of the form

$$- \Delta h^\varepsilon + \nabla p^\varepsilon = \varepsilon^{-3} \operatorname{div}\big(f(x)(F(x/\varepsilon) - \bar{F})\big), \quad \operatorname{div} h^\varepsilon = 0, \tag{60}$$

where $\bar{F}$ is the mean of F (in the periodic or random sense). Roughly, one has the correspondence

$$\varepsilon \approx n^{-\frac{1}{3}}, \quad F \approx \sum_{z \in \omega} \Psi^{\bar{\eta}}(\cdot - z), \quad \bar{F} \approx 1, \quad f \approx 1_{\mathcal{O}}\varphi. \tag{61}$$

The point is that for a model of type (60), the behaviour of the solution h^ε is well understood. For instance, in the periodic case one has

$$h^\varepsilon \approx \varepsilon^{-2} g(x) H(x/\varepsilon),$$

where $H = H(y)$ is a corrector, solving

$$- \Delta H + \nabla P = \operatorname{div} F, \quad \operatorname{div} H = 0, \tag{62}$$

with periodic boundary conditions. In particular, one can show that

$$\varepsilon^6 \int_{\mathbb{R}^3} |\nabla h^\varepsilon|^2 \to \int_{(0,1)^3} |\nabla H(y)|^2 \, dy \int_{\mathbb{R}^3} |g(x)|^2 \, dx.$$

As shown in [8] in the special case $\varphi = 1$, a similar situation holds in the context of system (58). There is a corrector problem similar to (62), that takes the form

$$- \Delta H^{\bar{\eta}} + \nabla P^{\bar{\eta}} = \operatorname{div} \sum_{z \in \omega} \Psi^{\bar{\eta}}(\cdot - z), \quad \operatorname{div} H^{\bar{\eta}} = 0 \quad \text{in } \mathbb{R}^3, \tag{63}$$

where $0 < \bar{\eta} < \frac{c}{2} m^{-\frac{1}{3}}$. This can be solved in both periodic and random stationary frameworks, as shown in [8]. Roughly,

(i)   in the periodic case, there is a unique (up to an additive constant) weak solution $H^{\bar{\eta}}$ in $H^1_{\mathrm{loc}}(\mathbb{R}^3)$, which is $\mathbb{Z}^d$ periodic and mean-free over a period;

(ii)   in the random stationary case, there is almost surely in ω a weak solution $H^{\bar{\eta}}(\cdot, \omega)$ in $H^1_{\mathrm{loc}}(\mathbb{R}^3)$ with a uniquely defined stationary and mean-free gradient. More precisely, $\nabla H^{\bar{\eta}}(y, \omega) = D_H(\omega - y)$, where D_H is an $\mathbb{R}^3$-valued L^2 random variable with zero mean, the unique solution of a probabilistic variational formulation; see [8] for details.

One can then express our mean field limit in terms of the energy of this corrector. This is the following proposition:

Proposition 4.4. *Let $\bar{\eta} < \frac{c}{2}m^{-\frac{1}{3}}$. Then, as $\varepsilon \to 0$, one has $n \to +\infty$, and*

$$\mathcal{W}_n[\varphi] \to \frac{25}{2}\left(-\frac{1}{m^2}\int_{(0,1)^3}|\nabla H^{\bar{\eta}}|^2 + \frac{1}{(\bar{\eta})^3m}\left(\int_{B(0,1)}|\nabla G^1_S|^2 + \frac{3}{10\pi}|S|^2\right)\right)\int_{\mathcal{O}}|\varphi|^2$$

in the periodic case, whereas

$$\mathcal{W}_n[\varphi] \to \frac{25}{2}\left(-\frac{1}{m^2}\mathbb{E}|\nabla H^{\bar{\eta}}(0,\cdot)|^2 + \frac{1}{(\bar{\eta})^3m}\left(\int_{B(0,1)}|\nabla G^1_S|^2 + \frac{3}{10\pi}|S|^2\right)\right)\int_{\mathcal{O}}|\varphi|^2$$

in the random stationary case. In particular, assumption (A2) *is satisfied with*

$$\mu_2 S : S = \frac{25\mu}{2}\left(-\frac{1}{m^2}\int_{(0,1)^3}|\nabla H^{\bar{\eta}}|^2 + \frac{1}{(\bar{\eta})^3m}\left(\int_{B(0,1)}|\nabla G^1_S|^2 + \frac{3}{10\pi}|S|^2\right)\right) \quad (64)$$

in the periodic case, whereas

$$\mu_2 S : S = \frac{25\mu}{2}\left(-\frac{1}{m^2}\mathbb{E}|\nabla H^{\bar{\eta}}(0,\cdot)|^2 + \frac{1}{(\bar{\eta})^3m}\left(\int_{B(0,1)}|\nabla G^1_S|^2 + \frac{3}{10\pi}|S|^2\right)\right) \quad (65)$$

in the random stationary case.

This proposition was established in [8] in the case $\varphi = 1$, and the proof extends without too much difficulty to the general case. Note that the right-hand sides that appear above do not actually depend on $\bar{\eta}$ (as long as $\bar{\eta} < \frac{c}{2}m^{-\frac{1}{3}}$), as the left-hand side $\mathcal{W}_n[\varphi]$ does not involve $\bar{\eta}$. See [8, Prop. 4.4] for a direct proof.

Let us briefly evoke the main steps of the proof of Proposition 4.4, restricting to the periodic case for brevity. Keeping in mind that $n \sim m\varepsilon^{-3}$ as $\varepsilon \to 0$, the point is to show that

$$\varepsilon^6 \int_{\mathbb{R}^3}|\nabla h^{\bar{\eta}\varepsilon}|^2 \to \int_{(0,1)^3}|\nabla H^{\bar{\eta}}|^2\int_{\mathcal{O}}|\varphi|^2.$$

The idea is to introduce an approximation of $h^{\bar{\eta}\varepsilon}$ using the corrector solution of (62). Namely, we introduce an approximate velocity field $h^{\varepsilon}_{\text{app}} \in H^1_{\text{loc}}(\mathbb{R}^3)$ and pressure field $p^{\varepsilon}_{\text{app}} \in L^2_{\text{loc}}(\mathbb{R}^3)$ defined by

$$h^{\varepsilon}_{\text{app}}(x) := \frac{1}{\varepsilon^2}\varphi(x)H^{\bar{\eta}}\left(\frac{x}{\varepsilon}\right) - \int_{\mathcal{O}}\frac{1}{\varepsilon^2}\varphi(x)H^{\bar{\eta}}\left(\frac{x}{\varepsilon}\right)dx, \quad x \in \mathcal{O},$$

$$p^{\varepsilon}_{\text{app}}(x) := \frac{1}{\varepsilon^3}\varphi(x)P^{\bar{\eta}}\left(\frac{x}{\varepsilon}\right) - \int_{\mathcal{O}}\frac{1}{\varepsilon^3}\varphi(x)P^{\bar{\eta}}\left(\frac{x}{\varepsilon}\right)dx, \quad x \in \mathcal{O},$$

and

$$-\Delta h^{\varepsilon}_{\text{app}} + \nabla p^{\varepsilon}_{\text{app}} = 0, \quad \text{div } h^{\varepsilon}_{\text{app}} = 0, \quad \text{in } \mathbb{R}^3 \setminus \bar{\mathcal{O}}.$$

One has straightforwardly

$$\varepsilon^6\int_{\mathcal{O}}|\nabla h^{\varepsilon}_{\text{app}}|^2 \to \int_{(0,1)^3}|\nabla H^{\bar{\eta}}|^2\int_{\mathcal{O}}|\varphi|^2, \quad \varepsilon \to 0.$$

Moreover, reasoning exactly as in [8], one shows that the $H^{\frac{1}{2}}(\partial\mathcal{O})$ norm of $h^\varepsilon_{\text{app}}$ goes to zero as $\varepsilon \to 0$, which implies

$$\|\nabla h^\varepsilon_{\text{app}}\|_{L^2(\mathbb{R}^3\setminus\bar{\mathcal{O}})} \to 0, \quad \varepsilon \to 0$$

so that

$$\varepsilon^6 \int_{\mathbb{R}^3} |\nabla h^\varepsilon_{\text{app}}|^2 \to \int_{(0,1)^3} |\nabla H^{\bar{\eta}}|^2 \int_{\mathcal{O}} |\varphi|^2, \quad \varepsilon \to 0.$$

The final step of the proof consists in showing that

$$\varepsilon^6 \int_{\mathbb{R}^3} |\nabla(h^{\bar{\eta}\varepsilon} - h^\varepsilon_{\text{app}})|^2 \to 0, \quad \varepsilon \to 0.$$

This is a consequence of an energy estimate, performed on the Stokes equation satisfied by the difference $h^{\bar{\eta}\varepsilon} - h^\varepsilon_{\text{app}}$. In the case $\varphi = 1$, all details are provided in [8]. For general φ, there are a few extra source terms in this Stokes equation, but they can be handled through similar ideas, so that we do not give further detail.

Step 3. Explicit computation of μ_2

Finally, as further explained in [8], one can make formulas (64) and (65) more explicit. In the periodic case, with $\omega = \{z_1, \ldots, z_m\} + \mathbb{Z}^3$, the following holds.

Proposition 4.5 (Periodic case; see [8, Props. 5.4 and 5.5]).

(1) *One has*

$$\mu_2 S : S = \frac{25\mu}{2m^2}\left(\sum_{i \neq j \in \{1,\ldots,m\}} S\nabla \cdot G_{S,1}(z_i - z_j) \right.$$

$$\left. + m \lim_{y \to 0} S\nabla \cdot (G_{S,1}(y) - G_S(y)) \right),$$

where $G_S(x) = (S\nabla) \cdot \mathcal{U}(x) = -\frac{3}{8\pi}(Sx \cdot x)\frac{x}{|x|^5}$ solves

$$-\Delta G_S + \nabla p_S = (S\nabla)\delta, \quad \text{div } G_S = 0, \quad \text{in } \mathbb{R}^3$$

and for all $L > 0$, $G_{S,L}$ is the periodic field with zero mean solution of

$$- \Delta G_{S,L} + \nabla p_{S,L} = (S\nabla) \sum_{z \in L\mathbb{Z}^3} \delta_z, \quad \text{div } G_{S,L} = 0, \quad \text{in } \mathbb{R}^3. \tag{66}$$

(2) *In the special case of a simple cubic lattice ($m = 1$), the formula simplifies to*

$$\mu_2 S : S = \mu\alpha \sum_i S_{ii}^2 + \mu\beta \sum_{i \neq j} S_{ij}^2,$$

with $\alpha = \frac{5}{2}(1 - 60a)$, $\beta = \frac{5}{2}(1 + 40a)$, and $a \approx -0.04655$.

In the ergodic stationary case, under the almost sure assumption (A1), a more expli-
cit expression can be obtained in terms of the two-point correlation function $\rho_2(x, y) = r(x - y)$. We recall that a point process in $\mathbb{R}^3$ has a two-point correlation function $\rho_2 \in L^1_{\text{loc}}(\mathbb{R}^3 \times \mathbb{R}^3)$ if for all bounded K and smooth functions F,

$$\mathbb{E} \sum_{z \neq z' \in K} F(z, z') = \int_{K \times K} F(x, y)\rho_2(x, y)\, dx\, dy.$$

Proposition 4.6 (Ergodic stationary case; see [8, Prop. 5.6]). *Assume that $\rho_2(x, y) = r(x - y)$ with $r \in L^1_{\text{loc}}(\mathbb{R}^3)$, zero near the origin. Then, for any $M > 0$, almost surely,*

$$\mu_2 S : S = \frac{25\mu}{2m^2} \lim_{L \to +\infty} \frac{1}{L^3} \int_{(M,L-M)^3 \times (M,L-M)^3} (S\nabla) \cdot G_{S,L}(z - z')r(z - z')\, dz\, dz',$$

where $G_{S,L}$ was introduced in (66).

One can actually push this calculation further in the case of an isotropic process. The
next proposition is the first justification of a formula that was derived formally in [18].

Proposition 4.7 (Isotropic point process). *Assume that $\rho_2(x, y) = r(x - y)$ with r zero near the origin, radial and such that $|r(x) - m^2| \leq \omega(|x|)$ for some bounded and decreasing function $\omega : \mathbb{R}_+ \to \mathbb{R}_+$ with $\int_{\mathbb{R}_+} \frac{\omega(t)}{t}\, dt < +\infty$. Then, almost surely,*

$$\mu_2 S : S = \frac{5}{2}\mu |S|^2.$$

Remark 4.1. Let us note that the formula above ($\mu_2 = \frac{5}{2}$) differs from the one derived
formally by Batchelor and Green ([2]) for isotropic random processes of Poisson type
with a no-penetration condition ($\mu_2 \approx 7.6$). The reason is that in the case considered in
[2], assumption (A1) is not satisfied. Roughly, it follows that the kernel $G_{S,L}(z - z')$
appearing in Proposition 4.6 must be modified: the appropriate kernel $G(z - z')$ is only
close to $G_{S,L}$ asymptotically as $|z - z'| \to +\infty$. This point was raised in [2, p. 418].
Since our paper was submitted for publication, rigorous results have been obtained on the
Batchelor–Green formula; see [3, 7].

Remark 4.2. The fact that r is radial in the proposition corresponds to the isotropy of the
point process. The weak decay condition on $r - m^2$ corresponds to a natural decorrela-
tion at large distances. See [3] for similar assumptions. Note that all assumptions of the
proposition are satisfied by the usual hardcore Poisson processes.

Proof of Proposition 4.7. For any $L > 2M \geq 0$, we denote $K_{M,L} := (M, L - M)^3$. By
Proposition 4.6, it is enough to prove that

$$\lim_{M \to +\infty} \lim_{L \to +\infty} \frac{1}{L^3} \int_{K_{M,L} \times K_{M,L}} (S\nabla) \cdot G_{S,L}(z - z')r(z - z')\, dz\, dz' = \frac{m^2}{5}|S|^2.$$

By a simple scaling argument, $G_{S,L}(x) = L^{-2}G_{S,1}(\frac{x}{L})$. It follows easily that

$$\frac{1}{L^3} \int_{K_{M,L} \times K_{M,L}} (S\nabla) \cdot G_{S,L}(z - z')r(z - z') \, dz \, dz'$$

$$= \int_{K_{M/L,1} \times K_{M/L,1}} (S\nabla) \cdot G_{S,1}(z - z')r(L(z - z')) \, dz \, dz'.$$

For $z, z' \in K_{M/L,1}$, we have $z - z' \in (-1 + 2M/L, 1 - 2M/L)^3$. Over this set, the periodic function $G_{S,1}$ can be decomposed as

$$G_{S,1} = \sum_{k \in \mathbb{Z}^3, |k_i| \leq 1} G_S(\cdot - k) + \tilde{G}_{S,1}, \tag{67}$$

where $\tilde{G}_{S,1}$ is smooth. From this decomposition, and the fact that $r(x) = 0$ for all $|x| \leq \delta$, $\delta > 0$ small enough, it follows that

$$\int_{(-1+2M/L, 1-2M/L)^3} |(S\nabla) \cdot G_{S,1}(u)r(Lu)| \, du$$

$$\leq C \int_{(-1+2M/L, 1-2M/L)^3 \cap \{|u| \geq \delta/L\}} \left(\sum_{k \in \mathbb{Z}^3, |k_i| \leq 1} \frac{1}{|u - k|^3} + 1 \right) du$$

$$\leq C_M \ln(L). \tag{68}$$

It follows in particular that

$$\int_{(K_{M/L,1} \setminus K_{2M/L,1}) \times K_{M/L,1}} (S\nabla) \cdot G_{S,1}(z - z')r(L(z - z')) \, dz \, dz'$$

$$\leq |K_{M/L,1} \setminus K_{2M/L,1}| C_M \ln L \leq C'_M \frac{\ln L}{L} \to 0, \quad L \to +\infty.$$

It remains to show that

$$\lim_{M \to +\infty} \lim_{L \to +\infty} \int_{K_{2M/L,1} \times K_{M/L,1}} (S\nabla) \cdot G_{S,1}(z - z')r(L(z - z')) \, dz \, dz' = \frac{m^2}{5}|S|^2.$$

We decompose

$$\int_{K_{2M/L,1} \times K_{M/L,1}} (S\nabla) \cdot G_{S,1}(z - z')r(L(z - z')) \, dz \, dz'$$

$$= \int_{K_{2M/L,1}} \int_{K_{M/L,1} \cap B(z',M/L)^c} (S\nabla) \cdot G_{S,1}(z - z')m^2 \, dz \, dz'$$

$$+ \int_{K_{2M/L,1}} \int_{K_{M/L,1} \cap B(z',M/L)^c} (S\nabla) \cdot G_{S,1}(z - z')(r(L(z - z')) - m^2) \, dz \, dz'$$

$$+ \int_{K_{2M/L,1}} \int_{B(z',M/L)} (S\nabla) \cdot G_{S,1}(z - z')r(L(z - z')) \, dz \, dz'$$

$$=: I_{M,L}^1 + I_{M,L}^2 + I_{M,L}^3.$$

We have used that for any $z' \in K_{2M/L,1}$, $K_{M/L,1} \cap B(z', M/L) = B(z', M/L)$. As regards the first term, we use (68) again, which yields

$$\int_{K_{2M/L,1}} \int_{K_{0,1} \setminus K_{M/L,1}} |(S\nabla) \cdot G_{S,1}(z - z')m^2| \, dz \, dz' \leq C_M \frac{\ln L}{L},$$

so that

$$I^1_{M,L} = \int_{K_{2M/L,1}} \int_{K_{0,1} \cap B(z',M/L)^c} (S\nabla) \cdot G_{S,1}(z - z')m^2 \, dz \, dz' + o(1),$$

where the term $o(1)$ goes to zero as $L \to +\infty$ for any fixed M. We then perform an integration by parts in the variable z. Note that for all $z' \in K_{2M/L,1}$, the boundary of the ball $B(z', M/L)$ is disjoint from the boundary of $K_{0,1}$. Taking into account that $(S\nabla) \cdot G_{S,1}(z - z')$ is $\mathbb{Z}^d$-periodic, the boundary term at $\partial K_{0,1}$ vanishes, and eventually

$$I^1_{M,L} = - \int_{K_{2M/L,1}} \int_{\partial B(z',M/L)} (S\nu) \cdot G_{S,1}(z - z') \, d\sigma(z) \, dz' \, m^2 + o(1)$$

$$= - \int_{K_{2M/L,1}} \int_{\partial B(z',M/L)} (S\nu) \cdot G_S(z - z') \, d\sigma(z) \, dz' \, m^2 + o(1)$$

$$= - \left(1 - \frac{4M}{L}\right)^3 \left(\int_{\partial B(0,1)} (S\nu) \cdot G_S \right) m^2 + o(1).$$

Using the explicit expression for G_S, we find that for any $M > 0$,

$$I^1_{M,L} \to \frac{3}{8\pi} \int_{\partial B_1} (Sn \cdot n)^2 m^2 = \frac{m^2}{5} |S|^2 \quad \text{as } L \to +\infty,$$

thanks to the identity $\int_{\partial B_1} n_i n_j n_k n_l = \frac{4\pi}{15} (\partial_{ij}\partial_{kl} + \partial_{ik}\partial_{jl} + \partial_{il}\partial_{jk})$. As regards the second term, using decomposition (67), we find the upper bound

$$|I^2_{M,L}|$$

$$\leq C \int_{\substack{(-1+2M/L,1-2M/L)^3 \\ \cap \{|u| \geq M/L\}}} \left(\sum_{k \in \mathbb{Z}^3, |k_i| \leq 1} \frac{1}{|u - k|^3} + 1 \right) |r(Lu) - m^2| \, du$$

$$\leq C \sum_{k \in \mathbb{Z}^3, |k_i| \leq 1} \int_{\substack{(-L+2M,L-2M)^3 \\ \cap \{|u'| \geq M\}}} \frac{|r(u') - m^2|}{|u' - Lk|^3} \, du' + C \sup_{|u'| \geq M} |r(u') - m^2|.$$

Clearly, the last term on the right-hand side is independent of L and goes to zero when $M \to +\infty$. Moreover, we claim that all integrals on the right-hand side go to zero when $M \to +\infty$, uniformly in L. Indeed, for $k = 0$, we simply write

$$\int_{\substack{(-L+2M,L-2M)^3 \\ \cap \{|u'| \geq M\}}} \frac{|r(u') - m^2|}{|u'|^3} \, du' \leq \int_{|t| \geq M} \frac{\omega(t)}{t} \, dt \to 0, \quad M \to +\infty.$$

For $k \neq 0$ in the sum, we write

$$\int_{\substack{(-L+2M,L-2M)^3 \\ \cap\{|u'|\geq M\}}} \frac{|r(u')-m^2|}{|u'-Lk|^3}\, du' \leq \int_{\substack{\{\sqrt{3}L\geq|u'|\geq M\} \\ \cap\{|u'-Lk|\geq 2M\}}} \frac{\omega(|u'|)}{|u'-Lk|^3}\, du'$$

$$\leq \int_{\substack{\{\sqrt{3}L\geq|u'|\geq M\} \\ \cap\{\frac{1}{2}L\geq|u'-Lk|\geq 2M\}}} \frac{\omega(|u'|)}{|u'-Lk|^3}\, du'$$

$$+ \int_{\substack{\{\sqrt{3}L\geq|u'|\geq M\} \\ \cap\{|u'-Lk|\geq\frac{1}{2}L\}}} \frac{\omega(|u'|)}{|u'-Lk|^3}\, du'.$$

For the second integral, we use the lower bound $|u'-Lk| \geq \frac{1}{2}L$ for the denominator, so that

$$\int_{\substack{\{\sqrt{3}L\geq|u'|\geq M\} \\ \cap\{|u'-Lk|\geq\frac{1}{2}L\}}} \frac{\omega(|u'|)}{|u'-Lk|^3}\, du' \leq C\omega(M).$$

For the first integral, we notice that

$$|u'| \geq |Lk| - |u'-Lk| \geq L - \tfrac{1}{2}L = \tfrac{1}{2}L \geq |u'-Lk|, \quad \text{so that } \omega(|u'|) \leq \omega(|u'-Lk|).$$

We infer

$$\int_{\substack{\{\sqrt{3}L\geq|u'|\geq M\} \\ \cap\{\frac{1}{2}L\geq|u'-Lk|\geq 2M\}}} \frac{\omega(|u'|)}{|u'-Lk|^3}\, du' \leq \int_{\{\frac{1}{2}L\geq|u'-Lk|\geq 2M\}} \frac{\omega(|u'-Lk|)}{|u'-Lk|^3}$$

$$\leq \int_{|t|\geq 2M} \frac{\omega(t)}{t}\, dt \to 0, \quad M \to +\infty.$$

Hence, $\lim_{M\to+\infty} \limsup_{L\to+\infty} |I_{M,L}^2| = 0$. Finally, for the last term, we find

$$I_{M,L}^3 = \int_{K_{2M/L,1}} \int_{B(z',M/L)} (S\nabla) \cdot G_S(z-z') r(L(z-z'))\, dz\, dz' + o(1),$$

where again the $o(1)$ refers to a quantity going to zero as $L \to +\infty$ for any fixed M. As r is radial and as the mean of $(S\nabla) \cdot G_S$ over spheres is zero, the integral in z on the right-hand side vanishes identically. This concludes the proof. ∎

References

[1] H. Ammari, P. Garapon, H. Kang, and H. Lee, Effective viscosity properties of dilute suspensions of arbitrarily shaped particles. *Asymptot. Anal.* **80** (2012), no. 3-4, 189–211 Zbl 1341.76033 MR 3025042

[2] G. K. Batchelor and J. T. Green, The determination of the bulk stress in a suspension of spherical particles to order c^2. *J. Fluid Mech.* **56** (1972), 401–427   Zbl 0246.76108

[3] M. Duerinckx and A. Gloria, On Einstein's effective viscosity formula. 2020, arXiv:2008.03837

[4] M. Duerinckx and A. Gloria, Corrector equations in fluid mechanics: effective viscosity of colloidal suspensions. *Arch. Ration. Mech. Anal.* **239** (2021), no. 2, 1025–1060   Zbl 1456.76134   MR 4201621

[5] A. Einstein, Eine neue Bestimmung der Moleküldimensionen. *Ann. Phys.* **19** (1906), 289–306   Zbl 37.0811.01

[6] G. P. Galdi, *An introduction to the mathematical theory of the Navier-Stokes equations*. 2nd edn., Springer Monogr. Math., Springer, New York, 2011   Zbl 1245.35002   MR 2808162

[7] D. Gérard-Varet, Derivation of the Batchelor-Green formula for random suspensions. *J. Math. Pures Appl. (9)* **152** (2021), 211–250   Zbl 07379402   MR 4280836

[8] D. Gérard-Varet and M. Hillairet, Analysis of the viscosity of dilute suspensions beyond Einstein's formula. *Arch. Ration. Mech. Anal.* **238** (2020), no. 3, 1349–1411   Zbl 1454.76101   MR 4160802

[9] D. Gérard-Varet and R. M. Höfer, Mild assumptions for the derivation of Einstein's effective viscosity formula. *Comm. Partial Differential Equations* **46** (2021), no. 4, 611–629   Zbl 07378230   MR 4260456

[10] B. M. Haines and A. L. Mazzucato, A proof of Einstein's effective viscosity for a dilute suspension of spheres. *SIAM J. Math. Anal.* **44** (2012), no. 3, 2120–2145   Zbl 1284.35339   MR 2982744

[11] M. Hillairet, On the homogenization of the Stokes problem in a perforated domain. *Arch. Ration. Mech. Anal.* **230** (2018), no. 3, 1179–1228   Zbl 1432.35018   MR 3851058

[12] M. Hillairet and D. Wu, Effective viscosity of a polydispersed suspension. *J. Math. Pures Appl. (9)* **138** (2020), 413–447   Zbl 1441.35197   MR 4098775

[13] V. V. Jikov, S. M. Kozlov, and O. A. Oleĭnik, *Homogenization of differential operators and integral functionals*. Springer, Berlin, 1994   Zbl 0838.35001   MR 1329546

[14] T. Lévy and E. Sánchez-Palencia, Einstein-like approximation for homogenization with small concentration. II. Navier-Stokes equation. *Nonlinear Anal.* **9** (1985), no. 11, 1255–1268   Zbl 0581.76097   MR 813657

[15] B. Niethammer and R. Schubert, A local version of Einstein's formula for the effective viscosity of suspensions. *SIAM J. Math. Anal.* **52** (2020), no. 3, 2561–2591   Zbl 07245266   MR 4102716

[16] E. Sánchez-Palencia, Einstein-like approximation for homogenization with small concentration. I. Elliptic problems. *Nonlinear Anal.* **9** (1985), no. 11, 1243–1254   Zbl 0538.76086   MR 813656

[17] S. Serfaty, *Coulomb gases and Ginzburg-Landau vortices*. Zurich Lect. Adv. Math., European Mathematical Society (EMS), Zürich, 2015   Zbl 1335.82002   MR 3309890

[18] L. J. Walpole, The elastic behaviour of a suspension of spherical particles. *Quart. J. Mech. Appl. Math.* **25** (1972), 153–160   Zbl 0266.73007

[19] M. Zuzovsky, P. M. Adler, and H. Brenner, Spatially periodic suspensions of convex particles in linear shear flows. III. Dilute arrays of spheres suspended in Newtonian fluids. *Phys. Fluids* **26** (1983), 1714–1723   Zbl 0622.76114

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