Rend. Lincei Mat. Appl. 19 (2008), 79–102

Differential geometry. — *Vanishing and conservativeness of harmonic forms of a non-compact CR manifold*, by JUN MASAMUNE.

ABSTRACT. — The self-adjointness of a sublaplacian and Kohn–Rossi laplacians on a non-compact strictly pseudoconvex CR manifold is proved. As applications to geometry, the vanishing and the conservativeness of harmonic forms are obtained.

KEY WORDS: CR manifold, sublaplacian, Kohn–Rossi laplacian, essentially self-adjoint, vanishing theorem, conservative.

MATHEMATICS SUBJECT CLASSIFICATION (2000): Primary 32V20, 53C17; Secondary 58A14, 14F15.

1. INTRODUCTION

The Hodge theory of a compact strictly pseudoconvex CR manifold, hereafter a *CR manifold* for short, has been successfully developed: the subellipticity of the Kohn–Rossi laplacian, Kohn's de-Rham–Hodge–Kodaira theorem [17], and a vanishing theorem [31]. In particular, *harmonic forms* are one of the main objects of interest in the theory.

On the other hand, there are important non-compact CR manifolds: the Heisenberg group, which plays a role in CR geometry similar to that of Euclidean space in Riemannian geometry [6], Sasakian space forms, spherical orbits, etc. (see Section 6).

In this paper, we are interested in the vanishing and a conservative principle for harmonic forms on a non-compact CR manifold M. We always assume that M is smooth, connected, strictly pseudoconvex, and of hypersurface type.

We define the domain $D(\overline{\partial}_b)$ of the tangential Cauchy–Riemann operator $\overline{\partial}_b$ to be the set of measurable forms α such that both α and $\overline{\partial}_b\alpha$ are square-integrable. Similarly, we define the domain $D(\delta)$ of the co-derivative δ to be the set of measurable forms α such that both α and $\delta\alpha$ are square-integrable. We then define the Kohn–Rossi laplacian $\Box_b^G$ with domain $D(\Box_b^G)$ given by

$$\{\alpha \in D(\overline{\partial}_b) \cap D(\delta) \cap A^q(M) : \overline{\partial}_b\alpha \in D(\delta) \text{ and } \delta\alpha \in D(\overline{\partial}_b)\},$$

where $A^q(M)$ is the set of smooth q-forms on M.

We assume that M has *negligible boundary*, which makes $\Box_b^G$ into a symmetric operator (see Definition 1 in Section 2). This class of manifolds is so large that it includes every complete manifold [22], in the sense that M is complete as a distance space with respect to the *intrinsic distance*, associated to M's canonical sub-Riemannian structure [22], [31], including all of the examples cited above.

The main result of this paper is the following.

MAIN RESULT. *Let M be a $(2n+1)$-dimensional strictly pseudoconvex CR manifold of hypersurface type with negligible boundary. Then the sublaplacian Δ^E on functions $L^2(M)$, and the Kohn–Rossi laplacian $\Box_b^G$ on $L^2(A^q)$ with $0 < q < n$, are essentially self-adjoint. Moreover, the Sobolev spaces S_0 and S (defined in Section 2) coincide, and the closure of $\Box_b^G$ coincides with the Friedrichs extension $\Box_b^F$ of the minimal Kohn–Rossi laplacian.*

Vanishing of cohomology: *If the Ricci operator* Ric *is non-negative on $L^2(A^q)$, then every L^2-harmonic q-form α is parallel for the Tanaka–Webster connection, the norm $|\alpha|$ is constant, and the bottom of the spectrum $\lambda^q_{\min}$ on $L^2(A^q)$ is estimated as*

$$\lambda^q_{\min} \geq \frac{n-q}{n} \inf\{\langle \mathrm{Ric}(\alpha), \alpha\rangle : \alpha \in A^q_0,\ \|\alpha\| = 1\}.$$

If, additionally, either Ric *is positive, or M has infinite volume, then α is identically* 0*, and the L^2-reduced cohomology group of degree q is trivial.*

Conservative principle: *Let r be the radius function from an arbitrary but fixed reference point in M. If*

$$e^{-ar} \in L^1(M) \quad \text{for every } a > 0, \tag{1}$$

then for any bounded harmonic q-form α, there exist $\alpha_\epsilon \in D(\overline{\partial}_b) \cap D(\delta)$ for $\epsilon > 0$ such that $\|\alpha_\epsilon\|_\infty \leq \|\alpha\|_\infty$ for every $\epsilon > 0$, and

$$\lim_{\epsilon\to 0} \alpha_\epsilon = \alpha \qquad \text{strongly},$$
$$\lim_{\epsilon\to 0} T_t\alpha_\epsilon = \alpha \qquad \text{weakly for every } t > 0,$$

where T_t is the semigroup generated by $\Box_b^F$.

Therefore, the self-adjointness, the vanishing theorem, and the spectrum estimate for the Kohn–Rossi laplacian on a compact CR manifold hold true on a CR manifold with negligible boundary. Let us point out that the self-adjointness, which is the foundation for the Main Result, does not necessarily hold if the manifold is incomplete; for example, if the Cauchy boundary $\partial_C M$, that is, the difference $\overline{M} \setminus M$ between the completion $\overline{M}$ and M, has positive capacity, then the Kohn–Rossi laplacian is not essentially self-adjoint (see Proposition 3 in Section 2).

Comparing to the Riemannian or Kählerian case ([34] and [9]), the most significant difference is the constant $(n-q)/n$ in the spectrum estimate. This appears already in the Weitzenböck formula (Theorem 3 in Section 4), and is a manifestation of the fact that the Tanaka–Webster connection is *not* torsion free.

In the technical contribution relating to the vanishing theorem, we consider the following. The standard proof of the vanishing theorems for a complete non-compact Riemannian and Kählerian manifold is to combine the Bochner method and Liouville property (e.g. [34]). The naive generalization of this method to CR manifolds does not work because the Weitzenböck formula for $\Delta^E|\alpha|^2$, where Δ^E is the *sublaplacian*, which is the real of the double of $\Box_b$ restricted to functions, involves an extra torsion term. To overcome this difficulty, we first work with the Bochner method on the compact case;

decompose the Kohn–Rossi laplacian into the sum of the connection laplacians $\nabla^{0,1*}\nabla^{0,1}$, $\nabla^{1,0*}\nabla^{1,0}$, and the Ricci operator (Weitzenböck formula: Theorem 3), so that if

$$\nabla^{0,1*}\nabla^{0,1} \text{ and } \nabla^{1,0*}\nabla^{1,0} \text{ are non-negative,} \tag{2}$$

then α is parallel for ∇, provided the Ricci operator is non-negative. Due to the standard divergence theorem, (2) holds true if α has compact support. Next, in order to extend (2) to an arbitrary L^2-harmonic form α, we approximate it by a form with compact support α_l. We do this by constructing a sequence $\{\alpha_l\}$ that converges to α in the Dirichlet norm, thus proving the essential self-adjointness of $\Box_b^G$ (Theorem 2 in Section 3). The assumption that M has negligible boundary is required for the self-adjointness.

Another issue related to harmonic forms which we study is the *conservative principle*. The concept of *conservative principle for differential forms* has been introduced by Vesentini [32], who proved it for complete Riemannian and Kähler manifolds under the volume growth condition (1) (see also [23]). In the present paper, we will study a similar type of conservativeness, which coincides with Vesentini's for a complete Riemannian manifold under certain curvature conditions (see Remark 6 in Section 5), and is rather similar to that of a Dirichlet form [7]: the canonical Dirichlet form on a CR manifold is conservative if there exist $u_\epsilon \in D(\nabla^E)$ with $0 \leq u_\epsilon \leq 1$ such that

$$\begin{aligned} \lim_{\epsilon\to 0} u_\epsilon &= u \qquad \text{strongly,} \\ \lim_{\epsilon\to 0} T_t u_\epsilon &= u \qquad \text{weakly for every } t > 0, \end{aligned}$$

where ∇^E is the subgradient and T_t is the semigroup generated by the sublaplacian Δ^E (see Section 2). If the Dirichlet form is conservative, then M is said to be *conservative* or *stochastically complete*.

Let us briefly discuss the conservativeness of M and that of forms. We note that if $\mathrm{Cap}(\partial_C M) = 0$ and M satisfies (1), then M is conservative (see Remark 7 in Section 5). If M has negligible boundary and satisfies (1), then $\mathrm{Cap}(\partial_C M) = 0$, and hence such a manifold is conservative (Corollary 3 in Section 3). This discussion is based on the fact that any function in $D(\nabla^E)$ has the *truncation property*. Obviously, differential forms do not have the corresponding property, but the assumption that α *is bounded* partially plays the role of it, which enables us to study the conservative principle for forms.

Finally, let us point out that our results extend to any holomorphic vector bundle valued forms over a weighted CR manifold (see e.g. [15]).

The paper is organized in the following manner.

In Section 2, we review some notions relating to CR manifolds, like CR structure, sublaplacian, intrinsic distance, Tanaka–Webster connection, tangential Cauchy–Riemann complex, and Kohn–Rossi laplacian.

In Section 3, we study the self-adjointness of the Kohn–Rossi laplacians $\Box_b^G$ and $\Box_b^M$ which have different domains. The self-adjointness of $\Box_b^G$ was proved in [22], but here we present an alternative, more direct proof. In Section 4, we prove the vanishing of L^2-harmonic forms and obtain an estimate of the bottom of the spectrum of $\Box_b^G$. We establish the Weitzenböck formulae for $\Box_b$, and together with the essential self-adjointness, we obtain the vanishing theorem.

In Section 5, we show the conservative principle for bounded harmonic forms.

In Section 6, we collect important examples of CR manifolds with negligible boundary. In the Appendix, we present a proof of the following L^2-Kohn–Hodge decomposition for a CR manifold with negligible boundary:

$$\mathcal{H}^q \simeq (\ker(\overline{\partial}_b) \cap L^2(A^q))/\overline{\mathrm{range}(\overline{\partial}_b)}^{L^2} \quad \text{for } 0 < q < n,$$

where $\mathcal{H}^q$ is the space of L^2-harmonic q-forms and the bar with L^2 indicates the L^2-closure.

2. Preliminaries

In this section, we recall some notions relating to CR manifolds. For further differential geometric study of CR manifolds, we refer the readers to [5] and [31]. As we will see, a CR manifold is sub-Riemannian and we shall employ some notions from sub-Riemannian geometry. For further study in that direction, we refer the readers to [30] and the references therein.

Let M be an oriented connected $(2n+1)$-dimensional smooth real manifold without boundary. We assume $n > 0$. A *CR structure* on M is a distinguished complex subbundle $T^{1,0}(M)$ of the complex tangential bundle $\mathbb{C}T(M)$ satisfying

$$T^{1,0}(M) \cap T^{0,1}(M) = \{0\}, \quad [T^{1,0}(M), T^{1,0}(M)] \subset T^{1,0}(M),$$

where $T^{0,1}(M) = \overline{T^{1,0}(M)}$ and the bar indicates complex conjugate. A CR manifold is said to be of *hypersurface type* if $\mathrm{rank}_{\mathbb{C}}\, T^{1,0}(M) = n$. Set

$$E = \mathrm{Re}(T^{1,0}(M) \oplus T^{0,1}(M)),$$

and let θ be its annihilator. As $T^{1,0}(M)$ is integrable,

$$[X, Y] - [JX, JY] \in \Gamma(E) \quad \text{for } X, Y \in \Gamma(E), \tag{3}$$

where J is the almost complex structure of E, that is, the unique homomorphism $J : E \to E$ such that

$$J^2 = -1.$$

We have

$$T^{1,0} = \{X - \sqrt{-1}\, JX : X \in E\}, \quad T^{0,1} = \{X + \sqrt{-1}\, JX : X \in E\}.$$

Due to (3), the bilinear form

$$g^E(X, Y) = -d\theta(JX, Y) \quad \text{for } X, Y \in \Gamma(E)$$

is symmetric. We assume that g^E is positive-definite on E. The CR structure with g^E positive-definite is called *strictly pseudoconvex*. In this case E satisfies the Hörmander condition and θ defines a contact structure on M. We assume M to be a strictly pseudoconvex CR manifold of hypersurface type without boundary (hereafter, a *CR manifold*). Note that $M = (M, E, g^E)$ is *sub-Riemannian* in the sense of [30].

The gradient of a CR manifold is the *horizontal gradient*

$$\nabla^E : C^\infty(M) \to \Gamma(E)$$

where $C^\infty(M)$ is the set of smooth functions on M, defined by

$$g^E(\nabla^E u, X) = Xu,$$

where $u \in C^\infty(M)$ and $X \in \Gamma(E)$, or locally,

$$\nabla^E u = \sum_{1 \le i \le n} \{(X_i u)X_i + (JX_i u)JX_i\},$$

where $\{X_i, JX_i\}_{1 \le i \le n}$ is an orthogonal Hörmander system of E with respect to g^E. Let ω be the volume form associated to θ defined by $\omega = (d\theta)^n \wedge \theta$. Then we have the *integration-by-parts formula*

$$\int_M u(\operatorname{div} X)\,\omega = -\int_M g^E(\nabla^E u, X)\,\omega,$$

where div is the divergence associated to ω, provided one of $u \in C^\infty(M)$ or $X \in \Gamma(E)$ has compact support. The *sublaplacian* Δ^E is defined by

$$\Delta^E := -\operatorname{div} \nabla^E,$$

or locally,

$$\Delta^E = -\sum_{1 \le i \le n} \{X_i^2 + JX_i^2\}. \tag{4}$$

The expression (4) shows, by the Hörmander theorem [13], that Δ^E is subelliptic of order $1/2$, and hence hypoelliptic [19], that is, whenever the distribution $\Delta^E u$ is smooth, then u is smooth.

Another important notion linked to the sub-Riemannian structure of a CR manifold is the *intrinsic distance* dist (e.g. [2]): for every $x, y \in M$,

$$\operatorname{dist}(x, y) = \sup\{\psi(x) - \psi(y) : \psi \in C^\infty(M) \text{ and } \sup_{x \in M} |\nabla^E \psi(x)| \le 1\}.$$

Since E satisfies the Hörmander condition, by the Chow theorem [4] (see also [16], [30], and [22]) we have

PROPOSITION 1. dist *is non-degenerate, and generates the original topology.*

We say that M is *complete* if the distance space (M, dist) is complete.

Let $\xi \in \Gamma(T(M))$ be the *characteristic direction* defined by

$$\begin{cases} \theta(\xi)(x) = 1 \text{ for every } x \in M, \\ \theta([\xi, X])(x) = 0 \text{ for every } x \in M \text{ and } X \in \Gamma(E). \end{cases}$$

A CR manifold carries a natural affine connection:

LEMMA 1 (Tanaka–Webster connection [31], [33]). *There is a unique affine connection*

$$\nabla : \Gamma(T(M)) \to \Gamma(T(M) \otimes T(M)^*)$$

on M satisfying the following conditions:

(i) *the subbundle E is parallel to ∇,*
(ii) *ξ, J, and the exterior differential of θ are parallel to ∇,*
(iii) *the torsion T of ∇ satisfies*

$$T(X, Y) = -\theta([X, Y])\xi, \qquad T(\xi, Y) = JT(\xi, JY)$$

for $X, Y \in \Gamma(E)$.

Note that condition (ii) in Lemma 1 implies $\nabla g^E = 0$. We extend ∇ to a differential operator

$$\Gamma(\mathbb{C}T(M)) \to \Gamma(\mathbb{C}T(M) \otimes \mathbb{C}T(M)^*)$$

in the canonical way, and call it the *Tanaka–Webster connection*.

Let Z_i, $i = 1, 2, \ldots$, be elements of $T_x^{1,0}(M)$. Since $\nabla J = 0$ implies

$$\nabla : \Gamma(T^{0,1}(M)) \to \Gamma(T^{0,1}(M) \otimes \mathbb{C}T(M)^*),$$

and ∇ is torsion free on $\Gamma(T^{0,1}(M))$, the *tangential Cauchy–Riemann operator*

$$\overline{\partial}_b : A^q(M) \to A^{q+1}(M),$$

where $A^q(M) = \Gamma(\bigwedge^q T^{0,1}(M)^*)$, is defined by

$$\begin{aligned}
\overline{\partial}_b\alpha(\overline{Z}_1, \ldots, \overline{Z}_{q+1}) := &\sum_{1\le i\le q+1} (-1)^{i+1}\overline{Z}_i(\alpha(\overline{Z}_1, \ldots, \hat{\overline{Z}}_i, \ldots, \overline{Z}_{q+1})) \\
&+ \sum_{1\le i<j\le q+1} (-1)^{i+j}\alpha([\overline{Z}_i, \overline{Z}_j], \overline{Z}_1, \ldots, \hat{\overline{Z}}_i, \ldots, \hat{\overline{Z}}_j, \cdots, \overline{Z}_{q+1}) \\
= &\sum_{1\le i\le q+1} (-1)^{i+1}(\nabla_{\overline{Z}_i}\alpha)(\overline{Z}_1, \ldots, \hat{\overline{Z}}_i, \ldots, \overline{Z}_{q+1}),
\end{aligned}$$

where a hat indicates suppression of the term. Set

$$A_0^q(M) = \{\alpha \in A^q(M) : \alpha \text{ has compact support}\},$$
$$A(M) = \bigoplus_q A^q, \qquad A_0(M) = \bigoplus_q A_0^q.$$

Let $e_1, \ldots, e_n$ be an orthonormal frame of $T^{1,0}(M)$, and θ^i be the dual of e_i for $1 \le i \le n$. For a q-form α, we have

$$\alpha = \sum_{I_q} \alpha_{I_q}\theta^{I_q},$$

where α_{I_q} is a function, $I_q = (i_1, \ldots, i_q)$, $1 \le i_1 < \cdots < i_q \le n$ and $\theta^{I_q} = \theta^{i_1} \wedge \cdots \wedge \theta^{i_q}$.

Define an inner product on $A_0^q(M)$ by

$$\langle \alpha, \beta \rangle := \int_M \langle \alpha, \beta \rangle(x)\, \omega(dx),$$

where

$$\langle \alpha, \beta \rangle(x) = \sum_{I_q} \alpha_{I_q}(x) \overline{\beta_{I_q}(x)} \quad \text{for every } x \in M.$$

We will frequently use the following (see [28, p. 158]):

LEMMA 2. *For arbitrary forms $\alpha, \beta \in A(M)$ we have the pointwise inequality*

$$|\alpha \wedge \beta| \leq |\alpha|\, |\beta|.$$

Define $|\cdot|(x) = \sqrt{\langle \cdot, \cdot \rangle(x)}$, $\|\cdot\| = \sqrt{\langle \cdot, \cdot \rangle}$, and $\|\cdot\|_\infty = \sup_{x \in M} \sqrt{\langle \cdot, \cdot \rangle(x)}$. Let $L^2(A^q)$ (resp. $L^2(A)$) be the completion of $A_0^q(M)$ (resp. $A_0(M)$) with respect to $\|\cdot\|$.

The associated Sobolev space S is the completion of $C^\infty(M)$ with respect to the metric $\|\cdot\|_{1,2} = \|\cdot\| + \|\nabla^E \cdot\|$. The space $S_0 \subset S$ is the closure of the space $C_0^\infty(M)$ of functions $u \in C^\infty(M)$ with compact support in S. Another Sobolev type space W is the set of measurable functions u such that both u and $\nabla^E u$ are in L^2. In the next section, we will show that $S = W$ (Proposition 4).

Define the first order differential operator $\delta : A^{q+1}(M) \to A^q(M)$ by

$$\delta\alpha(\overline{Z}_1, \ldots, \overline{Z}_q) := -\sum_{1 \leq i \leq n} \overline{e}_j \rfloor \nabla_{e_j}(\overline{Z}_1, \ldots, \overline{Z}_q) = -\sum_{1 \leq i \leq n} (\nabla_{e_i}\alpha)(\overline{e}_i, \overline{Z}_1, \ldots, \overline{Z}_q),$$

where $\rfloor : A^{q+1}(M) \to A^q(M)$ is the contraction operator.

The following can be proved by a direct calculation.

LEMMA 3. *The tangential Cauchy–Riemann operator $\overline{\partial}_b$ is the unique linear operator that satisfies:*

(i) $\overline{\partial}_b^2 = 0$,
(ii) $\overline{\partial}_b(\alpha \wedge \beta) = \overline{\partial}_b\alpha \wedge \beta + (-1)^q \alpha \wedge \overline{\partial}_b\beta$,
(iii) $\overline{\partial}_b u(\overline{Z}) = \overline{Z}(u)$,

for every $\alpha \in A^q(M)$, $\beta \in A(M)$, $u \in C^\infty(M)$, and $Z \in \Gamma(T^{1,0}(M))$.

For later purposes, we express $\overline{\partial}_b$ in a suitable form:

PROPOSITION 2. *We have*

$$\overline{\partial}_b = \sum_j \overline{\theta}^j \wedge \nabla_{\overline{e}_j}. \tag{5}$$

PROOF. It suffices to check that the right-hand side of (5) has the properties in Lemma 3. It is obvious that (iii) holds; (ii) may be shown by a straightforward calculation; and (i) holds true because ∇ is torsion free on $\Gamma(T^{0,1}(M))$. □

Finally, the *Kohn–Rossi laplacian* $\Box_b : A^q(M) \to A^q(M)$ is defined by

$$\Box_b := \overline{\partial}_b \cdot \delta + \delta \cdot \overline{\partial}_b.$$

Noting

$$\langle \overline{\partial}_b \alpha, \beta \rangle = \langle \alpha, \delta\beta \rangle + \int_M (\mathrm{div} X)\omega,$$

where X is the dual of the one-form γ given by $\gamma(Z) = \langle \alpha, Z \rfloor \beta \rangle$, we have

$$\langle \overline{\partial}_b \alpha, \beta \rangle = \langle \alpha, \delta\beta \rangle$$

provided one of the forms α, β has compact support. Hence, δ is the formal adjoint of $\overline{\partial}_b$, and $\Box_b$ is symmetric on $A_0(M)$. Since $\Box_b$ degenerates in ξ, it is not elliptic; however, we have:

LEMMA 4 ([18] and [19]). *The Kohn–Rossi laplacian on* $L^2(A^q)$ *with* $0 < q < n$ *is subelliptic of order* $1/2$*, and hence hypoelliptic.*

We say $\alpha \in L^2(A^q)$ is *harmonic* if $\overline{\partial}_b \alpha = 0$ and $\delta\alpha = 0$ pointwise, and denote by $\mathcal{H}^q$ the set of L^2-harmonic q-forms. The Kohn–Rossi laplacian $\Box_b$ relates to the sublaplacian Δ^E as

$$2\,\mathrm{Re}\,\Box_b u = \Delta^E u \quad \text{for } u \in C^\infty(M).$$

In fact, by Proposition 2,

$$\begin{aligned}\Box_b u &= \delta \sum_i (\overline{e}_i u)\overline{\theta}_i = -\sum_{i,j} \nabla_{e_j}(\overline{e}_i u \cdot \overline{\theta}_i(\overline{e}_j)) \\ &= -\frac{1}{2}\sum_i \{X_i^2 + JX_i^2 + \sqrt{-1}[X_i, JX_i]\}u = \frac{1}{2}\{\Delta^E - \sqrt{-1}\sum_i [X_i, JX_i]\}u.\end{aligned}$$

Following the Riemannian terminology [8], we introduce

DEFINITION 1. *We say M has* negligible boundary *if*

$$\langle \overline{\partial}_b \alpha, \beta \rangle = \langle \alpha, \delta\beta \rangle \quad \textit{for every } \alpha \in D(\overline{\partial}_b) \textit{ and } \beta \in D(\delta),$$

where $D(\overline{\partial}_b)$ (respectively, $D(\delta)$) is the completion of $A(M)$ with respect to the norm $\|\alpha\| + \|\overline{\partial}_b \alpha\|$ (respectively, $\|\alpha\| + \|\delta\alpha\|$). We define the Kohn–Rossi laplacian $\Box_b^G$ with domain

$$D(\Box_b^G) = \{\alpha \in D(\overline{\partial}_b) \cap D(\delta) \cap A^q(M) : \overline{\partial}_b \alpha \in D(\delta) \textit{ and } \delta\alpha \in D(\overline{\partial}_b)\}.$$

LEMMA 5 ([22]). *A complete CR manifold has negligible boundary.*

Let us close this section by discussing briefly the relationships between *almost polarity* and *negligibility* of the *Cauchy boundary*

$$\partial_C M := \overline{M} \setminus M,$$

where $\overline{M}$ is the completion of M with respect to dist. Define the capacity $\mathrm{Cap}(\partial_C M)$ of $\partial_C M$ as

$$\mathrm{Cap}(\partial_C M) = \begin{cases} \inf_{\partial_C M \subset O \in \mathcal{O}} \{\mathrm{Cap}(O)\} & \text{if } \mathcal{O} \neq \emptyset, \\ \infty & \text{otherwise,} \end{cases}$$

where $\mathcal{O}$ is the family of open sets of $\overline{M}$, and

$$\mathrm{Cap}(O) = \inf\{\|u\|_{1,2} : u \in S_0 \text{ such that } u = 1 \text{ on } M \cap O\}.$$

We say $\partial_C M$ is *almost polar* if it has null capacity. Then we have

PROPOSITION 3. *If* $0 < \mathrm{Cap}(\partial_C M) < \infty$, *then* M *does not have negligible boundary. If* M *has negligible boundary and any bounded set* B *has finite volume* $\mu(B)$, *then* $\partial_C M$ *is almost polar. In particular, the volume growth condition* (1) *yields* $\mu(B) < \infty$ *for any bounded set* B.

PROOF. Since $S_0 \neq S$ if and only if $0 < \mathrm{Cap}(\partial_C M) < \infty$ (e.g. [22]) we have, by Theorem 1, the first assertion.

Next, assume that M has negligible boundary, $\partial_C M \neq \emptyset$ and that any finite set $B \subset M$ has finite volume. If $\mathrm{Cap}(\partial_C M) > 0$, then there exists a ball $B = B(R) \subset \overline{M}$ of radius $R > 0$ such that $\mathrm{Cap}(\partial_C M \cap B) > 0$. Set

$$u(x) = \begin{cases} 1 & \text{if } x \in B, \\ 1 - r_B/R & \text{if } x \in B(2R) \setminus B, \\ 0 & \text{otherwise,} \end{cases}$$

where r_B is the radius function from B. Since $\mu(B(2R)) < \infty$, $u \in S$, but $u \notin S_0$ because $0 < \mathrm{Cap}(\partial_C M \cap B)$, which contradicts the fact that M has negligible boundary. In fact, on such a manifold $S_0 = S$ by Theorem 1 proved in the next section. □

REMARK 1. Other related results of almost polarity and negligibility of the Cauchy boundary for an algebraic variety $M \subset \mathbb{CP}^n$ with Bergman metric and with the singular set Σ removed are as follows:

- $\partial_C M = \Sigma$. If the real codimension of Σ is greater than 1, then Σ is almost polar ([20], [35], and [21]),
- there exists M such that $\partial_C M$ is almost polar but M does not have negligible boundary [11] (see also [3] and [24]),
- if M is Kähler and $\partial_C M$ is almost polar, then M has negligible boundary [10], [26].

The last item follows from an estimate of the cut-off function's gradient near the Cauchy boundary, which could be proved by applying the Kähler identity. Let us point out that Osawa and Sibony proved a Kähler identity on a Levi flat CR manifold [25].

3. SELF-ADJOINTNESS OF KOHN–ROSSI LAPLACIANS

Since the classical Kohn–Rossi laplacian which is defined on smooth functions is never self-adjoint, we extend it to a self-adjoint operator and apply the functional-analytic techniques. Here the natural question arises if the extension is unique. In general, if a symmetric operator defined on a Hilbert space has a unique self-adjoint extension, it is called *essentially self-adjoint*.

In this section, we discuss the essential self-adjointness of the sublaplacian Δ^E and the Kohn–Rossi laplacians $\Box_b^G$ and $\Box_b^M$. First we show the essential self-adjointness of Δ^E and $\Box_b^G$ on a CR manifold with negligible boundary. Next we show the self-adjointness of $\Box_b^M$ on a complete CR manifold. Recall that a complete CR manifold has negligible boundary, and the self-adjointness of $\Box_b^M$ implies the essential self-adjointness of $\Box_b^G$ [22].

We define the domain $D(\nabla^E)$ of ∇^E to be S. Similarly, the domain $D(\mathrm{div})$ of div is the set of vector fields $X \in \Gamma(E)$ such that both X and $\mathrm{div}\, X$ are square-integrable. Then the domain $D(\Delta^E)$ of Δ^E is

$$D(\Delta^E) = \{u \in D(\nabla^E) \cap C^\infty(M) : \nabla^E u \in D(\mathrm{div})\}.$$

The *minimal Kohn–Rossi laplacian* $\Box_b^M$ is the closure of $\Box_b$ defined on $A_0(M)$. Denote by $\Box_b^F$ the Friedrichs extension of $\Box_b^M$. The following result was proved in [22]. Here we present a more direct alternative proof.

THEOREM 1. *Assume that M has negligible boundary. Then the sublaplacian Δ^E on $L^2(M)$ functions, and the Kohn–Rossi laplacian $\Box_b^G$ on $L^2(A^q)$ with $0 < q < n$, are essentially self-adjoint. Moreover, $S_0 = S$ and the closure of $\Box_b^G$ coincides with $\Box_b^F$.*

PROOF. First we show that

$$\langle \nabla^E u, X\rangle = \langle u, -\mathrm{div}\, X\rangle \quad \text{for every } u \in D(\nabla^E) \text{ and } X \in D(\mathrm{div}), \tag{6}$$

because this is equivalent to the essential self-adjointness of Δ^E and to the coincidence of S_0 and S [22]. Since

$$|\bar\partial_b u|^2 = \sum |\bar e_i u|^2 = \frac{1}{2}|\nabla^E u|^2, \qquad |\delta\beta|^2 = \frac{1}{2}|\mathrm{div}\, X|^2$$

for $u \in D(\nabla^E)$ and $X \in D(\mathrm{div})$, where $X = \sum f^i X_i$ and $\beta = \sqrt{-1}\sum f^i \bar\theta^i$, it follows that $u \in D(\bar\partial_b)$ and $\beta \in D(\delta)$. Comparing the real and imaginary parts of

$$\langle \bar\partial_b u, \beta\rangle = \langle u, \delta\beta\rangle,$$

which holds true because M has negligible boundary, we have (6).

Next, consider $\Box_b^N = \bar\partial_b^* \bar\partial_b + \delta^*\delta$, where the domains of $\bar\partial_b$ and δ are given in Definition 1. By the von Neumann theorem, $\Box_b^N$ is self-adjoint. Set

$$\tilde\Box_b^N = \Box_b^N|_{A(M)}.$$

Let $\alpha \in D(\Box_b^N) \cap L^2(A^q)$ with $0 < q < n$. By Lemma 4,

$$e^{-t\Box_b^N}\alpha \in D(\tilde\Box_b^N) \quad \text{for every } t > 0.$$

Therefore, since

$$e^{-t\square_b^N}\alpha \to \alpha \quad \text{as } t \to 0 \text{ in the graph norm of } \square_b^N,$$

$\tilde{\square}_b^N$ is essentially self-adjoint. This implies the essential self-adjointness of $\square_b^G$ because $\tilde{\square}_b^N \subset \square_b^G$ and $\square_b^G$ is symmetric.

By comparing the domains of $\square_b^F$ and $\square_b^G$, we see that $\square_b^G \subset \square_b^F$. Since $\square_b^G$ is essentially self-adjoint and $\square_b^F$ is self-adjoint, the closure of $\square_b^G$ coincides with $\square_b^F$. □

Next, we study the same problem for $\square_b^M$ on a complete CR manifold. For that purpose we prepare some lemmae.

LEMMA 6 (Theorem X.26 of [27]). *Let A be a strictly positive symmetric operator. Then A is essentially self-adjoint if and only if* $\ker(A^*) = \{0\}$.

Let $\alpha \in \ker((\square_b^M + \lambda)^*)$ with $\lambda > 0$. In order to cut off α at infinity, we need a suitable cut-off function:

LEMMA 7 ([2] and [22]). *Denote by $B(r) \subset M$ the geodesic ball of radius $r > 0$ centered at some fixed point. For every $r > 0$ there exists a non-negative function $\chi_r \in W$ such that $\|\chi_r\|_\infty \le 1$, $\|\nabla^E \chi_r\|_\infty \le r^{-1}$, and*

$$\chi_r(x) = \begin{cases} 1 & \text{if } x \in B(r), \\ 0 & \text{if } x \notin B(2r). \end{cases}$$

The cut-off function χ_r is in S because of

PROPOSITION 4. *We have*

$$S = W.$$

PROOF. As $S \subset W$ is clear, we only check $W \subset S$. For an arbitrary $u \in W$, consider $e^{-t\Delta^E}u \in L^2 \cap C^\infty(M)$. Since $e^{-t\Delta^E}$ commutes with ∇^E, it follows that $\nabla^E e^{-t\Delta^E}u = e^{-t\Delta^E}\nabla^E u \in L^2$ and $e^{-t\Delta^E}u$ converges to u in S as $t \to 0$. By completeness of S, $u \in S$. □

LEMMA 8. *If M is complete, then $\chi_r^2\alpha \in D(\overline{\partial}_b) \cap D(\delta)$ for every $r > 0$.*

PROOF. To see this, we fix $r > 0$ and show that there exists a sequence $\beta_k \in A_0(M)$ which

- is a Cauchy sequence with respect to the $(\overline{\partial}_b + \delta)$-graph norm,
- converges to $\chi_r^2\alpha$ in $L^2(A)$ as $k \to \infty$.

If such a sequence exists, then $\chi_r^2\alpha = \lim_{k\to\infty}\beta_k \in D(\overline{\partial}_b) \cap D(\delta)$. Let $\chi_{r_k} \in C_0^\infty(M)$ be a sequence which converges to χ_r in S_0 as $k \to \infty$. Let $\eta \in C_0^\infty(M)$ be equal to 1 on $B(2r)$. Since $\eta\chi_{r_k}$ converges to χ_r in S_0 as $k \to \infty$, we may assume that $\mathrm{supp}(\chi_{r_k}) \subset K := \mathrm{supp}(\eta)$ for every $k > 0$.

Because $\Box_b$ is hypoelliptic, $\alpha \in A(M)$, and therefore $\beta_k := \chi_{r_k}^2 \alpha \in A_0(M)$. By the Lebesgue theorem, $\beta_k \to \chi_r^2 \alpha$ in $L^2(A)$ as $k \to \infty$.

Applying Proposition 3 and Lemma 2 gives

$$\begin{aligned}\|\overline{\partial}_b(\beta_k - \beta_l)\| &\le \|(\overline{\partial}_b(\chi_{r_k}^2 - \chi_{r_1}^2)) \wedge \alpha\| + \|(\chi_{r_k}^2 - \chi_{r_1}^2)\overline{\partial}_b \alpha\| \\ &\le \|\overline{\partial}_b(\chi_{r_k}^2 - \chi_{r_1}^2)\|_\infty \|\alpha\| + \|\chi_{r_k}^2 - \chi_{r_1}^2\| \cdot \|\overline{\partial}_b \alpha\|_{L^\infty(K)},\end{aligned}$$

where the last line tends to 0 as $k, l \to \infty$ because $\sqrt{2}|\overline{\partial}_b \chi_{r_k}| = |\nabla^E \chi_{r_k}|$ and by Lemma 7. Thus, β_k is a Cauchy sequence with respect to the $\overline{\partial}_b$-graph norm.

Next, since

$$\begin{aligned}|(\delta(\beta_k - \beta_l))_{I_{q-1}}| &\le |(\chi_{r_k}^2 - \chi_{r_k}^2)(\delta\beta)_{I_{q-1}}| + |\langle \overline{\partial}_b(\chi_{r_k}^2 - \chi_{r_k}^2), \beta \rfloor I_{q-1}\rangle| \\ &\le |\chi_{r_k}^2 - \chi_{r_k}^2||(\delta\beta)_{I_{q-1}}| + |\langle \overline{\partial}_b(\chi_{r_k}^2 - \chi_{r_k}^2)| \, |\beta|,\end{aligned}$$

we deduce that $\delta(\beta_k - \beta_l)$ tends to 0 in $L^2(A^{q-1})$ as $k, l \to \infty$.

Therefore, β_k is a Cauchy sequence with respect to the $(\overline{\partial}_b + \delta)$-graph norm, and hence, as explained at the beginning of the proof, $\chi_r^2 \alpha \in D(\overline{\partial}_b)$. □

LEMMA 9.

$$\langle \delta(\chi_r^2 \alpha), \delta\alpha \rangle = \|\chi_r \delta\alpha\|^2 - 2\langle \chi_r \alpha, \overline{\partial}_b \chi_r \wedge \delta\alpha \rangle.$$

PROOF. Indeed, we have pointwise

$$\begin{aligned}\langle \delta(\chi_r^2 \alpha), \delta\alpha \rangle &= \sum_{I_{q-1}} (\delta(\chi_r^2 \alpha))_{I_{q-1}} (\delta\alpha)_{I_{q-1}} \\ &= \sum_{I_{q-1}} |\chi_r (\delta\alpha)_{I_{q-1}}|^2 - 2 \sum_{I_{q-1}} \langle \chi_r \overline{\partial}_b \chi_r, \alpha \rfloor I_{q-1} \rangle (\delta\alpha)_{I_{q-1}} \\ &= |\chi_r \delta\alpha|^2 - 2\chi_r \sum_{I_q} \alpha_{I_q} \sum_{i_j} (\overline{\partial}_b \chi_r)_{i_j} (-1)^{j-1} (\delta\alpha)_{i_1 \dots \hat{i}_j \dots i_q} \\ &= |\chi_r \delta\alpha|^2 - 2\chi_r \sum_{I_q} \alpha_{I_q} (\overline{\partial}_b \chi_r \wedge \delta\alpha)_{I_q} = |\chi_r \delta\alpha|^2 - 2\langle \chi_r \alpha, \overline{\partial}_b \chi_r \wedge \delta\alpha \rangle. \quad \Box\end{aligned}$$

We are in a position to prove

THEOREM 2. *If M is complete, then the minimal Kohn–Rossi laplacian $\Box_b^M$ is self-adjoint on $L^2(A^q)$ with $0 < q < n$.*

PROOF. In view of Lemma 6, we prove the statement by showing $\alpha \equiv 0$. Let $\beta_k \in A_0(M)$ be the form which appeared in the proof of Lemma 8. Then

$$-\lambda \langle \beta_k, \alpha \rangle = \langle \Box_b \beta_k, \alpha \rangle = \langle \overline{\partial}_b \beta_k, \overline{\partial}_b \alpha \rangle + \langle \delta\beta_k, \delta\alpha \rangle. \tag{7}$$

By letting k tend to ∞, and applying Lemmae 7, 8 and 9, we obtain

$$\begin{aligned}-\lambda \langle \chi_r^2 \alpha, \alpha \rangle &= \langle \overline{\partial}_b(\chi_r^2 \alpha), \overline{\partial}_b \alpha \rangle + \langle \delta(\chi_r^2 \alpha), \delta\alpha \rangle \\ &= \|\chi_r \overline{\partial}_b \alpha\|^2 + 2\langle \chi_r(\overline{\partial}_b \chi_r) \wedge \alpha, \overline{\partial}_b \alpha \rangle + \|\chi_r \overline{\partial}_b \alpha\|^2 - 2\langle \chi_r \alpha, (\overline{\partial}_b \chi_r) \wedge \delta\alpha \rangle \\ &\ge \|\chi_r \overline{\partial}_b \alpha\|^2 + \|\chi_r \overline{\partial}_b \alpha\|^2 - 2\{\|\chi_r \overline{\partial}_b \alpha\| + \|\chi_r \delta\alpha\|\} \cdot \|\overline{\partial}_b \chi_r\|_{L^\infty} \|\alpha\| \\ &\ge \|\chi_r \overline{\partial}_b \alpha\|^2 + \|\chi_r \overline{\partial}_b \alpha\|^2 - \sqrt{2}\{\|\overline{\partial}_b \alpha\| + \|\delta\alpha\|\} \cdot \frac{\|\alpha\|}{r},\end{aligned}$$

hence, letting $r \to \infty$, we have

$$-\lambda\|\alpha\|^2 \geq \|\overline{\partial}_b\alpha\|^2 + \|\delta\alpha\|^2.$$

Since $\lambda > 0$ we deduce that $\alpha \equiv 0$. □

REMARK 2. Strichartz [29] proved the self-adjointness of the minimal Laplace–Beltrami operator of a complete Riemannian manifold in a similar way.

The following is a consequence of Theorems 1 and 2:

COROLLARY 1. *Assume that M has negligible boundary. Then, for any $\alpha \in D(\Box_b^G) \cap L^2(A^q)$ with $0 < q < n$, there exists $\alpha_l \in A_0^q$ such that*

$$\|\overline{\partial}_b(\alpha - \alpha_l)\| + \|\delta(\alpha - \alpha_l)\| \to 0 \quad \text{as } l \to \infty. \tag{8}$$

Moreover, α is L^2-harmonic if and only if $\alpha \in D(\Box_b^G)$ and $\Box_b^G\alpha = 0$.

If M is complete, then $\Box_b^G = \Box_b^M$ and (8) *becomes*

$$\|\alpha - \alpha_l\| + \|\overline{\partial}_b(\alpha - \alpha_l)\| + \|\delta(\alpha - \alpha_l)\| + \|\Box_b^G(\alpha - \alpha_l)\| \to 0 \quad \text{as } l \to \infty. \tag{9}$$

PROOF. For every $\alpha \in D(\Box_b^G) \cap L^2(A^q)$, there exists $\alpha_l \in A_0^q$ that satisfies (8) by the definition of $D(\Box_b^F)$ and Theorem 1. If α is harmonic, then by the definition of $\Box_b^G$, we have $\alpha \in D(\Box_b^G)$ and $\Box_b^G\alpha = 0$. Conversely, if $\alpha \in D(\Box_b^G)$ and $\Box_b^G\alpha = 0$, then

$$\|\overline{\partial}_b\alpha\|^2 + \|\delta\alpha\|^2 = \langle\Box_b^G\alpha, \alpha\rangle = 0,$$

so α is harmonic.

Next, assume that M is complete. Since both $\Box_b^M$ and $\Box_b^G$ are self-adjoint and $\Box_b^M \subset \Box_b^G$, it follows that $\Box_b^M = \Box_b^G$. As $\Box_b$ with domain A_0^q is essentially self-adjoint, there exists $\alpha_l \in A_0^q$ such that

$$\|\alpha - \alpha_l\| + \|\Box_b(\alpha - \alpha_l)\| \to 0 \quad \text{as } l \to \infty. \tag{10}$$

Since the right-hand side of

$$\|\overline{\partial}_b(\alpha - \alpha_l)\| + \|\delta(\alpha - \alpha_l)\| = \langle\Box_b(\alpha - \alpha_l), \alpha - \alpha_l\rangle \leq \|\Box_b(\alpha - \alpha_l)\| \, \|\alpha - \alpha_l\|$$

tends to 0 as $l \to \infty$, by (10) we have (9). □

REMARK 3. The second part of the corollary was proved for complete Riemannian manifolds by Andreotti and Vesentini [1].

4. VANISHING THEOREM AND SPECTRUM ESTIMATE

In this section, we establish a vanishing theorem and an estimate of the bottom of the Kohn–Rossi laplacian's spectrum of a CR manifold with negligible boundary. For that purpose, we first study the Weitzenböck formulae for $\Box_b$. Together with the self-adjointness, which was obtained in the previous section, we show the vanishing theorem and the spectrum estimate.

Let us start from

DEFINITION 2. *Define the* connection laplacians $\nabla^{0,1*}\nabla^{0,1}$ *and* $\nabla^{1,0*}\nabla^{1,0}$ *on* $A(M)$ *by*

$$\nabla^{0,1*}\nabla^{0,1} := -\sum_i \nabla^2_{e_i,\overline{e}_i} \; (= -\sum_i \{\nabla_{e_i}\nabla_{\overline{e}_i} - \nabla_{\nabla_{e_i}\overline{e}_i}\}),$$
$$\nabla^{1,0*}\nabla^{1,0} := -\sum_i \nabla^2_{\overline{e}_i,e_i},$$

where ∇^2 *is the second invariant covariant, associated to the Tanaka–Webster connection.*

PROPOSITION 5. *Both connection laplacians are non-negative on* $A_0(M)$.

PROOF. Since $\nabla g^E = 0$, for every $x \in M$ we have

$$\langle \nabla^{0,1*}\nabla^{0,1}\alpha, \beta\rangle(x) = \sum_i \langle \nabla_{\overline{e}_i}\alpha, \nabla_{\overline{e}_i}\beta\rangle(x) + \sum_i \{\langle \nabla_{\nabla_{e_i}\overline{e}_i}\alpha, \beta\rangle(x) - e_i\langle \nabla_{\overline{e}_i}\alpha, \beta\rangle(x)\}.$$

By the standard divergence theorem, the integral of the second line vanishes, provided one of the forms α, β has compact support. In the same way, one may show that $\nabla^{1,0*}\nabla^{1,0}$ is non-negative on $A_0(M)$. □

DEFINITION 3. *The* Ricci operator $\mathrm{Ric} : A(M) \to A(M)$ *is defined by*

$$\mathrm{Ric}(\alpha) = \sum_{i,j} \overline{\theta}^i \wedge (\overline{e}_j \rfloor R(e_j, \overline{e}_i)\alpha),$$

where R *is the curvature tensor associated to the Tanaka–Webster connection.*

In order to use the Ricci identity, we need

LEMMA 10.

$$\mathrm{Ric} = -\sum_i \mathrm{Ric}(e_i, \overline{e}_i). \tag{11}$$

PROOF. Note that

$$R(Z_1, \overline{Z}_2)\overline{Z}_3 = R(Z_1, \overline{Z}_3)\overline{Z}_2$$

for any $Z_i \in \Gamma(T^{1,0}(M))$. Thus,

$$\begin{aligned}\mathrm{Ric}(\alpha)(\overline{Z}_1, \dots, \overline{Z}_q) &\\ = -\frac{1}{(q-1)!}&\sum_{i,j,\sigma} \mathrm{sgn}(\sigma)\alpha(\overline{e}_i, \overline{Z}_{\sigma(2)}, \dots, R(e_j, \overline{Z}_{\sigma(1)})\overline{Z}_{\sigma(i)}, \dots, \overline{Z}_{\sigma(q)})\\ -\frac{1}{(q-1)!}&\sum_{j,\sigma} \mathrm{sgn}(\sigma)\alpha(R(e_j, \overline{e}_j)\overline{Z}_{\sigma(1)}, \dots),\end{aligned}$$

where the second line vanishes, and the third line equals the right-hand side of (11) with the variables $(\overline{Z}_1, \dots, \overline{Z}_q)$. This proves the lemma. □

The following is the Weitzenböck formula:

THEOREM 3. *We have*

$$\Box_b = \nabla^{0,1*}\nabla^{0,1} + \mathrm{Ric} - q\sqrt{-1}\nabla_\xi \tag{12}$$

$$= \nabla^{1,0*}\nabla^{1,0} + (n-q)\sqrt{-1}\nabla_\xi \tag{13}$$

$$= \frac{n-q}{n}\nabla^{0,1*}\nabla^{0,1} + \frac{q}{n}\nabla^{1,0*}\nabla^{1,0} + \frac{n-q}{n}\mathrm{Ric} \tag{14}$$

on $A^q(M)$ for $0 \le q \le n$.

PROOF. Let $\alpha \in A^q(M)$. Fix $x \in M$ and assume that $\nabla e_i(x) = 0$ for $1 \le i \le n$. By applying Proposition 2 and the Leibniz rule for the contraction operator, a direct calculation yields

$$\begin{aligned}
\Box_b\alpha = (\overline{\partial}_b\delta + \delta\overline{\partial}_b)\alpha &= \sum_i \overline{\theta}^i \wedge (\nabla_{\overline{e}_i}(\delta\alpha)) - \sum_j \overline{e}_j \rfloor (\nabla_{e_j}(\overline{\partial}_b\alpha))\\
&= -\sum_{i,j}\overline{\theta}^i \wedge (\overline{e}_j \rfloor \nabla_{\overline{e}_i}\nabla_{e_j}\alpha) - \sum_{i,j}\overline{e}_j \rfloor (\overline{\theta}^i \wedge \nabla_{e_j}\nabla_{\overline{e}_i}\alpha)\\
&= -\sum_{i,j}\overline{\theta}^i \wedge (\overline{e}_j \rfloor \nabla_{\overline{e}_i}\nabla_{e_j}\alpha) - \sum_i \nabla^2_{e_i,\overline{e}_i}\alpha + \sum_{i,j}\overline{\theta}^i \wedge (\overline{e}_j \rfloor \nabla_{e_j}\nabla_{\overline{e}_i}\alpha)\\
&= \nabla^{0,1*}\nabla^{0,1}\alpha + \sum_{i,j}\overline{\theta}^i \wedge (\overline{e}_j \rfloor (R(e_j,\overline{e}_i) - \nabla_{T(e_j,\overline{e}_i)})\alpha)\\
&= \nabla^{0,1*}\nabla^{0,1}\alpha + \sum_{i,j}\overline{\theta}^i \wedge (\overline{e}_j \rfloor (R(e_j,\overline{e}_i)\alpha) - q\sqrt{-1}\nabla_\xi\alpha;
\end{aligned}$$

in the last passage we have used the fact that $T(e_i, \overline{e}_j) = \delta_{ij}\sqrt{-1}\xi$. Therefore, by the Ricci identity,

$$\Box_b = \nabla^{1,0*}\nabla^{1,0} + (n-q)\sqrt{-1}\nabla_\xi \quad \text{on } A^q(M). \tag{15}$$

Combining (12) and (15) completes the proof of Theorem 3. □

A consequence is the following vanishing theorem for compact CR manifolds:

COROLLARY 2 (Tanaka's vanishing theorem [31]). *If* Ric *is non-negative on $A^q(M)$, then every harmonic q-form with compact support is parallel for the Tanaka–Webster connection.*

PROOF. Assume that Ric is non-negative on $A^q(M)$. By Proposition 5 and Theorem 3, for $\alpha \in \mathcal{H}^q \cap A_0(M)$ we have

$$\nabla_Z\alpha = \nabla_{\overline{Z}}\alpha = \nabla_\xi\alpha = 0 \quad \text{for every } Z \in \Gamma(T^{1,0}(M)).$$

Since $T(M) = \mathrm{Re}\{T^{1,0}(M) \oplus T^{0,1}(M)\} \oplus \mathbb{R}\xi$, it follows that $\nabla\alpha = 0$. □

REMARK 4. Tanaka [31] obtained Theorem 3 and Corollary 2 in terms of the operator $R_* : A^q(M) \to A^q(M)$:

$$(R_*\alpha)(\overline{Z}_1, \dots, \overline{Z}_q) = \sum_i \alpha(\overline{Z}_1, \dots, R_*\overline{Z}_i, \dots, \overline{Z}_q),$$

where $R_*X = -\sqrt{-1}\sum_i R(e_i, \overline{e}_i)JX$.

We extend Corollary 2 to

THEOREM 4. *Let M be a CR manifold with negligible boundary. Assume that* $0 < q < n$. *If* Ric *is non-negative on $A^q(M)$, then every L^2-harmonic q-form α is parallel for the Tanaka–Webster connection, the norm $|\alpha|$ is constant, and the bottom of the spectrum $\lambda^q_{\min}$ in $L^2(A^q)$ is estimated as*

$$\lambda^q_{\min} \ge \frac{n-q}{n} \inf\{\langle \mathrm{Ric}(\alpha), \alpha\rangle : \alpha \in A^q_0(M) \text{ and } \|\alpha\| = 1\}.$$

Additionally, if Ric *is positive, or M has infinite volume, then the L^2-reduced cohomology of degree q is trivial.*

PROOF. Let α be an L^2-harmonic q-form. By Corollary 1, there exists a sequence $\alpha_l \in A^q_0(M)$ which converges to α as $l \to \infty$ in the Dirichlet norm. By the Weitzenböck formula, for every α_l we have

$$n\langle \Box_b\alpha_l, \alpha_l\rangle = (n-q)\langle \nabla^{0,1*}\nabla^{0,1}\alpha_l, \alpha_l\rangle + q\langle \nabla^{1,0*}\nabla^{1,0}\alpha_l, \alpha_l\rangle + (n-q)\langle \mathrm{Ric}(\alpha_l), \alpha_l\rangle.$$

Because the left-hand side tends to 0 as $l \to \infty$ and Ric is non-negative, the first and second terms of the right-hand side tend to 0 as $l \to \infty$, since they are non-negative by Proposition 5. Therefore,

$$\nabla\alpha = \lim_{l\to\infty} \nabla\alpha_l = 0 \quad \text{on } T^{1,0}(M) \oplus T^{0,1}(M),$$

which implies also $\nabla\alpha = 0$. Because ∇ is a metric connection,

$$\nabla|\alpha|^2 = 0. \tag{16}$$

Moreover, since E satisfies the Hörmander condition, $\xi|\alpha|^2 = 0$, and hence, together with (16), we deduce that $|\alpha| \equiv$ const.

In addition, if M has infinite volume, then $|\alpha|$ must be 0 because $\alpha \in L^2(A^q)$. On the other hand, $\mathrm{Ric}(\alpha) = 0$ by (14), and thus, if Ric is positive, then $\alpha \equiv 0$.

The spectrum estimate is almost evident. Indeed, by Corollary 1 and the fact that Ric is symmetric,

$$\lambda^q_{\min} = \inf\{\langle \Box_b\alpha, \alpha\rangle : \alpha \in D(\Box^G_b) \text{ such that } \|\alpha\| = 1\}.$$

This implies the desired estimate. □

REMARK 5. If M is complete, then the fact that $|\alpha|$ is constant could be proved in the following way. Since

$$\begin{aligned}\Delta^E|\alpha|^2 &= -2\,\mathrm{Re}\sum_i e_i\bar{e}_i|\alpha|^2\\ &= -2\,\mathrm{Re}\sum_i\{\langle\nabla_{e_i}\nabla_{\bar{e}_i}\alpha,\alpha\rangle + |\nabla_{\bar{e}_i}\alpha|^2 + |\nabla_{e_i}\alpha|^2 + \langle\alpha,\nabla_{\bar{e}_i}\nabla_{e_i}\alpha\rangle\} = 0,\end{aligned}$$

and $\Delta^E|\alpha|^2 = 2\{|\nabla^E|\alpha|\,|^2 + |\alpha|\Delta^E|\alpha|\}$, $|\alpha|$ is subharmonic. As Δ^E is hypoelliptic, we may apply the argument in the proof of Theorem 2 to deduce that

$$0 \geq \langle\Delta^E|\alpha|, \chi_r^2|\alpha|\rangle = \langle\nabla^E|\alpha|, \nabla^E(\chi_r^2|\alpha|)\rangle = 2\langle\nabla^E|\alpha|, \chi_r|\alpha|\nabla^E\chi_r\rangle + \|\chi_r\nabla^E|\alpha|\,\|^2,$$

where the first term on the right-hand side tends to 0 as $r \to \infty$. Hence, $\nabla^E|\alpha| = 0$. The latter part is the Liouville property for a sub-Riemannian manifold.

The proof of the Liouville property of a sublaplacian is essentially due to Yau [34], who proved it for the Laplace–Beltrami operator on a complete Riemannian manifold. By combining the Liouville property and the Weitzenböck formula, he obtained the vanishing theorem for a complete Riemannian manifold. We first tried to follow his method, however, if we apply our Weitzenböck formula to $\Delta^E|\alpha|^2$, then an extra term

$$\langle(n-2q)\sqrt{-1}\nabla_\xi\alpha, \alpha\rangle$$

appears, which prevents us from deducing that $\Delta^E|\alpha|^2 = 0$ even if $\mathrm{Ric} \geq 0$. This is the reason why we previously proved $\nabla\alpha = 0$.

5. CONSERVATIVE PRINCIPLE FOR DIFFERERNTIAL FORMS

In this section, we will establish the conservative principle for differential forms, namely

THEOREM 5. *Assume that M has negligible boundary and satisfies the volume growth condition: $e^{-ar} \in L^1$ for every $a > 0$. Then for any bounded harmonic form α, there exist $\alpha_\epsilon \in D(\bar\partial_b) \cap D(\delta)$ such that $\|\alpha_\epsilon\|_\infty \leq \|\alpha\|$ for every $\epsilon > 0$, $\alpha_\epsilon \to \alpha$ as $\epsilon \to 0$, and*

$$\lim_{\epsilon\to 0} T_t\alpha_\epsilon = \alpha \quad \text{weakly for every } t > 0. \tag{17}$$

The following is the key estimate, which was proved for a complete Riemannian manifold by Vesentini [32].

LEMMA 11. *For a CR manifold with negligible boundary, and every $0 \leq b \leq 2\sqrt{2}$,*

$$\|e^{br/2}\psi\| \leq e^{b\sqrt{2}t}\|e^{br/2}\psi_0\| \quad \text{for every } \psi_0 \in A_0^q \text{ and } t \geq 0, \tag{18}$$

where $\psi = e^{-t\Box_b^G}\psi_0$.

PROOF. Let us first show

$$-\langle\bar\partial_b\psi, \bar\partial_b(e^{br}\psi)\rangle \leq \frac{b}{2\sqrt{2}}\|e^{br/2}\psi\|^2. \tag{19}$$

By Lemma 2,

$$
\begin{aligned}
-\langle\overline{\partial}_b\psi, \overline{\partial}_b(e^{br}\psi)\rangle &\leq |\langle\overline{\partial}_b\psi, \overline{\partial}_b e^{br}\wedge\psi)\rangle| - \langle\overline{\partial}_b\psi, e^{br}\overline{\partial}_b\psi\rangle \\
&\leq \frac{b}{2\sqrt{2}}(\|e^{br/2}\overline{\partial}_b\psi\|^2 + \|e^{br/2}\psi\|^2) - ||e^{br/2}\overline{\partial}_b\psi\|^2 \\
&\leq \left(\frac{b}{2\sqrt{2}} - 1\right)\|e^{br/2}\overline{\partial}_b\psi\|^2 + \frac{b}{2\sqrt{2}}\|e^{br/2}\psi\|^2 \leq \frac{b}{2\sqrt{2}}\|e^{br/2}\psi\|^2,
\end{aligned}
$$

provided $0 \leq b \leq 2\sqrt{2}$.

Next, we show

$$
-\langle\delta\psi, \delta(e^{br}\psi)\rangle \leq \frac{b}{2\sqrt{2}}\|e^{br/2}\psi\|^2. \tag{20}
$$

By Lemma 9 and the assumption $0 \leq b \leq 2\sqrt{2}$,

$$
\begin{aligned}
-\langle\delta\psi, \delta(e^{br}\psi)\rangle &= -\|e^{br/2}\delta\psi\|^2 + \langle\psi, \overline{\partial}_b e^{br}\wedge\delta\psi\rangle \\
&\leq -\|e^{br/2}\delta\psi\|^2 + \frac{b}{\sqrt{2}}\int|e^{br/2}\psi||e^{br/2}\delta\psi| \\
&\leq -\|e^{br/2}\delta\psi\|^2 + \frac{b}{2\sqrt{2}}(\|e^{br/2}\psi\|^2 + \|e^{br/2}\delta\psi\|^2) \\
&= \left(\frac{b}{2\sqrt{2}} - 1\right)\|e^{br/2}\delta\psi\|^2 + \frac{b}{2\sqrt{2}}\|e^{br/2}\psi\|^2 \leq \frac{b}{2\sqrt{2}}\|e^{br/2}\psi\|^2.
\end{aligned}
$$

Since M has negligible boundary, (19) and (20) yield

$$
\frac{\partial}{\partial t}\frac{1}{2}\|e^{br/2}\psi\|^2 = \langle\dot{\psi}, e^{br}\psi\rangle = -\langle\Box_b^G\psi, e^{br}\psi\rangle \leq \frac{b}{\sqrt{2}}\|e^{br/2}\psi\|^2. \tag{21}
$$

Thus, we have

$$
\|e^{br/2}\psi\| \leq e^{b\sqrt{2}t}\|e^{br/2}\psi_0\|. \qquad \Box
$$

Now, set

$$
\alpha_\epsilon = e^{-\epsilon r}\alpha \quad \text{for } \epsilon > 0.
$$

In order to prove Theorem 5, we need

LEMMA 12. *Assume that e^{-ar} is integrable for every $a > 0$. Then $\alpha_\epsilon \in D(\overline{\partial}_b) \cap D(\delta)$ for every bounded harmonic form α and $\epsilon > 0$.*

PROOF. By Definition 1, we only need to prove that both $\overline{\partial}_b\alpha_\epsilon$ and $\delta\alpha_\epsilon$ are square-integrable. However, this can be seen from

$$
|\overline{\partial}_b\alpha_\epsilon| = |ae^{-\epsilon r}(\overline{\partial}_b r)\wedge\alpha| \leq \epsilon\sqrt{\frac{q+1}{2}}e^{-\epsilon r}\|\alpha\|_\infty \in L^2, \tag{22}
$$

$$
|(\delta\alpha_\epsilon)_{I_{q-1}}| \leq \frac{\epsilon}{\sqrt{2}}e^{-\epsilon r}\|\alpha\|_\infty \in L^2. \qquad \Box \tag{23}
$$

PROOF OF THEOREM 5. By Lemma 12 and Theorem 1,

$$-\frac{\partial}{\partial t}\langle\psi,\alpha_\epsilon\rangle = \langle\overline{\partial}_b\psi,\overline{\partial}_b\alpha_\epsilon\rangle + \langle\delta\psi,\delta\alpha_\epsilon\rangle. \tag{24}$$

Integrating both sides of (24) from 0 to T yields

$$-\langle\psi,\alpha_\epsilon\rangle|_0^T = \int_0^T [\langle\overline{\partial}_b\psi,\overline{\partial}_b\alpha_\epsilon\rangle + \langle\delta\psi,\delta\alpha_\epsilon\rangle]\,dt. \tag{25}$$

We would like to show that the right-hand side of (25) tends to 0 as $\epsilon\to 0$, because this together with the facts that $e^{-\epsilon r}$ converges to 1 as $\epsilon\to 0$ and that T_t is symmetric on $L^2(A)$ will imply (17). As T_t commutes with $\overline{\partial}_b$, by applying (22) and (18), we obtain

$$\begin{aligned}\left|\int_0^T \langle\overline{\partial}_b\psi,\overline{\partial}_b\alpha_\epsilon\rangle\,dt\right| &\le \|e^{-r/2}\overline{\partial}_b\alpha_\epsilon\| \int_0^T \|e^{r/2}T_t(\overline{\partial}_b\psi_0)\|\,dt\\ &\le \frac{\epsilon}{\sqrt{2}}\|e^{-(1/2+\epsilon)r}\|\cdot\|\alpha\|_\infty \int_0^T \|e^{r/2}T_t(\overline{\partial}_b\psi_0)\|\,dt\\ &\le \frac{\epsilon}{\sqrt{2}}\|e^{-(1/2+\epsilon)r}\|\cdot\|\alpha\|_\infty\sqrt{T\int_0^T \|e^{r/2}T_t(\overline{\partial}_b\psi_0)\|^2\,dt}\\ &\le \frac{\epsilon}{\sqrt{2}}\|e^{-(1/2+\epsilon)r}\|\cdot\|\alpha\|_\infty\sqrt{T\int_0^T e^{\sqrt{2}t}\|e^{r/2}\overline{\partial}_b\psi_0\|^2\,dt},\end{aligned}$$

where the last line tends to 0 as $\epsilon\to 0$ because ψ_0 has compact support.

In the same way, by noting that δ commutes with T_t, it can be proved that

$$\lim_{\epsilon\to 0}\int_0^T \langle\delta\psi,\delta\alpha_\epsilon\rangle\,dt = 0.$$

Now we have completed the proof of Theorem 5. □

REMARK 6. Let α be a bounded harmonic form. Vesentini [32] proved that a complete Riemannian manifold with volume growth condition (1) satisfies

$$\langle T_t\psi,\alpha\rangle = \langle\psi,\alpha\rangle \quad \text{for every } \psi\in A_0^q \text{ and } t>0, \tag{26}$$

where $T_t = e^{-t\Delta}$. In order to make sense of the left-hand side of (26), $T_t\psi$ should be integrable, which is not clear on a CR manifold. This leads us to consider an additional condition (C): For an arbitrary bounded harmonic form α, there exist $\alpha_\epsilon\in D(\overline{\partial}_b)\cap D(\delta)$ such that $\|\alpha_\epsilon\|_\infty\le\|\alpha\|_\infty$ for every $\epsilon>0$, $\alpha_\epsilon\to\alpha$ as $\epsilon\to 0$, and

$$\lim_{\epsilon\to 0}\langle\psi,T_t\alpha_\epsilon\rangle = \langle\psi,\alpha\rangle \quad \text{for every } \psi\in A_0^q \text{ and } t>0. \tag{27}$$

Let us point out that (C) holds true if α is in $L^2(A^q)$, for example, if M has finite volume.

The conditions (26) and (C) relate as follows:

PROPOSITION 6. *Let M be either a CR or Riemannian manifold. If $T_t A_0^q \subset L^1(A^q)$, then (C) implies* (26). *Additionally, if M has almost polar Cauchy boundary, then* (26) *and (C) are equivalent.*

PROOF. Assume that $T_t A_0^q \in L^1(A)$ and (C) holds on M. Since T_t is symmetric on $L^2(A)$,

$$\langle T_t \psi, \alpha \rangle = \lim_{\epsilon \to 0} \langle T_t \psi, \alpha_\epsilon \rangle = \lim_{\epsilon \to 0} \langle \psi, T_t \alpha_\epsilon \rangle = \langle \psi, \alpha \rangle,$$

which is (26).

Conversely, assume that (26) holds and $\partial_C M$ is almost polar. Let χ_r be the cut-off function considered in Lemma 7 of Section 3. Since $\partial_C M$ is almost polar, there exists an open set $O \subset \overline{M}$ which contains $\partial_C M$ and has finite capacity. Let $B \subset \overline{M}$ be an arbitrary bounded set. Since $B = (B \cap O) \cup (B \cap O^c)$, B has finite volume. Therefore, by Lemma 8, $\chi_r \alpha \in D(d) \cap D(\delta)$ for every $r > 0$. Due to the property of χ_r, (C) holds with $\alpha_\epsilon = \chi_{1/\epsilon}\alpha$. □

REMARK 7. Let us recall a result of Grigor'yan [12]: A complete Riemannian manifold is conservative if

$$\int^\infty \frac{r\,dr}{\log \mathrm{vol}(B(r))} = \infty. \tag{28}$$

This was generalized to incomplete Riemannian manifolds in [21], and in essentially the same way, it can also be generalized to incomplete CR manifolds. It would be interesting to know if (1) could be replaced by (28). Indeed, on a complete Riemannian manifold which satisfies (28) we have the uniqueness of bounded solutions α to the Cauchy problem for the heat equation [23]. Recall that this is equivalent to the conservativeness when $p = 0$.

6. EXAMPLES

In this section we introduce some examples of CR manifolds with negligible boundary. We say that M is *Riemannian complete* if it is complete with respect to the Riemannian metric $g = g^E + \theta \otimes \theta$. Since the corresponding Riemannian distance is not greater than the intrinsic distance and they generate the same topology on M, we have

LEMMA 13. *If M is Riemannian complete, then it is complete.*

Therefore, by combining Lemmae 13 and 5, if M is Riemannian complete, it has negligible boundary. All of the following examples are Riemannian complete (see [15] for details).

EXAMPLE 1 (Heisenberg group). The *Heisenberg group $\mathcal{H}^{2n+1}$* is the space $\mathbb{R} \times \mathbb{C}^n$ with contact form

$$\theta = dt + 2 \sum (x^i dy^i - y^i dx^i),$$

where $t \in \mathbb{R}$ and $x^i + \sqrt{-1} y^i \in \mathbb{C}$ for $1 \leq i \leq n$.

EXAMPLE 2 (Sasakian space forms). There are exactly three Riemannian complete connected Sasakian space forms: S^{2n+1}, $\mathbb{R}^{2n+1}$, and $\Omega \times \mathbb{R}$, where $\Omega \subset \mathbb{C}$ is a simply connected bounded domain with the canonical Kähler form $d\omega$. The last two spaces are obviously non-compact, and carry the contact forms $dt - \sum y^i dx^i$ and $\omega + dt$, respectively.

EXAMPLE 3 (Spherical orbits). Let M be an n-dimensional non-homogeneous hyperbolic manifold with automorphism group of dimension n^2. The orbit $O(x)$, $x \in M$, is called *spherical* if each point of $O(x)$ has a neighborhood which is CR-equivalent to an open set of S^{2n-1}. A spherical orbit is CR-equivalent to one of the following hypersurfaces (see e.g. [14]):

(1) a lens space $S^{2n-1}/\mathbb{Z}_m$,
(2) $\sigma = \{(w, z) \in \mathbb{C}^{n-1} \times \mathbb{C} : \operatorname{Re} z = \|w\|^2\}$;
(3) $\sigma' = \{(w, z) \in \mathbb{C}^{n-1} \times \mathbb{C} : |z| = \exp(\|w\|^2)\}$;
(4) $\omega = \{(w, z) \in \mathbb{C}^{n-1} \times \mathbb{C} : \|w\|^2 + \exp(\operatorname{Re} z) = 1\}$;
(5) $\omega_\alpha = \{(w, z) \in \mathbb{C}^{n-1} \times \mathbb{C} : \|w\|^2 + |z|^\alpha = 1, z \neq 0\}$ for some $\alpha > 0$.

Then σ and σ' are non-compact and Riemannian complete [15].

7. APPENDIX

On a compact CR manifold, the Hodge decomposition holds for $A(M)$ [18], [6]. In this section, we study its generalization to CR manifolds with negligible boundary.

THEOREM 6. *For a CR manifold with negligible boundary,*

$$L^2(A^q) = \mathcal{H}^q \oplus \overline{\operatorname{range}(\overline{\partial}_b^{q-1})}^{L^2} \oplus \overline{\operatorname{range}(\delta^{q+1})}^{L^2} \quad \text{for } 0 < q < n,$$

where the domains of $\overline{\partial}_b$ *and* δ *are as in Definition* 1.

PROOF. Since M has negligible boundary, α is an L^2-harmonic q-form if and only if

$$\alpha \in D(\Box_b^G) \quad \text{and} \quad \Box_b^G \alpha = 0.$$

Thus, by a standard argument in Hilbert space theory,

$$L^2(A^q) = \mathcal{H}^q \oplus \overline{\operatorname{range}(\Box_b^G)}^{L^2} \cap L^2(A^q) \subset \mathcal{H}^q \oplus \overline{\operatorname{range}(\overline{\partial}_b)}^{L^2} \oplus \overline{\operatorname{range}(\delta)}^{L^2} \cap L^2(A^q).$$

Hence, we only need to show that $\mathcal{H}^q$ is perpendicular to both $\overline{\operatorname{range}(\overline{\partial}_b)}^{L^2}$ and $\overline{\operatorname{range}(\delta)}^{L^2}$. For $\omega \in \overline{\operatorname{range}(\overline{\partial}_b)}^{L^2}$, let $\gamma_l \in D(\overline{\partial}_b)$ be such that $\omega_l := \overline{\partial}_b \gamma_l \to \omega$ as $l \to \infty$ in L^2. Since

$$\langle \alpha, \omega_l \rangle = \langle \delta\alpha, \gamma_l \rangle = 0 \quad \text{for every } l > 0 \text{ and } \alpha \in \mathcal{H}^q,$$

by letting $l \to \infty$, we have $\langle \alpha, \omega \rangle = 0$. The same holds true if we change δ to $\overline{\partial}_b$, completing the proof. $\Box$

The following is a consequence:

COROLLARY 3. *For a CR manifold with negligible boundary,*

$$\mathcal{H}^q \simeq \ker(\overline{\partial}_b^q)/\overline{\mathrm{range}(\overline{\partial}_b^{q-1})}^{L^2} \qquad \textit{for } 0 < q < n.$$

PROOF. By Corollary 1,

$$\mathcal{H} \oplus \overline{\mathrm{range}(\overline{\partial}_b)}^{L^2} \subset \ker(\overline{\partial}_b).$$

In view of Theorem 6, in order to complete the proof, we only need to show that $\ker(\overline{\partial}_b)$ is perpendicular to $\overline{\mathrm{range}(\delta)}^{L^2}$. This could be done in the same way as in the proof of Theorem 6. □

REMARK 8. If Ric is positive, then

$$\mathrm{range}(\overline{\partial}_b) = \overline{\mathrm{range}(\overline{\partial}_b)}^{L^2} \quad \text{and} \quad \mathrm{range}(\delta) = \overline{\mathrm{range}(\delta)}^{L^2}.$$

ACKNOWLEDGMENTS. The author wishes to express his gratitude to Dr. Itoh and Mr. Saotome for providing him with interesting examples of non-compact CR manifolds that are studied from a different point of view in the joint work [15].

This work was partially supported by the Italian Education, University and Research Ministry (project Cofin 2000 "Proprietá geometriche delle varietá reali e complesse"), I.N.D.A.M, and Japan Society for the Promotion of Science.

REFERENCES

[1] A. ANDREOTTI - E. VESENTINI, *Carleman estimates for the Laplace–Beltrami equation on complex manifolds*. Publ. Math. IHES 25 (1965), 81–130.

[2] M. BIROLI - U. MOSCO, *A Saint-Venant type principle for Dirichlet forms on discontinuous media*. Ann. Mat. Pura Appl. 169 (1995), 125–181.

[3] J. CHEEGER, *On the Hodge theory of Riemannian pseudomanifolds*. In: Geometry of the Laplace Operator (Honolulu, 1979), Proc. Sympos. Pure Math. 36, Amer. Math. Soc., Providence, RI, 1980, 91–146.

[4] W. CHOW, *Über Systeme von linearen partiellen Differentialgleichungen erster Ordnung*. Math. Ann. 117 (1939), 98–105.

[5] G. B. FOLLAND - J. J. KOHN, *The Neumann Problem for the Cauchy–Riemann Complex*. Ann. of Math. Stud. 75, Princeton Univ. Press, Princeton, NJ, and Univ. of Tokyo Press, Tokyo, 1972.

[6] G. B. FOLLAND - E. M. STEIN, *Estimates for the $\overline{\partial}_b$-complex and analysis of the Heisenberg group*. Comm. Pure Appl. Math. 27 (1974), 429–522.

[7] M. FUKUSHIMA - Y. OSHIMA - M. TAKEDA, *Dirichlet Forms and Markov Processes*. North-Holland, 1980.

[8] M. P. GAFFNEY, *Hilbert space methods in the theory of harmonic integral*. Trans. Amer. Math. Soc. 78 (1955), 426–444.

[9] R. E. GREENE - H. WU, *Harmonic forms on noncompact Riemannian and Kaehler manifolds*. Michigan Math. J. 28 (1981), 63–83.

[10] D. GRIESER, *Local geometry of singular real analytic surfaces*. Trans. Amer. Math. Soc. 355 (2003), 1559–1577.

[11] D. GRIESER - M. LESCH, *On the L^2-Stokes theorem and Hodge theory for singular algebraic varieties*. Math. Nachr. 246/247 (2002), 68–82.

[12] A. GRIGOR'YAN, *Analytic and geometric background of recurrence and non-explosion of the Brownian motion on Riemannian manifolds*. Bull. Amer. Math. Soc. 36 (1999), 135–249.

[13] L. HÖRMANDER, *Hypoelliptic second order differential equations*. Acta Math. 119 (1967), 147–171.

[14] A. ISAEV, *Lectures on the Automorphism Groups of Kobayashi-Hyperbolic Manifolds*. Lecture Notes in Math. 1902, Springer, 2007.

[15] M. ITOH - J. MASAMUNE - T. SAOTOME, *The Serre duality theorem for a non-compact weighted CR manifold*. Proc. Amer. Math. Soc., to appear.

[16] D. JERISON - A. SÁNCHEZ-CALLE, *Subelliptic second order differential operators*. In: Complex Analysis, III (College Park, MD, 1985–86), Lecture Notes in Math. 1277, Springer, Berlin, 1987, 46–77.

[17] J. J. KOHN, *Harmonic integrals on strongly pseudoconvex manifolds, I*. Ann. of Math. 78 (1963), 112–148; II, ibid. 79 (1964), 450–472.

[18] J. J. KOHN, *Boundaries of complex manifolds*. In: Proc. Conference on Complex Manifolds (Minneapolis, 1964), Springer, 1965, 81–94.

[19] J. J. KOHN AND L. NIRENBERG, *Non-coercive boundary value problems*. Comm. Pure Appl. Math. 18 (1965), 443–492.

[20] P. LI - G. TIAN, *On the heat kernel of Bergman metric on algebraic variety*. J. Amer. Math. Soc. 8 (1995), 857–877.

[21] J. MASAMUNE, *Analysis of the Laplacian of an incomplete manifold with almost polar boundary*. Rend. Mat. Appl. (7) 25 (2005), 109–126.

[22] J. MASAMUNE, *Essential self-adjointness of sublaplacians via heat equation*. Comm. Partial Differential Equations 30 (2005), 1595–1609.

[23] J. MASAMUNE, *Conservative principle for differential forms*. Rend. Lincei Mat. Appl. 18 (2007), 351–358.

[24] J. MASAMUNE - J. TAKAHASHI, *Vanishing theorems for L^2 harmonic forms on incomplete weighted manifolds*. Preprint.

[25] T. OHSAWA - N. SIBONY, *Kähler identity on Levi flat manifolds and application to the embedding*. Nagoya Math. J. 158 (2000), 87–93.

[26] W. PARDON - M. STERN, *Pure Hodge structure on the L_2 cohomology of varieties with isolated singularities*. J. Reine Angew. Math. 533 (2001), 55–80.

[27] M. REED - B. SIMON, *Methods of Modern Mathematical Physics. II. Fourier Analysis, Self-Adjointness*. Academic Press, New York, 1975.

[28] G. DE RHAM, *Differentiable Manifolds*. Springer, Berlin, 1984.

[29] R. S. STRICHARTZ, *Analysis of the Laplacian on a complete Riemannian manifold*. J. Funct. Anal. 52 (1983), 48–79.

[30] R. S. STRICHARTZ, *Sub-Riemannian geometry*. J. Differential Geom. 24 (1986), 221–263.

[31] N. TANAKA, *A Differential Geometric Study on Strongly Pseudoconvex CR Manifolds*. Kinokuniya, Kyoto, 1975.

[32] E. VESENTINI, *Heat conservation on Riemannian manifolds*. Ann. Mat. Pura Appl. 182 (2003), 1–19.

[33] S. WEBSTER, *Pseudo hermitian geometry on a real hypersurface*. J. Differential Geom. 13 (1973), 25–41.

[34] S. T. YAU, *Some function-theoretic properties of complete Riemannian manifold and their applications to geometry*. Indiana Univ. Math. J. 25 (1976), 659–670.

[35] K. YOSHIKAWA, *Degeneration of algebraic manifolds and spectrum of Laplacians*. Nagoya Math. J. 146 (1997), 83–129.

Received 27 July 2007,
and in revised form 29 November 2007.

Department of Mathematical Sciences
Worcester Polytechnic Institute
100 Institute Road
WORCESTER, MA 01609-2280, U.S.A.
masamune@wpi.edu