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Anisotropy and stratification effects in the dynamics of fast rotating compressible fluids

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Abstract. The primary goal of this paper is to develop robust methods to handle two ubiquitous features appearing in the modeling of geophysical flows: (i) the *anisotropy* of the viscous stress tensor and (ii) *stratification* effects. We focus on the barotropic Navier–Stokes equations with Coriolis and gravitational forces. Two results are the main contributions of the paper. Firstly, we establish a local well-posedness result for finite-energy solutions, via a maximal regularity approach. This method allows us to circumvent the use of the effective viscous flux, which plays a key role in the weak solutions theories of Lions–Feireisl and Hoff, but seems to be restricted to isotropic viscous stress tensors. Moreover, our approach is sturdy enough to take into account nonconstant reference density states; this is crucial when dealing with stratification effects. Secondly, we study the structure of the solutions to the previous model in the regime when the Rossby, Mach and Froude numbers are of the same order of magnitude. We prove an error estimate on the relative entropy between actual solutions and their approximation by a large-scale quasi-geostrophic flow supplemented with Ekman boundary layers. Our analysis holds for a large class of barotropic pressure laws.

1. Introduction

This paper is devoted to the study of a class of barotropic Navier–Stokes systems. Our focus is on developing robust methods to handle two ubiquitous effects appearing in the modeling of geophysical fluids: (i) the strong *anisotropy* of the viscous stress tensor and (ii) *stratification* effects. As for the first aspect, we consider systems with anisotropic viscosity tensors

$$\Delta_{\mu,\varepsilon} := \mu\Delta_h + \varepsilon\partial_3^2 = \mu(\partial_1^2 + \partial_2^2) + \varepsilon\partial_3^2, \tag{1.1}$$

where the parameters μ and ε are dimensionless numbers such that

$$\mu > 0 \quad \text{and} \quad \varepsilon > 0, \quad \text{with } \varepsilon \ll \mu.$$

Concerning the second point, we consider highly rotating fluids with a strong Coriolis force $\frac{1}{\varepsilon}e_3 \times \rho u$ and a strong gravitational potential $\frac{1}{\varepsilon^2}\rho\nabla G = -\frac{1}{\varepsilon^2}\rho e_3$, where $u = u(x,t) \in \mathbb{R}^3$ denotes the velocity field of the fluid, $\rho = \rho(x,t) \in \mathbb{R}$ represents the density

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of the fluid and $e_3 = (0,0,1)^T$ is the unit vertical vector. We handle barotropic fluids, so the pressure of the fluid is assumed to be a function of the density only, i.e. $P = P(\rho)$; see (1.10) for precise assumptions on the pressure law. We consider the simplest possible geometrical setups and scalings enabling us to study nontrivial phenomena, such as boundary layers and vertical stratification.

Our goal in this paper is twofold. The first part of the paper is concerned with the existence and uniqueness of strong solutions for the barotropic Navier–Stokes equations with potential force ∇G:

$$\begin{cases} \partial_t \rho + \nabla \cdot (\rho u) = 0, \\ \partial_t(\rho u) + \nabla \cdot (\rho u \otimes u) + \nabla P(\rho) = \Delta_{\mu,\varepsilon} u + \lambda \nabla(\nabla \cdot u) + \rho \nabla G. \end{cases} \tag{1.2}$$

The term ∇G is responsible for stratification effects, thus the equilibrium density $\bar{\rho}$ of the previous system becomes nonconstant. As a consequence, there are two main difficulties in handling the well-posedness study for system (1.2): on the one hand, the strong anisotropy of the viscous tensor (see (1.1)) and, on the other hand, the fact that $\bar{\rho}$ introduces variable coefficients in the equations (see more details below). Our main result in this direction is Theorem 1. We set system (1.2) in the simple space-time domain $\mathbb{R}^3 \times (0,T)$, with $T > 0$; domains with boundaries, such as $(\mathbb{R}^2 \times (0,1)) \times (0,T)$, may be handled by the same method, at the price of more technical difficulties. Here we work with fixed values of the parameters $(\mu, \lambda, \varepsilon)$, and do not keep track of how the estimates depend on them: the existence of solutions to (1.2) is a challenge in itself. Our well-posedness result still holds if one incorporates a Coriolis force $\rho e_3 \times u$ on the left-hand side of (1.2).

The second part of the paper is devoted to the asymptotic analysis of the barotropic Navier–Stokes equations in presence of fast rotation and gravitational stratification. We consider the following system:

$$\begin{cases} \partial_t \rho + \nabla \cdot (\rho u) = 0, \\ \partial_t(\rho u) + \nabla \cdot (\rho u \otimes u) + \dfrac{\rho}{\varepsilon} e_3 \times u + \dfrac{\nabla P(\rho)}{\varepsilon^2} = \Delta_{\mu,\varepsilon} u + \lambda \nabla(\nabla \cdot u) + \dfrac{\rho}{\varepsilon^2} \nabla G, \end{cases} \tag{1.3}$$

set on the strip $(\mathbb{R}^2 \times (0,1)) \times (0,T)$, with the no-slip boundary conditions

$$u = 0 \quad \text{at } x_3 = 0, 1. \tag{1.4}$$

Here G is the gravitational potential, i.e. $G(x_3) = -x_3$. We describe the structure of weak solutions in the limit $\varepsilon \to 0$ for well-prepared data, analyze the Ekman boundary layers and their effect on the limit quasi-geostrophic flow and prove quantitative bounds based on relative entropy estimates. Our main result in this direction is Theorem 2. Our results hold for a large class of monotone pressure laws. Here our focus is on the asymptotic behavior for a family of weak solutions, under the assumption that such global-in-time weak solutions exist.

Let us emphasize a further aspect of the connection between the two parts of the paper. The quantitative stability estimates obtained in the second part lay the ground for a large-time well-posedness result for system (1.3) in the limit when $\varepsilon \to 0$. Inspired by previous results for incompressible flows ([8, 25, 40, 41]), we believe that strong solutions may be constructed for large data close to the two-dimensional limit quasi-geostrophic flow, by using a variation of the well-posedness result of the first part. In that perspective, our goal with the first part of this work is to have a robust local well-posedness result that can be adapted in subsequent works to handle the global existence for large data. In order to carry out this program, there are a number of issues to deal with, notably the adjustment of the local well-posedness result to the infinite slab $\mathbb{R}^2 \times (0,1)$. We give additional comments about this problem above Theorem 1. Therefore, we leave this work for further studies.

1.1. Modeling of geophysical flows

We consider mid-latitude and high-latitude motions of fast rotating compressible fluids, typically the Earth's atmosphere or oceans. System (1.3) is a particular case of the general nondimensional system

$$\begin{cases} \partial_t \rho + \nabla \cdot (\rho u) = 0, \\ \partial_t(\rho u) + \nabla \cdot (\rho u \otimes u) - \dfrac{1}{\mathtt{Re}_h}\Delta_h u - \dfrac{1}{\mathtt{Re}_3}\partial_3^2 u - \dfrac{1}{\mathtt{Re}}\nabla\nabla \cdot u \\ \qquad\qquad + \dfrac{1}{\mathtt{Ro}} e_3 \times (\rho u) + \dfrac{1}{\mathtt{Ma}^2}\nabla P(\rho) = \dfrac{1}{\mathtt{Fr}^2}\rho \nabla G. \end{cases} \tag{1.5}$$

As usual, the Reynolds number measures the ratio of inertial forces to viscous forces. System (1.5) modeling large-scale geophysical flows has, in particular, a horizontal Reynolds number $\mathtt{Re}_h = \frac{UL}{\nu_h}$ and a vertical Reynolds number $\mathtt{Re}_3 = \frac{UL}{\nu_3}$ that may be of different orders of magnitude; see below the comments about the anisotropy in the viscous stress tensor. Here U represents the typical speed of the flow, L the typical length, ν_h the horizontal viscosity and ν_3 the vertical one. The Mach number is defined as $\mathtt{Ma} = \frac{U}{c}$, where the constant c is the propagation speed of acoustic waves. For strong jet streams near the tropopause, $U = 50\,\mathrm{m\,s^{-2}}$, which corresponds to $\mathtt{Ma} = 0.15$; see [30]. The Rossby number, defined as $\mathtt{Ro} = \frac{U}{2\Omega L}$, measures the effect of the Earth's rotation; here $\Omega = 7.3 \cdot 10^{-5}\,\mathrm{s^{-1}}$ is the module of the Earth's angular velocity. In the case of the Gulf Stream, the length $L = 100\,\mathrm{km}$ and $U = 1\,\mathrm{m\,s^{-1}}$ are smaller than typical oceanic scales, and the Rossby number is about 0.07. Notice that here we consider the f-plane approximation of the Coriolis force.

Stratification. The gravitational force deriving from the geopotential $G = -gx_3$ tends to raise regions of fluid with lower density and lower regions of fluid with higher density. In the equilibrium configuration, the density profile decreases with respect to the vertical direction. The Froude number is then defined as $\mathtt{Fr} = \frac{U}{\sqrt{gH}}$, where $g = 9.81\,\mathrm{m\,s^{-2}}$ is the acceleration of gravity. It measures the ratio of inertial forces of a fluid element to its weight. The centrifugal force also derives from a potential. It is often neglected in models for the atmosphere ([12, 20]), but in certain regimes it can have a dramatic effect. The mathematical analysis of the centrifugal force poses different challenges that we do not dwell upon in the present work.

Anisotropy. In general, the horizontal and vertical viscosities are not equal, in particular when dealing with large-scale motions of geophysical flows. For instance, in the ocean the horizontal turbulent viscosity ν_h ranges from 10^3 to 10^8 $\mathrm{cm}^2\,\mathrm{s}^{-1}$, while the vertical viscosity ν_3 is much smaller and ranges from 1 to 10^3 $\mathrm{cm}^2\,\mathrm{s}^{-1}$. A justification of this fact can be seen in (i) the anisotropy between the horizontal and the vertical scales of the flows and (ii) the stabilizing effect of the Coriolis force, which makes the large-scale motion almost two-dimensional (see also Section 3). Hence, a frequent (and crude) modeling assumption is to suppose that the diffusion in the vertical direction is much weaker than the horizontal viscosity, which is enhanced by turbulent phenomena. For further insights about the physics of anisotropic diffusion, we refer to [45], in particular to [39, (2.122), Chapter 4], to [6] and [10].

Scaling. In (1.3), the dimensionless number ε denotes the Rossby number, which measures the strength of the rotation. We consider the scaling where the Rossby, the Mach and the Froude numbers are of the same order of magnitude, i.e. $\mathtt{Ro} = \mathtt{Ma} = \mathtt{Fr} = \varepsilon$. This is the richest scaling, since the effects due to the rotation are in balance with the compressible and gravitational effects. Notice however that other scalings are considered in the physical literature ([31]), for application to meteorology, as well as in the mathematical literature, see e.g. [16, 20].

Ekman layers. Ekman boundary layers are regions near horizontal boundaries with no-slip boundary condition where viscous effects balance the Coriolis force. The thickness of these boundary layers (see [11, Part I]) is

$$\delta_E = \left(\frac{\nu_3}{\Omega}\right)^{\frac{1}{2}},$$

which does not depend on the velocity. Note that the faster the rotation, the smaller is the layer affected by viscosity. See also [12, Chapter 8].

Pressure laws. We consider barotropic flows, for which the pressure is a function of the density only. A typical example is that of Boyle's law $P(\rho) = a\rho^\gamma$, with $a > 0$ and $\gamma \geq 1$. For the precise definition of the pressure law, we refer to (1.10).

1.2. Mathematical challenges related to anisotropy and stratification

We outline here some aspects of the study of compressible viscous fluids. We focus on two points in particular: (i) well-posedness and the difficulties related to the anisotropy in the viscosity and (ii) asymptotic analysis in the presence of stratification.

Well-posedness. For fluids modeled by the incompressible Navier–Stokes equations, the fact that the viscous stress tensor is isotropic or anisotropic does not affect the well-posedness theory. One can prove the existence of weak (Leray–Hopf), mild or strong solutions *regardless* of the structure of the viscous stress tensor.

For compressible fluids modeled by the compressible Navier–Stokes system though, anisotropy represents a major hurdle. Interestingly, the obstacle has similar roots for several well-posedness theories of weak solutions.

In the isotropic case, $\mu = \varepsilon$ in (1.2), one has a pointwise relation between the pressure and the divergence of u. Indeed, applying the divergence to the momentum equation and using the algebraic relation $\nabla \cdot (\nabla\nabla \cdot u) = \Delta\nabla \cdot u = \nabla \cdot \Delta u$, one gets

$$\Delta((\mu + \lambda)\nabla \cdot u - P(\rho) + P(1)) = \nabla \cdot (\rho\partial_t u + \rho u \cdot \nabla u). \tag{1.6}$$

This suggests defining the quantity

$$F := (\mu + \lambda)\nabla \cdot u - P(\rho) + P(1),$$

which is dubbed the *effective viscous flux*. According to (1.6), one obtains the relation

$$\Delta F = \nabla \cdot (\rho\partial_t u + \rho u \cdot \nabla u). \tag{1.7}$$

Property (1.7) is key to the existence of global finite-energy weak solutions in the Lions ([32]) and Feireisl ([19, 38]) theory on the one hand, and to the existence of global weak solutions with bounded density in the Hoff theory ([29]) (see also [13]).

In the anisotropic case $\mu \neq \varepsilon$, the above analysis breaks down, and the relation between $\nabla \cdot u$ and $P(\rho)$ becomes nonlocal. Indeed, (1.7) becomes

$$(\Delta_{\mu,\varepsilon} + \lambda\Delta)\nabla \cdot u - \Delta P = \nabla \cdot (\rho\partial_t u + \rho u \cdot \nabla u),$$

so that the definition of a modified effective flux should read

$$F_{\mathrm{ani}} := \Delta^{-1}(\Delta_{\mu,\varepsilon} + \lambda\Delta)\nabla \cdot u - P(\rho) + P(1).$$

The nonlocal operator $\Delta^{-1}(\Delta_{\mu,\varepsilon} + \lambda\Delta)$ changes the picture dramatically. In view of the existence of Hoff-type solutions with bounded density ([13,29]), one of the major flaws of F_{ani} is the lack of boundedness of $\Delta^{-1}(\Delta_{\mu,\varepsilon} + \lambda\Delta)$ on L^∞. Similarly, for the existence of global finite-energy weak solutions, the nonlocality is a major obstacle to getting the compactness of a sequence of approximate solutions; see [6, 7]. In the breakthrough work of Bresch and Jabin ([7]), a totally new compactness criterion was proved that enables the existence of global weak solutions to the compressible system (1.2) to be proved in the anisotropic case. There remains a restriction that $|\mu - \varepsilon| < \lambda - \frac{\mu}{3}$ which is compatible with the modeling of large-scale geophysical flows. A more important limitation of the result in [7] is on the pressure law $\gamma \geq 2 + \frac{\sqrt{10}}{2}$, where γ is defined in (1.10). On that subject, Bresch and Burtea proved recently in [4] the global existence of weak solutions for the quasi-stationary compressible Stokes equations and to the stationary compressible Navier–Stokes system ([5]) with an anisotropic viscous tensor. Their approach is based on the control of defect measures.

The anisotropy prompts us to consider the framework of strong solutions, in particular those with minimal regularity assumptions as in [13]. In that perspective, a further challenge for the well-posedness theory is the presence of nonconstant reference density states, due to the gravitational term. Indeed, the fact that the densities are perturbations of a *constant* state plays a major role in the analysis of [13]. This question seems to remain broadly unexplored in general. In our work we are able to incorporate nonconstant reference density states in the approach of [13].

Asymptotic analysis. We review here some of the literature concerned with the quantitative analysis of viscous barotropic fluids in high rotation (for an overview of this topic for incompressible fluids, we refer to [11]). For systems of type (1.3), results on the combined low Mach and low Rossby limits were obtained in several directions: well-prepared data (data close to the kernel of the penalization operator; see (3.57) for what it means in our context), ill-prepared data, slip or no-slip boundary data, with or without stratification (centrifugal force or gravitational force), and different scaling regimes. We do not attempt to be exhaustive here, but select some works that are relevant to our study here.

In the well-prepared case, [6] studies the limit for the same scaling as in (1.3), with no-slip boundary conditions and $G = 0$. They study the Ekman layers and prove stability estimates in the limit $\varepsilon \to 0$ in the case of pressure laws with $\gamma = 2$. As far as we know, this seems to be the only work concerned with the study of Ekman layers for compressible fluids. Extending it to more general pressure laws, and taking into account gravitational stratification effect, is an obvious motivation for our present work. The analysis with a gravitational potential was considered in [23], for fluids slipping on the boundary, using the relative entropy method.

In the ill-prepared case, the fact that the initial data is away from the kernel of the penalization operator is responsible for the propagation of high frequency acoustic-Poincaré waves. Different scaling regimes of $\mathtt{Ma}$ and $\mathtt{Ro}$ were analyzed in [16, 17, 20, 35, 37]. All of these works are concerned with fluids in the whole space or in an infinite slab satisfying the slip boundary condition, hence no boundary layers are needed in the asymptotic expansions. Let us point out that [16] manages to handle the centrifugal force.

1.3. Novelty of our results

We comment here on the main theorems of the paper.

Well-posedness of system (1.2): Theorem 1. We prove the short-time existence of finite-energy solutions to system (1.2). We introduce a simple and sturdy method based on a priori bounds obtained via maximal regularity estimates, following the approach of [13, 43]. However, our techniques are robust enough to deal with both effects mentioned above: (i) the anisotropy in the viscous stress tensor and (ii) the presence of a nonconstant reference density state. Thus, our result represents a generalization of [13, 43] in both directions. However, as already explained in Section 1.2, the anisotropy makes it impossible to use Hoff's effective viscous flux as in [13], so we need to look for higher-regularity estimates for the density, in order to get compactness for passing to the limit in the pressure term. For this reason, we require some regularity on the initial data: roughly, the initial density ρ_{in} is sufficiently well localized around the reference profile $\bar{\rho}$, namely $\rho_{\text{in}} - \bar{\rho} \in H^2$, while the initial velocity u_{in} is taken in $H^{3/2+}$. So the solutions that we construct are strong solutions with finite energy; roughly, they are halfway between the Hoff solutions ([13, 29]) with bounded density and the strong solutions of Matsumura–Nishida ([34]) (which require $\rho_{\text{in}} - 1 \in H^3$).

We emphasize that the initial density ρ_{in} can be taken close to an arbitrary reference density profile $\bar\rho = \bar\rho(x_3) \in W^{3,\infty}(\mathbb{R})$. Thus, setting $G = G(x_3) = H'(\bar\rho(x_3))$, with H defined in (1.11) below, we infer that $\bar\rho$ is a static state of system (1.2), namely $\bar\rho$ satisfies the logistic equation

$$\nabla P(\bar\rho) = \bar\rho \nabla G. \tag{1.8}$$

However, the smallness condition rests only on the quantity $\|\rho_{\text{in}} - \bar\rho\|_{L^\infty}$, and not on higher-order derivatives. Finally, we point out that the assumption $\bar\rho = \bar\rho(x_3)$ is made only for modeling purposes (we have in mind the case when G is the gravitational potential), but our method works also in the more general situation $\bar\rho = \bar\rho(x)$, $x \in \mathbb{R}^3$. So our theorem opens the way to achieving the well-posedness of systems with stratification effects, such as (1.3), in the strip $\mathbb{R}^2 \times (0,1)$ with the physical gravitational potential $G = -x_3$.

Asymptotic analysis of system (1.3): Theorem 2. We build an asymptotic expansion for the solutions of the compressible system (1.3) when $\varepsilon \to 0$ and prove quantitative estimates on the errors. We consider well-prepared initial data, i.e. close to the kernel of the penalization operator. The scaling $\mathtt{Ro} = \mathtt{Ma} = \mathtt{Fr} = \varepsilon$ which is considered in (1.3) is the richest scaling, in the sense that the Coriolis force, the pressure and the gravitational force balance each other. In addition, we analyze the effect of boundary layers on the limiting two-dimensional quasi-geostrophic equation. This limit equation, see (3.36) below, represents the large-scale dynamics of the bulk flow. It is a two-dimensional incompressible Navier–Stokes equation written in terms of the stream function Q, where $u = \nabla^\perp Q$ is the limit velocity. Frictional effects dissipate energy in the Ekman boundary layer flow, so a damping term appears in (3.36). Such effects have been pointed out for incompressible ([10, 11, 26, 33]) or compressible ([6]) fluids in high rotation.

As far as we know, [6] is the only work dealing with the asymptotic analysis of compressible fluids in high rotation in the presence of Ekman layers. Our result extends the state of the art in two main directions: (i) we take into account stratification effects due to gravitation, hence we handle nonconstant reference density states, and (ii) we consider general pressure laws $P(\rho) \sim \rho^\gamma$, with $\gamma \geq \frac{3}{2}$ (see (1.10) below). Concerning (i), let us stress that our work seems to be the first one able to tackle the combined effect of the gravitational force and boundary layers. As for (ii), [6] handles the case $\gamma = 2$. Incidentally, the threshold $\frac{3}{2}$ for the number γ is the same as that for the Lions–Feireisl theory of weak solutions.

Our approach is based on relative entropy estimates for system (1.3); see [18, 22, 23, 27]. The progress achieved in this paper is made possible thanks to the introduction of a simple tool, which seems to be new in this context: we rely on *anisotropic Sobolev embeddings*; see Lemma 3.12 below. This enables us to compensate for the lack of coercivity for $\partial_3 u_h$, due to the strong anisotropy in the Lamé operator (1.9). Doing so, we are able to extend the range of values for the parameter γ, and also to improve the quantitative error bounds.

1.4. Outline of the paper

The paper consists of two parts. The first one, treated in Section 2, is devoted to the proof of the well-posedness result for system (1.2). The main result is Theorem 1. The proof relies on maximal regularity estimates for a parabolic equation related to an anisotropic and variable coefficient Lamé operator; see Proposition 2.3. The second part, which is the matter of Section 3, is concerned with the proof of the quantitative estimates for (1.3) in the limit $\varepsilon \to 0$. We first build an expansion based on a formal multi-scale analysis. Second, we derive a relative entropy inequality. Finally, we carry out the quantitative estimates, using in particular the anisotropic Sobolev embeddings of Lemma 3.12.

1.5. Main notation and definitions

Since our interest is in real fluid flows, the whole paper is written in space dimension $d = 3$. Notice though that some results, in particular those of Section 2 such as the maximal regularity statement, can easily be extended to higher-dimensional systems. The domain Ω denotes an open set, usually $\Omega = \mathbb{R}^3$ or $\mathbb{R}^2 \times (0,1)$ in this paper.

When appropriate, we use Einstein's convention on repeated indices for summation.

Given a two-dimensional vector $v = (v_1, v_2)$, we define $v^\perp := (-v_2, v_1)$. Given a vector field $v \in \mathbb{R}^3$, we will often use the notation $\nabla \cdot v$ and $\nabla \times v$ to denote respectively $\operatorname{div} v$ and $\operatorname{curl} v$. For a vector $x \in \mathbb{R}^3$, we often use the notation $x = (x_h, x_3) \in \mathbb{R}^3$ to denote the horizontal component $x_h \in \mathbb{R}^2$ and the vertical component $x_3 \in \mathbb{R}$. According to this decomposition, we define the horizontal differential operators ∇_h, Δ_h and $\nabla_h \cdot$ as usual; we also set $\nabla_h^\perp := (-\partial_2, \partial_1)$. These operators act just on the x_h variables. Notice that the third component of the vector $\nabla \times v$ is $\partial_1 v_2 - \partial_2 v_1$: this quantity will be denoted by $\nabla_h^\perp \cdot v_h$.

We introduce $\Delta_{\mu,\varepsilon}$ as in (1.1). Similarly, we define the modified gradient operator $\nabla_{\mu,\varepsilon} := (\sqrt{\mu}\,\partial_1, \sqrt{\mu}\,\partial_2, \sqrt{\varepsilon}\,\partial_3)$. The anisotropic *Lamé operator* $\mathcal{L}$ is defined by

$$\mathcal{L}u = -\Delta_{\mu,\varepsilon} u - \lambda \nabla\nabla \cdot u. \tag{1.9}$$

Throughout this paper, given a Banach space X and a sequence $(a^\varepsilon)_\varepsilon$ of elements of X, the notation $(a^\varepsilon)_\varepsilon \subseteq X$ is to be understood as the fact that the sequence $(a^\varepsilon)_\varepsilon$ is uniformly bounded in X. We will often denote, for any $p \in [1, \infty]$ and any Banach space X, $L^p_T(X) := L^p((0,T); X(\Omega))$. When $T = +\infty$, we will simply write $L^p(X)$.

For the definition and basic properties of Besov spaces, we refer to [2, Chapter 2] or to [13, Sections 2.2 and 2.3].

Pressure law. We consider barotropic flows, for which the pressure P is supposed to be a smooth function of the density only. We assume (see e.g. [17, 18, 20, 22]) that $P \in \mathcal{C}([0,\infty)) \cap \mathcal{C}^3((0,\infty))$ enjoys

$$P(0) = 0, \quad P'(\rho) > 0 \ \ \forall \rho > 0, \quad \lim_{\rho \to \infty} \frac{P'(\rho)}{\rho^{\gamma-1}} = a > 0, \tag{1.10}$$

for some $\gamma \geq 1$. Given P, we define the *internal energy function* H by the formula

$$H(\rho) := \rho \int_1^\rho \frac{P(z)}{z^2}\, dz \quad \text{for all } \rho \in (0, \infty). \tag{1.11}$$

Notice that the relation $\rho H''(\rho) = P'(\rho)$ holds for all $\rho > 0$.

2. A well-posedness result in the presence of a strongly anisotropic viscous stress tensor and stratification

In this section we show a well-posedness result for the barotropic Navier–Stokes system (1.2). As explained in Section 1.2, there are two main difficulties. The first one is to handle the anisotropy of the viscous stress tensor. It prevents one from using classical compactness techniques to prove the existence of weak solutions having finite energy (see e.g. [21, 32, 38] and the comments in [6, 7]). It is also a major obstacle to the use of the effective flux (1.6) as in [13, 29]. The second one is the nonconstant reference density state $\bar{\rho}$, due to the potential force ∇G.

Our approach is reminiscent of [13,43]. It is based on maximal regularity estimates for the velocity field. This approach enables us to fully exploit the parabolic gain of regularity due to the momentum equation, and to use it in the mass equation in order to transport higher-order Sobolev norms of the density function. Moreover, it allows us to consider nonconstant density reference states, which is crucial in view of studying stratification effects; see system (1.3) and Section 3.

It is not clear whether or not the whole method of [13] works in the presence of an anisotropic Lamé operator, since it makes substantial use of Hoff's effective flux (1.6) and algebraic cancellations appearing in its equation, see (1.7). Thus, compared to [13], we will work with solutions in the energy space, so with less integrability, but with more regularity. We are also able to deal with parabolic operators with variable coefficients when applying the maximal regularity results.

Our main result in this direction is the following statement. Our aim is to give a streamlined method for well-posedness, appropriate for proving the existence and uniqueness of finite-energy solutions in the presence of (i) an anisotropic viscous stress tensor given by (1.1) and (ii) nonconstant reference density states. A small compromise consists, on the one hand, in the fact that we do not strive for optimality in the assumptions of the theorem and, on the other hand, in the fact that we work in the space domain $\Omega = \mathbb{R}^3$. We believe that our strategy is robust enough to be adapted to problems in bounded domains. There are however a number of technical difficulties. One is related to the technique that we use in the whole space case. Indeed, since we do not have maximal regularity estimates for the Lamé operator in domains, we carry out the study of maximal regularity estimates for the divergence-free and potential parts of the solutions. This approach does not seem to be directly applicable to the case of a bounded domain, because it is unclear which would be the right boundary conditions for the two parabolic problems.

We remark also that we fix reference density profiles $\bar\rho = \bar\rho(x_3)$ depending only on the vertical variable, since this corresponds to the physically relevant case where the external force G is the gravity. As a matter of fact, our analysis works exactly the same for more general profiles $\bar\rho$ depending on all the variables, at the price of dealing with more complicated commutator terms.

Theorem 1. *Let $\gamma \geq 1$. Let $\bar\rho = \bar\rho(x_3) \in W^{3,\infty}(\mathbb{R})$. Assume that $\bar\rho$ is uniformly bounded from below, i.e. $\bar\rho \geq \kappa > 0$. We define the potential by $G = H'(\bar\rho)$, where H is defined by* (1.11)*. Then for any $\eta > 0$ which verifies*

$$\eta \leq \min\Big(\frac{1}{8C_0}, \frac{\kappa}{8}\Big), \tag{2.1}$$

where C_0 is the constant given by Proposition 2.3, the property below holds.

Consider system (1.2)*, supplemented with the initial datum $(\rho, u)_{|t=0} = (\rho_{\rm in}, u_{\rm in})$. For any $(\rho_{\rm in}, u_{\rm in})$, with $\rho_{\rm in} - \bar\rho \in H^2$ and $u_{\rm in} \in B^{3/2}_{2,4/3}$ and such that*

$$\|\rho_{\rm in} - \bar\rho\|_{L^\infty} \leq \eta,$$

there exist a time $T^*(\gamma, \mu, \varepsilon, \lambda, \|\bar\rho\|_{W^{3,\infty}}, \kappa, \|u_{\rm in}\|_{B^{3/2}_{2,4/3}}, \|\rho_{\rm in} - \bar\rho\|_{H^2}) > 0$ and a unique solution (ρ, u) to (1.2) *on $[0, T^*] \times \Omega$, such that*

1. *$\rho - \bar\rho \in \mathcal{C}_{T^*}(H^2)$, with $\|\rho - \bar\rho\|_{L^\infty_{T^*}(L^\infty)} \leq 4\eta$;*
2. *$u \in L^\infty_T(L^2) \cap L^2_{T^*}(L^\infty)$, with in addition $\nabla u \in L^4_{T^*}(L^2) \cap L^2_{T^*}(L^\infty)$, $\nabla^2 u \in L^4_{T^*}(L^2)$, $\nabla^3 u \in L^{4/3}_{T^*}(L^2)$ and $\partial_t u \in L^{4/3}_{T^*}(H^1)$;*
3. *(ρ, u) satisfies the classical energy inequality (see estimate* (3.42) *below).*

Remark 2.1. Notice that we have some freedom in the regularity of the initial datum. In this sense, we take $B^{3/2}_{2,4/3}$ regularity for u_0 for simplicity of presentation, but this condition can be somehow weakened.

Notice also that $H^s \hookrightarrow B^{3/2}_{2,4/3}$ for any $s > \frac{3}{2}$.

Remark 2.2. Going along the lines of the proof of Theorem 1, it is possible to see that our estimates, applied to the rescaled system (1.3), are *not* uniform in the small parameter ε because of the presence of the Coriolis term (and possible remainders arising from the pressure term, when looking at higher-order estimates). Thus, the solutions exist on small time intervals $[0, T_\varepsilon)$, with $T_\varepsilon \to 0^+$ when $\varepsilon \to 0^+$.

The rest of this section is devoted to the proof of Theorem 1. The proof is based only on elementary methods, namely energy estimates and maximal regularity. We will limit ourselves to showing a priori estimates for smooth solutions to system (1.2). The proof of existence by an approximation scheme is rather standard; see e.g. [13] and references therein.

2.1. Maximal regularity for an anisotropic Lamé operator

Here we prove the following maximal regularity result for a parabolic equation with an anisotropic Lamé operator and vertical stratification due to the coefficient $\bar{\rho}$. Pay attention to the fact that, in the definition of $W^{2,1}_{p_2,r_2}$ below, the indices for space and time are in reverse order, in order to stick to the classical definition (see e.g. [42]).

Proposition 2.3. *Let $\bar{\rho} \in W^{1,\infty}(\mathbb{R})$. Assume that $\bar{\rho}$ is uniformly bounded from below, i.e. $\bar{\rho} \geq \kappa > 0$. Let $((p_j, r_j))_{j=0,1,2}$ satisfy $1 < p_2, r_2 < +\infty$, $r_2 < r_0 < +\infty$, $r_2 < r_1 < +\infty$, $p_0 \geq p_2$, $p_1 \geq p_2$, together with the relations*

$$\frac{2}{r_2} + \frac{3}{p_2} = 1 + \frac{2}{r_1} + \frac{3}{p_1} \quad \text{and} \quad \frac{2}{r_2} + \frac{3}{p_2} = 2 + \frac{2}{r_0} + \frac{3}{p_0}.$$

Let h_{in} be in $\dot{B}^{s_2}_{p_2,r_2}$ with $s_2 := 2 - \frac{2}{r_2}$, and let f be in $L^{r_2}_{\mathrm{loc}}(\mathbb{R}_+; L^{p_2}(\mathbb{R}^d))$.

Then there exists a constant $C_0 = C_0(\mu, \varepsilon, \lambda, r_0, p_0, r_1, p_1, r_2, p_2, \|\bar{\rho}\|_{W^{1,\infty}}, \kappa) > 0$ such that, for all $T > 0$ and all $h \in L^{r_2}_T(W^{1,p_2})$, with h a solution to the Lamé system

$$\begin{cases} \bar{\rho}(x_3)\partial_t h + \mathcal{L} h = f, \\ h_{|t=0} = h_{\mathrm{in}}, \end{cases} \tag{2.2}$$

where $\mathcal{L}$ is defined by (1.9), one has the properties $h \in L^{r_0}([0,T]; L^{p_0})$ and $\nabla h \in L^{r_1}([0,T]; L^{p_1})$, together with the estimate

$$\begin{aligned} \|h\|_{L^\infty_T(\dot{B}^{s_2}_{p_2,r_2})} + \|h\|_{L^{r_0}_T(L^{p_0})} + \|\nabla h\|_{L^{r_1}_T(L^{p_1})} + \|(\partial_t h, \nabla^2 h)\|_{L^{r_2}_T(L^{p_2})} \\ \leq C_0(\|h_{\mathrm{in}}\|_{\dot{B}^{s_2}_{p_2,r_2}} + \|f\|_{L^{r_2}_T(L^{p_2})} + \|h\|_{L^{r_2}_T(L^{p_2})} + \|\nabla h\|_{L^{r_2}_T(L^{p_2})}). \end{aligned} \tag{2.3}$$

To prove this result, we follow the idea of [13]. We apply the Leray projector $\mathbb{P}$ to system (2.2) and rely on the maximal regularity for a divergence-form parabolic equation. Because of the stratification, additional commutators involving $\mathbb{P}$ and the reference density $\bar{\rho}$ have to be analyzed. We do not strive for making the dependence of C_0 explicit in ε, μ, λ, $\|\bar{\rho}\|_{W^{1,\infty}}$ and κ, because our aim is to obtain a well-posedness theorem for a fixed set of parameters; see Theorem 1.

Remark 2.4. Notice that the two last terms on the right-hand side of estimate (2.3) cannot be got rid of. Their appearance is due to the commutator terms involving the Leray projector. It is not possible to swallow them in the left-hand side of (2.3). In spite of this, estimate (2.3) can be used as such to prove the well-posedness result stated in Theorem 1. Indeed, the terms $\|h\|_{L^{r_2}_T(L^{p_2})}$ and $\|\nabla h\|_{L^{r_2}_T(L^{p_2})}$ are of lower order and can be handled by interpolating with the finite energy. Such an analysis will also be done for the source term f, which contains in particular the nonlinear term $\rho u \cdot \nabla u$; see below in Section 2.4.

Proof of Proposition 2.3. We proceed in several steps.

Step 1: reduction to the heat equation. We first rewrite system (2.2) in divergence form: we have

$$\partial_t h - \nabla_{\mu,\varepsilon}\cdot\Big(\frac{1}{\bar\rho(x_3)}\nabla_{\mu,\varepsilon} h\Big) - \lambda\nabla\Big(\frac{1}{\bar\rho(x_3)}\nabla\cdot h\Big) = F, \tag{2.4}$$

where, for simplicity of notation, we have defined

$$F := \frac{1}{\bar\rho(x_3)} f + \varepsilon\frac{\bar\rho'(x_3)}{\bar\rho(x_3)^2}\partial_3 h + \lambda\frac{\bar\rho'(x_3)}{\bar\rho(x_3)^2}(\nabla\cdot h)e_3. \tag{2.5}$$

As in [13], we apply the Leray projector $\mathbb{P}$ on the equation. The difference compared to [13] is that we now have commutator terms appearing. Recall that

$$\mathbb{P} = \mathrm{Id} - \mathbb{Q}, \quad \mathbb{Q} := -\nabla(-\Delta)^{-1}\nabla\cdot, \tag{2.6}$$

where the formulas have to be interpreted in the sense of Fourier multipliers. Applying $\mathbb{P}$ to (2.4) we get

$$\partial_t \mathbb{P}h - \nabla_{\mu,\varepsilon}\cdot\Big(\frac{1}{\bar\rho(x_3)}\nabla_{\mu,\varepsilon}\mathbb{P}h\Big) = \mathbb{P}F + \mathbf{C}, \tag{2.7}$$

where the commutator term $\mathbf{C}$ is defined by

$$\mathbf{C} := -\nabla_{\mu,\varepsilon}\cdot\Big(\Big[\frac{1}{\bar\rho(x_3)},\mathbb{P}\Big]\nabla_{\mu,\varepsilon} h\Big). \tag{2.8}$$

Here above, the symbol $[A,B] := AB - BA$ denotes the commutator between two operators A and B.

We now compute the equation for $\mathbb{Q}h = h - \mathbb{P}h$. Notice that, using (2.6), we have

$$\nabla\nabla\cdot h = \nabla\nabla\cdot\mathbb{Q}h = \Delta\mathbb{Q}h.$$

Therefore, rewriting (2.4) in the form

$$\partial_t h - \nabla_{\mu,\varepsilon}\cdot\Big(\frac{1}{\bar\rho(x_3)}\nabla_{\mu,\varepsilon} h\Big) - \lambda\frac{1}{\bar\rho(x_3)}\nabla\nabla\cdot h = \frac{1}{\bar\rho(x_3)} f + \varepsilon\frac{\bar\rho'(x_3)}{\bar\rho(x_3)^2}\partial_3 h$$

and taking the difference of this equation with (2.7), we immediately find

$$\begin{aligned}
&\partial_t\mathbb{Q}h - \nabla_{\mu+\lambda,\varepsilon+\lambda}\cdot\Big(\frac{1}{\bar\rho(x_3)}\nabla_{\mu+\lambda,\varepsilon+\lambda}\mathbb{Q}h\Big)\\
&\quad = \mathbb{Q}\Big(\frac{1}{\bar\rho(x_3)} f\Big) - \mathbb{P}\Big(\varepsilon\frac{\bar\rho'(x_3)}{\bar\rho(x_3)^2}\partial_3 h + \lambda\frac{\bar\rho'(x_3)}{\bar\rho(x_3)^2}(\nabla\cdot h)e_3\Big)\\
&\qquad - \mathbf{C} + \lambda\frac{\bar\rho'(x_3)}{\bar\rho(x_3)^2}\partial_3\mathbb{Q}h.
\end{aligned} \tag{2.9}$$

Both equations (2.7) and (2.9) are heat-type equations with a variable coefficient. The next step will show that all the quantities appearing in their right-hand sides are of lower order in h.

Step 2: computation of the commutators. Our goal is now to compute the commutator term $\mathbf{C}$, defined by (2.8). Owing to (2.6) and switching the position of the constant factors, we have

$$\mathbf{C} := -\nabla_{\mu,\varepsilon} \cdot \Big(\Big[\frac{1}{\bar{\rho}(x_3)}, \mathbb{P}\Big]\nabla_{\mu,\varepsilon} h\Big) = \nabla \cdot \Big(\Big[\frac{1}{\bar{\rho}(x_3)}, \mathbb{Q}\Big]\nabla_{\mu^2,\varepsilon^2} h\Big).$$

Recall the definition of $\mathbb{Q}$ in (2.6). We start by applying the divergence operator to the various quantities: after observing that $\nabla \cdot \nabla_{\mu^2,\varepsilon^2} = \Delta_{\mu,\varepsilon}$, we find

$$\Big[\frac{1}{\bar{\rho}}, \mathbb{Q}\Big]\nabla_{\mu^2,\varepsilon^2} h = -\frac{1}{\bar{\rho}}\nabla(-\Delta)^{-1}\Delta_{\mu,\varepsilon} h + \nabla(-\Delta)^{-1}\nabla \cdot \Big(\frac{1}{\bar{\rho}}\nabla_{\mu^2,\varepsilon^2} h\Big).$$

Therefore, we find

$$\begin{aligned}
\mathbf{C} &= \nabla \cdot \Big(\Big[\frac{1}{\bar{\rho}(x_3)}, \mathbb{Q}\Big]\nabla_{\mu^2,\varepsilon^2} h\Big) \\
&= -\nabla \cdot \Big(\frac{1}{\bar{\rho}(x_3)}\nabla(-\Delta)^{-1}\Delta_{\mu,\varepsilon} h\Big) + \nabla \cdot \Big(\nabla(-\Delta)^{-1}\nabla \cdot \Big(\frac{1}{\bar{\rho}(x_3)}\nabla_{\mu^2,\varepsilon^2} h\Big)\Big) \\
&= \frac{\bar{\rho}'(x_3)}{\bar{\rho}(x_3)^2}\partial_3(-\Delta)^{-1}\Delta_{\mu,\varepsilon} h + \frac{1}{\bar{\rho}(x_3)}\Delta_{\mu,\varepsilon} h - \nabla \cdot \Big(\frac{1}{\bar{\rho}(x_3)}\nabla_{\mu^2,\varepsilon^2} h\Big),
\end{aligned}$$

which in the end leads us to

$$\mathbf{C} = \frac{\bar{\rho}'(x_3)}{\bar{\rho}(x_3)^2}\big(\partial_3(-\Delta)^{-1}\Delta_{\mu,\varepsilon} h + \varepsilon\partial_3 h\big). \tag{2.10}$$

Since the operator $(-\Delta)^{-1}\Delta_{\mu,\varepsilon}$ is a singular integral operator, whose symbol is homogeneous of degree 0, the previous computations show that $\mathbf{C}$ is indeed a lower-order term, as claimed.

Step 3: estimates via maximal regularity. We apply the maximal regularity estimates of [28] for the divergence-form parabolic operator

$$\partial_t(\cdot) - \nabla_{\mu,\varepsilon} \cdot (\bar{\rho}^{-1}\nabla_{\mu,\varepsilon}(\cdot)) \tag{2.11}$$

to equation (2.7). We obtain

$$\|(\partial_t \mathbb{P}h, \nabla^2\mathbb{P}h)\|_{L^{r_2}_T(L^{p_2})} \leq C(\|\mathbb{P}h_{\rm in}\|_{\dot{B}^{s_2}_{p_2,r_2}} + \|\mathbb{P}F + \mathbf{C}\|_{L^{r_2}_T(L^{p_2})}).$$

Using the continuity of $\mathbb{P}$ and $(-\Delta)^{-1}\Delta_{\mu,\varepsilon}$ on L^{p_2} (since $1 < p_2 < +\infty$) and keeping in mind definition (2.5) of F and (2.10), it is easy to bound

$$\|\mathbb{P}F + \mathbf{C}\|_{L^{r_2}_T(L^{p_2})} \leq C(\|f\|_{L^{r_2}_T(L^{p_2})} + \|\nabla h\|_{L^{r_2}_T(L^{p_2})}).$$

So, in the end we find

$$\|(\partial_t \mathbb{P}h, \nabla^2\mathbb{P}h)\|_{L^{r_2}_T(L^{p_2})} \leq C_0(\|\mathbb{P}h_{\rm in}\|_{\dot{B}^{s_2}_{p_2,r_2}} + \|f\|_{L^{r_2}_T(L^{p_2})} + \|\nabla h\|_{L^{r_2}_T(L^{p_2})}). \tag{2.12}$$

Next, thanks to the Gaussian bounds on the fundamental solution of the parabolic equation (2.11) (see [1, 36]), arguing as in [13, Lemma 2.4] we infer that

$$\|\mathbb{P}h\|_{L^{r_0}_T(L^{p_0})} \leq C_0(\|\mathbb{P}h_{\mathrm{in}}\|_{\dot{B}^{s_2}_{p_2,r_2}} + \|f\|_{L^{r_2}_T(L^{p_2})} + \|\nabla h\|_{L^{r_2}_T(L^{p_2})}). \tag{2.13}$$

Thus, in order to complete the proof, it remains for us to bound the $L^{r_1}_T(L^{p_1})$ norm of $\nabla\mathbb{P}h$. This is the goal of the next step.

Step 4: a functional inequality. We claim that, for any $w \in W^{2,1}_{p_2,r_2}(\mathbb{R}^3 \times (0,T))$, one has the estimate

$$\|\nabla w\|_{L^{r_1}_T(L^{p_1})} \leq C(\|\nabla^2 w\|_{L^{r_2}_T L^{p_2}(\mathbb{R}^3\times(0,T))} + \|\partial_t w\|_{L^{r_2}_T L^{p_2}(\mathbb{R}^3\times(0,T))}). \tag{2.14}$$

In fact, estimate (2.14) is a functional inequality, which does not use the parabolic equation. We are going to deduce it from the corresponding inequality in a bounded domain; see in particular [44, Proposition 2.1].

In order to prove (2.14), we take $\varphi \in C^\infty_c(B(0,1))$ such that $\varphi = 1$ on $B(0,\frac{1}{2})$. For $k \in \mathbb{N}$, we consider $w_k := \varphi(\frac{\cdot}{k})w$. Notice that w_k is supported in $B(0,k)$. Then we consider the rescaled function w_k^λ for $\lambda \geq k$, according to the parabolic scaling

$$w_k^\lambda = w_k(\lambda\cdot,\lambda^2\cdot) = \varphi\big(\tfrac{\lambda}{k}\cdot\big)w(\lambda\cdot,\lambda^2\cdot).$$

Notice that w_k^λ is supported in $B(0,1)$. By the mixed norm parabolic Sobolev embedding in the bounded domain $B(0,1)$, we have

$$\|\nabla w_k^\lambda\|_{L^{r_1}_T L^{p_1}(B(0,1)\times(0,T/\lambda^2))} \leq C\|w_k^\lambda\|_{W^{2,1}_{p_2,r_2}(B(0,1)\times(0,T/\lambda^2))},$$

where C is a constant independent of T. Then, by rescaling, we obtain

$$\begin{aligned}
&\frac{\lambda}{\lambda^{\frac{3}{p_1}+\frac{2}{r_1}}}\|\nabla w_k\|_{L^{r_1}_T L^{p_1}(B(0,k)\times(0,T))} \\
&\quad\leq C\Big(\frac{1}{\lambda^{\frac{3}{p_2}+\frac{2}{r_2}}}\|w_k\|_{L^{r_2}_T L^{p_2}(B(0,k)\times(0,T))} + \frac{\lambda}{\lambda^{\frac{3}{p_2}+\frac{2}{r_2}}}\|\nabla w_k\|_{L^{r_2}_T L^{p_2}(B(0,k)\times(0,T))} \\
&\qquad + \frac{\lambda^2}{\lambda^{\frac{3}{p_2}+\frac{2}{r_2}}}\|\nabla^2 w_k\|_{L^{r_2}_T L^{p_2}(B(0,k)\times(0,T))} + \frac{\lambda^2}{\lambda^{\frac{3}{p_2}+\frac{2}{r_2}}}\|\partial_t w_k\|_{L^{r_2}_T L^{p_2}(B(0,k)\times(0,T))}\Big),
\end{aligned}$$

and letting $\lambda \to \infty$ we deduce

$$\begin{aligned}
&\|\nabla w_k\|_{L^{r_1}_T L^{p_1}(\mathbb{R}^3\times(0,T))} \\
&\qquad\leq C(\|\nabla^2 w_k\|_{L^{r_2}_T L^{p_2}(\mathbb{R}^3\times(0,T))} + \|\partial_t w_k\|_{L^{r_2}_T L^{p_2}(\mathbb{R}^3\times(0,T))}).
\end{aligned} \tag{2.15}$$

From estimate (2.15) and the fact that $w \in W^{2,1}_{p_2,r_2}$, we get that ∇w_k is a Cauchy sequence in $L^{r_1}_T(L^{p_1})$. Hence we can pass to the limit in (2.15): thus, we obtain (2.14), as claimed.

Step 5: end of the proof. Owing to the fact that $1 < p_2 < +\infty$ and to (2.12), we have that $\mathbb{P}h$ belongs to $W^{2,1}_{p_2,r_2}(\mathbb{R}^3 \times (0,T))$. Therefore, we can apply inequality (2.14) to $w = \mathbb{P}h$. We find

$$\|\nabla \mathbb{P}h\|_{L^{r_1}_T(L^{p_1})} \leq C_0(\|\mathbb{P}h_{\rm in}\|_{\dot{B}^{s_2}_{p_2,r_2}} + \|f\|_{L^{r_2}_T(L^{p_2})} + \|\nabla h\|_{L^{r_2}_T(L^{p_2})}), \tag{2.16}$$

for a possibly new constant $C_0 > 0$. In addition, since $\mathbb{Q}h$ solves equation (2.9), which is analogous to (2.7), estimates similar to (2.12), (2.13) and (2.16) hold true for $\mathbb{Q}h$ and its derivatives. Then writing $h = \mathbb{P}h + \mathbb{Q}h$ completes the proof of the proposition. ■

2.2. Basic energy estimates

Let us take a smooth solution (ρ, u) to system (1.2), such that $\rho - \bar{\rho}$ and u decay sufficiently fast at space infinity. We want to find a priori estimates in suitable norms. For this, we are going to work with the variables

$$r(t) := \rho(t) - \bar{\rho} \quad \text{and} \quad u.$$

In the same way, we set $r_{\rm in} = \rho_{\rm in} - \bar{\rho}$. Recall that

$$\|r_{\rm in}\|_{L^\infty} \leq \eta. \tag{2.17}$$

We start by performing classical energy estimates, which provides us with a bound for the low frequencies of the velocity field. Namely, by multiplying the momentum equation in (1.2) by u and integrating by parts we get, in a standard way, the control

$$\|\sqrt{\rho}\, u\|_{L^\infty(L^2)} + \|\nabla u\|_{L^2(L^2)} \leq C_{\rm energy}. \tag{2.18}$$

See Section 3.2 for similar bounds. Of course, the constant $C_{\rm energy}$ depends also on the (fixed) values of the coefficients $(\mu, \varepsilon, \lambda)$. From this control and

$$\int_\Omega \bar{\rho}|u|^2 = \int_\Omega \rho|u|^2 - \int_\Omega r|u|^2,$$

we infer that, for any $T > 0$,

$$\|u\|_{L^\infty_T(L^2)} \leq \kappa^{-1}(C_{\rm energy} + \|r\|_{L^\infty_T(L^\infty)}\|u\|_{L^\infty_T(L^2)}), \tag{2.19}$$

where we recall that κ is a lower bound for $\bar{\rho}$. Hence, if $T > 0$ is such that

$$\|r\|_{L^\infty_T(L^\infty)} \leq 4\eta < \frac{\kappa}{2}, \tag{2.20}$$

where $\eta > 0$ is the size of the initial datum in the L^∞ norm (recall (2.17)) we deduce

$$\|u\|_{L^\infty_T(L^2)} \leq 2C_{\rm energy}\kappa^{-1}. \tag{2.21}$$

The next goal is to exhibit a control on the density variation function r. We will work in higher Sobolev norms, namely in H^2. However, we are going to bound its L^∞ norm independently (i.e. without using Sobolev embeddings), in order to get, in view of (2.20), a smallness condition only on $\|r_{\rm in}\|_{L^\infty}$, and not on the higher-order norm of $r_{\rm in}$.

2.3. Estimates for the density function

In this subsection we find transport estimates for the density variation function r. First of all, from the mass equation in (1.2) we find that r fulfills

$$\partial_t r + u \cdot \nabla r + r \nabla \cdot u = -u_3 \bar{\rho}' - \bar{\rho} \nabla \cdot u. \tag{2.22}$$

A basic L^p estimate for this equation gives, for any $t > 0$,

$$\begin{aligned}\|r(t)\|_{L^p} \leq \|r_{\rm in}\|_{L^p} + \left(1 - \frac{1}{p}\right) \int_0^t \|r(\tau)\|_{L^p} \|\nabla \cdot u(\tau)\|_{L^\infty}\, d\tau \\ + \int_0^t \|\bar{\rho}'\|_{L^\infty} (\|u_3(\tau)\|_{L^p} + \|\nabla \cdot u(\tau)\|_{L^p})\, d\tau.\end{aligned}$$

On the one hand, in the case $p = 2$, we get

$$\begin{aligned}\|r\|_{L^\infty_T(L^2)} \leq \|r_{\rm in}\|_{L^2} + \frac{1}{2} \int_0^T \|r\|_{L^2} \|\nabla \cdot u\|_{L^\infty}\, d\tau \\ + \int_0^T \|\bar{\rho}'\|_{L^\infty} (\|u_3(\tau)\|_{L^2} + \|\nabla \cdot u(\tau)\|_{L^2})\, d\tau\end{aligned} \tag{2.23}$$

for any time $T > 0$. In turn, Grönwall's lemma gives the bound

$$\begin{aligned}\|r\|_{L^\infty_T(L^2)} \leq e^{\int_0^T \|\nabla \cdot u(\tau)\|_{L^\infty}\, d\tau} \\ \times \left(\|r_{\rm in}\|_{L^2} + \int_0^T \|\bar{\rho}'\|_{L^\infty} (\|u_3(\tau)\|_{L^2} + \|\nabla \cdot u(\tau)\|_{L^2})\, d\tau \right).\end{aligned} \tag{2.24}$$

On the other hand, by letting $p \to +\infty$, from Grönwall's lemma again we deduce

$$\begin{aligned}\|r\|_{L^\infty_T(L^\infty)} \leq e^{\int_0^T \|\nabla \cdot u(\tau)\|_{L^\infty}\, d\tau} \\ \times \left(\|r_{\rm in}\|_{L^\infty} + \int_0^T \|\bar{\rho}'\|_{L^\infty} (\|u_3(\tau)\|_{L^\infty} + \|\nabla \cdot u(\tau)\|_{L^\infty})\, d\tau \right).\end{aligned} \tag{2.25}$$

We now define the time $T > 0$ as

$$\begin{aligned}T := \sup\Big\{ t > 0 \mid \textstyle\int_0^t \|\bar{\rho}'\|_{L^\infty} \|u_3(\tau)\|_{L^\infty}\, d\tau \\ + \textstyle\int_0^t (1 + \|\bar{\rho}\|_{L^\infty}) \|\nabla \cdot u(\tau)\|_{L^\infty}\, d\tau \leq \min\{\eta, \log 2\} \Big\},\end{aligned} \tag{2.26}$$

where $\eta > 0$ is as in (2.17). From the previous bound we gather then

$$\|r\|_{L^\infty_T(L^\infty)} \leq 2\|r_{\rm in}\|_{L^\infty} + 2\eta \leq 4\eta. \tag{2.27}$$

We now differentiate equation (2.22) with respect to x_j, for any $j \in \{1, 2, 3\}$. We get that $\partial_j r$ verifies

$$\begin{aligned}\partial_t \partial_j r + u \cdot \nabla \partial_j r + \partial_j r \nabla \cdot u = -\partial_j u \cdot \nabla r - r \nabla \cdot \partial_j u \\ - \partial_j u_3 \bar{\rho}' - u_3 \partial_j \bar{\rho}' - \partial_j \bar{\rho} \nabla \cdot u - \bar{\rho} \nabla \cdot \partial_j u.\end{aligned} \tag{2.28}$$

The same L^2 energy estimate for the continuity equation as above yields

$$\begin{aligned}\|\partial_j r(t)\|_{L^2} \leq \|\partial_j r_{\text{in}}\|_{L^2} + \frac{1}{2}\int_0^t \|\partial_j r\|_{L^2}\|\nabla\cdot u\|_{L^\infty}\,d\tau \\ + \int_0^t (\|r\|_{L^\infty}\|\nabla\cdot\partial_j u\|_{L^2} + \|\nabla r\|_{L^2}\|\partial_j u\|_{L^\infty} + \|\partial_j\bar\rho'\|_{L^\infty}\|u_3\|_{L^2} \\ + \|\bar\rho'\|_{L^\infty}\|\partial_j u_3\|_{L^2} + \|\partial_j\bar\rho\|_{L^\infty}\|\nabla\cdot u\|_{L^2} + \|\bar\rho\|_{L^\infty}\|\nabla\cdot\partial_j u\|_{L^2})\,d\tau,\end{aligned}$$

for any $t \geq 0$. Hence, for all $t \geq 0$ one has

$$\begin{aligned}\|\nabla r(t)\|_{L^2} \leq \|\nabla r_{\text{in}}\|_{L^2} \\ + C\int_0^t (\|r\|_{L^\infty}\|\nabla^2 u\|_{L^2} + \|\nabla r\|_{L^2}\|\nabla u\|_{L^\infty} + \|\bar\rho\|_{W^{2,\infty}}\|u\|_{H^2})\,d\tau.\end{aligned} \tag{2.29}$$

We do not apply Grönwall's lemma directly on this inequality. It is better to first bound the second-order derivatives of r. For this, let us differentiate equation (2.28) with respect to x_k, for any $k \in \{1,2,3\}$. We deduce the following equation for the quantity $r_{jk} := \partial^2_{kj} r$:

$$\begin{aligned}\partial_t r_{jk} + u\cdot\nabla r_{jk} + r_{jk}\nabla\cdot u = -(\partial^2_{jk}u\cdot\nabla r + \partial_j u\cdot\nabla\partial_k r + \partial_k r\cdot\nabla\partial_j u - r\nabla\cdot\partial^2_{kj}u) \\ - \partial_k(\partial_j u_3\bar\rho' + u_3\partial_j\bar\rho' + \partial_j\bar\rho\nabla\cdot u + \bar\rho\nabla\cdot\partial_j u).\end{aligned}$$

Performing another energy estimate, we infer, for any $t \geq 0$, the inequality

$$\begin{aligned}\|r_{jk}(t)\|_{L^2} \leq \|r_{\text{in},jk}\|_{L^2} + \frac{1}{2}\int_0^t \|r_{jk}\|_{L^2}\|\nabla\cdot u\|_{L^\infty}\,d\tau \\ + \int_0^t (\|\nabla r\|_{L^4}\|\nabla^2 u\|_{L^4} + \|\nabla u\|_{L^\infty}\|\nabla^2 r\|_{L^2} \\ + \|r\|_{L^\infty}\|\nabla^3 u\|_{L^2} + \|\bar\rho\|_{W^{3,\infty}}\|u\|_{H^3})\,d\tau,\end{aligned}$$

where we have defined $r_{\text{in},jk} := \partial^2_{jk} r_{\text{in}}$. After using the interpolation inequality

$$\|\phi\|_{L^4} \leq \|\phi\|_{L^2}^{1/4}\|\phi\|_{L^6}^{3/4}, \tag{2.30}$$

together with the Sobolev embedding $\dot H^1(\mathbb{R}^3) \hookrightarrow L^6(\mathbb{R}^3)$, we obtain

$$\begin{aligned}\|\nabla r\|_{L^4}\|\nabla^2 u\|_{L^4} &\leq \|\nabla r\|_{L^2}^{1/4}\|\nabla^2 r\|_{L^2}^{3/4}\|\nabla^2 u\|_{L^2}^{1/4}\|\nabla^3 u\|_{L^2}^{3/4} \\ &\leq (\|\nabla r\|_{L^2} + \|\nabla^2 r\|_{L^2})(\|\nabla^2 u\|_{L^2} + \|\nabla^3 u\|_{L^2}).\end{aligned}$$

In view of this inequality, the previous bound on r_{jk} yields, on the time interval $[0, T]$,

$$\begin{aligned}\|\nabla^2 r\|_{L^\infty_T(L^2)} \leq \|\nabla^2 r_{\text{in}}\|_{L^2} \\ + C\int_0^t \big(\|r\|_{L^\infty}\|\nabla^3 u\|_{L^2} + \|\bar\rho\|_{W^{3,\infty}}\|u\|_{H^3} \\ + (\|\nabla r\|_{L^2} + \|\nabla^2 r\|_{L^2})(\|\nabla u\|_{L^\infty} + \|\nabla^2 u\|_{L^2} + \|\nabla^3 u\|_{L^2})\big)\,d\tau.\end{aligned} \tag{2.31}$$

Let us now introduce the notation

$$\mathcal{R}(t) := \|r\|_{L_t^\infty(H^2)} + \|r\|_{L_t^\infty(L^\infty)}.$$

Summing (2.24), (2.27), (2.29) and (2.31) we deduce the following bound:

$$\mathcal{R}(T) \le C\left(\|r_{\rm in}\|_{H^2} + \int_0^T (\|\bar\rho\|_{W^{3,\infty}} + \mathcal{R}(\tau))(\|\nabla u\|_{L^\infty} + \|u\|_{H^3})\, d\tau\right). \tag{2.32}$$

To end this part, we observe that, for almost all $t \in [0,T]$ and $x \in \mathbb{R}^3$, $|r(x,t)| \le 4\eta$. So, by assumptions (1.10) on the pressure, there exists a constant $C > 0$, depending on the function P and $\|\bar\rho\|_{L^\infty}$, such that, for any parameter $s \in [0,1]$, on $[0,T]\times\mathbb{R}^3$ one has

$$|P(\bar\rho + rs)| + |P'(\bar\rho + rs)| + |P''(\bar\rho + rs)| + |P'''(\bar\rho + rs)| \le C. \tag{2.33}$$

2.4. Maximal regularity estimates for the velocity field

It remains to find a bound on u. For this, we mimic an approach used in [13], based on the maximal regularity estimates of Section 2.1. To begin with, we recast the equation for u as

$$\bar\rho\partial_t u + \mathcal{L}u = -r\partial_t u - f, \tag{2.34}$$

where the anisotropic Lamé operator $\mathcal{L}$ is defined by (1.9), and where we have set

$$f := (\bar\rho + r)u\cdot\nabla u + \nabla P(\rho) - \rho\nabla G. \tag{2.35}$$

In the computations below, when convenient, we will resort to the notation $\bar\rho + r = \rho$, and use the bounds (2.18) and (2.27) for $\sqrt{\rho}\,u$ and $\sqrt{\rho}$ respectively.

In view of (2.32) above, we are interested in H^3 bounds for u: for this, we will apply Proposition 2.3 to both u and ∇u. To this end, we fix the following values of the parameters:

$$(p_2, r_2) = (2, \tfrac{4}{3}), \quad (p_0, r_0) = (+\infty, 2), \quad (p_1, r_1) = (2, 4). \tag{2.36}$$

With these choices, all the hypotheses in Proposition 2.3 are satisfied. Indeed, thanks to the energy estimates (2.18) and (2.21) and maximal regularity (2.12), one can easily verify that, a priori, both u and ∇u belong to the space $W^{2,1}_{2,4/3}$. We choose $p_0 = +\infty$ in order to have, thanks to (3.53) above, a control on u in any L^p, $p \in [2,+\infty]$. Notice also that we have some freedom for the values of r_2 and p_1, which then determine r_0 and r_1. Here we take the simple choice $r_2 = \frac{4}{3}$ and $p_1 = 2$. This implies $s_2 = \frac{1}{2}$.

For $t \ge 0$, let us introduce the quantity

$$\begin{aligned}\mathcal{U}(t) := {}& \|u\|_{L_t^\infty(L^2)\cap L_t^2(L^\infty)} + \|\nabla u\|_{L_t^4(L^2)\cap L_t^2(L^\infty)} \\ &+ \|\nabla^2 u\|_{L_t^4(L^2)} + \|\nabla^3 u\|_{L_t^{4/3}(L^2)} + \|\partial_t u\|_{L_t^{4/3}(H^1)}.\end{aligned}$$

On $[0,T]$, where $T > 0$ is the time defined in (2.26), we have

$$\begin{aligned}\mathcal{U}(T) \le C_0(&\|u_{\rm in}\|_{\dot B^{1/2}_{2,4/3}} + \|\nabla u_{\rm in}\|_{\dot B^{1/2}_{2,4/3}} \\ &+ \|r\partial_t u + f\|_{L_T^{4/3}(H^1)} + \|u\|_{L_T^{4/3}(L^2)} + \|\nabla u\|_{L_T^{4/3}(L^2)} + \|\nabla^2 u\|_{L_T^{4/3}(L^2)}),\end{aligned} \tag{2.37}$$

where f is defined in (2.35). Notice that

$$\|u_{\rm in}\|_{\dot B^{1/2}_{2,4/3}} + \|\nabla u_{\rm in}\|_{\dot B^{1/2}_{2,4/3}} \le \|u_{\rm in}\|_{\dot B^{1/2}_{2,4/3}\cap \dot B^{3/2}_{2,4/3}} \le \|u_{\rm in}\|_{B^{3/2}_{2,4/3}},$$

where the last inequality holds in view of the fact that $B^s_{p,r} = \dot B^s_{p,r} \cap L^p$ for any $s > 0$; see also [2, Chapter 2].

Next we bound the term containing the time derivative. Notice that for any $j \in \{1,2,3\}$ we have

$$\partial_j(r\partial_t u) = r\partial_t\partial_j u + \partial_j r\partial_t u.$$

Therefore we have

$$\|r\partial_t u\|_{L^{4/3}_T(L^2)} + \|r\partial_t \nabla u\|_{L^{4/3}_T(L^2)} \le 4\eta\|\partial_t u\|_{L^{4/3}_T(H^1)},$$

so this term can be absorbed into the left-hand side of (2.37), if $\eta > 0$ is fixed so that condition (2.1) is fulfilled. As for the remaining term $\partial_j r\partial_t u$, we notice first of all that from (2.27), the lower bound for $\bar\rho$, the momentum equation in (1.2) and (2.18), we can bound

$$\|\partial_t u\|_{L^2_T(L^2)} \le C\|\rho\partial_t u\|_{L^2_T(L^2)} \le C\|\mathcal{L}u + \rho u\cdot\nabla u + \nabla P - \rho\nabla G\|_{L^2_T(L^2)}.$$

At this point, we obviously have

$$\|\mathcal{L}u\|_{L^2_T(L^2)} \le T^{(1/2)-(1/4)}\|\mathcal{L}u\|_{L^4_T(L^2)} \le CT^{1/4}\mathcal{U}(T).$$

In addition, one has

$$\|\rho u\cdot\nabla u\|_{L^2_T(L^2)} \le CT^{(1/2)-(1/2)}\|\sqrt{\rho}u\|_{L^\infty_T(L^2)}\|\nabla u\|_{L^2_T(L^\infty)} \le C\,\mathcal{U}(T). \tag{2.38}$$

Let us now turn to the pressure and potential force terms. In view of (1.8) we have

$$\nabla P(\rho) - \rho\nabla G = \nabla(P(\rho) - P(\bar\rho)) - (\rho - \bar\rho)\nabla G = \nabla(P'(\bar\rho + sr)r) - r\nabla G, \tag{2.39}$$

for some $s \in\,]0,1[$. On the one hand, since $G = H'(\bar\rho)$, G is bounded; so, for a constant $C > 0$ depending on $\|\bar\rho\|_{W^{1,\infty}}$ and on the function P, we have

$$\|r\nabla G\|_{L^2_T(L^2)} \le CT^{1/2}\|r\|_{L^\infty_T(L^2)}.$$

On the other hand, by writing

$$\nabla(P'(\bar\rho + sr)r) = P'(\bar\rho + sr)\nabla r + P''(\bar\rho + sr)r\nabla\bar\rho + sP''(\bar\rho + sr)r\nabla r \tag{2.40}$$

and making use of (2.33) and (2.27), direct computations show that

$$\|\nabla(P'(\bar\rho + sr)r)\|_{L^2_T(L^2)} \le CT^{1/2}\|r\|_{L^\infty_T(H^1)}.$$

Here, the constant $C > 0$ depends on the constant appearing in (2.33), on η and $\|\bar\rho\|_{W^{1,\infty}}$. Assuming without loss of generality that $T \le 1$, we have thus proved that

$$\|\partial_t u\|_{L^2_T(L^2)} \le C(\mathcal{R}(T) + \mathcal{U}(T)). \tag{2.41}$$

With this control at hand, let us return to the bound of $\partial_j r \partial_t u$ in $L^{4/3}_T(L^2)$, for $j \in \{1,2,3\}$. By resorting again to the interpolation inequality (2.30), in view of the Sobolev embedding $\dot{H}^1 \hookrightarrow L^6$ and of Young's inequality we get

$$\begin{aligned}\|\partial_j r \partial_t u\|_{L^{4/3}_T(L^2)} &\leq \|\nabla r\|^{1/4}_{L^\infty_T(L^2)} \|\nabla^2 r\|^{3/4}_{L^\infty_T(L^2)} \|\partial_t u\|^{1/4}_{L^{4/3}_T(L^2)} \|\partial_t \nabla u\|^{3/4}_{L^{4/3}_T(L^2)} \\ &\leq C\mathcal{R}(T)(T^{(3/4)-(1/2)}\|\partial_t u\|_{L^2_T(L^2)})^{1/4}\|\partial_t \nabla u\|^{3/4}_{L^{4/3}_T(L^2)} \\ &\leq C T^{1/16}\mathcal{R}(T)(\mathcal{R}(T)+\mathcal{U}(T))^{1/4}(\mathcal{U}(T))^{3/4} \\ &\leq C T^{1/16}(\mathcal{R}(T)+\mathcal{U}(T))^2.\end{aligned}$$

Inserting all the previous bounds in (2.37), under condition (2.1) we deduce that

$$\begin{aligned}\mathcal{U}(T) &\leq C\big(\|u_{\rm in}\|_{B^{3/2}_{2,4/3}} + T^{1/16}(\mathcal{R}(T)+\mathcal{U}(T))^2 + \|f\|_{L^{4/3}_T(H^1)} + \|u\|_{L^{4/3}_T(H^2)}\big) \\ &\leq C\big(\|u_{\rm in}\|_{B^{3/2}_{2,4/3}} + T^{1/16}(1+\mathcal{R}(T)+\mathcal{U}(T))^2 + \|f\|_{L^{4/3}_T(H^1)}\big). \qquad (2.42)\end{aligned}$$

Our next goal is to bound f in the $L^{4/3}_T(H^1)$ norm. First of all, let us focus on the $L^{4/3}_T(L^2)$ norm. Repeating the computations which led to (2.41), we easily find that

$$\|f\|_{L^{4/3}_T(L^2)} \leq C(T^{3/4}\mathcal{R}(T) + T^{1/4}\mathcal{U}(T)) \leq C T^{1/4}(\mathcal{R}(T)+\mathcal{U}(T)),$$

where again, without loss of generality, we have assumed that $T \leq 1$. Next, for $j \in \{1,2,3\}$, from the definition (2.35) of f, we get

$$\partial_j f = \partial_j \rho u\cdot\nabla u + \rho\partial_j u\cdot\nabla u + \rho u\cdot\nabla\partial_j u + \partial_j(\nabla P(\rho) - \rho\nabla G). \qquad (2.43)$$

We first bound the convective terms in (2.43). By (2.27), interpolation between Lebesgue spaces and (2.30), we have

$$\begin{aligned}\|\rho u\cdot\nabla\partial_j u\|_{L^{4/3}_T(L^2)} &\leq C T^{(3/4)-(1/2)}\|u\|_{L^4_T(L^4)}\|\nabla^2 u\|_{L^4_T(L^4)} \\ &\leq C T^{1/4}\|u\|^{1/2}_{L^\infty_T(L^2)}\|u\|^{1/2}_{L^2_T(L^\infty)}\|\nabla^2 u\|^{1/4}_{L^4_T(L^2)}\|\nabla^3 u\|^{3/4}_{L^4_T(L^2)} \\ &\leq C T^{1/4}(\mathcal{U}(T))^2.\end{aligned}$$

Arguing in a similar way, we obtain

$$\begin{aligned}\|\rho\partial_j u\cdot\nabla u\|_{L^{4/3}_T(L^2)} &\leq C T^{(3/4)-(1/2)}\|\nabla u\|^2_{L^4_T(L^4)} \\ &\leq C T^{1/4}(\|\nabla u\|^{1/4}_{L^4_T(L^2)}\|\nabla^2 u\|^{3/4}_{L^4_T(L^2)})^2 \leq C T^{1/4}(\mathcal{U}(T))^2.\end{aligned}$$

Finally, writing $\rho = \bar\rho + r$, we split the term $\partial_j\rho u\cdot\nabla u = \partial_j\bar\rho u\cdot\nabla u + \partial_j r u\cdot\nabla u$. We estimate on the one hand, similarly to (2.38),

$$\|\partial_j\bar\rho u\cdot\nabla u\|_{L^{4/3}_T(L^2)} \leq C T^{\frac14}\mathcal{U}(T),$$

and on the other hand,

$$\begin{aligned}\|\partial_j r u\cdot\nabla u\|_{L_T^{4/3}(L^2)} &\le C T^{(3/4)-(1/4)-(5/12)}\|\nabla r\|_{L_T^\infty(L^4)}\|\nabla u\|_{L_T^4(L^6)}\|u\|_{L_T^{12/5}(L^{12})}\\ &\le C T^{1/12}\|\nabla r\|_{L_T^\infty(L^2)}^{1/4}\|\nabla^2 r\|_{L_T^\infty(L^2)}^{3/4}\\ &\quad\times\|\nabla^2 u\|_{L_T^4(L^2)}\|u\|_{L_T^\infty(L^2)}^{1/6}\|u\|_{L_T^2(L^\infty)}^{5/6}\\ &\le C T^{1/12}\mathcal{R}(T)(\mathcal{U}(T))^2.\end{aligned}$$

We now turn to the last term in (2.43). Thanks to (2.39) and (2.40) we have

$$\begin{aligned}\partial_j(\nabla P(\rho)-\rho\nabla G) = {}& P''(\partial_j\bar\rho+s\partial_j r)\nabla r + P'\partial_j\nabla r + P'''(\partial_j\bar\rho+s\partial_j r)r\nabla\bar\rho\\ &+P''\partial_j r\nabla\bar\rho + P'' r\partial_j\nabla\bar\rho + sP'''(\partial_j\bar\rho+s\partial_j r)r\nabla r\\ &+ sP''\partial_j r\nabla r + sP'' r\partial_j\nabla r - \partial_j r\nabla G - r\partial_j\nabla G,\end{aligned}$$

where all the functions P', P'' and P''' are computed at the point $\bar\rho + sr$. Making repeated use of (2.33), (2.27) and (2.30),

$$\|\partial_j(\nabla P(\rho)-\rho\nabla G)\|_{L_T^{4/3}(L^2)} \le C T^{3/4}\big(\mathcal{R}(T)+(\mathcal{R}(T))^2\big),$$

where $C = C(P, \|\bar\rho\|_{W^{3,\infty}}) > 0$.

Putting all those bounds together, assuming again $T \le 1$, we deduce that

$$\|\nabla f\|_{L_T^{4/3}(L^2)} \le C T^{1/12}(1+\mathcal{R}(T)+\mathcal{U}(T))^3,$$

which in turn implies that

$$\|f\|_{L_T^{4/3}(H^1)} \le C T^{1/12}(1+\mathcal{R}(T)+\mathcal{U}(T))^3. \tag{2.44}$$

In the end, inserting (2.44) into (2.42), we find

$$\mathcal{U}(T) \le C\big(\|u_{\rm in}\|_{B^{3/2}_{2,4/3}} + T^{1/16}(1+\mathcal{R}(T)+\mathcal{U}(T))^3\big). \tag{2.45}$$

On the other hand, the integral term in (2.32) can be bounded by

$$(\|\bar\rho\|_{W^{3,\infty}}+\mathcal{R}(T))(T^{1/2}\|\nabla u\|_{L_T^2(L^\infty)} + T^{3/4}\|u\|_{L_T^4(H^2)} + T^{1/4}\|\nabla^3 u\|_{L_T^{4/3}(L^2)}),$$

which implies that

$$\mathcal{R}(T) \le C\big(\|r_{\rm in}\|_{H^2} + T^{1/4}(1+\mathcal{R}(T)+\mathcal{U}(T))^2\big). \tag{2.46}$$

Define now, for all $t \ge 0$, the quantity

$$\mathcal{N}(t) := \mathcal{R}(t) + \mathcal{U}(t).$$

Summing estimates (2.45) and (2.46), we infer that

$$\mathcal{N}(T) \le C\big(\|u_{\rm in}\|_{B^{3/2}_{2,4/3}} + \|r_{\rm in}\|_{H^2} + T^{1/16}(1+\mathcal{N}(T))^3\big).$$

From this inequality, it is a standard matter to deduce the existence of a time $T^*(\gamma,\mu,\varepsilon,\lambda,$ $\|\bar\rho\|_{W^{3,\infty}},\kappa,\|u_{\rm in}\|_{B^{3/2}_{2,4/3}},\|r_{\rm in}\|_{H^2})>0$, with $T^*\leq \min\{1,T\}$, such that

$$\mathcal{N}(T^*)\leq 2C(\|u_{\rm in}\|_{B^{3/2}_{2,4/3}}+\|r_{\rm in}\|_{H^2}).$$

The a priori estimates are hence proved in the interval $[0,T^*]$.

2.5. Uniqueness

The uniqueness of solutions, claimed in Theorem 1, is a consequence of the following statement.

Proposition 2.5. *Let $\gamma\geq 1$. Let $\bar\rho\in W^{3,\infty}(\mathbb{R})$. Assume that $\bar\rho$ is uniformly bounded from below, i.e. $\bar\rho\geq\kappa>0$. We define the potential by $G=H'(\bar\rho)$, where H is defined by (1.11). Let $(\rho_{\rm in},u_{\rm in})$ be such that $\rho_{\rm in}-\bar\rho\in H^2$ and $u_{\rm in}\in B^{3/2}_{2,4/3}$, with $\|\rho_{\rm in}-\bar\rho\|_{L^\infty}\leq\eta$, for some $\eta>0$ satisfying (2.1). Assume that (ρ^1,u^1) and (ρ^2,u^2) are two solutions to system (1.2), related to the same initial datum $(\rho_{\rm in},u_{\rm in})$ and belonging to the space*

$$X_T:=\big\{(\rho,u)\in L^\infty_T(L^\infty)\times L^\infty_T(L^2)\ |\ \textit{Properties (1)–(2) of Theorem 1 hold true}\big\},$$

for some $T>0$.

Then $\rho^1=\rho^2$ and $u^1=u^2$ almost everywhere in $[0,T]\times\Omega$.

First we show a simple lemma.

Lemma 2.6. *Let $(\rho_{\rm in},u_{\rm in})$ be as in Proposition 2.5 above, and let (ρ,u) be a solution to system (1.2), related to the initial datum $(\rho_{\rm in},u_{\rm in})$ and belonging to the space X_T, for some $T>0$.*

Then $\rho\in\mathcal{C}([0,T]\times\Omega)$ and $u\in\mathcal{C}([0,T];H^1(\Omega))$. In addition, the following estimate holds true, for a "universal" constant $C>0$:

$$\|u\|^2_{L^\infty_T(H^1)}\leq C(\|u_{\rm in}\|^2_{B^{3/2}_{2,4/3}}+\|\partial_t u\|_{L^2_T(L^2)}\|u\|_{L^2_T(H^2)}).$$

Proof of Lemma 2.6. By definition of the space X_T, we know that $r:=\rho-\bar\rho$ belongs to $\mathcal{C}_T(H^2)$, which, by Sobolev embeddings, is continuously embedded in $\mathcal{C}_T(\mathcal{C}\cap L^\infty)$. Thus $\rho=\bar\rho+r$ is continuous with respect to both x and t.

Next let us consider u. We showed in (2.41) above that $\partial_t u\in L^2_T(L^2)$ and that $u\in L^2_T(H^2)$. From these properties, one easily derives (see e.g. [15, Section 5.9]) that $u\in\mathcal{C}_T(H^1)$. The quantitative estimate is a simple consequence of the bound given in [15, Theorem 3, p. 305], combined with the embedding $B^{3/2}_{2,4/3}\hookrightarrow H^1$. The lemma is hence proved. ∎

We can now prove uniqueness.

Proof of Proposition 2.5. For $j=1,2$, let us define $r^j:=\rho^j-\bar\rho$. We also set

$$\delta r:=r^1-r^2=\rho^1-\rho^2\quad\text{and}\quad\delta u:=u^1-u^2.$$

Let us deal with δr first. Its equation reads as follows:

$$\partial_t \delta r + u^1 \cdot \nabla \delta r + \delta r \nabla \cdot u^1 = -(\bar{\rho} \nabla \cdot \delta u + \delta u_3 \bar{\rho}' + \delta u \cdot \nabla r^2 + r^2 \nabla \cdot \delta u).$$

An L^2 estimate for this equation (see the beginning of Section 2.3 above) yields, for all $t \in [0, T]$, the bound

$$\begin{aligned} \|\delta r(t)\|_{L^2} \leq \|\delta r_{\text{in}}\|_{L^2} &+ \frac{1}{2} \int_0^t \|\delta r\|_{L^2} \|\nabla \cdot u^1\|_{L^\infty} \, d\tau \\ &+ \int_0^t (\|r^2\|_{L^\infty} \|\nabla \cdot \delta u\|_{L^2} + \|\delta u\|_{L^\infty} \|\nabla r^2\|_{L^2}) \, d\tau \\ &+ \int_0^t (\|\bar{\rho}'\|_{L^\infty} \|\delta u_3\|_{L^2} + \|\bar{\rho}\|_{L^\infty} \|\nabla \cdot \delta u\|_{L^2}) \, d\tau. \end{aligned}$$

We also look for an L^2 bound on $\nabla \delta r$. To this end, we differentiate the equation for δr with respect to the space variables and we perform an energy estimate. We get

$$\begin{aligned} \|\nabla \delta r(t)\|_{L^2} \leq \|\nabla \delta r_{\text{in}}\|_{L^2} &+ \frac{1}{2} \int_0^t \|\nabla \delta r\|_{L^2} \|\nabla \cdot u^1\|_{L^\infty} \, d\tau \\ &+ \int_0^t (\|\nabla u^1\|_{L^\infty} \|\nabla \delta r\|_{L^2} + \|\nabla \nabla \cdot u^1\|_{L^4} \|\delta r\|_{L^4} \\ &\quad + \|\nabla \nabla \cdot \delta u\|_{L^2} \|r^2\|_{L^\infty} + \|\nabla \delta u\|_{L^4} \|\nabla r^2\|_{L^4} \\ &\quad + \|\delta u\|_{L^\infty} \|\nabla^2 r^2\|_{L^2} + \|\bar{\rho}\|_{W^{2,\infty}} \|\delta u\|_{H^2}) \, d\tau. \end{aligned} \tag{2.47}$$

Using the embedding $H^1(\mathbb{R}^3) \hookrightarrow L^4(\mathbb{R}^3)$ (see (2.30) above), and summing the previous estimate with (2.47), we finally get

$$\begin{aligned} \|\delta r(t)\|_{H^1} \leq \|\delta r_{\text{in}}\|_{H^1} &+ C \int_0^t \|\delta r\|_{H^1} (\|\nabla u^1\|_{L^\infty} + \|\nabla^2 u^1\|_{L^4}) \, d\tau \\ &+ C \int_0^t \Big((\|r^2\|_{L^\infty} + \|\nabla r^2\|_{L^4})(\|\nabla \delta u\|_{L^2} + \|\nabla^2 \delta u\|_{L^2}) \\ &\quad + \|\delta u\|_{L^\infty} \|\nabla^2 r^2\|_{L^2} + \|\bar{\rho}\|_{W^{2,\infty}} \|\delta u\|_{H^2} \Big) \, d\tau. \end{aligned} \tag{2.48}$$

Observe that, by definition of the space X_T, we have $\nabla u^1 \in L^2_T(L^\infty)$. In addition, from the control (2.30) applied to $\|\nabla^2 u^1\|_{L^4}$ and Young's inequality, we infer that $\nabla^2 u^1 \in L^{4/3}_T(L^4)$. By the same token we get $\nabla r^2 \in L^\infty_T(L^4)$, while we already know that $r^2 \in L^\infty_T(L^\infty) \cap L^\infty_T(H^2)$. Hence, estimate (2.48) above tells us that the quantity

$$\theta(t) := \sup_{\tau \in [0,t]} \|\delta r(\tau)\|_{H^1}$$

verifies the following bound, for a suitable exponent $\alpha > 0$:

$$\theta(t) \leq C \left(t^\alpha \theta(t) + \int_0^t (\|\delta u\|_{H^2} + \|\delta u\|_{L^\infty}) \, d\tau \right) \tag{2.49}$$

for all $t \in [0, \min\{T, 1\}]$. Notice that here we used the fact that $\delta r_{\text{in}} = 0$.

We now turn to velocity estimates. Taking the difference of the equations for u^1 and u^2 yields the following equation for δu:

$$\rho^1\partial_t\delta u+\rho^1u^1\cdot\nabla\delta u+\mathcal{L}\delta u$$
$$=-\delta r\partial_t u^2-\nabla(P(\rho^1)-P(\rho^2))+\delta r\nabla G-(\rho^1u^1-\rho^2u^2)\cdot\nabla u^2. \tag{2.50}$$

We observe that

$$(\rho^1u^1-\rho^2u^2)\cdot\nabla u^2=\rho^1\delta u\cdot\nabla u^2+\delta r u^2\cdot\nabla u^2,$$
$$\nabla(P(\rho^1)-P(\rho^2))=P'(\rho^1)\nabla\delta r+P''(\zeta)\delta r\nabla(\bar\rho+r^2),$$

for some $\zeta=\zeta(x,t)$ in between the values of $\rho^1(x,t)$ and $\rho^2(x,t)$. A basic energy estimate for the equation then gives, for any $t\in[0,T]$, the bound

$$\frac{1}{2}\frac{d}{dt}\|\sqrt{\rho^1}\,\delta u\|_{L^2}^2+\int_\Omega|\nabla_{\mu,\varepsilon}\delta u|^2+\lambda\int_\Omega|\nabla\cdot\delta u|^2$$
$$\leq\|\delta u\|_{L^2}(\|\delta r\|_{L^4}\|\partial_t u^2\|_{L^4}+\|\rho^1\|_{L^\infty}\|\delta u\|_{L^2}\|\nabla u^2\|_{L^\infty}$$
$$+\|\delta r\|_{L^4}\|u^2\|_{L^4}\|\nabla u^2\|_{L^\infty}+C\|\nabla\delta r\|_{L^2}+C\|\delta r\|_{L^2}$$
$$+C\|\delta r\|_{L^4}\|\nabla r^2\|_{L^4}),$$

where we have also used the $L^\infty_T(L^\infty)$ boundedness of ρ^1 and ρ^2 in order to control the terms involving derivatives of the pressure function. Let us forget about the viscosity terms for a while. A simple argument allows one to deduce the following control, for any $t\in[0,T]$:

$$\|\delta u\|_{L^\infty_t(L^2)}\leq C\int_0^t(\|\delta u\|_{L^2}\|\nabla u^2\|_{L^\infty}$$
$$+\theta(\tau)(1+\|\partial_t u^2\|_{H^1}+\|u^2\|_{L^4}\|\nabla u^2\|_{L^\infty}+\|\nabla r^2\|_{L^4}))\,d\tau,$$

where we have used the fact that $\delta u_{\rm in}=0$. At this point, we observe that u^2 and ∇u^2 both belong to $L^2_T(L^\infty)$. Moreover, by (2.30) and the fact that $u^2\in L^\infty_T(L^2)$ and $\nabla u^2\in L^4_T(L^2)$, one gathers that $u^2\in L^{16/3}_T(L^4)$. Finally, $\partial_t u^2\in L^{4/3}_T(H^1)$. In the end, similarly to what is done in (2.49), we deduce the existence of a positive exponent, that we keep calling α without loss of generality, such that

$$\|\delta u\|_{L^\infty_t(L^2)}\leq Ct^\alpha(\theta(t)+\|\delta u\|_{L^\infty_t(L^2)}) \tag{2.51}$$

for all $t\in[0,\min\{T,1\}]$.

As a last step, we rewrite equation (2.50) in the form

$$\bar\rho\partial_t\delta u+\mathcal{L}\delta u=-(r^1\partial_t\delta u+\delta r\partial_t u^2+\delta f), \tag{2.52}$$

where we have defined

$$\delta f:=\rho^1u^1\cdot\nabla\delta u+\rho^1\delta u\cdot\nabla u^2+\delta r u^2\cdot\nabla u^2$$
$$+P'(\rho^1)\nabla\delta r+P''(\zeta)\delta r\nabla(\bar\rho+r^2)-\delta r\nabla G.$$

Applying Proposition 2.3 to equation (2.52), with the choice (2.36) of the parameters, and using the smallness condition (2.1), we get, for any $t \in [0, T]$, the inequality

$$\begin{aligned}
&\|\delta u\|_{L^2_t(L^\infty)} + \|\nabla \delta u\|_{L^4_t(L^2)} + \|(\partial_t \delta u, \nabla^2 \delta u)\|_{L^{4/3}_t(L^2)} \\
&\quad \leq C(\|\delta u_{\mathrm{in}}\|_{\dot{B}^{1/2}_{2,4/3}} + \|\delta r \partial_t u^2\|_{L^{4/3}_t(L^2)} + \|\delta f\|_{L^{4/3}_t(L^2)} \\
&\qquad + \|\delta u\|_{L^{4/3}_T(L^2)} + \|\nabla \delta u\|_{L^{4/3}_T(L^2)}). \qquad (2.53)
\end{aligned}$$

Recall that $\partial_t u^2 \in L^2_T(L^2)$ by virtue of (2.41) and $\partial_t \nabla u^2 \in L^{4/3}_T(L^2)$ by definition of X_T. Hence, using Sobolev embedding and interpolation (2.30), we deduce that $\partial_t u^2 \in L^{16/11}_T(L^4)$. Therefore, we can bound

$$\begin{aligned}
\|\delta r \partial_t u^2\|_{L^{4/3}_t(L^2)} &\leq t^{3/4-11/16} \|\delta r\|_{L^\infty_t(L^4)} \|\partial_t u^2\|_{L^{16/11}_t(L^4)} \\
&\leq t^{1/16} \|\delta r\|_{L^\infty_t(H^1)} \|\partial_t u^2\|^{1/4}_{L^2_T(L^2)} \|\partial_t \nabla u^2\|^{3/4}_{L^{4/3}_T(L^2)} \leq C t^{1/16} \theta(t).
\end{aligned}$$

Next we are going to bound δf in $L^{4/3}_t(L^2)$. This has already been done for the energy estimate (2.51) above, but here we have to take special care of the integrability in time. For the pressure terms and the potential force term we have

$$\begin{aligned}
\|P''(\zeta)\delta r \nabla(\bar{\rho} + r^2)\|_{L^{4/3}_t(L^2)} &\leq C t^{3/4} (\|\delta r\|_{L^\infty_t(L^2)} \|\nabla \bar{\rho}\|_{L^\infty_T(L^\infty)} \\
&\qquad + \|\delta r\|_{L^\infty_t(L^4)} \|\nabla r^2\|_{L^\infty_T(L^4)}) \leq C t^{3/4} \theta(t), \\
\|P'(\rho^1) \nabla \delta r\|_{L^{4/3}_t(L^2)} &\leq C t^{3/4} \|\nabla \delta r\|_{L^\infty_t(L^2)} \leq C t^{3/4} \theta(t), \\
\|\delta r \nabla G\|_{L^{4/3}_t(L^2)} &\leq C t^{3/4} \theta(t).
\end{aligned}$$

Thus, it remains for us to bound the convective terms. First of all, recall that we showed above that $\|u^2\|_{L^4} \|\nabla u^2\|_{L^\infty}$ belongs to $L^{16/11}_T$. Therefore,

$$\|\delta r u^2 \cdot \nabla u^2\|_{L^{4/3}_t(L^2)} \leq t^{(3/4)-(11/16)} \|\delta r\|_{L^\infty_t(L^4)} \|u^2 \cdot \nabla u^2\|_{L^{16/11}_T(L^4)} \leq C t^{1/16} \theta(t).$$

In addition, we have

$$\begin{aligned}
\|\rho^1 \delta u \cdot \nabla u^2\|_{L^{4/3}_t(L^2)} &\leq t^{(3/4)-(1/2)} \|\rho^1\|_{L^\infty_T(L^\infty)} \|\delta u\|_{L^\infty_t(L^2)} \|\nabla u^2\|_{L^2_T(L^\infty)} \\
&\leq C t^{1/4} \|\delta u\|_{L^\infty_t(L^2)}.
\end{aligned}$$

Finally, for the last term we use the following Gagliardo–Nirenberg-type inequality (see e.g. [24, Lemma II.3.3]): $\|u\|_{L^\infty} \leq \|u\|^{1/4}_{L^2} \|\nabla^2 u\|^{3/4}_{L^2}$. This, combined with Young's inequality, implies $u^2 \in L^4_T(L^\infty)$ (recall the definition of the space X_T). Using this bound, we can estimate

$$\begin{aligned}
\|\rho^1 u^1 \cdot \nabla \delta u\|_{L^{4/3}_t(L^2)} &\leq t^{(3/4)-(1/2)} \|\rho^1\|_{L^\infty_T(L^\infty)} \|u^1\|_{L^4_T(L^\infty)} \|\nabla \delta u\|_{L^4_t(L^2)} \\
&\leq C t^{1/4} \|\nabla \delta u\|_{L^4_t(L^2)}.
\end{aligned}$$

Putting all the previous estimates into (2.53), we have shown that there exists a positive exponent $\alpha > 0$ for which, for all $t \in [0, \min\{1, T\}]$, one has

$$\begin{aligned} &\|\delta u\|_{L^2_t(L^\infty)} + \|\nabla \delta u\|_{L^4_t(L^2)} + \|(\partial_t \delta u, \nabla^2 \delta u)\|_{L^{4/3}_t(L^2)} \\ &\quad \leq C t^\alpha (\theta(t) + \|\delta u\|_{L^\infty_t(L^2)} + \|\nabla \delta u\|_{L^4_t(L^2)}). \end{aligned} \tag{2.54}$$

Let us now introduce the quantity

$$\mathcal{D}(t) := \|\delta u\|_{L^\infty_t(L^2)} + \|\delta u\|_{L^2_t(L^\infty)} + \|\nabla \delta u\|_{L^4_t(L^2)} + \|(\partial_t \delta u, \nabla^2 \delta u)\|_{L^{4/3}_t(L^2)}.$$

Summing inequalities (2.49), (2.51) and (2.54), we finally deduce that, for all $t \in [0, \min\{1, T\}]$, we have

$$\theta(t) + \mathcal{D}(t) \leq C t^\alpha (\theta(t) + \mathcal{D}(t)),$$

for a suitable exponent $\alpha > 0$. Therefore, if t is now small enough, we can absorb the right-hand side into the left-hand side, deducing that both $\theta(t)$ and $\mathcal{D}(t)$ have to be 0. In particular, we deduce that $\rho^1 \equiv \rho^2$ and $u^1 \equiv u^2$ almost everywhere on $[0, t] \times \Omega$. In this way, we also infer that

$$\|\nabla \delta u\|_{L^2_t(L^\infty)} + \|(\nabla^3 \delta u, \partial_t \nabla \delta u)\|_{L^{4/3}_t(L^2)} = 0,$$

from which we deduce that

$$\|\delta u\|_{L^\infty_t(H^1)} = 0,$$

where we have also used (2.41) and the estimate in Lemma 2.6.

To complete the argument, let us define the set

$$I := \left\{ t \in [0, T] \mid \|\rho^1(\tau) - \rho^2(\tau)\|_{H^1} + \|u^1(\tau) - u^2(\tau)\|_{H^1} = 0 \ \forall \tau \in [0, t] \right\}.$$

Of course, $I \neq \emptyset$, since $0 \in I$. In addition, the previous argument, combined with Lemma 2.6, shows that I is open. On the other hand, by continuity in time of the norms appearing in the definition of the space X_T (again see Lemma 2.6 above), we infer that I is also closed. Then, by connectedness, we must have $I = [0, T]$. This completes the proof of the proposition. ∎

3. Quantitative asymptotic analysis with stratification effects and anisotropic diffusion

The goal of this section is to analyze the structure of the solutions to the highly rotating compressible system (1.3) with vertical stratification. The main result of this part is contained in Theorem 2 (see Section 3.3): there we derive an asymptotic expansion and quantify the error in terms of the parameter ε. This stability result relies on relative entropy estimates.

3.1. Formal asymptotic expansion

In this section we perform formal computations in order to have a grasp on the structure of the solutions to system (1.3). Because of the no-slip boundary conditions (1.4), boundary layers appear in the limit $\varepsilon \to 0$ both in the vicinity of the top boundary $\mathbb{R}^2 \times \{1\}$ and of the bottom boundary $\mathbb{R}^2 \times \{0\}$.

We will specify later the precise hypotheses on the initial conditions

$$\rho_{|t=0} = \rho_{\text{in}} \quad \text{and} \quad u_{|t=0} = u_{\text{in}}$$

at time $t = 0$. For the purpose of the formal analysis, let us say in a loose way that we impose the following *far field conditions*, for $|x| \to \infty$:

$$\rho_{\text{in}}(x) \to \bar{\rho}(x_3) \quad \text{and} \quad u_{\text{in}}(x) \to 0,$$

where $\bar{\rho}$ is a strictly positive function satisfying the logistic equation

$$\nabla P(\bar{\rho}) = \bar{\rho} \nabla G. \tag{3.1}$$

In addition, the initial densities are assumed to be far away from vacuum. Moreover, we focus on *well-prepared initial data*, in the sense specified in Section 3.3 below.

3.1.1. Construction of the ansatz. We start by expanding the solution $(u^\varepsilon, \rho^\varepsilon)$ to (1.3) as

$$\begin{aligned}
u^\varepsilon &= u_0(x_h, x_3, t) + u^{\text{bl}}_{0,b}\big(x_h, \tfrac{x_3}{\varepsilon}, t\big) + u^{\text{bl}}_{0,t}\big(x_h, \tfrac{1-x_3}{\varepsilon}, t\big) \\
&\quad + \varepsilon\big(u_1(x_h, x_3, t) + u^{\text{bl}}_{1,b}\big(x_h, \tfrac{x_3}{\varepsilon}, t\big) + u^{\text{bl}}_{1,t}\big(x_h, \tfrac{1-x_3}{\varepsilon}, t\big)\big) + O(\varepsilon^2), \\
\rho^\varepsilon &= \rho_0(x_h, x_3, t) + \rho^{\text{bl}}_{0,b}\big(x_h, \tfrac{x_3}{\varepsilon}, t\big) + \rho^{\text{bl}}_{0,t}\big(x_h, \tfrac{1-x_3}{\varepsilon}, t\big) \\
&\quad + \varepsilon\big(\rho_1(x_h, x_3, t) + \rho^{\text{bl}}_{1,b}\big(x_h, \tfrac{x_3}{\varepsilon}, t\big) + \rho^{\text{bl}}_{1,t}\big(x_h, \tfrac{1-x_3}{\varepsilon}, t\big)\big) \\
&\quad + \varepsilon^2(\rho_2(x_h, x_3, t) + \rho^{\text{bl}}_{2,b}\big(x_h, \tfrac{x_3}{\varepsilon}, t\big) + \rho^{\text{bl}}_{2,t}\big(x_h, \tfrac{1-x_3}{\varepsilon}, t\big)) + O(\varepsilon^3).
\end{aligned} \tag{3.2}$$

The superscript bl stands for "boundary layer", while the subscripts b and t stand for "bottom" and "top" respectively. For simplicity of presentation, in the next computations we are going to consider only the boundary layer near the bottom, since the terms related to the top boundary layer are dealt with in the exact same way. Therefore, from now on we omit the subscript b for the boundary layer terms. However, when needed, we will explicitly write t or b subscripts to avoid confusion.

Below we denote by $\zeta = \frac{x_3}{\varepsilon}$ the fast vertical variable in the boundary layer. The boundary layer profiles are supposed to decay to 0 at an exponential rate when $\zeta \to \infty$, since their effect is almost negligible in the interior of the domain: we will use this fact repeatedly in the following computations.

We remark that, at this level, (3.2) is just a formal ansatz. As is usual, we will first formally derive the equations for the profiles: this is the purpose of the present section. After that, we will prove quantitative estimates for the difference between the solution and the profiles we have constructed, using the relative entropy method: this will be done in Section 3.

Identification of the profiles. In order to identify the profiles, we plug the ansatz (3.2) into (1.3) and identify the terms of the same order of magnitude in ε. We immediately notice that the highest-order term is of order ε^{-3}, which appears in the third component of the momentum equation:

$$P'(\rho_0 + \rho_0^{\mathrm{bl}})\partial_\zeta \rho_0^{\mathrm{bl}} = 0.$$

We assume that $\rho_0 + \rho_0^{\mathrm{bl}}$ stays bounded away from zero. This hypothesis is fully justified here below. In view of hypothesis (1.10) on the pressure and the fact that ρ_0^{bl} has to vanish for $\zeta \to \infty$, we immediately deduce that $\rho_0^{\mathrm{bl}} \equiv 0$. Thanks to that property, and ignoring terms of order $O(\varepsilon^2)$, which have been neglected in (3.2) in the expansion of the velocity fields, we find the following cascade of equations: from the conservation of mass equation, we get

$$\rho_0 \partial_\zeta u_{0,3}^{\mathrm{bl}} = 0, \tag{mass-ε^{-1}}$$

$$\begin{aligned} &\partial_t \rho_0 + \nabla_h \cdot (\rho_0(u_{0,h} + u_{0,h}^{\mathrm{bl}})) + \partial_3(\rho_0 u_{0,3}) + \partial_3 \rho_0 u_{0,3}^{\mathrm{bl}} \\ &\qquad + \rho_1 \partial_\zeta u_{0,3}^{\mathrm{bl}} + \partial_\zeta(\rho_1^{\mathrm{bl}} u_{0,3}^{\mathrm{bl}}) + \partial_\zeta \rho_1^{\mathrm{bl}} u_{0,3} + \rho_0 \partial_\zeta u_{1,3}^{\mathrm{bl}} = 0, \end{aligned} \tag{mass-ε^{0}}$$

and from the momentum equation we get

$$\nabla P(\rho_0) + \begin{pmatrix} 0 \\ P'(\rho_0)\partial_\zeta \rho_1^{\mathrm{bl}} \end{pmatrix} = \begin{pmatrix} 0 \\ \lambda \partial_\zeta^2 u_{0,3}^{\mathrm{bl}} \end{pmatrix} + \rho_0 \nabla G, \tag{mom-ε^{-2}}$$

$$\begin{aligned} &\rho_0(u_{0,3} + u_{0,3}^{\mathrm{bl}}) \cdot \partial_\zeta u_0^{\mathrm{bl}} + e_3 \times \rho_0(u_0 + u_0^{\mathrm{bl}}) \\ &\quad + \begin{pmatrix} \nabla_h(P'(\rho_0)(\rho_1 + \rho_1^{\mathrm{bl}})) \\ \partial_3(P'(\rho_0)\rho_1) + \partial_3(P'(\rho_0))\rho_1^{\mathrm{bl}} + P''(\rho_0)\rho_1^{\mathrm{bl}}\partial_\zeta \rho_1^{\mathrm{bl}} + P'(\rho_0)\partial_\zeta \rho_2^{\mathrm{bl}} \end{pmatrix} \\ &\quad = \partial_\zeta^2 \begin{pmatrix} u_{0,h}^{\mathrm{bl}} \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} \nabla_h \partial_\zeta u_{0,3}^{\mathrm{bl}} \\ \partial_\zeta \nabla_h \cdot u_{0,h}^{\mathrm{bl}} + \partial_\zeta^2 u_{1,3}^{\mathrm{bl}} \end{pmatrix} + (\rho_1 + \rho_1^{\mathrm{bl}})\nabla G. \end{aligned} \tag{mom-ε^{-1}}$$

We will examine the equation of order $O(\varepsilon^0)$ coming from the momentum equation later. Let us first infer some properties for the profiles.

The terms in the interior. Recall that the boundary layer profiles are expected to go to zero when $\zeta \to \infty$. Therefore, it follows from (mom-ε^{-2}) that

$$\nabla P(\rho_0) = \rho_0 \nabla G, \tag{3.3}$$

which yields, by using (1.10), the properties

$$H'(\rho_0) = G + c(t) \quad \text{and} \quad \nabla_h \rho_0 = 0. \tag{3.4}$$

Hence ρ_0 is independent of x_h, namely $\rho_0 = \rho_0(x_3, t)$, and satisfies the ODE

$$P'(\rho_0)\partial_3 \rho_0 = -\rho_0. \tag{3.5}$$

Since $P' \in \mathcal{C}^1((0,\infty))$ and is nonzero, we can use the Cauchy–Lipschitz theorem to get that $\rho_0(t) \in \mathcal{C}^1((0,1))$, and is hence bounded. Moreover, from (mom-ε^{-1}) we infer

$$\rho_0 \begin{pmatrix} u_{0,h}^{\perp} \\ 0 \end{pmatrix} + \begin{pmatrix} P'(\rho_0)\nabla_h \rho_1 \\ \partial_3(P'(\rho_0)\rho_1) \end{pmatrix} = \rho_1 \nabla G.$$

This equation is called *geostrophic balance*; it implies the Taylor–Proudman theorem (see Section 1). In particular, its third component reads

$$\partial_3(P'(\rho_0)\rho_1) = -\rho_1.$$

Using the previous relation together with (3.3), we get

$$\partial_3\Big(\frac{P'(\rho_0)}{\rho_0}\rho_1\Big) = 0, \tag{3.6}$$

hence the quantity

$$Q := \frac{P'(\rho_0)}{\rho_0}\rho_1 \quad \text{verifies} \quad Q = Q(t, x_h), \tag{3.7}$$

i.e. Q is independent of the vertical variable. From the horizontal component, instead we get (recall that $\nabla_h \rho_0 = 0$)

$$u_{0,h} = \nabla_h^{\perp}\Big(\frac{P'(\rho_0)}{\rho_0}\rho_1\Big) = \nabla_h^{\perp} Q. \tag{3.8}$$

In particular, we deduce that $u_{0,h} = u_{0,h}(x_h, t)$, which justifies the introduction of boundary layer terms in order to enforce the no-slip boundary conditions on $x_3 = 0, 1$. In addition, applying the horizontal divergence we obtain

$$\nabla_h \cdot u_{0,h} = \nabla_h \cdot \nabla_h^{\perp}\Big(\frac{P'(\rho_0)}{\rho_0}\rho_1\Big) = 0,$$

so that $u_{0,h}$ is a two-dimensional horizontal divergence-free vector field.

We now exploit (mass-ε^0): considering it in the interior of the domain (i.e. neglecting the boundary terms) and using the inequalities just proved, after an integration in the vertical variable we infer that

$$\int_0^1 \partial_t \rho_0 \, dx_3 = -\int_0^1 \partial_3(\rho_0 u_{0,3}) \, dx_3 = 0. \tag{3.9}$$

By taking the time derivative of (3.4) and using (1.10) we have

$$\partial_t \rho_0 = \frac{\partial_t c}{H''(\rho_0)}.$$

We integrate in the vertical variable and, from (3.9), we get $\partial_t c = 0$, hence $\partial_t \rho_0 = 0$. This implies that ρ_0 has to be independent of time also, and hence it is equal to a positive function $\bar\rho(x_3)$, solution of (3.3), or equivalently (3.1). Thanks to this fact, we now have that $\partial_3(\rho_0 u_{0,3}) = 0$. Using the no-slip boundary condition and the positivity of $\rho_0 = \bar\rho$, we find $u_{0,3} \equiv 0$.

From now on $\bar\rho$ denotes ρ_0. Let us now consider the equations outside the boundary layers: at order $O(\varepsilon)$ in the mass equation,

$$\partial_t \rho_1 + \nabla\cdot(\bar\rho u_1) + \nabla_h\cdot(\rho_1 u_{0,h}) = 0, \qquad \text{(mass-}\varepsilon^1)$$

and at order $O(\varepsilon^0)$ in the momentum equation,

$$\begin{aligned}\bar\rho\partial_t u_0 + \nabla\cdot(\bar\rho u_0\otimes u_0) + e_3\times(\rho_1 u_0 + \bar\rho u_1)&\\ + \nabla\Big(\frac{P''(\bar\rho)}{2}\rho_1^2 + P'(\bar\rho)\rho_2\Big) &= \mu\Delta_h u_0 + \lambda\nabla(\nabla\cdot u_0) + \rho_2\nabla G. \qquad \text{(mom-}\varepsilon^0)\end{aligned}$$

Recall that $u_0 = (u_{0,h}(t,x_h),0)$. Taking the curl of the horizontal component in (mom-ε^0), we obtain an equation for the horizontal vorticity $\omega_0 = \nabla_h^\perp\cdot u_{0,h}$:

$$\bar\rho\partial_t\omega_0 + \bar\rho u_{0,h}\cdot\nabla_h\omega_0 + \nabla_h\cdot(\rho_1 u_{0,h}) + \nabla_h\cdot(\bar\rho u_{1,h}) - \mu\Delta_h\omega_0 = 0.$$

Notice that, by (3.8), we get

$$\omega_0 = \omega_0(t,x_h) = \Delta_h Q,$$

where Q is defined in (3.7); from the previous relation it follows that

$$\bar\rho\partial_t\Delta_h Q + \bar\rho\nabla_h^\perp Q\cdot\nabla_h\Delta_h Q + \nabla_h\cdot(\bar\rho u_{1,h}) - \mu\Delta_h^2 Q = 0, \qquad (3.10)$$

where we have used the cancellation

$$\nabla_h\cdot(\rho_1\nabla_h^\perp\rho_1) = \frac{1}{2}\nabla_h\cdot\nabla_h^\perp(\rho_1^2) = 0 \qquad (3.11)$$

in order to get rid of the term $\nabla_h\cdot(\rho_1 u_{0,h})$. In order to compute the term $\nabla_h\cdot(\bar\rho u_{1,h})$ in (3.10), we use equation (mass-ε^1) and cancellation (3.11) again: we find

$$\nabla_h\cdot(\bar\rho u_{1,h}) = -\partial_t\rho_1 - \nabla_h\cdot(\rho_1 u_{0,h}) - \partial_3(\bar\rho u_{1,3}) = -\partial_t\rho_1 - \partial_3(\bar\rho u_{1,3}). \qquad (3.12)$$

After integrating both (3.10) and (3.12) in x_3 and summing the resulting expressions, we eventually obtain

$$\begin{aligned}\partial_t\Big(\langle\bar\rho\rangle\Delta_h Q - \Big\langle\frac{\bar\rho}{P'(\bar\rho)}\Big\rangle Q\Big) + \langle\bar\rho\rangle\nabla_h^\perp Q\cdot\nabla_h\Delta_h Q - \mu\Delta_h^2 Q&\\ = \bar\rho(1)u_{1,3}(x_h,1,t) - \bar\rho(0)u_{1,3}(x_h,0,t),& \qquad (3.13)\end{aligned}$$

where $\langle f\rangle = \int_0^1 f(x_3)\,dx_3$ denotes the vertical mean of f.

Boundary layer terms. We now consider the boundary layer terms. These terms are crucial to compute the right-hand side of (3.13): indeed

$$u^{\mathrm{bl}}_{j,3,b}(x_h,0,t) = -u_{j,3}(x_h,0,t) \quad \text{and} \quad u^{\mathrm{bl}}_{j,3,t}(x_h,0,t) = -u_{j,3}(x_h,1,t) \tag{3.14}$$

for $j = 0, 1$, in order to enforce the no-slip boundary condition on the bottom and top boundaries.

First of all, (mass-ε^{-1}) yields $u^{\mathrm{bl}}_{0,3} = u^{\mathrm{bl}}_{0,3}(x_h,t)$, and hence

$$u^{\mathrm{bl}}_{0,3} \equiv 0. \tag{3.15}$$

Using (3.15), we obtain from (mom-ε^{-2}) that

$$P'(\bar{\rho})\partial_\zeta \rho^{\mathrm{bl}}_1 = \lambda \partial^2_\zeta u^{\mathrm{bl}}_{0,3} = 0.$$

Hence, thanks to (1.10), $\rho^{\mathrm{bl}}_1 = \rho^{\mathrm{bl}}_1(x_h,t)$ is constant in the boundary layer and goes to zero when $\zeta \to \infty$, therefore $\rho^{\mathrm{bl}}_1 \equiv 0$. Taking into account this last equality and reading the horizontal component of (mom-ε^{-1}), one has

$$\bar{\rho}(u^{\mathrm{bl}}_{0,h})^{\perp} = \partial^2_\zeta u^{\mathrm{bl}}_{0,h}. \tag{3.16}$$

Notice that, in (3.16), x_h is a parameter. We use the Taylor formula of first order

$$\bar{\rho}(x_3) = \bar{\rho}(0) + x_3 \int_0^1 \partial_3 \bar{\rho}(s x_3)\, ds$$

to write (3.16) as

$$\bar{\rho}(0)(u^{\mathrm{bl}}_{0,h})^{\perp} + \left(x_3 \int_0^1 \partial_3 \bar{\rho}(s x_3)\, ds \right) (u^{\mathrm{bl}}_{0,h})^{\perp} = \partial^2_\zeta u^{\mathrm{bl}}_{0,h}. \tag{3.17}$$

Let us now consider the equation

$$\bar{\rho}(0)(u^{\mathrm{bl}}_{0,h})^{\perp} = \partial^2_\zeta u^{\mathrm{bl}}_{0,h}, \tag{3.18}$$

supplemented with the boundary condition

$$u^{\mathrm{bl}}_{0,h}(x_h,0,t) = -u_{0,h}(x_h,t) \tag{3.19}$$

at $\zeta = 0$, in view of (1.4) and (3.14). We remark that the system of ODEs (3.18)–(3.19) is the same (here in general $\bar{\rho}(0) \neq 1$) as in the incompressible case; see e.g. [11, Chapter 7] and references therein. Its solutions are exponentially decaying and have a spiral structure. Indeed, we have the following formula:

$$u^{\mathrm{bl}}_{0,h,b}(x_h,\zeta,t) = -\begin{pmatrix} e^{-\zeta\sqrt{\frac{\bar{\rho}(0)}{2}}}\left[u_{0,1}(x_h,t)\cos\left(\zeta\sqrt{\frac{\bar{\rho}(0)}{2}}\right) + u_{0,2}(x_h,t)\sin\left(\zeta\sqrt{\frac{\bar{\rho}(0)}{2}}\right)\right] \\ e^{-\zeta\sqrt{\frac{\bar{\rho}(0)}{2}}}\left[-u_{0,1}(x_h,t)\sin\left(\zeta\sqrt{\frac{\bar{\rho}(0)}{2}}\right) + u_{0,2}(x_h,t)\cos\left(\zeta\sqrt{\frac{\bar{\rho}(0)}{2}}\right)\right] \end{pmatrix}.$$

Let us move further. The vertical component in (mom-ε^{-1}) is

$$0 = \lambda(\partial_\zeta \nabla_h \cdot u^{\mathrm{bl}}_{0,h} + \partial^2_\zeta u^{\mathrm{bl}}_{1,3}) + P'(\bar\rho)\partial_\zeta \rho^{\mathrm{bl}}_2. \tag{3.20}$$

Equation (mass-ε^0), together with the fact the $\bar\rho$ is strictly positive, yields

$$\nabla_h \cdot u^{\mathrm{bl}}_{0,h} + \partial_\zeta u^{\mathrm{bl}}_{1,3} = 0. \tag{3.21}$$

Hence $P'(\bar\rho)\partial_\zeta \rho^{\mathrm{bl}}_2 = 0$ and, similarly to the argument used for ρ^{bl}_1, we get $\rho^{\mathrm{bl}}_2 \equiv 0$. The previous equality (3.21) determines $u^{\mathrm{bl}}_{1,3}$ up to a constant in ζ, which we take so that $u^{\mathrm{bl}}_{1,3}$ converges to zero when $\zeta \to \infty$:

$$u^{\mathrm{bl}}_{1,3,b}(x_h,\zeta,t) = -\frac{e^{-\zeta\sqrt{\frac{\bar\rho(0)}{2}}}}{\sqrt{2\bar\rho(0)}}\left(\cos\left(\zeta\sqrt{\frac{\bar\rho(0)}{2}}\right) + \sin\left(\zeta\sqrt{\frac{\bar\rho(0)}{2}}\right)\right)\nabla_h^\perp \cdot u_{0,h}(x_h,t).$$

Similar computations can be done for the top boundary layers. Indeed, denoting by $\eta = \frac{1-x_3}{\varepsilon}$ the fast vertical variable in the upper boundary layer, we use the Taylor formula at first order

$$\bar\rho(x_3) = \bar\rho(1) - (1-x_3)\int_0^1 \partial_3\bar\rho(1-s(1-x_3))\,ds \tag{3.22}$$

to define $u^{\mathrm{bl}}_{0,h,t}$ as the solution to the equation

$$\bar\rho(1)(u^{\mathrm{bl}}_{0,h,t})^\perp = \partial^2_\eta u^{\mathrm{bl}}_{0,h,t}, \tag{3.23}$$

supplemented with the boundary condition

$$u^{\mathrm{bl}}_{0,h,t}(x_h,0,t) = -u_{0,h}(x_h,t) \tag{3.24}$$

at $\eta = 0$; recall (3.14). We have

$$u^{\mathrm{bl}}_{0,h,t}(x_h,\eta,t) = -\begin{pmatrix} e^{-\eta\sqrt{\frac{\bar\rho(1)}{2}}}\left[u_{0,1}(x_h,t)\cos\left(\eta\sqrt{\frac{\bar\rho(1)}{2}}\right) + u_{0,2}(x_h,t)\sin\left(\eta\sqrt{\frac{\bar\rho(1)}{2}}\right)\right] \\ e^{-\eta\sqrt{\frac{\bar\rho(1)}{2}}}\left[-u_{0,1}(x_h,t)\sin\left(\eta\sqrt{\frac{\bar\rho(1)}{2}}\right) + u_{0,2}(x_h,t)\cos\left(\eta\sqrt{\frac{\bar\rho(1)}{2}}\right)\right]\end{pmatrix}.$$

and, from (3.21) with ∂_ζ replaced by $-\partial_\eta$,

$$u^{\mathrm{bl}}_{1,3,t}(x_h,\eta,t) = \frac{e^{-\eta\sqrt{\frac{\bar\rho(1)}{2}}}}{\sqrt{2\bar\rho(1)}}\left(\cos\left(\eta\sqrt{\frac{\bar\rho(1)}{2}}\right) + \sin\left(\eta\sqrt{\frac{\bar\rho(1)}{2}}\right)\right)\nabla_h^\perp \cdot u_{0,h}(x_h,t).$$

Hence, using (3.14), one can now compute the right-hand side of equation (3.13):

$$\begin{aligned}\bar\rho(1)u_{1,3}(x_h,1,t) - \bar\rho(0)u_{1,3}(x_h,0,t) &= -\bar\rho(1)u^{\mathrm{bl}}_{1,3,t}(x_h,0,t) + \bar\rho(0)u^{\mathrm{bl}}_{1,3,b}(x_h,0,t) \\ &= -\frac{\sqrt{\bar\rho(0)}+\sqrt{\bar\rho(1)}}{\sqrt2}\omega_0 \\ &= -\frac{\sqrt{\bar\rho(0)}+\sqrt{\bar\rho(1)}}{\sqrt2}\Delta_h Q. \end{aligned} \tag{3.25}$$

This is the so-called *Ekman pumping term*, which represents the secondary (global) circulation created by the boundary layer. It appears as a damping term for the quasi-geostrophic dynamics, described by equation (3.13).

Final choices for correctors. It remains to choose the functions ρ_2, u_1 and $u^{\text{bl}}_{1,h}$. These terms are auxiliary terms which do not appear in the final result.

We choose the interior terms in order to make the terms of order $O(\varepsilon)$ in the mass equation and the terms of order $O(\varepsilon^0)$ in the momentum equation vanish identically. Notice that (mom-ε^0) determines $u_{1,h}$ in terms of u_0, ρ_1 and ρ_2, and hence, through relation (3.8), in terms of ρ_1 and ρ_2 only. Specifically,

$$u_{1,h} := \frac{1}{\bar{\rho}}\Big(- \mu\Delta_h u^{\perp}_{0,h} + \bar{\rho}\partial_t u^{\perp}_{0,h} + \bar{\rho} u_{0,h}\cdot\nabla_h u^{\perp}_{0,h} - u_{0,h}\rho_1 + \nabla^{\perp}_h\Big(P'(\bar{\rho})\rho_2 + \frac{P''(\bar{\rho})}{2}\rho_1^2\Big)\Big). \tag{3.26}$$

Next, the vertical component of (mom-ε^0) reads

$$\partial_3\Big(P'(\bar{\rho})\rho_2 + \frac{P''(\bar{\rho})}{2}\rho_1^2\Big) = -\rho_2, \tag{3.27}$$

where we have used that $u_{0,3} \equiv 0$. Since, by (1.10), $P'(\bar{\rho}) > 0$, ρ_2 can be defined as the solution of the ODE

$$\partial_3\rho_2 + \frac{\partial_3(P'(\bar{\rho})) + 1}{P'(\bar{\rho})}\rho_2 = -\frac{\partial_3(P''(\bar{\rho})\rho_1^2)}{2P'(\bar{\rho})}, \tag{3.28}$$

up to an arbitrary constant $c(x_h, t)$, that we take equal to zero for simplicity. We remark that this choice does not affect the choice of the other quantities since ρ_2 appears only in (mom-ε^0) or higher-order equations. Moreover, since ρ_1 and $\nabla\rho_1$ are bounded in time and space (Q, defined in (3.7), satisfies the quasi-geostrophic equation (3.13), which admits regular solutions; see Lemma 3.3 later), ρ_2 and $\nabla\rho_2$ are also bounded in time and space.

Next, equation (mass-ε^1) determines $u_{1,3}$ up to a constant in x_3, which we take equal to $-u^{\text{bl}}_{1,3}(x_h, 0, t)$ in order to enforce the no-slip boundary condition for the vertical component at order $O(\varepsilon)$. Therefore, thanks to (3.12) we get

$$\bar{\rho}(x_3)u_{1,3}(x_h, x_3, t) = -\bar{\rho}(0)u^{\text{bl}}_{1,3}(x_h, 0, t) - \int_0^{x_3} (\partial_t\rho_1 + \bar{\rho}\nabla_h\cdot u_{1,h})\,dz. \tag{3.29}$$

Differently from the case without the gravitational potential, this term does not have an affine structure as in the incompressible case (see again [11, Chapter 7]), since $u_{1,h}$ does not depend only on x_h.

In order to enforce the no-slip boundary condition at order $O(\varepsilon)$ for the horizontal component also, we impose

$$u^{\text{bl}}_{1,h}(x_h, 0, t) = -u_{1,h}(x_h, 0, t) \tag{3.30}$$

at $\zeta = 0$. It remains to choose the boundary layer term $u^{\mathrm{bl}}_{1,h}$. The specifications for the boundary layer term $u^{\mathrm{bl}}_{1,h}$ are that it is exponentially decaying to 0 for $\zeta \to \infty$ and satisfies (3.30) at the boundary $\zeta = 0$. Hence, we define $u^{\mathrm{bl}}_{1,h,b}$ in the following way: for all $\zeta \in [0, \infty)$ and $x_h \in \mathbb{R}^2$,

$$u^{\mathrm{bl}}_{1,h,b}(x_h, \zeta, t) := -u_{1,h}(x_h, 0, t) e^{-\zeta \sqrt{\frac{\bar{\rho}(0)}{2}}}.$$

Analogously, $u^{\mathrm{bl}}_{1,h,t}$ is defined for all $\eta \in [0, \infty)$ and $x_h \in \mathbb{R}^2$ by

$$u^{\mathrm{bl}}_{1,h,t}(x_h, \eta, t) := -u_{1,h}(x_h, 1, t) e^{-\eta \sqrt{\frac{\bar{\rho}(1)}{2}}}.$$

Remark 3.1. Contrary to the interior terms, it is not possible to make the terms of order $O(\varepsilon)$ in the mass equation and the terms of order $O(\varepsilon^0)$ in the momentum equation vanish identically. Indeed that would come down to imposing

$$\begin{aligned} \bar{\rho} \nabla_h \cdot u^{\mathrm{bl}}_{1,h} &= - \nabla_h \rho_1 \cdot u^{\mathrm{bl}}_{0,h} - \partial_3 \bar{\rho} u^{\mathrm{bl}}_{1,3}, \\ \lambda \partial_\zeta (\nabla_h \cdot u^{\mathrm{bl}}_{1,h}) &= - \partial^2_\zeta u^{\mathrm{bl}}_{1,3} = \partial_\zeta (\nabla_h \cdot u^{\mathrm{bl}}_{0,h}), \end{aligned} \tag{3.31}$$

which is overdetermined. This fact is due to the lack of higher-order correctors, since we truncate the expansion at order 1 in ε.

Notice that, due to exponential decay to zero in the interior of the domain, the boundary layer terms are small. Moreover, we can exploit their decay by relying on Hardy's inequality (see the computations in Section 3.3.3). The final stability estimate, though, will be worse than in the absence of boundary layer phenomena (as e.g. for complete slip boundary conditions). Improving this estimate would require considering higher-order correctors in the ansatz (3.2).

Notice also that, using (3.14), we have at the bottom $x_3 = 0$,

$$\begin{aligned} u_{0,h}(x_h, t) + u^{\mathrm{bl}}_{0,h,b}(x_h, 0, t) + u^{\mathrm{bl}}_{0,h,t}\big(x_h, \tfrac{1}{\varepsilon}, t\big) &= u^{\mathrm{bl}}_{0,h,t}\big(x_h, \tfrac{1}{\varepsilon}, t\big), \\ u_1(x_h, 0, t) + u^{\mathrm{bl}}_{1,b}(x_h, 0, t) + u^{\mathrm{bl}}_{1,t}\big(x_h, \tfrac{1}{\varepsilon}, t\big) &= u^{\mathrm{bl}}_{1,t}\big(x_h, \tfrac{1}{\varepsilon}, t\big), \end{aligned} \tag{3.32}$$

and at the top $x_3 = 1$,

$$\begin{aligned} u_{0,h}(x_h, t) + u^{\mathrm{bl}}_{0,h,b}\big(x_h, \tfrac{1}{\varepsilon}, t\big) + u^{\mathrm{bl}}_{0,h,t}(x_h, 0, t) &= u^{\mathrm{bl}}_{0,h,b}\big(x_h, \tfrac{1}{\varepsilon}, t\big), \\ u_1(x_h, 1, t) + u^{\mathrm{bl}}_{1,b}\big(x_h, \tfrac{1}{\varepsilon}, t\big) + u^{\mathrm{bl}}_{1,t}(x_h, 0, t) &= u^{\mathrm{bl}}_{1,b}\big(x_h, \tfrac{1}{\varepsilon}, t\big). \end{aligned} \tag{3.33}$$

It means that we have an (exponentially small, but still nonzero) trace of the top boundary layer on the bottom boundary and vice versa. Hence, we will add corrector terms in the ansatz (3.34) below, in order to keep homogeneous boundary conditions. This is a technical point, but needed to apply Hardy's inequality later.

The ansatz. To put it in a nutshell, we have obtained the following ansatz for the structure of the solutions to (1.3)–(1.4):

$$
\begin{aligned}
&\rho^{\varepsilon}_{\mathrm{app}}(x_h,x_3,t) = \bar\rho(x_3) + \varepsilon\rho_1(x_h,x_3,t) + \varepsilon^2\rho_2(x_h,x_3,t),\\
&u^{\varepsilon}_{\mathrm{app}}(x_h,x_3,\zeta,\eta,t)\\
&\quad= \begin{pmatrix} \nabla_h^{\perp}Q(x_h,t) + u^{\mathrm{bl}}_{0,h,b}(x_h,\zeta,t) + u^{\mathrm{bl}}_{0,h,t}(x_h,\eta,t) - u^{\mathrm{bl}}_{0,h,1/\varepsilon}(x_h,x_3,t)\\ 0\end{pmatrix}\\
&\qquad+\varepsilon\begin{pmatrix} u_{1,h}(x_h,x_3,t) + u^{\mathrm{bl}}_{1,h,b}(x_h,\zeta,t) + u^{\mathrm{bl}}_{1,h,t}(x_h,\eta,t) - u^{\mathrm{bl}}_{1,h,1/\varepsilon}(x_h,x_3,t)\\ u_{1,3}(x_h,x_3,t) + u^{\mathrm{bl}}_{1,3,b}(x_h,\zeta,t) + u^{\mathrm{bl}}_{1,3,t}(x_h,\eta,t) - u^{\mathrm{bl}}_{1,3,1/\varepsilon}(x_h,x_3,t)\end{pmatrix}
\end{aligned}
\tag{3.34}
$$

with Q defined in (3.7) and

$$
\begin{aligned}
u^{\mathrm{bl}}_{0,h,1/\varepsilon}(x_h,x_3,t) &= x_3 u^{\mathrm{bl}}_{0,h,b}\big(x_h,\tfrac{1}{\varepsilon},t\big) + (1-x_3)u^{\mathrm{bl}}_{0,h,t}\big(x_h,\tfrac{1}{\varepsilon},t\big),\\
u^{\mathrm{bl}}_{1,1/\varepsilon}(x_h,x_3,t) &= x_3 u^{\mathrm{bl}}_{1,b}\big(x_h,\tfrac{1}{\varepsilon},t\big) + (1-x_3)u^{\mathrm{bl}}_{1,t}\big(x_h,\tfrac{1}{\varepsilon},t\big).
\end{aligned}
\tag{3.35}
$$

In addition, it follows from (3.13) that

$$
\begin{aligned}
&\partial_t\Big(\Big\langle\frac{\bar\rho}{P'(\bar\rho)}\Big\rangle Q - \langle\bar\rho\rangle\Delta_h Q\Big) - \langle\bar\rho\rangle\nabla_h^{\perp}Q\cdot\nabla_h\Delta_h Q + \mu\Delta_h^2 Q\\
&\qquad - \frac{\sqrt{\bar\rho(0)}+\sqrt{\bar\rho(1)}}{\sqrt{2}}\Delta_h Q = 0.
\end{aligned}
\tag{3.36}
$$

This is the *quasi-geostrophic equation*. Similar limit equations without damping term have been shown in e.g. [14], [17] and [23], where the boundary layers do not appear due to the complete slip condition. Notice that in [23] the parabolic term disappears, since the authors also consider the inviscid limit. We state here the well-posedness and the regularity results for the quasi-geostrophic equation (3.36), whose detailed proofs are given in [3].

Proposition 3.2. *Let $Q_{\mathrm{in}} \in H^1(\mathbb{R}^2)$. Then there exists a unique global weak solution Q to the quasi-geostrophic equation (3.36) with initial datum Q_{in}, such that*

$$
Q \in \mathcal{C}(\mathbb{R}_+;H^1(\mathbb{R}^2)) \cap L^{\infty}(\mathbb{R}_+;H^1(\mathbb{R}^2)) \quad \textit{and} \quad \nabla_h Q \in L^2(\mathbb{R}_+;H^1(\mathbb{R}^2)).
$$

Lemma 3.3. *Let $n \geq 1$ be an integer and $Q_{\mathrm{in}} \in H^n(\mathbb{R}^2)$. Then, there exists a constant $C_{n-1} > 0$ such that any weak solution to (3.36) with initial datum Q_{in} satisfies the following inequality for all $t \geq 0$:*

$$
\begin{aligned}
&\sum_{j=0}^{n-1}(\|\nabla_h^j Q(t)\|^2_{L^2} + \|\nabla_h^{j+1}Q(t)\|^2_{L^2}) + \sum_{j=0}^{n-1}\Big(\int_0^t \|\nabla_h^{j+1}Q\|^2_{L^2} + \|\nabla_h^{j+2}Q\|^2_{L^2}\Big)\\
&\qquad\leq C_{n-1}\sum_{j=0}^{n-1}(\|\nabla_h^j Q_{\mathrm{in}}\|_{L^2} + \|\nabla_h^{j+1}Q_{\mathrm{in}}\|^2_{L^2}),
\end{aligned}
\tag{3.37}
$$

where $C_0 = C_1 = 1$ and $C_{n-1} = C_{n-1}(\|Q_{\mathrm{in}}\|_{H^{n-1}})$ for $n-1 \geq 2$.

The boundary layer profiles $u^{\mathrm{bl}}_{0,h,b}$ and $u^{\mathrm{bl}}_{0,h,t}$ are solutions of the systems (3.18)–(3.19) and (3.23)–(3.24) respectively. We refer to the previous computations for the precise definitions of the higher-order terms.

We conclude this part by remarking that, according to the previous computations, we have that $(\rho^\varepsilon_{\mathrm{app}}, u^\varepsilon_{\mathrm{app}})$ solves the following system:

$$\begin{cases} \partial_t \rho^\varepsilon_{\mathrm{app}} + \nabla \cdot (\rho^\varepsilon_{\mathrm{app}} u^\varepsilon_{\mathrm{app}}) = \varepsilon R^{\mathrm{bl}} + \varepsilon^2 R^\varepsilon, \\ \rho^\varepsilon_{\mathrm{app}} \partial_t u^\varepsilon_{\mathrm{app}} + \rho^\varepsilon_{\mathrm{app}} u^\varepsilon_{\mathrm{app}} \cdot \nabla u^\varepsilon_{\mathrm{app}} + \dfrac{1}{\varepsilon} e_3 \times \rho^\varepsilon_{\mathrm{app}} u^\varepsilon_{\mathrm{app}} + \dfrac{1}{\varepsilon^2} \nabla P(\rho^\varepsilon_{\mathrm{app}}) \\ \quad = \dfrac{1}{\varepsilon^2} \rho^\varepsilon_{\mathrm{app}} \nabla G + \dfrac{x_3}{\varepsilon} \displaystyle\int_0^1 \partial_3 \bar\rho(s x_3)\, ds\, e_3 \times u^{\mathrm{bl}}_{0,h,b} \\ \qquad - \dfrac{1 - x_3}{\varepsilon} \displaystyle\int_0^1 \partial_3 \bar\rho(1 - s(1 - x_3))\, ds\, e_3 \times u^{\mathrm{bl}}_{0,h,t} \\ \qquad + \Delta_{\mu,\varepsilon} u^\varepsilon_{\mathrm{app}} + \lambda \nabla(\nabla \cdot u^\varepsilon_{\mathrm{app}}) + S^{\mathrm{bl}} + \varepsilon S^\varepsilon \end{cases} \tag{3.38}$$

in the slab Ω with no-slip boundary conditions (1.4). The remainder terms R^ε and S^ε are of the form

$$R^\varepsilon = R^\varepsilon\big(x_h, x_3, \tfrac{x_3}{\varepsilon}, \tfrac{1-x_3}{\varepsilon}, t\big) \quad \text{and} \quad S^\varepsilon = S^\varepsilon\big(x_h, x_3, \tfrac{x_3}{\varepsilon}, \tfrac{1-x_3}{\varepsilon}, t\big)$$

while the boundary layer terms are

$$\begin{aligned} R^{\mathrm{bl}}\big(x_h, x_3, \tfrac{x_3}{\varepsilon}, \tfrac{1-x_3}{\varepsilon}, t\big) &= \bar\rho \nabla_h \cdot (u^{\mathrm{bl}}_{1,h,b} + u^{\mathrm{bl}}_{1,h,t}) \\ &\quad + \nabla_h \rho_1 \cdot (u^{\mathrm{bl}}_{0,h,b} + u^{\mathrm{bl}}_{0,h,t}) + \partial_3 \bar\rho (u^{\mathrm{bl}}_{1,3,b} + u^{\mathrm{bl}}_{1,3,t}), \\ S^{\mathrm{bl}}\big(x_h, x_3, \tfrac{x_3}{\varepsilon}, \tfrac{1-x_3}{\varepsilon}, t\big) &= \bar\rho \partial_t (u^{\mathrm{bl}}_{0,b} + u^{\mathrm{bl}}_{0,t}) + \bar\rho u_{0,h} \cdot \nabla_h (u^{\mathrm{bl}}_{0,b} + u^{\mathrm{bl}}_{0,t}) \\ &\quad + \bar\rho (u^{\mathrm{bl}}_{0,b} + u^{\mathrm{bl}}_{0,t}) \cdot \nabla_h u_0 + \bar\rho (u^{\mathrm{bl}}_{0,b} + u^{\mathrm{bl}}_{0,t}) \cdot \nabla_h (u^{\mathrm{bl}}_{0,b} + u^{\mathrm{bl}}_{0,t}) \\ &\quad + \bar\rho (u_{1,3} + u^{\mathrm{bl}}_{1,3,b} + u^{\mathrm{bl}}_{1,3,t}) \partial_\eta (u^{\mathrm{bl}}_{0,b} + u^{\mathrm{bl}}_{0,t}) \\ &\quad - \mu \Delta_h (u^{\mathrm{bl}}_{0,t} + u^{\mathrm{bl}}_{0,b}) - \partial^2_\eta (u^{\mathrm{bl}}_{1,b} + u^{\mathrm{bl}}_{1,t}) \\ &\quad - \lambda \begin{pmatrix} 0 \\ \partial_\eta \nabla_h \cdot (u^{\mathrm{bl}}_{1,h,b} + u^{\mathrm{bl}}_{1,h,t}) \end{pmatrix} \\ &\quad + e_3 \times (\rho_1 (u^{\mathrm{bl}}_{0,b} + u^{\mathrm{bl}}_{0,t}) + \bar\rho (u^{\mathrm{bl}}_{1,b} + u^{\mathrm{bl}}_{1,t})). \end{aligned}$$

The remainders $\varepsilon^2 R^\varepsilon$ and $\varepsilon S^\varepsilon$ also contain the terms of order $O(e^{-1/\varepsilon})$ coming from the correctors $u^{\mathrm{bl}}_{0,h,1/\varepsilon}$ and $u^{\mathrm{bl}}_{1,1/\varepsilon}$, defined in (3.35). Notice that S^{bl} appears at order $O(1)$, but has fast, exponential, decay inside Ω: more precisely, we have $\|S^{\mathrm{bl}}\|_{L^p} \leq C \varepsilon^{\frac{1}{p}}$ for all $p \in [1, \infty]$.

The choice of the regularity of the initial datum Q_{in} guarantees enough regularity for the approximate solution $(\rho^\varepsilon_{\mathrm{app}}, u^\varepsilon_{\mathrm{app}})$ in order to derive the stability estimates later in Section 3.3. This is stated in the following lemma, which is a straightforward consequence of Lemma 3.3 above.

Lemma 3.4. *The approximated density $\rho^\varepsilon_{\mathrm{app}}$ and velocity field $u^\varepsilon_{\mathrm{app}}$ can be written as*

$$\rho^\varepsilon_{\mathrm{app}}(x_h, x_3, t) = \bar\rho(x_3) + Q(t, x_h)\frac{\bar\rho}{P'(\bar\rho)}(x_3) + Q(t, x_h) l(x_3),$$

$$u^\varepsilon_{\mathrm{app}}(x_h, x_3, \zeta, \eta, t) = \sum_{i=1}^{N} f_i(t, x_h) g_i(x_3) h_i(\zeta) w_i(\eta),$$

for some $N \geq 1$, with $\bar\rho, l, g_i \in C^1([0,1])$ and $h_i, w_i \in C^\infty(\mathbb{R}_+)$. In addition, for $Q_{\mathrm{in}} \in H^5(\mathbb{R}^2)$, we have

$$Q \in L^\infty(\mathbb{R}_+; H^5(\mathbb{R}^2)), \quad f_i \in L^\infty(\mathbb{R}_+; H^{k_i}(\mathbb{R}^2)) \ \textit{with}\ k_i \geq 1.$$

3.1.2. Large-scale quasi-geostrophic equation. We recover here the equation for u_0 from (3.36). For this we need the following standard lemma, which gives the Helmholtz decomposition for two-dimensional vector fields.

Lemma 3.5. *Let $p \in (1, \infty)$. Let a and b be two scalar fields in $L^p(\mathbb{R}^2)$.*

Then there exists a unique vector field F, belonging to the homogeneous Sobolev space $\dot W^{1,p}(\mathbb{R}^2; \mathbb{R}^2)$, which solves the system

$$\begin{cases} \nabla_h^\perp \cdot F = a, \\ \nabla_h \cdot F = b. \end{cases} \tag{3.39}$$

Moreover, the following formula holds:

$$F = -\nabla_h^\perp(-\Delta_h)^{-1} a - \nabla_h(-\Delta_h)^{-1} b.$$

The previous result being classical, we do not give the proof here: rather, we refer to [9, Sections 1.2 and 1.3] and [19, Section 10.6] for details. We just give some explanation about the uniqueness, which will be needed below. By linearity, let us suppose that F solves (3.39) with $a = b = 0$. In particular, $\nabla \times F = 0$, hence (see [9, Corollary 1.2.1]) $F = \nabla h$, for some $h \in L^p$. But from $\nabla \cdot F = 0$, we deduce that $-\Delta h = 0$, which admits the only solution $h = 0$ in L^p.

Now, let $\pi \in \dot H^1(\mathbb{R}^2)$ be the (unique, up to additive constants) solution to

$$-\Delta_h \pi = \langle\bar\rho\rangle \nabla_h \cdot (u_{0,h} \cdot \nabla_h u_{0,h}) = \langle\bar\rho\rangle \nabla_h u_{0,h} : \nabla_h u_{0,h}. \tag{3.40}$$

We then define $F(\cdot, t) \in L^2(\mathbb{R}^2; \mathbb{R}^2)$ for almost every $t \geq 0$ by the formula

$$F := \langle\bar\rho\rangle \partial_t u_{0,h} + \langle\bar\rho\rangle u_{0,h} \cdot \nabla_h u_{0,h} - \mu \Delta_h u_{0,h} + \frac{\sqrt{\bar\rho(0)} + \sqrt{\bar\rho(1)}}{\sqrt{2}} u_{0,h} + \nabla_h \pi.$$

Notice that, thanks to equations (3.40) and (3.36) and the divergence-free condition $\nabla_h \cdot u_{0,h} = 0$, we have

$$\nabla_h^\perp \cdot F = \left\langle \frac{\bar\rho}{P'(\bar\rho)} \right\rangle \partial_t Q \quad \text{and} \quad \nabla_h \cdot F = 0.$$

Therefore, the uniqueness part of Lemma 3.5 implies that

$$F = \nabla_h^{\perp}(\Delta_h)^{-1}\Big\langle\frac{\bar\rho}{P'(\bar\rho)}\Big\rangle\partial_t Q = \Big\langle\frac{\bar\rho}{P'(\bar\rho)}\Big\rangle\partial_t(\Delta_h)^{-1}u_{0,h},$$

where we have also used (3.8). Eventually, we find that $u_{0,h}$ solves the system

$$\begin{cases} \partial_t\Big(\langle\bar\rho\rangle - \Big\langle\frac{\bar\rho}{P'(\bar\rho)}\Big\rangle(\Delta_h)^{-1}\Big)u_{0,h} \\ \quad +\langle\bar\rho\rangle u_{0,h}\cdot\nabla_h u_{0,h} - \mu\Delta_h u_{0,h} + \dfrac{\sqrt{\bar\rho(0)}+\sqrt{\bar\rho(1)}}{\sqrt{2}}u_{0,h} + \nabla_h\pi = 0, \\ \nabla_h\cdot u_{0,h} = 0, \end{cases} \tag{3.41}$$

in $\mathbb{R}^2$. The second term appearing in the time derivative is a consequence of the combination of the effects due to density stratification and fast rotation. Notice that both (3.13) and (3.41) are averaged (in x_3) versions of (mom-ε^0).

3.2. Weak solutions and uniform a priori bounds

We recall here some basics about *finite energy weak solutions* to system (1.3). We refer e.g. to [32], [38] and [19] for details.

Definition 3.6. Let $\bar\rho > 0$ be the solution to the logistic equation (3.1), and let $(\rho_{\mathrm{in}}, u_{\mathrm{in}})$ verify

$$\int_\Omega\Big(\frac{1}{2}\rho_{\mathrm{in}}|u_{\mathrm{in}}|^2 + \frac{1}{\varepsilon^2}E(\rho_{\mathrm{in}},\bar\rho)\Big)\,dx < \infty.$$

A couple (ρ, u) is a *finite-energy weak solution* to system (1.3) on $[0,T]\times\Omega$, related to the initial datum $(\rho_{\mathrm{in}}, u_{\mathrm{in}})$, if the following conditions are satisfied:

- $\rho \geq 0$, with $\rho - \bar\rho \in L^\infty((0,T);(L^2+L^\gamma)(\Omega))$, with $\gamma > 1$ appearing in (1.10), and $u \in L^2((0,T);H^1(\Omega;\mathbb{R}^3))$;
- the mass equation is satisfied in the weak sense: namely, for any test function $\varphi \in \mathcal{C}_0^\infty([0,T)\times\Omega)$, one has

$$-\int_0^T\int_\Omega(\rho\partial_t\varphi + \rho u\cdot\nabla\varphi)\,dx\,dt = \int_\Omega \rho_{\mathrm{in}}\varphi(0)\,dx;$$

- $P(\rho) \in L^1_{\mathrm{loc}}((0,T)\times\Omega)$, and the momentum equation is verified in the weak sense: for any $\psi \in \mathcal{C}_0^\infty([0,T)\times\Omega;\mathbb{R}^3)$, one has

$$\int_0^T\int_\Omega\Big(-\rho u\cdot\partial_t\psi - \rho u\otimes u:\nabla\psi + \frac{1}{\varepsilon}e^3\times(\rho u)\cdot\psi - \frac{1}{\varepsilon^2}P(\rho)\nabla\cdot\psi \\ + \nabla_{\mu,\varepsilon}u:\nabla_{\mu,\varepsilon}\psi + \lambda\nabla\cdot u\nabla\cdot\psi - \frac{1}{\varepsilon^2}\rho\nabla G\cdot\psi\Big)\,dx\,dt = \int_\Omega\rho_{\mathrm{in}}u_{\mathrm{in}}\cdot\psi(0);$$

- the following energy inequality holds true for almost every $t \in (0, T)$:

$$\int_\Omega \Big(\frac{1}{2}\rho(t)|u(t)|^2 + \frac{1}{\varepsilon^2}E(\rho(t),\bar\rho)\Big)\,dx \\ + \int_0^t \int_\Omega (\mu|\nabla_h u|^2 + \varepsilon|\partial_3 u|^2 + \lambda|\nabla\cdot u|^2)\,dx\,d\tau \\ \le \int_\Omega \Big(\frac{1}{2}\rho_{\rm in}|u_{\rm in}|^2 + \frac{1}{\varepsilon^2}E(\rho_{\rm in},\bar\rho)\Big)\,dx, \tag{3.42}$$

where we have defined the *relative energy functional*

$$E(\rho,\bar\rho) := H(\rho) - H(\bar\rho) - H'(\bar\rho)(\rho-\bar\rho). \tag{3.43}$$

The solution is said to be *global* if the previous conditions hold true for all $T > 0$.

Consider now a family of global-in-time finite-energy weak solutions $(\rho^\varepsilon, u^\varepsilon)_\varepsilon$ to system (1.3). Recall that the existence of such a family is only assumed here, but open in general, because of anisotropy of the viscous stress tensor. We collect here some uniform bounds verified by that family. These bounds will be important in the next subsection, when proving stability estimates.

By assumption, for any $\varepsilon \in (0, 1]$ the energy inequality

$$\int_\Omega \Big(\rho^\varepsilon(t)|u^\varepsilon(t)|^2 + \frac{1}{\varepsilon^2}E(\rho^\varepsilon(t),\bar\rho)\Big) + \int_0^t \int_\Omega (\mu|\nabla_h u^\varepsilon|^2 + \varepsilon|\partial_3 u^\varepsilon|^2 + \lambda|\nabla\cdot u^\varepsilon|^2) \\ \le \int_\Omega \Big(\rho^\varepsilon_{\rm in}|u^\varepsilon_{\rm in}|^2 + \frac{2}{\varepsilon^2}E(\rho^\varepsilon_{\rm in},\bar\rho)\Big) \tag{3.44}$$

holds for almost every $t > 0$. According to [22, inequality (4.15)], we have the following control, which holds for any positive scalar functions $\rho(x,t)$ and $r(x,t)$, with $0 < r_- \le r(x,t) \le r_+$, for some real numbers r_-, r_+: there exist constants $c_1, c_2 > 0$ such that, for almost all $(x,t) \in \Omega \times \mathbb{R}_+$, one has

$$c_1(|\rho(x,t) - r(x,t)|^2 \mathbf{1}_{\{|\rho-r|(\cdot,t)<1\}} + |\rho(x,t) - r(x,t)|^\gamma \mathbf{1}_{\{|\rho-r|(\cdot,t)\ge1\}}) \\ \le E(\rho(x,t), r(x,t)) \\ \le c_2(|\rho(x,t) - r(x,t)|^2 \mathbf{1}_{\{|\rho-r|(\cdot,t)<1\}} + |\rho(x,t) - r(x,t)|^\gamma \mathbf{1}_{\{|\rho-r|(\cdot,t)\ge1\}}), \tag{3.45}$$

where the notation $\{|\rho - r|(\cdot,t) < 1\}$ stands for the set of $x \in \Omega$ such that $|\rho(x,t) - r(x,t)| < 1$ (and analogously for the $\ge$ symbol) and $\mathbf{1}_A$ denotes the characteristic function of a set $A \subseteq \Omega$. Notice that the same inequalities hold if we replace 1 by any constant $M > 0$, up to changing the value of the constants c_1 and c_2.

Now, following [19, Chapters 4 and 5], let us introduce the *essential set* and the *residual set* as follows: for almost every $t > 0$, we set

$$\Omega_{\rm ess}(t) := \big\{x \in \Omega \mid |\rho^\varepsilon(x,t) - \bar\rho(x_3)| < \sigma\big\} \quad \text{and} \quad \Omega_{\rm res}(t) := \Omega \setminus \Omega_{\rm ess}(t), \tag{3.46}$$

for some σ (to be fixed later) such that

$$0 < \sigma < \inf_{(0,1)} \bar\rho.$$

Accordingly, given any function h, we define its *essential part* and *residual part* as

$$[h]_{\mathrm{ess}} := h\mathbf{1}_{\Omega_{\mathrm{ess}}} \quad \text{and} \quad [h]_{\mathrm{res}} := h\mathbf{1}_{\Omega_{\mathrm{res}}} = h - [h]_{\mathrm{ess}}.$$

Keep in mind that such a decomposition depends on ρ^ε.

After this preparation, let us establish uniform bounds for $(\rho^\varepsilon, u^\varepsilon)_\varepsilon$. First of all, in view of the assumptions we will fix on the initial data $(\rho^\varepsilon_{\mathrm{in}}, u^\varepsilon_{\mathrm{in}})_\varepsilon$ in the next subsection, we can assume that the right-hand side of (3.44) is uniformly bounded for $\varepsilon \in (0, 1]$. Then, using (3.45), we deduce the existence of a constant $C > 0$ such that, for all $T > 0$ fixed and all $0 < \varepsilon \le 1$, one has

$$\|\sqrt{\rho^\varepsilon}\, u^\varepsilon\|_{L^\infty_T(L^2)}, \le C \tag{3.47}$$

$$\frac{1}{\varepsilon}\|[\rho^\varepsilon - \bar{\rho}]_{\mathrm{ess}}\|_{L^\infty_T(L^2)} \le C, \tag{3.48}$$

$$\sup_{t\in[0,T]} \mathcal{L}(\Omega_{\mathrm{res}}(t)) + \|[\rho^\varepsilon]_{\mathrm{res}}\|^\gamma_{L^\infty_T(L^\gamma)} \le C\varepsilon^2, \tag{3.49}$$

where $\mathcal{L}(A)$ denotes the Lebesgue measure of a set $A \subseteq \Omega$. We refer to [17, Section 2] and [20, Section 4] for details.

Next, let us consider the viscosity terms: recalling that $\mu > 0$ and $\lambda > 0$, from (3.44) we immediately get

$$\|\nabla_h u^\varepsilon\|_{L^2_T(L^2)} + \|\nabla\cdot u^\varepsilon\|_{L^2_T(L^2)} \le C, \tag{3.50}$$

$$\sqrt{\varepsilon}\,\|\partial_3 u^\varepsilon_h\|_{L^2_T(L^2)} \le C, \tag{3.51}$$

for some universal constant $C > 0$ independent of ε and of the fixed time $T > 0$. In addition, owing to the identity

$$\partial_3 u^\varepsilon_3 = \nabla\cdot u^\varepsilon - \nabla_h \cdot u^\varepsilon_h,$$

we also deduce that

$$\|\partial_3 u^\varepsilon_3\|_{L^2_T(L^2)} \le C. \tag{3.52}$$

Finally, arguing exactly as in [17, Section 2], we deduce that there exists a constant $C > 0$ such that, for all $\varepsilon > 0$ and all $T > 0$, one has

$$\|u^\varepsilon\|_{L^2_T(L^2)} \le C. \tag{3.53}$$

3.3. Stability estimates

This section is devoted to estimating the error between weak solutions to (1.3) and their smooth approximation built in Section 3.1. We consider *well-prepared* initial data. Specifically, the initial density $(\rho^\varepsilon_{\mathrm{in}})_\varepsilon$ and velocity fields $(u^\varepsilon_{\mathrm{in}})_\varepsilon$ satisfy the following requirements:

- for all $\varepsilon \in (0, 1]$, one has

$$\rho^\varepsilon_{\mathrm{in}} = \bar{\rho} + \varepsilon r^\varepsilon_{\mathrm{in}}, \quad \text{with } (r^\varepsilon_{\mathrm{in}})_\varepsilon \subseteq (L^2 \cap L^\infty)(\Omega); \tag{3.54}$$

- we have $(u_{\text{in}}^\varepsilon)_\varepsilon \subseteq L^2(\Omega)$;
- there exists $Q_{\text{in}} \in H^5(\mathbb{R}^2)$ such that, after defining

$$r_{\text{in}} := \frac{\bar{\rho}}{P'(\bar{\rho})} Q_{\text{in}} \quad \text{and} \quad u_{\text{in}} := (-\partial_2 Q_{\text{in}}, \partial_1 Q_{\text{in}}, 0), \tag{3.55}$$

we have the strong convergence properties

$$r_{\text{in}}^\varepsilon \to r_{\text{in}} \quad \text{and} \quad u_{\text{in}}^\varepsilon \to u_{\text{in}} \quad \text{in } L^2(\Omega). \tag{3.56}$$

Remark 3.7. Condition (3.55) implies in particular that

$$\bar{\rho} \begin{pmatrix} u_{\text{in},h}^\perp \\ 0 \end{pmatrix} + \begin{pmatrix} P'(\bar{\rho}) \nabla_h r_{\text{in}} \\ \partial_3 (P'(\bar{\rho}) r_{\text{in}}) \end{pmatrix} = r_{\text{in}} \nabla G. \tag{3.57}$$

Remark 3.8. Our analysis does not hold for *ill-prepared* initial data. In this case, in fact, a careful analysis of the fast time oscillations of the solutions must be performed. This would correspond to adding a fast variable $\frac{t}{\varepsilon}$ with respect to time in the asymptotic expansion of the approximated profiles.

In the incompressible case, a key point of the analysis for ill-prepared data (see e.g. [11]) is to show that dispersive effects linked to Strichartz estimates are not destroyed by the presence of the boundary layer. In the compressible case, dispersive estimates are less precise (see e.g. [17]), which makes the problem harder.

From the uniform bounds in Section 3.2, it is classical to derive that, up to extraction of a suitable subsequence, weak solutions $(\rho^\varepsilon, u^\varepsilon)_\varepsilon$ converge to a limit state $(\bar{\rho}, \bar{u})$ which belongs to the kernel of the singular perturbation operator. We refer to e.g. [16, 17, 20] for details. The goal of the present subsection is to make this convergence quantitative, to show the general structure of the solutions and to take into account the correctors due to Ekman boundary layers. We aim to prove the following result. Recall that the relative entropy E is defined in (3.43).

Theorem 2. *For $\gamma \geq \frac{3}{2}$, suppose that there exists a finite-energy weak solution $(\rho^\varepsilon, u^\varepsilon)_\varepsilon$ to (1.3) with well-prepared initial data $(\rho_{\text{in}}^\varepsilon, u_{\text{in}}^\varepsilon)_\varepsilon \in L^\infty \times L^2$ verifying hypotheses (3.54), (3.55) and (3.56). Let $(\rho_{\text{app}}^\varepsilon, u_{\text{app}}^\varepsilon)_\varepsilon$ be defined as in (3.34), and define $\delta u^\varepsilon = u^\varepsilon - u_{\text{app}}^\varepsilon$. Then there exist functions $C_1(t), C_2(t) \in L^1([0,T))$ for all $T > 0$, and constants $C > 0$ and $\varepsilon_0 \in (0,1)$ such that, for all $\varepsilon \in (0, \varepsilon_0)$, the following estimate holds, for almost every $t > 0$:*

$$\begin{aligned} &\int_\Omega \rho^\varepsilon(t) |\delta u^\varepsilon(t)|^2 \, dx + \frac{1}{\varepsilon^2} \int_\Omega E(\rho^\varepsilon(t), \rho_{\text{app}}^\varepsilon(t)) \, dx \\ &\quad + \int_0^t \int_\Omega (\mu |\nabla_h \delta u^\varepsilon|^2 + \varepsilon |\partial_3 \delta u^\varepsilon|^2 + \lambda |\nabla \cdot \delta u^\varepsilon|^2) \, dx \\ &\leq C e^{2 \int_0^t C_1(s) \, ds} \left(\int_\Omega \rho_{\text{in}}^\varepsilon |\delta u_{\text{in}}^\varepsilon|^2 \, dx + \frac{1}{\varepsilon^2} \int_\Omega E(\rho_{\text{in}}^\varepsilon, \rho_{\text{in,app}}^\varepsilon) \, dx + \varepsilon \int_0^t C_2(\tau) \, d\tau \right). \end{aligned}$$

Remark 3.9. The lower bound for the exponent γ comes from the control of the source term in the relative entropy inequality (3.64): in particular, in (3.83) we need $\gamma \geq \frac{3}{2}$ to apply Hölder's inequality and get estimate (3.84).

In order to prove the previous result, we resort to the technique of the *relative entropy/relative energy* inequality; see e.g. [18, 20, 22, 23, 27]. The relative entropy estimate of these works is directly applicable in our framework, but it is not immediately clear how to take advantage of the small remainders in (3.38). Instead, we directly derive the entropy inequality on the system for $(\delta\rho^\varepsilon, \delta u^\varepsilon)$ and take into account from the beginning that $(\rho^\varepsilon_{\mathrm{app}}, u^\varepsilon_{\mathrm{app}})$ is almost a solution to (1.3). On the contrary, the relative entropy inequality of e.g. [20] holds for a much wider class of smooth functions.

3.3.1. The relative entropy inequality. We set

$$\delta\rho^\varepsilon := \rho^\varepsilon - \rho^\varepsilon_{\mathrm{app}} \quad \text{and} \quad \delta u^\varepsilon := u^\varepsilon - u^\varepsilon_{\mathrm{app}}.$$

From systems (1.3) and (3.38), it is easy to find an equation for $\delta\rho^\varepsilon$ and δu^ε: after setting $\delta P^\varepsilon := P(\rho^\varepsilon) - P(\rho^\varepsilon_{\mathrm{app}})$, we get

$$\partial_t \delta\rho^\varepsilon + \nabla \cdot (u^\varepsilon_{\mathrm{app}} \delta\rho^\varepsilon) = -\nabla \cdot (\rho^\varepsilon \delta u^\varepsilon) - \varepsilon R^{\mathrm{bl}} - \varepsilon^2 R^\varepsilon, \tag{3.58}$$

$$\begin{aligned}
&\rho^\varepsilon \partial_t \delta u^\varepsilon + \rho^\varepsilon u^\varepsilon \cdot \nabla \delta u^\varepsilon + \frac{1}{\varepsilon} e_3 \times \rho^\varepsilon \delta u^\varepsilon + \frac{1}{\varepsilon^2} \nabla \delta P^\varepsilon - \Delta_{\mu,\varepsilon} \delta u^\varepsilon - \lambda \nabla\nabla \cdot \delta u^\varepsilon \\
&\quad = \frac{1}{\varepsilon^2} \delta\rho^\varepsilon \nabla G - \delta\rho^\varepsilon \partial_t u^\varepsilon_{\mathrm{app}} + (\rho^\varepsilon_{\mathrm{app}} u^\varepsilon_{\mathrm{app}} - \rho^\varepsilon u^\varepsilon) \cdot \nabla u^\varepsilon_{\mathrm{app}} \\
&\qquad - \frac{1}{\varepsilon} e_3 \times \delta\rho^\varepsilon u^\varepsilon_{\mathrm{app}} - S^{\mathrm{bl}} - \varepsilon S^\varepsilon - \frac{x_3}{\varepsilon} \int_0^1 \partial_3 \bar\rho(s x_3)\, ds\, e_3 \times u^{\mathrm{bl}}_{0,h,b} \\
&\qquad + \frac{1 - x_3}{\varepsilon} \int_0^1 \partial_3 \bar\rho(1 - s(1 - x_3))\, ds\, e_3 \times u^{\mathrm{bl}}_{0,h,t}.
\end{aligned} \tag{3.59}$$

From the point of view of energy estimates, the main term to work on is the difference of the pressure terms. Testing it against δu^ε yields

$$\begin{aligned}
\int_\Omega \nabla \delta P^\varepsilon \cdot \delta u^\varepsilon \, dx &= \int_\Omega \nabla P(\rho^\varepsilon) \cdot u^\varepsilon \, dx - \int_\Omega \nabla P(\rho^\varepsilon_{\mathrm{app}}) \cdot u^\varepsilon_{\mathrm{app}} \, dx \\
&\quad + \int_\Omega \nabla \cdot u^\varepsilon_{\mathrm{app}} \delta P^\varepsilon \, dx - \int_\Omega \nabla P(\rho^\varepsilon_{\mathrm{app}}) \cdot \delta u^\varepsilon \, dx.
\end{aligned} \tag{3.60}$$

By standard computations, using the mass equation in (1.3), we get

$$\int_\Omega \nabla P(\rho^\varepsilon) \cdot u^\varepsilon \, dx = \int_\Omega \nabla (H'(\rho^\varepsilon)) \cdot \rho^\varepsilon u^\varepsilon \, dx = \frac{d}{dt} \int_\Omega H(\rho^\varepsilon) \, dx.$$

Similarly, from the first equation in (3.38) we gather

$$\int_\Omega \nabla P(\rho^\varepsilon_{\mathrm{app}}) \cdot u^\varepsilon_{\mathrm{app}} \, dx = \frac{d}{dt} \int_\Omega H(\rho^\varepsilon_{\mathrm{app}}) \, dx - \varepsilon \int_\Omega H'(\rho^\varepsilon_{\mathrm{app}})(R^{\mathrm{bl}} + \varepsilon R^\varepsilon) \, dx.$$

In identity (3.60), we now add and subtract the term $\frac{d}{dt}\int H'(\rho^\varepsilon_{\text{app}})\delta\rho^\varepsilon\,dx$, in order to make the relative entropy $E(\rho^\varepsilon(t), \rho^\varepsilon_{\text{app}}(t))$ appear. Then, from (3.60) and the previous computations, we infer

$$\begin{aligned}\int_\Omega \nabla\delta P^\varepsilon\cdot\delta u^\varepsilon\,dx &= \frac{d}{dt}\int_\Omega E(\rho^\varepsilon,\rho^\varepsilon_{\text{app}})\,dx + \int_\Omega \nabla\cdot u^\varepsilon_{\text{app}}\delta P^\varepsilon\,dx - \int_\Omega \nabla P(\rho^\varepsilon_{\text{app}})\cdot\delta u^\varepsilon\,dx\\ &\quad + \frac{d}{dt}\int_\Omega H'(\rho^\varepsilon_{\text{app}})\delta\rho^\varepsilon\,dx + \varepsilon\int_\Omega H'(\rho^\varepsilon_{\text{app}})(R^{\text{bl}}+\varepsilon R^\varepsilon)\,dx.\end{aligned}$$

Using the mass equations in (1.3) and (3.38) again, we get

$$\begin{aligned}\frac{d}{dt}\int_\Omega H'(\rho^\varepsilon_{\text{app}})\delta\rho^\varepsilon\,dx &= \int_\Omega \partial_t H'(\rho^\varepsilon_{\text{app}})\delta\rho^\varepsilon\,dx + \int_\Omega H'(\rho^\varepsilon_{\text{app}})\partial_t\delta\rho^\varepsilon\,dx\\ &= \int_\Omega \partial_t H'(\rho^\varepsilon_{\text{app}})\delta\rho^\varepsilon\,dx + \int_\Omega \nabla H'(\rho^\varepsilon_{\text{app}})\cdot(\rho^\varepsilon u^\varepsilon - \rho^\varepsilon_{\text{app}}u^\varepsilon_{\text{app}})\,dx\\ &\quad - \varepsilon\int_\Omega H'(\rho^\varepsilon_{\text{app}})(R^{\text{bl}}+\varepsilon R^\varepsilon)\,dx.\end{aligned}$$

This relation yields

$$\int_\Omega \nabla\delta P^\varepsilon\cdot\delta u^\varepsilon\,dx = \frac{d}{dt}\int_\Omega E(\rho^\varepsilon,\rho^\varepsilon_{\text{app}})\,dx - \int_\Omega \nabla P(\rho^\varepsilon_{\text{app}})\cdot\delta u^\varepsilon\,dx + I, \tag{3.61}$$

where we have defined

$$I := \int_\Omega \nabla\cdot u^\varepsilon_{\text{app}}\delta P^\varepsilon\,dx + \int_\Omega \partial_t H'(\rho^\varepsilon_{\text{app}})\delta\rho^\varepsilon\,dx + \int_\Omega \nabla H'(\rho^\varepsilon_{\text{app}})\cdot(\rho^\varepsilon u^\varepsilon - \rho^\varepsilon_{\text{app}}u^\varepsilon_{\text{app}})\,dx.$$

Let us work on this term for a while. We use the Taylor expansion

$$\begin{aligned}P(\rho^\varepsilon,\rho^\varepsilon_{\text{app}}) &:= P(\rho^\varepsilon) - P(\rho^\varepsilon_{\text{app}}) - P'(\rho^\varepsilon_{\text{app}})\delta\rho^\varepsilon\\ &= \frac{1}{2}(\delta\rho^\varepsilon)^2\int_0^1 (1-s)P''(\rho^\varepsilon_{\text{app}}+s\delta\rho^\varepsilon)\,ds,\end{aligned} \tag{3.62}$$

and the fact that $H''(z) = \frac{P'(z)}{z}$ according to (1.11), to obtain the next series of equalities:

$$\begin{aligned}I &= \int_\Omega \nabla\cdot u^\varepsilon_{\text{app}}P'(\rho^\varepsilon_{\text{app}})\delta\rho^\varepsilon\,dx + \int_\Omega \nabla\cdot u^\varepsilon_{\text{app}}P(\rho^\varepsilon,\rho^\varepsilon_{\text{app}})\,dx\\ &\quad + \int_\Omega H''(\rho^\varepsilon_{\text{app}})\partial_t\rho^\varepsilon_{\text{app}}\delta\rho^\varepsilon\,dx + \int_\Omega H''(\rho^\varepsilon_{\text{app}})\nabla\rho^\varepsilon_{\text{app}}\cdot(\rho^\varepsilon u^\varepsilon - \rho^\varepsilon_{\text{app}}u^\varepsilon_{\text{app}})\,dx\\ &= \int_\Omega \nabla\cdot u^\varepsilon_{\text{app}}P'(\rho^\varepsilon_{\text{app}})\delta\rho^\varepsilon\,dx - \int_\Omega H''(\rho^\varepsilon_{\text{app}})\rho^\varepsilon_{\text{app}}\nabla\cdot u^\varepsilon_{\text{app}}\delta\rho^\varepsilon\,dx\\ &\quad + \int_\Omega H''(\rho^\varepsilon_{\text{app}})(\partial_t\rho^\varepsilon_{\text{app}} + \nabla\cdot(\rho^\varepsilon_{\text{app}}u^\varepsilon_{\text{app}}))\delta\rho^\varepsilon\,dx\\ &\quad + \int_\Omega H''(\rho^\varepsilon_{\text{app}})\nabla\rho^\varepsilon_{\text{app}}\cdot\delta u^\varepsilon\rho^\varepsilon\,dx + \int_\Omega \nabla\cdot u^\varepsilon_{\text{app}}P(\rho^\varepsilon,\rho^\varepsilon_{\text{app}})\,dx\\ &= \int_\Omega H''(\rho^\varepsilon_{\text{app}})\nabla\rho^\varepsilon_{\text{app}}\cdot\delta u^\varepsilon\rho^\varepsilon\,dx + \varepsilon\int_\Omega H''(\rho^\varepsilon_{\text{app}})\delta\rho^\varepsilon(R^{\text{bl}}+\varepsilon R^\varepsilon)\,dx\\ &\quad + \int_\Omega \nabla\cdot u^\varepsilon_{\text{app}}P(\rho^\varepsilon,\rho^\varepsilon_{\text{app}})\,dx.\end{aligned}$$

The last two terms in the above identity are small (in a sense to be made precise later). So let us focus on the first term on the right-hand side: we have

$$\int_\Omega H''(\rho^\varepsilon_{\rm app})\nabla\rho^\varepsilon_{\rm app}\cdot\delta u^\varepsilon\rho^\varepsilon\,dx-\int_\Omega\nabla P(\rho^\varepsilon_{\rm app})\cdot\delta u^\varepsilon\,dx=\int_\Omega H''(\rho^\varepsilon_{\rm app})\nabla\rho^\varepsilon_{\rm app}\cdot\delta u^\varepsilon\delta\rho^\varepsilon\,dx.$$

Inserting this expression into the last equality for I, from (3.61) we finally find

$$\begin{aligned}\int_\Omega\nabla\delta P^\varepsilon\cdot\delta u^\varepsilon\,dx&=\frac{d}{dt}\int_\Omega E(\rho^\varepsilon,\rho^\varepsilon_{\rm app})\,dx+\int_\Omega H''(\rho^\varepsilon_{\rm app})\nabla\rho^\varepsilon_{\rm app}\cdot\delta u^\varepsilon\delta\rho^\varepsilon\,dx\\&\quad+\varepsilon\int_\Omega H''(\rho^\varepsilon_{\rm app})\delta\rho^\varepsilon(R^{\rm bl}+\varepsilon R^\varepsilon)\,dx+\int_\Omega\nabla\cdot u^\varepsilon_{\rm app}P(\rho^\varepsilon,\rho^\varepsilon_{\rm app})\,dx.\end{aligned}\tag{3.63}$$

At this point, we can perform energy estimates directly on equations (3.58)–(3.59). Using (3.63) above, we obtain

$$\begin{aligned}&\frac{d}{dt}\int_\Omega\Big(\frac12\rho^\varepsilon|\delta u^\varepsilon|^2+\frac{1}{\varepsilon^2}E(\rho^\varepsilon,\rho^\varepsilon_{\rm app})\Big)\,dx\\&\qquad+\mu\int_\Omega|\nabla_h\delta u^\varepsilon|^2\,dx+\varepsilon\int_\Omega|\partial_3\delta u^\varepsilon|^2\,dx+\lambda\int_\Omega|\nabla\cdot\delta u^\varepsilon|^2\,dx\\&\quad\le\frac{1}{\varepsilon^2}\int_\Omega\delta\rho^\varepsilon\nabla G\cdot\delta u^\varepsilon\,dx-\frac{1}{\varepsilon^2}\int_\Omega H''(\rho^\varepsilon_{\rm app})\nabla\rho^\varepsilon_{\rm app}\cdot\delta u^\varepsilon\delta\rho^\varepsilon\,dx\\&\qquad-\frac1\varepsilon\int_\Omega H''(\rho^\varepsilon_{\rm app})\delta\rho^\varepsilon(R^{\rm bl}+\varepsilon R^\varepsilon)\,dx-\frac{1}{\varepsilon^2}\int_\Omega\nabla\cdot u^\varepsilon_{\rm app}P(\rho^\varepsilon,\rho^\varepsilon_{\rm app})\,dx\\&\qquad-\frac1\varepsilon\int_\Omega e_3\times\delta\rho^\varepsilon u^\varepsilon_{\rm app}\cdot\delta u^\varepsilon\,dx-\int_\Omega\delta\rho^\varepsilon\partial_t u^\varepsilon_{\rm app}\cdot\delta u^\varepsilon\,dx\\&\qquad+\int_\Omega(\rho^\varepsilon_{\rm app}u^\varepsilon_{\rm app}-\rho^\varepsilon u^\varepsilon)\cdot\nabla u^\varepsilon_{\rm app}\cdot\delta u^\varepsilon\,dx-\int_\Omega S^{\rm bl}\cdot\delta u^\varepsilon\,dx-\varepsilon\int_\Omega S^\varepsilon\cdot\delta u^\varepsilon\,dx\\&\qquad-\frac1\varepsilon\int_\Omega x_3\int_0^1\partial_3\bar\rho(sx_3)\,ds\,(u^{\rm bl}_{0,h,b})^\perp\cdot\delta u^\varepsilon_h\,dx\\&\qquad+\frac1\varepsilon\int_\Omega(1-x_3)\int_0^1\partial_3\bar\rho(1-s(1-x_3))\,ds\,(u^{\rm bl}_{0,h,t})^\perp\cdot\delta u^\varepsilon_h\,dx=\sum_{j=1}^{11}I_j.\end{aligned}\tag{3.64}$$

Remark 3.10. In order to rigorously justify the relative entropy inequality (3.64), where the equality holds if the solutions are regular enough, one may either proceed as in [18], or use a regularization argument (see for instance [22] and [27]).

Our next goal is to bound each term appearing in the sum $\sum_{j=1}^{11}I_j$ on the right-hand side of (3.64). Before doing that, let us remark that, since $\rho_1,\rho_2\in L^\infty(\Omega\times\mathbb{R}_+)$, up to restricting our attention to all $\varepsilon\le\varepsilon_0$, with ε_0 depending on $\|\rho_1\|_{L^\infty_{t,x}}$ and $\|\rho_2\|_{L^\infty_{t,x}}$, we can assume that $-\frac{\sigma}{2}\le\varepsilon\rho_1+\varepsilon^2\rho_2\le\frac{\sigma}{2}$ with $\sigma>0$ as in (3.46). Consequently, we can suppose that

$$0<\rho^-_{\rm app}\le\rho^\varepsilon_{\rm app}(x,t)\le\rho^+_{\rm app}\quad\text{for all }\varepsilon>0,$$

with $\rho^-_{\rm app}=\inf_{(0,1)}\bar\rho-\sigma$ and $\rho^+_{\rm app}=\sup_{(0,1)}\bar\rho+\sigma$. Then, in view of (3.45), we have the following control:

$$E(\rho^\varepsilon,\rho^\varepsilon_{\rm app})(x,t)\ge c(|\delta\rho^\varepsilon(x,t)|^2\mathbf{1}_{\{|\delta\rho^\varepsilon|(\cdot,t)<1\}}+|\delta\rho^\varepsilon(x,t)|^\gamma\mathbf{1}_{\{|\delta\rho^\varepsilon|(\cdot,t)\ge1\}}).\tag{3.65}$$

Resorting to definitions (3.46), from (3.65) we derive the following lower bound.

Lemma 3.11. *There exist $\sigma > 0$ small enough (depending on $\inf_{(0,1)} \bar{\rho}$) and a positive constant $c > 0$, independent of $\varepsilon \in]0, \varepsilon_0]$, such that, for almost all $(x,t) \in \Omega \times \mathbb{R}_+$, the following bound holds:*

$$E(\rho^\varepsilon(x,t), \rho^\varepsilon_{\mathrm{app}}(x,t)) \geq c([\delta\rho^\varepsilon]^2_{\mathrm{ess}}(x,t) + \mathbf{1}_{\Omega_{\mathrm{res}}(t)}(x)). \tag{3.66}$$

Proof. We divide the proof of the inequality into two steps. First we show that

$$E(\rho^\varepsilon(x,t), \rho^\varepsilon_{\mathrm{app}}(x,t)) \geq c|\delta\rho^\varepsilon(x,t)|^2 \mathbf{1}_{\{|\delta\rho^\varepsilon(x,t)|<1\}}$$

implies the lower bound

$$E(\rho^\varepsilon(x,t), \rho^\varepsilon_{\mathrm{app}}(x,t)) \geq c[\delta\rho^\varepsilon]^2_{\mathrm{ess}}(x,t). \tag{3.67}$$

For this, we just need to show that $\Omega_{\mathrm{ess}}(t) \subseteq \{|\delta\rho^\varepsilon(x,t)| < 1\}$. Let $x \in \Omega_{\mathrm{ess}}(t)$; then

$$-\frac{3}{2}\sigma \leq -\sigma - \varepsilon\rho_1(x,t) - \varepsilon^2\rho_2(x,t) < \delta\rho^\varepsilon(x,t) < -\varepsilon\rho_1(x,t) - \varepsilon^2\rho_2(x,t) + \sigma \leq \frac{3}{2}\sigma,$$

where we have used that $-\frac{\sigma}{2} \leq \varepsilon\rho_1(x,t) + \varepsilon^2\rho_2(x,t) \leq \frac{\sigma}{2}$. By choosing σ such that $\sigma < \min(\frac{2}{3}, \inf_{(0,1)} \bar{\rho})$, we deduce that $|\delta\rho^\varepsilon(x,t)| < 1$. Thus, (3.67) is proved.

Afterwards, we prove that, for $x \in \Omega_{\mathrm{res}}(t)$, one has

$$E(\rho^\varepsilon(x,t), \rho^\varepsilon_{\mathrm{app}}(x,t)) \geq c, \tag{3.68}$$

where c is a positive constant independent of ε, t and x. By the definition of $\Omega_{\mathrm{res}}(t)$, either $\rho^\varepsilon(x,t) \leq \bar{\rho}(x_3) - \sigma$ or $\rho^\varepsilon(x,t) \geq \bar{\rho}(x_3) + \sigma$. Hence, since $E(\cdot, \rho^\varepsilon_{\mathrm{app}}(x,t))$ is strictly decreasing before $\rho^\varepsilon_{\mathrm{app}}(x,t)$ and strictly increasing after $\rho^\varepsilon_{\mathrm{app}}(x,t)$, we get

$$E(\rho^\varepsilon(x,t), \rho^\varepsilon_{\mathrm{app}}(x,t)) \geq E(\bar{\rho}(x_3) - \sigma, \rho^\varepsilon_{\mathrm{app}}(x,t))$$

if $\rho^\varepsilon(x,t) \leq \bar{\rho}(x_3) - \sigma$, and

$$E(\rho^\varepsilon(x,t), \rho^\varepsilon_{\mathrm{app}}(x,t)) \geq E(\bar{\rho}(x_3) + \sigma, \rho^\varepsilon_{\mathrm{app}}(x,t))$$

if $\rho^\varepsilon(x,t) \geq \bar{\rho}(x_3) + \sigma$. Now, by Taylor's formula, up to taking a smaller σ (which amounts to choosing a smaller ε_0), we have

$$\begin{aligned}
E(\bar{\rho}(x_3) - \sigma, \rho^\varepsilon_{\mathrm{app}}(x,t)) &\geq \frac{H''(\rho^\varepsilon_{\mathrm{app}}(x,t))}{4}(-\sigma - \varepsilon\rho_1(x,t) - \varepsilon^2\rho_2(x,t))^2 \\
&\geq \frac{H''(\rho^\varepsilon_{\mathrm{app}}(x,t))\sigma^2}{16}, \\
E(\bar{\rho}(x_3) + \sigma, \rho^\varepsilon_{\mathrm{app}}(x,t)) &\geq \frac{H''(\rho^\varepsilon_{\mathrm{app}}(x,t))}{4}(\sigma - \varepsilon\rho_1(x,t) - \varepsilon^2\rho_2(x,t))^2 \\
&\geq \frac{H''(\rho^\varepsilon_{\mathrm{app}}(x,t))\sigma^2}{16}.
\end{aligned}$$

Then, using the uniform boundedness in time and space of $\rho^\varepsilon_{\mathrm{app}}$ and hypothesis (1.10), we get (3.68). The lemma is proved. ∎

Notice that $[|\delta\rho^\varepsilon|]_{\rm ess}$ is uniformly bounded. Next we claim that there exists a constant $C > 0$ such that, for all $T > 0$ fixed, one has

$$\|[\delta\rho^\varepsilon]_{\rm res}\|_{L^\infty_T(L^p)} \le C\varepsilon^{2/p} \quad \forall p \in [1,\gamma]. \tag{3.69}$$

Indeed, by Hölder's inequality, the L^∞ control on $\rho^\varepsilon_{\rm app}$ and (3.49), we deduce

$$\begin{aligned}\int_\Omega |[\delta\rho^\varepsilon]_{\rm res}| &\le \int_\Omega [\rho^\varepsilon]_{\rm res} + \int_\Omega [\rho^\varepsilon_{\rm app}]_{\rm res}\\ &\le \Big(\int_\Omega (\rho^\varepsilon)^\gamma \mathbf{1}_{\Omega_{\rm res}}\Big)^{1/\gamma} (\mathcal{L}(\Omega_{\rm res}))^{1/\gamma'} + C\mathcal{L}(\Omega_{\rm res}) \le C\varepsilon^2,\end{aligned}$$

which yields (3.69) for $p = 1$. As for the L^γ norm, we write

$$\Omega_{\rm res}(t) = \big\{0 < \rho^\varepsilon(x,t) \le \bar\rho(x_3) - \sigma\big\} \cup \big\{\rho^\varepsilon(x,t) \ge \bar\rho(x_3) + \sigma\big\}. \tag{3.70}$$

For the first set, we just apply (3.49) again, since ρ^ε is bounded therein. For the second set, we use the fact that, for $a \ge \delta$ and $b \ge 0$, with $b \le b^*$, one has $|a-b|^\gamma \le (a+b)^\gamma \le C_{\delta,b^*}(a^\gamma + 1)$. The case $1 < p < \gamma$ follows from interpolation.

3.3.2. Anisotropic Sobolev embedding. We introduce an anisotropic version of the standard Sobolev embedding $\dot H^1 \hookrightarrow L^6$. This estimate enables us to handle the anisotropy of the viscosity in the stability estimates below; see in particular the treatment of I_7.

Lemma 3.12 (Anisotropic Sobolev embedding). *Let $\Omega = \mathbb{R}^2 \times (0,1)$. There exists a universal constant $C > 0$ such that, for all $\kappa > 0$ and all $u \in H^1_0(\Omega)$, one has*

$$\|u\|_{L^6(\Omega)} \le C(\kappa^{-\frac12}\|\nabla_h u\|_{L^2(\Omega)} + \kappa\|\partial_3 u\|_{L^2(\Omega)}). \tag{3.71}$$

Proof. Let $\kappa > 0$ and $u \in H^1_0(\Omega)$. We first extend u by zero on $\mathbb{R}^3 \setminus \Omega$ and still denote the extended function by u. Now $u \in H^1(\mathbb{R}^3)$. We then consider the rescaled function

$$u_\kappa(y_h, y_3) = u\Big(\frac{y_h}{\kappa^{\frac12}}, \kappa y_3\Big), \quad (y_h, y_3) \in \mathbb{R}^3.$$

By Sobolev's inequality [24, estimate (II.3.7)] for the whole space, there exists a universal constant $C \in (0,\infty)$ such that

$$\|u_\kappa\|_{L^6(\mathbb{R}^3)} \le C\|\nabla u_\kappa\|_{L^2(\mathbb{R}^3)}.$$

Estimate (3.71) then follows by a change of variables and the fact that u is zero outside the strip $\mathbb{R}^2 \times (0,1)$,

$$\begin{aligned}\|u_\kappa\|_{L^6(\mathbb{R}^3)} &= \|u\|_{L^6(\Omega)},\\ \|\nabla u_\kappa\|_{L^2(\mathbb{R}^3)} &= \kappa^{-\frac12}\|\nabla_h u\|_{L^2(\Omega)} + \kappa\|\partial_3 u\|_{L^2(\Omega)}.\end{aligned}$$

This concludes the proof. ∎

3.3.3. Conclusion of the stability estimates. Below we estimate every source term I_j appearing in (3.64), for $0 < \varepsilon \le \varepsilon_0$ (ε_0 is given by Lemma 3.11). For the terms I_1, I_2, I_3, I_5 and I_6 we need to treat the cases $\gamma \ge 2$ and $\frac{3}{2} \le \gamma < 2$ separately, since we use different estimates, whereas the terms I_4, I_8, I_9, I_{10} and I_{11} can be controlled in the same way for any $\gamma \ge \frac{3}{2}$. Term I_7 is more intricate; it is written as the sum of five terms: for some of them, again we need to distinguish the cases $\gamma \ge 2$ and $\frac{3}{2} \le \gamma < 2$.

The easiest terms to handle are I_3, I_4, I_6 and I_9. Terms I_1 and I_2 are combined with the Coriolis term I_5; For the remaining part of I_5, we rely on Hardy's inequality, which is also useful for dealing with I_7, I_8, I_{10} and I_{11}. The basic idea, borrowed from [6], is that whenever there is a boundary layer term $G^{\text{bl}}(\frac{x_3}{\varepsilon})$ we gain one additional ε by using the decay of G^{bl} in ζ. The price to pay is a ∂_3 derivative on δu^ε, which however can be swallowed by the third term on the left-hand side of (3.64).

For every term, we decompose $u^\varepsilon_{\text{app}}$ according to (3.34). The terms which require more care are those of order $O(1)$, which involve in general $u_{0,h}$ and $u^{\text{bl}}_{0,h}$, except for I_7 where the product $u^\varepsilon_{\text{app}} \cdot \nabla u^\varepsilon_{\text{app}}$ also involves $u_{1,3}$ and $u^{\text{bl}}_{1,3}$ at order $O(1)$. For the terms which are not of order $O(1)$ the analysis can be always reduced to the cases I_3, I_4, I_6, I_9, and the same estimates are used.

For every term involving a boundary layer, one has to consider the top and bottom boundary layers equally; again, for simplicity, we focus on the boundary layer at the bottom only. In the computations below, U and U^{bl} generically denote remainder terms in the expansion for $u^\varepsilon_{\text{app}}$ or its derivatives. The definitions of these remainder terms may change from the estimate of one I_i to another I_j.

First, we deal with the terms for which estimates hold for any γ.

Term I_4. We start by considering I_4, when restricted to the essential set. Using (3.62), the assumptions on the pressure function and the fact that $[|\delta\rho^\varepsilon|]_{\text{ess}}$ is uniformly bounded, we can estimate

$$\begin{aligned}\left|\frac{1}{\varepsilon^2}\int_\Omega \nabla\cdot u^\varepsilon_{\text{app}}[P(\rho^\varepsilon,\rho^\varepsilon_{\text{app}})]_{\text{ess}}\right| &\le \frac{1}{\varepsilon^2}\|\nabla\cdot u^\varepsilon_{\text{app}}\|_{L^\infty}\|[\delta\rho^\varepsilon]_{\text{ess}}\|^2_{L^2}\\ &\le C\varepsilon\frac{1}{\varepsilon^2}\int_\Omega E(\rho^\varepsilon,\rho^\varepsilon_{\text{app}}),\end{aligned}$$

where we have also used that $\nabla\cdot u^\varepsilon_{\text{app}} = \varepsilon(\nabla\cdot u_1 + \nabla_h\cdot u^{\text{bl}}_{1,h})$.

Let us consider the integral over the residual set. By (3.62) again, we have $[P]_{\text{res}} = [P(\rho^\varepsilon) - P(\rho^\varepsilon_{\text{app}})]_{\text{res}} - P'(\rho^\varepsilon_{\text{app}})[\delta\rho^\varepsilon]_{\text{res}}$. The second term can be easily controlled, in view of the uniform boundedness of $\rho^\varepsilon_{\text{app}}$ and the L^1 estimate in (3.69). For the first term, we use decomposition (3.70): when ρ^ε is bounded, the same argument as above applies. On the set $\{\rho^\varepsilon \ge \bar\rho + \sigma\}$, instead we use hypothesis (1.10), the uniform boundedness of $\rho^\varepsilon_{\text{app}}$ and the controls in (3.49) to get

$$\begin{aligned}\frac{1}{\varepsilon^2}\int_\Omega |\nabla\cdot u^\varepsilon_{\text{app}}|\,|P(\rho^\varepsilon) - P(\rho^\varepsilon_{\text{app}})|\mathbf{1}_{\{\rho^\varepsilon\ge\bar\rho+\sigma\}} &\le \frac{C}{\varepsilon}\int_\Omega |P(\rho^\varepsilon) - P(\rho^\varepsilon_{\text{app}})|\mathbf{1}_{\{\rho^\varepsilon\ge\bar\rho+\sigma\}}\\ &\le \frac{C}{\varepsilon}(\|[\rho^\varepsilon]_{\text{res}}\|^\gamma_{L^\gamma} + \mathcal{L}(\Omega_{\text{res}})) \le C\varepsilon.\end{aligned}$$

Putting everything together, we finally infer that

$$|I_4| \le C\varepsilon + C\varepsilon\frac{1}{\varepsilon^2}\int_\Omega E(\rho^\varepsilon, \rho^\varepsilon_{\text{app}}), \tag{3.72}$$

where the last term will be handled by Grönwall's lemma.

Term I_9. The control of I_9 is direct, as no ρ^ε or $\delta\rho^\varepsilon$ enters into play. We get

$$|I_9| \le \varepsilon\|S^\varepsilon\|_{L^2}\|\delta u^\varepsilon\|_{L^2} \le C\varepsilon K_2(t), \tag{3.73}$$

where the function $K_2(t) = \|u^\varepsilon(t)\|_{L^2} + \|u^\varepsilon_{\text{app}}(t)\|_{L^2}$ belongs to $L^2([0,T))$ for all $T > 0$.

Terms I_8, I_{10} and I_{11}. We deal with I_8 using Hardy's inequality. This gives

$$\begin{aligned}|I_8| &= \varepsilon\left|\int_\Omega \frac{x_3}{\varepsilon}S^{\text{bl}}\Big(\frac{x_3}{\varepsilon}\Big)\cdot\frac{\delta u^\varepsilon}{x_3}\right| \\ &\le C_\delta\varepsilon\|\zeta S^{\text{bl}}\|^2_{L^2} + \delta\varepsilon\|\partial_3\delta u^\varepsilon\|^2_{L^2} \le C_\delta\varepsilon^2 + \delta\varepsilon\|\partial_3\delta u^\varepsilon\|^2_{L^2},\end{aligned} \tag{3.74}$$

for some small $\delta > 0$ to be chosen later. The same holds for I_{10} and I_{11}: since $\partial_3\bar\rho$ is uniformly bounded we have

$$\begin{aligned}|I_{10}| &= \varepsilon\left|\int_\Omega \frac{x_3^2}{\varepsilon^2}\left(\int_0^1 \partial_3\bar\rho(sx_3)\,ds\right)(u^{\text{bl}}_{0,h,b})^\perp\cdot\frac{\delta u^\varepsilon_h}{x_3}\right| \\ &\le C_\delta\varepsilon\|\zeta^2 u^{\text{bl}}_{0,h,b}\|^2_{L^2} + \delta\varepsilon\|\partial_3\delta u^\varepsilon_h\|^2_{L^2} \le C_\delta\varepsilon^2 + \delta\varepsilon\|\partial_3\delta u^\varepsilon_h\|^2_{L^2},\end{aligned} \tag{3.75}$$

$$\begin{aligned}|I_{11}| &= \varepsilon\left|\int_\Omega \frac{(1-x_3)^2}{\varepsilon^2}\left(\int_0^1 \partial_3\bar\rho(1-s(1-x_3))\,ds\right)(u^{\text{bl}}_{0,h,t})^\perp\cdot\frac{\delta u^\varepsilon_h}{1-x_3}\right| \\ &\le C_\delta\varepsilon\|\eta^2 u^{\text{bl}}_{0,h,b}\|^2_{L^2} + \delta\varepsilon\|\partial_3\delta u^\varepsilon_h\|^2_{L^2} \le C_\delta\varepsilon^2 + \delta\varepsilon\|\partial_3\delta u^\varepsilon_h\|^2_{L^2}.\end{aligned} \tag{3.76}$$

In the estimates above, we have used the fact that the terms $\|\zeta S^{\text{bl}}\|^2_{L^2}$, $\|\zeta^2 u^{\text{bl}}_{0,h,b}\|^2_{L^2}$ and $\|\eta^2 u^{\text{bl}}_{0,h,t}\|^2_{L^2}$ are $O(\varepsilon^2)$.

Now we consider the terms whose bounds must be treated differently if $\gamma \ge 2$ or $\frac{3}{2} \le \gamma < 2$.

Term I_3. First of all, observe that $\|R^{\text{bl}}(\frac{x_3}{\varepsilon})\|^2_{L^2_x} = O(\varepsilon)$. Thus, we can estimate

$$\begin{aligned}\frac{1}{\varepsilon}\left|\int_\Omega H''(\rho^\varepsilon_{\text{app}})[\delta\rho^\varepsilon]_{\text{ess}}(R^{\text{bl}} + \varepsilon R^\varepsilon)\right| &\le C\frac{1}{\varepsilon}\|R^{\text{bl}} + \varepsilon R^\varepsilon\|_{L^2}\|[\delta\rho^\varepsilon]_{\text{ess}}\|_{L^2} \\ &\le C\varepsilon + \frac{C}{\varepsilon^2}\int_\Omega E(\rho^\varepsilon, \rho^\varepsilon_{\text{app}}),\end{aligned}$$

where we have used also (3.66). As for the residual part, in view of (3.69), we can argue in exactly the same way if $\gamma \ge 2$. If $\frac{3}{2} \le \gamma < 2$, instead we put the L^∞ norm on the remainder terms and use the L^1 bound of (3.69) to get

$$\frac{1}{\varepsilon}\left|\int_\Omega H''(\rho^\varepsilon_{\text{app}})[\delta\rho^\varepsilon]_{\text{res}}(R^{\text{bl}} + \varepsilon R^\varepsilon)\right| \le C\varepsilon.$$

In any case, in the end we arrive at the bound

$$|I_3| \le C\varepsilon + \frac{C}{\varepsilon^2}\int_\Omega E(\rho^\varepsilon, \rho^\varepsilon_{\text{app}}). \tag{3.77}$$

Term I_6. Once again, we use the decomposition of $\delta\rho^\varepsilon$ into essential and residual parts. For the term involving the essential part, thanks to Young's inequality and to the controls (3.53) and (3.66), one has

$$\left|\int_\Omega [\delta\rho^\varepsilon]_{\mathrm{ess}} \partial_t u^\varepsilon_{\mathrm{app}} \cdot \delta u^\varepsilon\right| \le \|[\delta\rho^\varepsilon]_{\mathrm{ess}}\|_{L^2} \|\partial_t u^\varepsilon_{\mathrm{app}}\|_{L^\infty} \|\delta u^\varepsilon\|_{L^2}$$
$$\le C\varepsilon^2 K_1(t) + \frac{1}{\varepsilon^2} E(\rho^\varepsilon, \rho^\varepsilon_{\mathrm{app}}),$$

where the function $K_1 = \|u^\varepsilon\|^2_{L^2} + \|u^\varepsilon_{\mathrm{app}}\|^2_{L^2}$ belongs to $L^1([0,T))$ for all $T > 0$.

Next let us consider the term involving the residual part: when $\gamma \ge 2$, we can argue exactly as above, in view of (3.69). If instead $\frac{3}{2} \le \gamma < 2$, we start by writing

$$\int_\Omega [\delta\rho^\varepsilon]_{\mathrm{res}} \partial_t u^\varepsilon_{\mathrm{app}} \cdot \delta u^\varepsilon = \int_\Omega [\rho^\varepsilon]_{\mathrm{res}} \partial_t u^\varepsilon_{\mathrm{app}} \cdot \delta u^\varepsilon - \int_\Omega [\rho^\varepsilon_{\mathrm{app}}]_{\mathrm{res}} \partial_t u^\varepsilon_{\mathrm{app}} \cdot \delta u^\varepsilon.$$

For the second term, we use the uniform boundedness of $\rho^\varepsilon_{\mathrm{app}}$ and estimate (3.49) to gather, for some function $K_2 \in L^2([0,T))$ for all $T > 0$, the inequality

$$\left|\int_\Omega [\rho^\varepsilon_{\mathrm{app}}]_{\mathrm{res}} \partial_t u^\varepsilon_{\mathrm{app}} \cdot \delta u^\varepsilon\right| \le C \|\delta u^\varepsilon\|_{L^2} (\mathcal{L}(\Omega_{\mathrm{res}}))^{1/2} \le C\varepsilon K_2(t).$$

For the term involving $[\rho^\varepsilon]_{\mathrm{res}}$, we use decomposition (3.70) for the residual set. The integral over the first set can be treated exactly as just done for $\rho^\varepsilon_{\mathrm{app}}$ (because ρ^ε is uniformly bounded therein). Concerning the integral over the second set, we have

$$\left|\int_\Omega \rho^\varepsilon \mathbf{1}_{\{\rho^\varepsilon \ge \bar\rho + \sigma\}} \partial_t u^\varepsilon_{\mathrm{app}} \cdot \delta u^\varepsilon\right| \le C \left\|\sqrt{\rho^\varepsilon}\, \delta u^\varepsilon\right\|_{L^2} \left(\int_\Omega \rho^\varepsilon \mathbf{1}_{\{\rho^\varepsilon \ge \bar\rho + \sigma\}}\right)^{1/2}$$
$$\le C \left\|\sqrt{\rho^\varepsilon}\, \delta u^\varepsilon\right\|^2_{L^2} + C\varepsilon^2,$$

since the last integral in the first line can be bounded by the integral over the residual set, for which we can use (3.49).

Let us introduce the following notation: we set $\delta_{2^-}(\gamma) = 1$ if $\frac{3}{2} \le \gamma < 2$, $\delta_{2^-}(\gamma) = 0$ otherwise. In the end, from the previous computations we get

$$|I_6| \le C\varepsilon(\varepsilon K_1(t) + \delta_{2^-}(\gamma) K_2(t) + \delta_{2^-}(\gamma)\varepsilon)$$
$$+ C\left(\frac{1}{\varepsilon^2}\int_\Omega E(\rho^\varepsilon, \rho^\varepsilon_{\mathrm{app}}) + \delta_{2^-}(\gamma) \left\|\sqrt{\rho^\varepsilon}\, \delta u^\varepsilon\right\|^2_{L^2}\right). \tag{3.78}$$

Terms I_1, I_2 and I_5. Terms I_1, I_2 and I_5 have to be combined, enabling us to see a cancellation at the highest order in ε. Such a cancellation is a key point in [20]. After setting $U := u^\varepsilon_{\mathrm{app}} - (u_{0,h} + u^{\mathrm{bl}}_{0,h}, 0)$, we can write

$$I_1 + I_2 + I_5 = \frac{1}{\varepsilon^2}\int_\Omega \delta\rho^\varepsilon \nabla G \cdot \delta u^\varepsilon - \frac{1}{\varepsilon^2}\int_\Omega H''(\rho^\varepsilon_{\mathrm{app}}) \partial_3 \bar\rho \delta u^\varepsilon_3 \delta\rho^\varepsilon$$
$$- \frac{1}{\varepsilon}\int_\Omega H''(\rho^\varepsilon_{\mathrm{app}}) \nabla\rho_1 \cdot \delta u^\varepsilon \delta\rho^\varepsilon - \frac{1}{\varepsilon}\int_\Omega \delta\rho^\varepsilon \frac{P'(\bar\rho)}{\bar\rho} (\nabla_h^\perp \rho_1)^\perp \cdot \delta u^\varepsilon_h$$
$$- \frac{1}{\varepsilon}\int_\Omega \delta\rho^\varepsilon (u^{\mathrm{bl}}_{0,h})^\perp \cdot \delta u^\varepsilon_h - \int_\Omega \delta\rho^\varepsilon e_3 \times U\big(x_h, x_3, \tfrac{x_3}{\varepsilon}, \tfrac{1-x_3}{\varepsilon}, t\big) \cdot \delta u^\varepsilon.$$

Notice that $H''(\bar\rho) = \frac{P'(\bar\rho)}{\bar\rho}$ and $(\nabla_h^\perp \rho_1)^\perp = -\nabla_h \rho_1$. Moreover, from (3.1) we get

$$\bar\rho \nabla G = P'(\bar\rho)\nabla\bar\rho.$$

Therefore, we find

$$\begin{aligned} I_1 + I_2 + I_5 = &-\frac{1}{\varepsilon^2}\int_\Omega (H''(\rho^\varepsilon_{\mathrm{app}}) - H''(\bar\rho))\partial_3\bar\rho\,\delta u^\varepsilon_3\delta\rho^\varepsilon - \frac{1}{\varepsilon}\int_\Omega H''(\bar\rho)\partial_3\rho_1\delta u^\varepsilon_3\delta\rho^\varepsilon \\ &-\frac{1}{\varepsilon}\int_\Omega (H''(\rho^\varepsilon_{\mathrm{app}}) - H''(\bar\rho))\nabla\rho_1\cdot\delta u^\varepsilon\delta\rho^\varepsilon - \frac{1}{\varepsilon}\int_\Omega \delta\rho^\varepsilon (u^{\mathrm{bl}}_{0,h})^\perp\cdot\delta u^\varepsilon_h \\ &-\int_\Omega \delta\rho^\varepsilon U_h^\perp\big(x_h, x_3, \tfrac{x_3}{\varepsilon}, \tfrac{1-x_3}{\varepsilon}, t\big)\cdot\delta u^\varepsilon_h = J_1 + J_2 + J_3 + J_4 + J_5. \end{aligned}$$

Using a Taylor expansion for $h(z) = H''(z)$ with integral remainder, we can write

$$\begin{aligned} J_1 + J_2 = &-\frac{1}{\varepsilon}\int_\Omega (h'(\bar\rho)\rho_1\partial_3\bar\rho + h(\bar\rho)\partial_3\rho_1)\delta u^\varepsilon_3\delta\rho^\varepsilon \\ &-\int_\Omega \rho_1^2\Big(\int_0^1 (1-s)h''(\bar\rho + s\varepsilon\rho_1)\,ds\Big)\partial_3\bar\rho\delta u^\varepsilon_3\delta\rho^\varepsilon \\ = &-\int_\Omega \rho_1^2\Big(\int_0^1 (1-s)h''(\bar\rho + s\varepsilon\rho_1)\,ds\Big)\partial_3\bar\rho\delta u^\varepsilon_3\delta\rho^\varepsilon, \end{aligned}$$

where we have used (3.6) in the last equality. Since ρ_1, $\bar\rho$ and $\partial_3\bar\rho$ are $L^\infty_{t,x}$, in view of (1.10) the control of $J_1 + J_2$ becomes similar to that exhibited for I_6. In the same way, after noticing that $\nabla_h\rho_1$ and $\varepsilon^{-1}(H''(\rho^\varepsilon_{\mathrm{app}}) - H''(\bar\rho))$ are uniformly bounded in time and space, the control of J_3 is obtained. Then $J_1 + J_2$ and J_3 verify estimate (3.78). The same can be said about J_5, because in addition U_h belongs to $L^\infty_{t,x}$.

Therefore, it remains to deal with J_4, for which we rely on Hardy's inequality. More precisely, let us start, as usual, by dealing with the essential part: we have

$$\begin{aligned} \left|\frac{1}{\varepsilon}\int_\Omega [\delta\rho^\varepsilon]_{\mathrm{ess}}(u^{\mathrm{bl}}_{0,h})^\perp\cdot\delta u^\varepsilon_h\right| &= \left|\int_\Omega [\delta\rho^\varepsilon]_{\mathrm{ess}}\frac{x_3}{\varepsilon}(u^{\mathrm{bl}}_{0,h})^\perp\cdot\frac{\delta u^\varepsilon_h}{x_3}\right| \\ &\le \|[\delta\rho^\varepsilon]_{\mathrm{ess}}\|_{L^2}\|\zeta u^{\mathrm{bl}}_{0,h}(t, x_h, \zeta)\|_{L^\infty_{t,x,\zeta}}\|\partial_3\delta u^\varepsilon_h\|_{L^2} \\ &\le \frac{C}{\varepsilon^2}\int_\Omega E(\rho^\varepsilon, \rho^\varepsilon_{\mathrm{app}}) + C\varepsilon^2\|\zeta u^{\mathrm{bl}}_{0,h}\|^2_{L^\infty_{t,x,\zeta}}\|\partial_3\delta u^\varepsilon_h\|^2_{L^2}. \end{aligned}$$

Notice that, for ε small enough, the second term can be swallowed by the third term on the left-hand side of (3.64). As for the control of the residual part, suppose that $\gamma \ge 2$ for a while: in this case, we can argue in the exact same way and obtain, in view of (3.69), that

$$\begin{aligned} \left|\frac{1}{\varepsilon}\int_\Omega [\delta\rho^\varepsilon]_{\mathrm{res}}(u^{\mathrm{bl}}_{0,h})^\perp\cdot\delta u^\varepsilon_h\right| &\le \|[\delta\rho^\varepsilon]_{\mathrm{res}}\|_{L^2}\|\zeta u^{\mathrm{bl}}_{0,h}\|_{L^\infty_{t,x,\zeta}}\|\partial_3\delta u^\varepsilon_h\|_{L^2} \\ &\le \frac{C}{\varepsilon^2}\int_\Omega E(\rho^\varepsilon, \rho^\varepsilon_{\mathrm{app}}) + C\varepsilon^2\|\zeta u^{\mathrm{bl}}_{0,h}\|^2_{L^\infty_{t,x,\zeta}}\|\partial_3\delta u^\varepsilon_h\|^2_{L^2}. \end{aligned}$$

The case $\frac{3}{2} \leq \gamma < 2$ is slightly more involved. The control over $\{0 < \rho^\varepsilon \leq \bar{\rho} - \sigma\}$ does not present any special difficulty, since we have uniform bounds for ρ_ε (and obviously for $\rho^\varepsilon_{\rm app}$) on that set: then we can argue as for controlling the essential part. Hence, let us focus on $\{\rho^\varepsilon \geq \bar{\rho} + \sigma\}$. First of all, using that $\sqrt{a+b} \leq \sqrt{a} + \sqrt{b}$, we notice that

$$\left| \frac{1}{\varepsilon} \int_\Omega [\delta\rho^\varepsilon]_{\rm res} (u^{\rm bl}_{0,h})^\perp \cdot \delta u^\varepsilon_h \right| \leq \frac{1}{\varepsilon} \int \sqrt{\delta\rho^\varepsilon}\, \mathbf{1}_{\{\rho^\varepsilon \geq \bar{\rho}+\sigma\}} |u^{\rm bl}_{0,h}| \sqrt{\rho_\varepsilon}\, |\delta u^\varepsilon_h| + \frac{1}{\varepsilon} \int \sqrt{\delta\rho^\varepsilon}\, \mathbf{1}_{\{\rho^\varepsilon \geq \bar{\rho}+\sigma\}} |u^{\rm bl}_{0,h}| \sqrt{\rho^\varepsilon_{\rm app}}\, |\delta u^\varepsilon_h|. \tag{3.79}$$

For the first term on the right-hand side of (3.79), we proceed in the following way:

$$\int_\Omega \sqrt{\delta\rho^\varepsilon}\, \mathbf{1}_{\{\rho^\varepsilon \geq \bar{\rho}+\sigma\}} |u^{\rm bl}_{0,h}| \sqrt{\rho_\varepsilon}\, |\delta u^\varepsilon_h| \leq \| [\delta\rho^\varepsilon]_{\rm res} \|^{1/2}_{L^\gamma} \| u^{\rm bl}_{0,h} \|_{L^\infty} \| \sqrt{\rho_\varepsilon}\, \delta u^\varepsilon_h \|_{L^2} (\mathcal{L}(\Omega_{\rm res}))^{1/q},$$

where $\frac{1}{2\gamma} + \frac{1}{2} + \frac{1}{q} = 1$. Using (3.65) and (3.66), we deduce that

$$\begin{aligned} \frac{1}{\varepsilon} \int_\Omega \sqrt{\delta\rho^\varepsilon}\, \mathbf{1}_{\{\rho^\varepsilon \geq \bar{\rho}+\sigma\}} |u^{\rm bl}_{0,h}| \sqrt{\rho_\varepsilon}\, |\delta u^\varepsilon_h| &\leq \varepsilon^{1/\gamma + 2/q - 1} \left(\frac{1}{\varepsilon^2} E(\rho^\varepsilon, \rho^\varepsilon_{\rm app}) \right)^{1/(2\gamma)+1/q} \| u^{\rm bl}_{0,h} \|_{L^\infty} \| \sqrt{\rho_\varepsilon}\, \delta u^\varepsilon_h \|_{L^2} \\ &= \left(\frac{1}{\varepsilon^2} E(\rho^\varepsilon, \rho^\varepsilon_{\rm app}) \right)^{1/2} \| u^{\rm bl}_{0,h} \|_{L^\infty} \| \sqrt{\rho_\varepsilon}\, \delta u^\varepsilon_h \|_{L^2}. \end{aligned}$$

After applying Young's inequality, this term can be controlled by Grönwall's lemma in the final estimate. For the last term in (3.79), we argue in the following way:

$$\begin{aligned} \int \sqrt{\delta\rho^\varepsilon}\, \mathbf{1}_{\{\rho^\varepsilon \geq \bar{\rho}+\sigma\}} |u^{\rm bl}_{0,h}| \sqrt{\rho^\varepsilon_{\rm app}}\, |\delta u^\varepsilon_h| &= \varepsilon \int \sqrt{\delta\rho^\varepsilon}\, \mathbf{1}_{\{\rho^\varepsilon \geq \bar{\rho}+\sigma\}} \left| \frac{x_3}{\varepsilon} u^{\rm bl}_{0,h} \right| \sqrt{\rho^\varepsilon_{\rm app}} \left| \frac{\delta u^\varepsilon_h}{x_3} \right| \\ &\leq \varepsilon \int \left(\sqrt{\rho^\varepsilon}\, \mathbf{1}_{\{\rho^\varepsilon \geq \bar{\rho}+\sigma\}} \left| \frac{x_3}{\varepsilon} u^{\rm bl}_{0,h} \right| \sqrt{\rho^\varepsilon_{\rm app}} \left| \frac{\delta u^\varepsilon_h}{x_3} \right| + \rho^\varepsilon_{\rm app} \mathbf{1}_{\{\rho^\varepsilon \geq \bar{\rho}+\sigma\}} \left| \frac{x_3}{\varepsilon} u^{\rm bl}_{0,h} \right| \left| \frac{1}{x_3} \delta u^\varepsilon_h \right| \right) \\ &\leq \varepsilon \| \partial_3 \delta u^\varepsilon_h \|_{L^2} \| \zeta u^{\rm bl}_{0,h} \|_{L^\infty} \\ &\quad \times \left(\| \rho^\varepsilon_{\rm app} \|^{1/2}_{L^\infty} \| [\rho^\varepsilon]_{\rm res} \|^{1/2}_{L^\gamma} (\mathcal{L}(\Omega_{\rm res}))^{1/q} + \| \rho^\varepsilon_{\rm app} \|_{L^\infty} (\mathcal{L}(\Omega_{\rm res}))^{1/2} \right), \end{aligned}$$

where q is defined as above. Notice that, in view of (3.49), we have $\| [\rho^\varepsilon]_{\rm res} \|_{L^\gamma} = O(\varepsilon^{2/\gamma})$ and $\mathcal{L}(\Omega_{\rm res}) = O(\varepsilon^2)$. Therefore, we finally find

$$\frac{1}{\varepsilon} \int \sqrt{\delta\rho^\varepsilon}\, \mathbf{1}_{\{\rho^\varepsilon \geq \bar{\rho}+\sigma\}} |u^{\rm bl}_{0,h}| \sqrt{\rho^\varepsilon_{\rm app}}\, |\delta u^\varepsilon_h| \leq C_\delta \varepsilon + \delta \varepsilon \| \partial_3 \delta u^\varepsilon_h \|^2_{L^2}.$$

In the end, we deduce the following control:

$$\begin{aligned} |I_1 + I_2 + I_5| \leq \frac{C}{\varepsilon^2} \int_\Omega E(\rho^\varepsilon, \rho^\varepsilon_{\rm app}) + C \delta_{2^-}(\gamma) \| \sqrt{\rho^\varepsilon}\, \delta u^\varepsilon_h \|^2_{L^2} \\ + C\varepsilon (\varepsilon K_1(t) + \delta_{2^-}(\gamma) K_2(t) + \delta_{2^-}(\gamma)) \\ + (C\varepsilon^2 + \delta_{2^-}(\gamma) \delta\varepsilon) \| \partial_3 \delta u^\varepsilon_h \|^2_{L^2}, \end{aligned} \tag{3.80}$$

where the last term on the right-hand side can be absorbed into the left-hand side of the relative entropy inequality (3.64).

Finally, let us deal with I_7.

Term I_7. We start by considering the following decomposition:

$$\begin{aligned} I_7 = &- \int_\Omega \delta\rho^\varepsilon u^\varepsilon_{\mathrm{app}} \cdot \nabla u^\varepsilon_{\mathrm{app}} \cdot \delta u^\varepsilon - \frac{1}{\varepsilon}\int_\Omega \rho^\varepsilon \delta u^\varepsilon_3 \partial_\zeta u^{\mathrm{bl}}_{0,h}\big(\tfrac{x_3}{\varepsilon}\big) \cdot \delta u^\varepsilon_h \\ &- \int_\Omega \rho^\varepsilon \delta u^\varepsilon_3 \partial_\zeta u^{\mathrm{bl}}_1\big(\tfrac{x_3}{\varepsilon}\big) \cdot \delta u^\varepsilon - \int_\Omega \rho^\varepsilon \delta u^\varepsilon_h \cdot \nabla_h (u_{0,h} + u^{\mathrm{bl}}_{0,h}) \cdot \delta u^\varepsilon_h \\ &- \varepsilon \int_\Omega \rho^\varepsilon \delta u^\varepsilon \cdot U \cdot \delta u^\varepsilon \\ = &\ J_6 + J_7 + J_8 + J_9 + J_{10}, \end{aligned}$$

where $\varepsilon U = \varepsilon(\nabla u_1 + \binom{\nabla_h}{0} u^{\mathrm{bl}}_1)$ is the remainder term in the expansion for $\nabla u^\varepsilon_{\mathrm{app}}$.

The first term J_6 can be handled as done with I_6. Indeed, one has

$$u^\varepsilon_{\mathrm{app}} \cdot \nabla u^\varepsilon_{\mathrm{app}} = (u_{0,h} + u^{\mathrm{bl}}_{0,h}) \cdot \nabla_h (u_{0,h} + u^{\mathrm{bl}}_{0,h}) + (u_{1,3} + u^{\mathrm{bl}}_{1,3}) \partial_\zeta u^{\mathrm{bl}}_{0,h}\big(\tfrac{x_3}{\varepsilon}\big) + \text{h.o.t.},$$

where h.o.t. represents higher-order terms in ε. Then $u^\varepsilon_{\mathrm{app}} \cdot \nabla u^\varepsilon_{\mathrm{app}}$ is uniformly bounded in $L^\infty_{t,x}$. Therefore, J_6 verifies an inequality similar to (3.78) above.

Terms J_8, J_9 and J_{10} can be simply bounded as

$$\begin{aligned} |J_8| &\le C \|\partial_\zeta u^{\mathrm{bl}}_1\|_{L^\infty_{t,x}} \|\sqrt{\rho^\varepsilon}\, \delta u^\varepsilon\|^2_{L^2}, \\ |J_9| &\le C \|\nabla_h (u_{0,h} + u^{\mathrm{bl}}_{0,h})\|_{L^\infty_{t,x}} \|\sqrt{\rho^\varepsilon}\, \delta u^\varepsilon\|^2_{L^2}, \\ |J_{10}| &\le C\varepsilon \|U\|_{L^\infty_{t,x}} \|\sqrt{\rho^\varepsilon}\, \delta u^\varepsilon\|^2_{L^2}. \end{aligned}$$

We remark that these estimates hold for $\gamma \ge \frac{3}{2}$.

We now focus on the remaining term J_7, which is the most difficult one to deal with. The difficulties come from the need to gain smallness in ε (by using Hardy's inequality as above) from the low integrability of the residual part and from the fact that this term is quadratic in δu^ε. We first decompose

$$\rho^\varepsilon = [\delta\rho^\varepsilon]_{\mathrm{ess}} + [\delta\rho^\varepsilon]_{\mathrm{res}} + \rho^\varepsilon_{\mathrm{app}}. \tag{3.81}$$

The essential part is easy to bound: owing to the boundedness of ρ^ε on that set and to an application of Hardy's inequality, we get

$$\begin{aligned} \frac{1}{\varepsilon}\left| \int_\Omega [\delta\rho^\varepsilon]_{\mathrm{ess}} \delta u^\varepsilon_3 \partial_\zeta u^{\mathrm{bl}}_{0,h}\big(\tfrac{x_3}{\varepsilon}\big) \cdot \delta u^\varepsilon_h \right| &\le C\varepsilon \|\zeta^2 \partial_\zeta u^{\mathrm{bl}}_{0,h}\|_{L^\infty_{t,x}} \left\| \frac{\delta u^\varepsilon_3}{x_3} \right\|_{L^2} \left\| \frac{\delta u^\varepsilon_h}{x_3} \right\|_{L^2} \\ &\le C\varepsilon \|\partial_3 \delta u^\varepsilon_3\|_{L^2} \|\partial_3 \delta u^\varepsilon_h\|_{L^2} \\ &\le C\varepsilon^{3/2} \|\partial_3 \delta u^\varepsilon_h\|^2_{L^2} + C\varepsilon^{1/2} \|\partial_3 \delta u^\varepsilon_3\|^2_{L^2}. \end{aligned}$$

The control of the part involving $\rho^\varepsilon_{\rm app}$ is similar, so let us turn to the residual part. Two different estimates are computed if γ is larger or smaller than the critical exponent 2. For $\gamma \geq 2$, we write, for $\alpha \in (0,1)$ to be chosen later on,

$$\delta u_3^\varepsilon = (\delta u_3^\varepsilon)^{1-\alpha} \frac{(\delta u_3^\varepsilon)^\alpha}{x_3^\alpha} x_3^\alpha$$

and then apply Sobolev's and Hardy's inequalities: this yields

$$\|(\delta u_3^\varepsilon)^{1-\alpha}\|_{L^{\frac{6}{1-\alpha}}} \leq C\|\nabla \delta u_3^\varepsilon\|_{L^2}^{1-\alpha} \quad \text{and} \quad \left\|\frac{(\delta u_3^\varepsilon)^\alpha}{x_3^\alpha}\right\|_{L^{\frac{2}{\alpha}}} \leq C\|\partial_3 \delta u_3^\varepsilon\|_{L^2}^\alpha. \tag{3.82}$$

We use the same technique for δu_h^ε with $\beta \in (0,1)$. Then, choosing α, β such that $\alpha + \beta = \frac{1}{2}$, we have, for all $\delta > 0$ to be chosen later,

$$\begin{aligned}
&\frac{1}{\varepsilon}\left|\int_\Omega [\delta\rho^\varepsilon]_{\rm res} \delta u_3^\varepsilon \partial_\zeta u_{0,h}^{\rm bl}\big(\tfrac{x_3}{\varepsilon}\big)\cdot \delta u_h^\varepsilon\right| \\
&\quad = \varepsilon^{\alpha+\beta-1}\left|\int_\Omega [\delta\rho^\varepsilon]_{\rm res}(\delta u_3^\varepsilon)^{1-\alpha}\frac{(\delta u_3^\varepsilon)^\alpha}{x_3^\alpha}\frac{x_3^{\alpha+\beta}}{\varepsilon^{\alpha+\beta}}\partial_\zeta u_{0,h}^{\rm bl}\big(\tfrac{x_3}{\varepsilon}\big)\cdot(\delta u_h^\varepsilon)^{1-\beta}\frac{(\delta u_h^\varepsilon)^\beta}{x_3^\beta}\right| \\
&\quad \leq \varepsilon^{-1/2}\|[\delta\rho^\varepsilon]_{\rm res}\|_{L^2}\|\nabla\delta u_3^\varepsilon\|_{L^2}^{1-\alpha}\|\partial_3\delta u_3^\varepsilon\|_{L^2}^\alpha\|\nabla\delta u_h^\varepsilon\|_{L^2}^{1-\beta} \\
&\qquad \times \|\partial_3\delta u_h^\varepsilon\|_{L^2}^\beta\|\zeta^{\alpha+\beta}\partial_\zeta u_{0,h}^{\rm bl}\|_{L^\infty} \\
&\quad \leq C\varepsilon^{1/2}\left(\frac{1}{\varepsilon^2}\int_\Omega E(\rho^\varepsilon,\rho^\varepsilon_{\rm app})\right)^{1/2}\|\nabla\delta u_3^\varepsilon\|_{L^2}\|\nabla\delta u_h^\varepsilon\|_{L^2} \\
&\quad \leq \frac{C_\delta}{\varepsilon^2}K_1(t)\int_\Omega E(\rho^\varepsilon,\rho^\varepsilon_{\rm app}) + \delta\varepsilon\|\nabla_h\delta u_h^\varepsilon\|_{L^2}^2 + \delta\varepsilon\|\partial_3\delta u_h^\varepsilon\|_{L^2}^2,
\end{aligned}$$

where $K_1(t) = \|\nabla u_3^\varepsilon(t)\|_{L^2}^2 + \|\nabla u^\varepsilon_{{\rm app},3}(t)\|_{L^2}^2$ belongs to $L^1([0,T))$ for all $T>0$. In the second inequality we have used the lower bound

$$E(\rho^\varepsilon(x,t),\rho^\varepsilon_{\rm app}(x,t)) \geq c|\delta\rho^\varepsilon(x,t)|^2,$$

which comes from (3.65) when $\gamma \geq 2$.

For $\frac{3}{2} \leq \gamma < 2$, we use the same argument as in the case $\gamma \geq 2$ for δu_3^ε. The control of δu_h^ε is instead done via the anisotropic Sobolev embedding given in Lemma 3.12. Hence, for $\alpha \in [0,1]$ such that $2 - \frac{3}{\gamma} = \alpha$, Hölder's inequality gives, using (3.82) for δu_3^ε and (3.71) for δu_h^ε with $\kappa = \varepsilon^{(-\frac{1}{2}+\frac{1}{\gamma})_+}$,

$$\begin{aligned}
&\frac{1}{\varepsilon}\left|\int_\Omega [\delta\rho^\varepsilon]_{\rm res}\delta u_3^\varepsilon\partial_\zeta u_{0,h}^{\rm bl}\big(\tfrac{x_3}{\varepsilon}\big)\cdot\delta u_h^\varepsilon\right| \\
&\quad = \varepsilon^{\alpha-1}\left|\int_\Omega [\delta\rho^\varepsilon]_{\rm res}(\delta u_3^\varepsilon)^{1-\alpha}\frac{(\delta u_3^\varepsilon)^\alpha}{x_3^\alpha}\frac{x_3^\alpha}{\varepsilon^\alpha}\partial_\zeta u_{0,h}^{\rm bl}\big(\tfrac{x_3}{\varepsilon}\big)\cdot\delta u_h^\varepsilon\right| \\
&\quad \leq C\varepsilon^{\alpha-1}\|[\delta\rho^\varepsilon]_{\rm res}\|_{L^\gamma}\|\nabla\delta u_3^\varepsilon\|_{L^2}^{1-\alpha}\|\partial_3\delta u_3^\varepsilon\|_{L^2}^\alpha \\
&\qquad \times (\kappa^{-\frac{1}{2}}\|\nabla_h\delta u_h^\varepsilon\|_{L^2} + \kappa\|\partial_3\delta u_h^\varepsilon\|_{L^2})\|\zeta^\alpha\partial_\zeta u_{0,h}^{\rm bl}\|_{L^\infty}
\end{aligned}$$

$$
\begin{aligned}
&\leq C\varepsilon^{1-\frac{1}{\gamma}}\|\nabla\delta u_3^\varepsilon\|_{L^2}(\kappa^{-\frac{1}{2}}\|\nabla_h\delta u_h^\varepsilon\|_{L^2}+\kappa\|\partial_3\delta u_h^\varepsilon\|_{L^2})\\
&\leq \tfrac{\min(\mu,\lambda)}{10}\|\nabla\delta u_3^\varepsilon\|_{L^2}^2+C(\mu,\lambda)\varepsilon^{2-\frac{2}{\gamma}}(\kappa^{-1}\|\nabla_h\delta u_h^\varepsilon\|_{L^2}^2+\kappa^2\|\partial_3\delta u_h^\varepsilon\|_{L^2}^2)\\
&\leq \tfrac{\min(\mu,\lambda)}{10}\|\nabla\delta u_3^\varepsilon\|_{L^2}^2+C(\mu,\lambda)\varepsilon^{(\frac{5}{2}-\frac{3}{\gamma})-}\|\nabla_h\delta u_h^\varepsilon\|_{L^2}^2\\
&\quad+C(\mu,\lambda)\varepsilon^{1+}\|\partial_3\delta u_h^\varepsilon\|_{L^2}^2.
\end{aligned}
\tag{3.83}
$$

Hence we can swallow the whole right-hand side on the conditions that $\gamma>\frac{6}{5}$ (which is the case, since $\gamma\geq\frac{3}{2}$) and ε is sufficiently small.

To put it in a nutshell, we obtain the following bound on J_7:

$$
\begin{aligned}
|J_7|\leq{}&\frac{C_\delta}{\varepsilon^2}K_1(t)\int_\Omega E(\rho^\varepsilon,\rho^\varepsilon_{\mathrm{app}})+(\delta\varepsilon+C\varepsilon^{\frac{3}{2}}+\delta_{2^-}(\gamma)C(\mu,\lambda)\varepsilon^{1+})\|\partial_3\delta u_h^\varepsilon\|_{L^2}^2\\
&+(\delta\varepsilon+\delta_{2^-}(\gamma)C(\mu,\lambda)\varepsilon^{(\frac{5}{2}-\frac{3}{\gamma})-})\|\nabla_h\delta u_h^\varepsilon\|_{L^2}^2\\
&+\big(\delta_{2^-}(\gamma)\tfrac{\min(\mu,\lambda)}{10}+C\varepsilon^{\frac{1}{2}}\big)\|\nabla\delta u_3^\varepsilon\|_{L^2}^2.
\end{aligned}
$$

Therefore, we finally get the following estimate for I_7:

$$
\begin{aligned}
|I_7|\leq{}&C\varepsilon(\varepsilon K_1(t)+\delta_{2^-}(\gamma)K_2(t)+\delta_{2^-}(\gamma)\varepsilon)+\frac{C+C_\delta K_1(t)}{\varepsilon^2}\int_\Omega E(\rho^\varepsilon,\rho^\varepsilon_{\mathrm{app}})\\
&+(\delta_{2^-}(\gamma)C+C_1+C_2\varepsilon)\|\sqrt{\rho^\varepsilon}\,\delta u^\varepsilon\|_{L^2}^2\\
&+(\delta\varepsilon+C\varepsilon^{\frac{3}{2}}+\delta_{2^-}(\gamma)C(\mu,\lambda)\varepsilon^{1+})\|\partial_3\delta u_h^\varepsilon\|_{L^2}^2\\
&+(\delta\varepsilon+\delta_{2^-}(\gamma)C(\mu,\lambda)\varepsilon^{(\frac{5}{2}-\frac{3}{\gamma})-})\|\nabla_h\delta u_h^\varepsilon\|_{L^2}^2\\
&+\big(\delta_{2^-}(\gamma)\tfrac{\min(\mu,\lambda)}{10}+C\varepsilon^{\frac{1}{2}}\big)\|\nabla\delta u_3^\varepsilon\|_{L^2}^2.
\end{aligned}
\tag{3.84}
$$

Remark 3.13. The anisotropic Sobolev embedding in Lemma 3.12 can be used to provide better estimates only for γ small. For instance, in (3.79), using Lemma 3.12 we get a remainder term of order ε^α, with $0<\alpha<1$ for $\frac{3}{2}\leq\gamma<2$ and $\alpha>1$ only for $\gamma<\frac{12}{11}$, while by using the smallness of the Lebesgue measure of Ω_{res} we get a remainder term of order ε for $\frac{3}{2}\leq\gamma<2$.

In the end, summing our estimates, we get from (3.64) the following differential inequality: there exist functions $C_1(t),C_2(t)\in L^1([0,T))$, and constants $C_3>0$ and $\varepsilon_0\in(0,1)$, such that, for all $\varepsilon\in(0,\varepsilon_0)$, all $t\in(0,T)$ and all $\delta>0$, one has

$$
\frac{d}{dt}\left(\frac{1}{2}\int_\Omega\rho^\varepsilon|\delta u^\varepsilon|^2\,dx+\frac{1}{\varepsilon^2}\int_\Omega E(\rho^\varepsilon,\rho^\varepsilon_{\mathrm{app}})\,dx\right)
\tag{3.85}
$$

$$
\begin{aligned}
&+\mu\int_\Omega|\nabla_h\delta u^\varepsilon|^2\,dx+\varepsilon\int_\Omega|\partial_3\delta u^\varepsilon|^2\,dx+\lambda\int_\Omega|\nabla\cdot\delta u^\varepsilon|^2\,dx\\
\leq{}&C_1(t)\left(\int_\Omega\rho^\varepsilon|\delta u^\varepsilon|^2\,dx+\frac{1}{\varepsilon^2}\int_\Omega E(\rho^\varepsilon,\rho^\varepsilon_{\mathrm{app}})\,dx\right)+\varepsilon C_2(t)\\
&+\big(\tfrac{\min(\mu,\lambda)}{10}+C\varepsilon^{(\frac{5}{2}-\frac{3}{\gamma})-}\big)\|\nabla_h\delta u^\varepsilon\|_{L^2}^2+C_3(\delta\varepsilon+\varepsilon^{1+})\|\partial_3\delta u^\varepsilon\|_{L^2}^2\\
&+\big(\tfrac{\min(\mu,\lambda)}{10}+C\varepsilon^{\frac{1}{2}}\big)\|\partial_3\delta u_3^\varepsilon\|_{L^2}^2.
\end{aligned}
\tag{3.86}
$$

Let us stress that $C_1(t)$, $C_2(t)$, C_3 and ε_0 do not depend on ε. The quantities these constants depend on have been written explicitly in the computations above; in particular, $C_1(t)$ and $C_2(t)$ contain the functions $K_1(t)$ and $K_2(t)$.

Choosing δ small enough and using the identity $\partial_3 \delta u_3^\varepsilon = \nabla \cdot \delta u^\varepsilon - \nabla_h \cdot \delta u_h^\varepsilon$, the last three terms in (3.86) can be swallowed in the left-hand side. The estimate in Theorem 2 follows from Grönwall's lemma.

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References

[1] D. G. Aronson, Bounds for the fundamental solution of a parabolic equation. *Bull. Amer. Math. Soc.* **73** (1967), 890–896 Zbl 0153.42002 MR 217444

[2] H. Bahouri, J.-Y. Chemin, and R. Danchin, *Fourier analysis and nonlinear partial differential equations*. Grundlehren Math. Wiss. 343, Springer, Heidelberg, 2011 Zbl 1227.35004 MR 2768550

[3] E. Bocchi, *Compressible-incompressible transitions in fluid mechanics: waves-structures inter-action and rotating fluids*. Ph.D. thesis, University of Bordeaux, 2019

[4] D. Bresch and C. Burtea, Global existence of weak solutions for the anisotropic compressible Stokes system. *Ann. Inst. H. Poincaré Anal. Non Linéaire* **37** (2020), no. 6, 1271–1297 Zbl 1456.35160 MR 4168917

[5] D. Bresch and C. Burtea, Weak solutions for the stationary anisotropic and nonlocal compressible Navier-Stokes system. *J. Math. Pures Appl. (9)* **146** (2021), 183–217 Zbl 1459.76120 MR 4197285

[6] D. Bresch, B. Desjardins, and D. Gérard-Varet, Rotating fluids in a cylinder. *Discrete Contin. Dyn. Syst.* **11** (2004), no. 1, 47–82 Zbl 1138.76446 MR 2073946

[7] D. Bresch and P.-E. Jabin, Global existence of weak solutions for compressible Navier-Stokes equations: thermodynamically unstable pressure and anisotropic viscous stress tensor. *Ann. of Math. (2)* **188** (2018), no. 2, 577–684 Zbl 1405.35133 MR 3862947

[8] F. Charve, Global well-posedness and asymptotics for a geophysical fluid system. *Comm. Partial Differential Equations* **29** (2004), no. 11-12, 1919–1940 Zbl 1156.35449 MR 2106072

[9] J.-Y. Chemin, Fluides parfaits incompressibles. *Astérisque* (1995), no. 230 Zbl 0829.76003 MR 1340046

[10] J.-Y. Chemin, B. Desjardins, I. Gallagher, and E. Grenier, Ekman boundary layers in rotating fluids. ESAIM Control Optim. Calc. Var. **8** (2002), 441–466 Zbl 1070.35505 MR 1932959

[11] J.-Y. Chemin, B. Desjardins, I. Gallagher, and E. Grenier, *Mathematical geophysics: an introduction to rotating fluids and the Navier-Stokes equations*. Oxford Lecture Ser. Math. Appl. 32, The Clarendon Press, Oxford University Press, Oxford, 2006 Zbl 1205.86001 MR 2228849

[12] B. Cushman-Roisin, *Introduction to geophysical fluid dynamics*. Prentice-Hall, Englewood Cliffs, 1994

[13] R. Danchin, F. Fanelli, and M. Paicu, A well-posedness result for viscous compressible fluids with only bounded density. *Anal. PDE* **13** (2020), no. 1, 275–316 Zbl 1434.35046 MR 4047647

[14] F. De Anna and F. Fanelli, Global well-posedness and long-time dynamics for a higher order quasi-geostrophic type equation. *J. Funct. Anal.* **274** (2018), no. 8, 2291–2355 Zbl 1384.35083 MR 3767435

[15] L. C. Evans, *Partial differential equations*. 2nd edn., Grad. Stud. Math. 19, American Mathematical Society, Providence, RI, 2010 Zbl 1194.35001 MR 2597943

[16] E. Feireisl, I. Gallagher, D. Gerard-Varet, and A. Novotný, Multi-scale analysis of compressible viscous and rotating fluids. *Comm. Math. Phys.* **314** (2012), no. 3, 641–670 Zbl 1251.35069 MR 2964771

[17] E. Feireisl, I. Gallagher, and A. Novotný, A singular limit for compressible rotating fluids. *SIAM J. Math. Anal.* **44** (2012), no. 1, 192–205 Zbl 1263.76077 MR 2888285

[18] E. Feireisl, B. J. Jin, and A. Novotný, Relative entropies, suitable weak solutions, and weak-strong uniqueness for the compressible Navier-Stokes system. *J. Math. Fluid Mech.* **14** (2012), no. 4, 717–730 Zbl 1256.35054 MR 2992037

[19] E. Feireisl and A. Novotný, *Singular limits in thermodynamics of viscous fluids*. Adv. Math. Fluid Mech., Birkhäuser, Basel, 2009 Zbl 1176.35126 MR 2499296

[20] E. Feireisl and A. Novotný, Multiple scales and singular limits for compressible rotating fluids with general initial data. *Comm. Partial Differential Equations* **39** (2014), no. 6, 1104–1127 Zbl 1307.35220 MR 3200090

[21] E. Feireisl, A. Novotný, and H. Petzeltová, On the existence of globally defined weak solutions to the Navier-Stokes equations. *J. Math. Fluid Mech.* **3** (2001), no. 4, 358–392 Zbl 0997.35043 MR 1867887

[22] E. Feireisl, A. Novotný, and Y. Sun, Suitable weak solutions to the Navier-Stokes equations of compressible viscous fluids. *Indiana Univ. Math. J.* **60** (2011), no. 2, 611–631 Zbl 1248.35143 MR 2963786

[23] E. Feireisl, L. Yong, and A. Novotný, Rotating compressible fluids under strong stratification. *Nonlinear Anal. Real World Appl.* **19** (2014), 11–18 Zbl 1367.76063 MR 3206654

[24] G. P. Galdi, *An introduction to the mathematical theory of the Navier-Stokes equations: steady-state problems*. 2nd edn., Springer Monogr. Math., Springer, New York, 2011 Zbl 1245.35002 MR 2808162

[25] I. Gallagher, The tridimensional Navier-Stokes equations with almost bidimensional data: stability, uniqueness, and life span. *Internat. Math. Res. Notices* (1997), no. 18, 919–935 Zbl 0893.35098 MR 1481611

[26] D. Gérard-Varet, Highly rotating fluids in rough domains. *J. Math. Pures Appl. (9)* **82** (2003), no. 11, 1453–1498 Zbl 1033.76008 MR 2020807

[27] P. Germain, Weak-strong uniqueness for the isentropic compressible Navier-Stokes system. *J. Math. Fluid Mech.* **13** (2011), no. 1, 137–146 Zbl 1270.35342 MR 2784900

[28] M. Hieber and J. Prüss, Heat kernels and maximal L^p-L^q estimates for parabolic evolution equations. *Comm. Partial Differential Equations* **22** (1997), no. 9-10, 1647–1669 Zbl 0886.35030 MR 1469585

[29] D. Hoff, Global solutions of the Navier-Stokes equations for multidimensional compressible flow with discontinuous initial data. *J. Differential Equations* **120** (1995), no. 1, 215–254 Zbl 0836.35120 MR 1339675
[30] R. Klein, Multiple spatial scales in engineering and atmospheric low Mach number flows. *M2AN Math. Model. Numer. Anal.* **39** (2005), no. 3, 537–559 Zbl 1130.76326 MR 2157149
[31] R. Klein, Scale-dependent models for atmospheric flows. In *Annual review of fluid mechanics. Vol. 42*, edited by S. H. Davis et al., pp. 249–274, Annual Reviews, Palo Alto, CA, 2010 Zbl 1213.86002 MR 2647596
[32] P.-L. Lions, *Mathematical topics in fluid mechanics: compressible models. Vol. 2*. Oxford Lecture Ser. Math. Appl. 10, The Clarendon Press, Oxford University Press, New York, 1998 Zbl 0908.76004 MR 1637634
[33] N. Masmoudi, Ekman layers of rotating fluids: the case of general initial data. *Comm. Pure Appl. Math.* **53** (2000), no. 4, 432–483 Zbl 1047.76124 MR 1733696
[34] A. Matsumura and T. Nishida, The initial value problem for the equations of motion of viscous and heat-conductive gases. *J. Math. Kyoto Univ.* **20** (1980), no. 1, 67–104 Zbl 0429.76040 MR 564670
[35] P. R. Mensah, A multi-scale limit of a randomly forced rotating 3-D compressible fluid. *J. Math. Fluid Mech.* **22** (2020), no. 3, Paper No. 30 Zbl 1451.35267 MR 4108618
[36] J. Nash, Continuity of solutions of parabolic and elliptic equations. *Amer. J. Math.* **80** (1958), 931–954 Zbl 0096.06902 MR 100158
[37] V.-S. Ngo and S. Scrobogna, Dispersive effects of weakly compressible and fast rotating inviscid fluids. *Discrete Contin. Dyn. Syst.* **38** (2018), no. 2, 749–789 Zbl 1379.35227 MR 3721875
[38] A. Novotný and I. Straškraba, *Introduction to the mathematical theory of compressible flow*. Oxford Lect. Ser. Math. Appl. 27, Oxford University Press, Oxford, 2004 Zbl 1088.35051 MR 2084891
[39] J. Pedlosky, *Geophysical fluid dynamics*. Springer, New York, 1987 Zbl 0713.76005
[40] G. Ponce, R. Racke, T. C. Sideris, and E. S. Titi, Global stability of large solutions to the 3D Navier-Stokes equations. *Comm. Math. Phys.* **159** (1994), no. 2, 329–341 Zbl 0795.35082 MR 1256992
[41] S. Scrobogna, Highly rotating fluids with vertical stratification for periodic data and vanishing vertical viscosity. *Rev. Mat. Iberoam.* **34** (2018), no. 1, 1–58 Zbl 1387.35469 MR 3763341
[42] G. A. Seregin, New version of the Ladyzhenskaya-Prodi-Serrin condition. *St Petersburg Math.* **18** (2007), no. 1, 80–103 Zbl 1129.35060 MR 2225215
[43] Y. Shibata and Y. Enomoto, Global existence of classical solutions and optimal decay rate for compressible flows via the theory of semigroups. In *Handbook of mathematical analysis in mechanics of viscous fluids*, pp. 2085–2181, Springer, Cham, 2018 Zbl 1386.76001 MR 3916811
[44] V. A. Solonnikov, Estimates of solutions of the Stokes equations in S. L. Sobolev spaces with a mixed norm. *J. Math. Sci. (N.Y.)* **123** (2004), no. 6, 4637–4653 Zbl 1068.35093 MR 1923551
[45] R. Temam and M. Ziane, Some mathematical problems in geophysical fluid dynamics. In *Handbook of mathematical fluid dynamics. Vol. III*, pp. 535–657, North-Holland, Amsterdam, 2004 Zbl 1222.35145 MR 2099038

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