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Lyapunov inequalities for Neumann boundary conditions at higher eigenvalues

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Abstract. This paper is devoted to the study of Lyapunov-type inequalities for Neumann boundary conditions at higher eigenvalues. Our main result is derived from a detailed analysis of the number and distribution of zeros of nontrivial solutions and their first derivatives, together with the use of suitable minimization problems. This method of proof yields new information on Lyapunov constants. For instance, we prove that as in the classical result by Lyapunov, the best constant is not attained. Additionally, we exploit the relation between Neumann boundary conditions and difocality to provide new nonresonance conditions at higher eigenvalues.

Keywords. Neumann boundary value problems, resonance, higher eigenvalues, Lyapunov inequalities, existence and uniqueness

1. Introduction

The classical L^1 Lyapunov inequality for the Neumann boundary problem

$$u''(x) + a(x)u(x) = 0, \quad x \in (0, L), \quad u'(0) = u'(L) = 0 \quad (1.1)$$

states that if

$$a \in L^1(0, L) \setminus \{0\}, \quad \int_0^L a(x) dx \geq 0, \quad (1.2)$$

is such that (1.1) has nontrivial solutions, then $\int_0^L a^+(x) dx > 4/L$, where $a^+(x) = \max\{a(x), 0\}$ ([5], [13]). In [1] and [14] the authors generalize this result by providing, for each p with $1 \leq p \leq \infty$, optimal necessary conditions for the boundary value problem (1.1) to have nontrivial solutions, given in terms of the L^p norm of the function a^+ . In particular, if $p = \infty$, it is proved that (1.1) has only the trivial solution if the function a satisfies

$$a \in L^\infty(0, L) \setminus \{0\}, \quad \int_0^L a \geq 0, \quad a^+ < \pi^2/L^2, \quad (1.3)$$

where for $c, d \in L^1(0, L)$, we write $c < d$ if $c(x) \leq d(x)$ for a.e. $x \in [0, L]$ and $c(x) < d(x)$ on a set of positive measure. This is a very well known result which is usually

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called the nonuniform nonresonance condition with respect to the first two eigenvalues $\lambda_0 = 0$ and $\lambda_1 = \pi^2/L^2$ of the eigenvalue problem

$$u''(x) + \lambda u(x) = 0, \quad x \in (0, L), \quad u'(0) = u'(L) = 0 \quad (1.4)$$

(see [8], [9] and [11]). From this point of view, it may be stated that the nonuniform nonresonance condition (1.3) is in fact the L^∞ Lyapunov inequality at the first two eigenvalues λ_0 and λ_1 .

On the other hand, the set of eigenvalues of (1.4) is given by $\lambda_n = n^2\pi^2/L^2$, $n \in \mathbb{N} \cup \{0\}$, and by using a general result due to Dolph [4], it can be proved that if for some $n \geq 1$ the function a satisfies

$$\lambda_n < a < \lambda_{n+1} \quad (1.5)$$

then (1.1) has only the trivial solution (see [10, Lemma 2.1] for some generalizations of (1.5) to more general boundary value problems). It is clear that condition (1.5) cannot be obtained from the L^p Lyapunov inequalities given in [1] and [14].

The previous observations motivate this article where, for any given natural number $n \geq 1$ and function a satisfying $\lambda_n < a$, we obtain the L^1 Lyapunov inequality (the case of L^p with $1 < p < \infty$ has some special features and will be considered in a forthcoming paper). In particular, we prove that, as in the classical Lyapunov inequality, the best constant is not attained for any value of n . To the best of our knowledge this result is new if $n \geq 1$. In the L^∞ case, the Lyapunov inequality is exactly (1.5) and in this sense, it is natural to say that this paper deals with Lyapunov inequalities at higher eigenvalues.

One of the main results of our paper is given by Lemma 2.2 below where we discuss in detail the number and distribution of zeros of u and u' , where u is any nontrivial solution of the linear boundary value problem (1.1).

In the second section we study the L^1 Lyapunov inequality when $\lambda_n < a$. The case where a satisfies $A \leq a(x) \leq B$ a.e. in $(0, L)$, where $\lambda_k < A < \lambda_{k+1} \leq B$ for some $k \in \mathbb{N} \cup \{0\}$, has been considered in [6]. In that paper the authors use optimal control theory methods, specially Pontryagin's maximum principle.

In the last section we use the natural relation between Neumann boundary conditions and disfocality, given by Lemma 2.2, to obtain new results on the existence and uniqueness of solutions for linear resonant problems with Neumann boundary conditions. We use L^1 and L^∞ Lyapunov constants. For example, by using Lemma 2.2 and the L^∞ Lyapunov inequality, we can prove (Theorem 3.1) that if $a \in L^\infty(0, L)$, $\lambda_n < a$ and there exists $0 = y_0 < y_1 < \dots < y_{2n+1} < y_{2n+2} = L$ such that

$$\max_{0 \leq i \leq 2n+1} \{(y_{i+1} - y_i)^2 \|a\|_{L^\infty(y_i, y_{i+1})}\} \leq \pi^2/4 \quad (1.6)$$

and, in addition, a is not the constant $\pi^2/4(y_{i+1} - y_i)^2$ at least in one of the intervals $[y_i, y_{i+1}]$, $0 \leq i \leq 2n + 1$, then (1.1) has only the trivial solution (this kind of functions a are usually named $2(n + 1)$ -step potentials).

The above hypothesis is optimal in the sense that if a is the constant $\pi^2/4(y_{i+1} - y_i)^2$ in each of the intervals (y_i, y_{i+1}) , $0 \leq i \leq 2n + 1$, then (1.1) has nontrivial solutions (see Remark 7 in Section 3). If $y_i = iL/2(n + 1)$, $0 \leq i \leq 2n + 2$, we have the so called

nonuniform nonresonance conditions at higher eigenvalues ([4], [10]) but if for instance $y_{j+1} - y_j < L/2(n+1)$ for some j , $0 \leq j \leq 2n+1$, then a can satisfy $\|a\|_{L^\infty(y_j, y_{j+1})} = \pi^2/4(y_{j+1} - y_j)^2$ (which is a quantity greater than $\lambda_{n+1} = (n+1)^2\pi^2/L^2$) as long as a satisfies (1.6) for each $i \neq j$.

Additionally, as has been done in [1], [2], [3] and [6], the linear study can be combined with the Schauder fixed point theorem to provide new conditions for the existence and uniqueness of solutions for resonant nonlinear problems (Theorem 3.3). Also, we may deal with other boundary value problems. Finally, one can expect that some results hold true in the case of the Neumann boundary value problem for partial differential equations

$$\Delta u(x) + a(x)u(x) = 0, \quad x \in \Omega, \quad \frac{\partial u(x)}{\partial n} = 0, \quad x \in \partial\Omega, \quad (1.7)$$

where Ω is a bounded and regular domain in \mathbb{R}^N , but here the role played by the dimension N may be important (see [2]).

2. Lyapunov inequality at higher eigenvalues

If $n \in \mathbb{N}$ is fixed, we introduce the set

$$\Lambda_n = \{a \in L^1(0, L) : \lambda_n < a \text{ and (1.1) has nontrivial solutions}\} \quad (2.1)$$

Here $u \in H^1(0, L)$, the usual Sobolev space. If we define

$$\beta_{1,n} \equiv \inf_{a \in \Lambda_n} \|a - \lambda_n\|_{L^1(0,L)}, \quad (2.2)$$

the main result of this section is the following.

Theorem 2.1.

$$\beta_{1,n} = \frac{2\pi n(n+1)}{L} \cot \frac{\pi n}{2(n+1)}.$$

Moreover, $\beta_{1,n}$ is not attained.

Proof. It is based on some lemmas. In the first one we perform a careful analysis of the number and distribution of zeros of nontrivial solutions u of (1.1). Since $a \in \Lambda_n$, it is clear that between two consecutive zeros of u there must exist a zero of u' , and between two consecutive zeros of u' there must exist a zero of u . More precisely, we have the following lemma.

Lemma 2.2. *Let $a \in \Lambda_n$ be given and u any nontrivial solution of (1.1). If the zeros of u' in $[0, L]$ are denoted by $0 = x_0 < x_2 < \dots < x_{2m} = L$ and the zeros of u in $(0, L)$ are denoted by $x_1 < x_3 < \dots < x_{2m-1}$, then:*

- (1) $x_{i+1} - x_i \leq L/2n$ for all $0 \leq i \leq 2m-1$. Moreover, at least one of these inequalities is strict.
- (2) $m \geq n+1$. Moreover, any value $m \geq n+1$ is possible.

(3) Let i , $0 \leq i \leq 2m - 1$, be given. Then the functions a and u satisfy

$$\|a - \lambda_n\|_{L^1(x_i, x_{i+1})} \geq \frac{\int_{x_i}^{x_{i+1}} u'^2 - \lambda_n \int_{x_i}^{x_{i+1}} u^2}{u^2(x_{i+1})} \quad \text{if } i \text{ is odd,} \quad (2.3)$$

$$\|a - \lambda_n\|_{L^1(x_i, x_{i+1})} \geq \frac{\int_{x_i}^{x_{i+1}} u'^2 - \lambda_n \int_{x_i}^{x_{i+1}} u^2}{u^2(x_i)} \quad \text{if } i \text{ is even.} \quad (2.4)$$

Proof. Let i , $0 \leq i \leq 2m - 1$, be given. Then u satisfies either the problem

$$u''(x) + a(x)u(x) = 0, \quad x \in (x_i, x_{i+1}), \quad u(x_i) = 0, \quad u'(x_{i+1}) = 0, \quad (2.5)$$

or the problem

$$u''(x) + a(x)u(x) = 0, \quad x \in (x_i, x_{i+1}), \quad u'(x_i) = 0, \quad u(x_{i+1}) = 0. \quad (2.6)$$

Let us assume the first case. The reasoning in the second case is similar. Note that u may be chosen such that $u(x) > 0$ for all $x \in (x_i, x_{i+1})$. Let us denote by μ_1^i and φ_1^i , respectively, the principal eigenvalue and eigenfunction of the eigenvalue problem

$$v''(x) + \mu v(x) = 0, \quad x \in (x_i, x_{i+1}), \quad v(x_i) = 0, \quad v'(x_{i+1}) = 0. \quad (2.7)$$

It is known that

$$\mu_1^i = \frac{\pi^2}{4(x_{i+1} - x_i)^2}, \quad \varphi_1^i(x) = \sin \frac{\pi(x - x_i)}{2(x_{i+1} - x_i)}. \quad (2.8)$$

Choosing φ_1^i as test function in the weak formulation of (2.5) and u as test function in the weak formulation of (2.7) for $\mu = \mu_1^i$ and $v = \varphi_1^i$, we obtain

$$\int_{x_i}^{x_{i+1}} (a(x) - \mu_1^i) u \varphi_1^i(x) dx = 0. \quad (2.9)$$

Then, if $x_{i+1} - x_i > L/2n$, we have

$$\mu_1^i = \frac{\pi^2 L^2}{4(x_{i+1} - x_i)^2 L^2} < \frac{n^2 \pi^2}{L^2} = \lambda_n \leq a(x) \quad \text{a.e. in } (x_i, x_{i+1}),$$

which is a contradiction with (2.9). Consequently, $x_{i+1} - x_i \leq L/2n$ for all $0 \leq i \leq 2m - 1$. Also, since $\lambda_n < a$ in $(0, L)$, we must have $\lambda_n < a$ in some subinterval (x_j, x_{j+1}) . If $x_{j+1} - x_j = L/2n$, it follows that $\mu_1^j < a$ in (x_j, x_{j+1}) , and this again contradicts (2.9). This completes the proof of the first part of the lemma. For the second part, let us observe that

$$L = \sum_{i=0}^{2m-1} (x_{i+1} - x_i) < 2m \frac{L}{2n}.$$

Consequently, $m > n$. Also, note that for any given natural number $q \geq n + 1$, the function $a(x) \equiv \lambda_q$ belongs to Λ_n and for $u(x) = \cos(q\pi x/L)$, we have $m = q$.

Lastly, if i with $0 \leq i \leq 2m - 1$ is given and u satisfies (2.5), then

$$\int_{x_i}^{x_{i+1}} u'^2(x) = \int_{x_i}^{x_{i+1}} a(x)u^2(x) = \int_{x_i}^{x_{i+1}} (a(x) - \lambda_n)u^2(x) + \int_{x_i}^{x_{i+1}} \lambda_n u^2(x).$$

Therefore,

$$\int_{x_i}^{x_{i+1}} u'^2(x) - \lambda_n \int_{x_i}^{x_{i+1}} u^2(x) \leq \|a - \lambda_n\|_{L^1(x_i, x_{i+1})} \|u^2\|_{L^\infty(x_i, x_{i+1})}$$

Since u' has no zeros in (x_i, x_{i+1}) and $u(x_i) = 0$, we have $\|u^2\|_{L^\infty(x_i, x_{i+1})} = u^2(x_{i+1})$. This proves the third part of the lemma when u satisfies (2.5). The reasoning is similar if u satisfies (2.6). \square

Lemma 2.3. Assume that $a < b$ and $0 < M \leq \pi^2/4(b-a)^2$ are given real numbers. Let $H = \{u \in H^1(a, b) : u(a) = 0, u(b) \neq 0\}$. If $J : H \rightarrow \mathbb{R}$ is defined by

$$J(u) = \frac{\int_a^b u'^2 - M \int_a^b u^2}{u^2(b)} \quad (2.10)$$

and $m \equiv \inf_{u \in H} J(u)$, then m is attained. Moreover,

$$m = M^{1/2} \cot(M^{1/2}(b-a)), \quad (2.11)$$

and if $u \in H$, then $J(u) = m \Leftrightarrow u(x) = k \sin(M^{1/2}(x-a))/\sin(M^{1/2}(b-a))$ for some nonzero constant k .

Proof. Remember that $\delta_1 = \pi^2/4(b-a)^2$ is the principal eigenvalue of the eigenvalue problem $v''(x) + \delta v(x) = 0$, $v(a) = 0$, $v'(b) = 0$, with associated eigenfunction $w(x) = \sin \frac{\pi(x-a)}{2(b-a)}$. Therefore, if $M = \pi^2/4(b-a)^2$, then $m = 0$ and it is attained at the function w .

If $M < \delta_1 = \pi^2/4(b-a)^2$, there exists some positive constant c such that

$$\int_a^b u'^2 - M \int_a^b u^2 \geq c \int_a^b u'^2, \quad \forall u \in H. \quad (2.12)$$

If $\{u_n\} \subset H$ is a minimizing sequence for J , since the sequence $\{k_n u_n\}$, $k_n \neq 0$, is also a minimizing sequence, we can assume without loss of generality that $u_n(b) = 1$. From (2.12) we deduce that $\int_a^b u_n'^2$ is bounded. So, we can suppose that, up to a subsequence, $u_n \rightharpoonup u_0$ in $H^1(a, b)$ and $u_n \rightarrow u_0$ in $C[a, b]$ (with the uniform norm). The strong convergence in $C[a, b]$ gives us $u_0(b) = 1$. The weak convergence in H implies $J(u_0) \leq \liminf J(u_n) = m$. Thus u_0 is a minimizer.

Since $J(u_0) = \min\{J(v) : v \in H^1(a, b), v(a) = 0, v(b) = 1\}$, the Lagrange multiplier theorem implies that there are real numbers α_1, α_2 such that

$$2 \int_a^b u_0' v' - 2M \int_a^b u_0 v - \alpha_1 v(b) - \alpha_2 v(a) = 0, \quad \forall v \in H^1(a, b).$$

In particular,

$$\int_a^b u_0' v' - M \int_a^b u_0 v = 0, \quad \forall v \in H^1(a, b) : v(a) = v(b) = 0.$$

We conclude that u_0 satisfies the problem

$$u_0''(x) + M u_0(x) = 0, \quad x \in (a, b), \quad u_0(a) = 0, \quad u_0(b) = 1. \quad (2.13)$$

Note that since $M < \pi^2/(b-a)^2$, (2.13) has a unique solution, which is given by

$$u_0(x) = \frac{\sin(M^{1/2}(x-a))}{\sin(M^{1/2}(b-a))}. \quad (2.14)$$

Finally, an elementary calculation gives $J(u_0) = M^{1/2} \cot(M^{1/2}(b-a))$. This proves the lemma. \square

Now, we combine Lemmas 2.2 and 2.3 to obtain the following result.

Lemma 2.4. *Let $a \in \Lambda_n$ be given and u any nontrivial solution of (1.1). If the zeros of u' are denoted by $0 = x_0 < x_2 < \dots < x_{2m} = L$ and the zeros of u are denoted by $x_1 < x_3 < \dots < x_{2m-1}$, then*

$$\|a - \lambda_n\|_{L^1(0,L)} \geq \frac{n\pi}{L} \sum_{i=0}^{2m-1} \cot\left(\frac{n\pi}{L}(x_{i+1} - x_i)\right). \quad (2.15)$$

The previous reasoning motivates the study of a special minimization problem given in the following lemma.

Lemma 2.5. *Given any $r \in \mathbb{N}$ and $S \in \mathbb{R}^+$ satisfying $r\pi > 2S$, let*

$$Z = \left\{ z = (z_0, z_1, \dots, z_{r-1}) \in (0, \pi/2]^r : \sum_{i=0}^{r-1} z_i = S \right\}.$$

If $F : Z \rightarrow \mathbb{R}$ is defined by

$$F(z) = \sum_{i=0}^{r-1} \cot z_i,$$

then $\inf_{z \in Z} F(z)$ is attained and its value is $r \cot(S/r)$. Moreover, $z \in Z$ is a minimizer if and only if $z_i = S/r$ for all $0 \leq i \leq r-1$.

Proof. Let us observe that for all $z \in Z$, $\cot z_i \geq 0$ for $0 \leq i \leq r-1$. Moreover, if $z_i \rightarrow 0^+$ for some $0 \leq i \leq r-1$, then $\cot z_i \rightarrow +\infty$. Also, since $r\pi > 2S$, if $z \in Z$ is such that $z_i = \pi/2$ for some $0 \leq i \leq r-1$, then there must exist some $0 \leq j \leq r-1$ such that $z_j < \pi/2$. Let us choose the point $z^* \in Z$ defined (for $\delta > 0$ sufficiently small) by $z_k^* = z_k$, if $k \neq i$ and $k \neq j$, $z_i^* = \pi/2 - \delta$, $z_j^* = z_j + \delta$. An elementary calculation shows

$$F(z^*) - F(z) = \frac{\cot z_j (1 - \cot z_j \cot \delta)}{\cot \delta (\cot z_j + \cot \delta)},$$

which is a negative number for δ sufficiently small. Consequently, there exists a sufficiently small positive constant ε_1 such that

$$\inf_{z \in Z} F(z) = \min_{z \in [\varepsilon_1, \pi/2]^r} F(z) = \min_{z \in (\varepsilon_1, \pi/2)^r} F(z).$$

Then, if $z \in Z$ is any minimizer of F , the Lagrange multiplier theorem implies that there is $\lambda \in \mathbb{R}$ such that

$$\frac{-1}{\sin^2 z_i} + \lambda = 0, \quad 0 \leq i \leq r-1, \quad \sum_{i=0}^{r-1} z_i = S.$$

We conclude that $z_i = S/r$, $0 \leq i \leq r-1$, and the lemma is proved. \square

From the previous two lemmas, we obtain the following one.

Lemma 2.6.

$$\beta_{1,n} \geq \frac{n\pi}{L} 2(n+1) \cot \frac{n\pi}{2(n+1)}. \quad (2.16)$$

Proof. Let $a \in \Lambda_n$ be given and u any nontrivial solution of (1.1). If the zeros of u' are denoted by $0 = x_0 < x_2 < \dots < x_{2m} = L$ and the zeros of u are denoted by $x_1 < x_3 < \dots < x_{2m-1}$, then from Lemmas 2.4 and 2.5 (with $r = 2m$, $S = n\pi$ and $z_i = \frac{n\pi}{L}(x_{i+1} - x_i)$) we obtain

$$\|a - \lambda_n\|_{L^1(0,L)} \geq \frac{n\pi}{L} \sum_{i=0}^{2m-1} \cot\left(\frac{n\pi}{L}(x_{i+1} - x_i)\right) \geq \frac{n\pi}{L} 2m \cot \frac{n\pi}{2m}. \quad (2.17)$$

Finally, taking into account that

$$\text{the function } 2m \cot \frac{n\pi}{2m} \text{ is strictly increasing with respect to } m \quad (P)$$

and that $m \geq n+1$, we deduce (2.16). \square

In the next lemma, we define a minimizing sequence for $\beta_{1,n}$.

Lemma 2.7. Let $\varepsilon > 0$ be sufficiently small. Define $u_\varepsilon : [0, L] \rightarrow \mathbb{R}$ by

$$u_\varepsilon(x) = \begin{cases} -\sin\left(\frac{n\pi}{L}\left(x - \frac{L}{2(n+1)}\right)\right) + \frac{n\pi}{L} \frac{(x-\varepsilon)^3}{3\varepsilon^2} \cos\left(\frac{n\pi}{2(n+1)}\right) & \text{if } 0 \leq x \leq \varepsilon, \\ -\sin\left(\frac{n\pi}{L}\left(x - \frac{L}{2(n+1)}\right)\right) & \text{if } \varepsilon \leq x \leq \frac{L}{2(n+1)}, \\ -u_\varepsilon\left(\frac{2L}{2(n+1)} - x\right) & \text{if } \frac{L}{2(n+1)} \leq x \leq \frac{2L}{2(n+1)}, \\ u_\varepsilon\left(\frac{4L}{2(n+1)} - x\right) & \text{if } \frac{2L}{2(n+1)} \leq x \leq \frac{4L}{2(n+1)}, \\ -u_\varepsilon\left(\frac{6L}{2(n+1)} - x\right) & \text{if } \frac{4L}{2(n+1)} \leq x \leq \frac{6L}{2(n+1)}, \\ \dots & \dots \end{cases} \quad (2.18)$$

Then $u_\varepsilon \in C^2[0, L]$, the function $a_\varepsilon(x) \equiv -u_\varepsilon''(x)/u_\varepsilon(x)$ for all $x \in [0, L]$, $x \neq \frac{(2k-1)L}{2(n+1)}$, $1 \leq k \leq n+1$, belongs to Λ_n and

$$\liminf_{\varepsilon \rightarrow 0^+} \|a_\varepsilon - \lambda_n\|_{L^1(0,L)} = \frac{n\pi}{L} 2(n+1) \cot \frac{n\pi}{2(n+1)}. \quad (2.19)$$

Proof. We claim that for each $0 \leq i \leq 2n+1$, the function a_ε satisfies

$$\lambda_n < a_\varepsilon \quad \text{in} \left(\frac{iL}{2(n+1)}, \frac{(i+1)L}{2(n+1)} \right), \quad (2.20)$$

and

$$\liminf_{\varepsilon \rightarrow 0^+} \|a_\varepsilon - \lambda_n\|_{L^1(\frac{iL}{2(n+1)}, \frac{(i+1)L}{2(n+1)})} = \frac{n\pi}{L} \cot \frac{n\pi}{2(n+1)}. \quad (2.21)$$

It is trivial that (2.19) follows from (2.20) and (2.21). Moreover, taking into account the definition of u_ε , it is clear that it is sufficient to prove the claim in the case $i = 0$. Now, if $x \in (0, L/2(n+1))$ we can distinguish two cases:

- (1) $x \in (\varepsilon, L/2(n+1))$. Then $a_\varepsilon(x) = -u_\varepsilon''(x)/u_\varepsilon(x) \equiv \lambda_n$.
- (2) $x \in (0, \varepsilon)$. Then

$$a_\varepsilon(x) - \lambda_n = \frac{-2 \frac{x-\varepsilon}{\varepsilon^2} \frac{n\pi}{L} \cos \frac{n\pi}{2(n+1)} - \frac{(x-\varepsilon)^3}{3\varepsilon^2} \frac{n^3\pi^3}{L^3} \cos \frac{n\pi}{2(n+1)}}{-\sin\left(\frac{n\pi}{L}\left(x - \frac{L}{2(n+1)}\right)\right) + \frac{(x-\varepsilon)^3}{3\varepsilon^2} \frac{n\pi}{L} \cos \frac{n\pi}{2(n+1)}} > 0.$$

Therefore $a_\varepsilon \in \Lambda_n$. Moreover, if $\varepsilon \rightarrow 0^+$, then

$$\frac{-\frac{(x-\varepsilon)^3}{3\varepsilon^2} \frac{n^3\pi^3}{L^3} \cos \frac{n\pi}{2(n+1)}}{-\sin\left(\frac{n\pi}{L}\left(x - \frac{L}{2(n+1)}\right)\right) + \frac{(x-\varepsilon)^3}{3\varepsilon^2} \frac{n\pi}{L} \cos \frac{n\pi}{2(n+1)}} \rightarrow 0,$$

uniformly in $x \in (0, \varepsilon)$.

Finally, since

$$\lim_{\varepsilon \rightarrow 0^+} \int_0^\varepsilon \left[\frac{-2 \frac{x-\varepsilon}{\varepsilon^2} \frac{n\pi}{L} \cos \frac{n\pi}{2(n+1)}}{-\sin\left(\frac{n\pi}{L}\left(x - \frac{L}{2(n+1)}\right)\right) + \frac{(x-\varepsilon)^3}{3\varepsilon^2} \frac{n\pi}{L} \cos \frac{n\pi}{2(n+1)}} - \frac{-2 \frac{x-\varepsilon}{\varepsilon^2} \frac{n\pi}{L} \cos \frac{n\pi}{2(n+1)}}{-\sin\left(\frac{n\pi}{L}\left(x - \frac{L}{2(n+1)}\right)\right)} \right] dx = 0,$$

and

$$-\sin\left(\frac{n\pi}{L}\left(x - \frac{L}{2(n+1)}\right)\right) \rightarrow \sin \frac{n\pi}{2(n+1)}$$

uniformly in $x \in (0, \varepsilon)$ when $\varepsilon \rightarrow 0^+$, we deduce

$$\begin{aligned} \liminf_{\varepsilon \rightarrow 0^+} \|a_\varepsilon - \lambda_n\|_{L^1(0, \frac{L}{2(n+1)})} &= \liminf_{\varepsilon \rightarrow 0^+} \frac{n\pi}{L} \cot \frac{n\pi}{2(n+1)} \frac{2}{\varepsilon^2} \int_0^\varepsilon (\varepsilon - x) dx \\ &= \frac{n\pi}{L} \cot \frac{n\pi}{2(n+1)}, \end{aligned}$$

which is (2.21) for the case $i = 0$. \square

Lemma 2.8. $\beta_{1,n}$ is not attained.

Proof. Let $a \in \Lambda_n$ be such that $\|a - \lambda_n\|_{L^1(0,L)} = \beta_{1,n}$. Let u be any nontrivial solution of (1.1) associated to the function a . As previously, we denote the zeros of u' by $0 = x_0 < x_2 < \dots < x_{2m} = L$ and the zeros of u by $x_1 < x_3 < \dots < x_{2m-1}$. By using Lemmas 2.4, 2.5 and 2.7, we have

$$\begin{aligned} \beta_{1,n} &= \|a - \lambda_n\|_{L^1(0,L)} = \sum_{i=0}^{2m-1} \|a - \lambda_n\|_{L^1(x_i, x_{i+1})} \\ &\geq \sum_{i=0}^{2m-1} J_i(u) \geq \frac{n\pi}{L} \sum_{i=0}^{2m-1} \cot \frac{n\pi(x_{i+1} - x_i)}{L} \\ &\geq \frac{n\pi}{L} 2m \cot \frac{n\pi}{2m} \geq \frac{n\pi}{L} 2(n+1) \cot \frac{n\pi}{2(n+1)} = \beta_{1,n} \end{aligned} \quad (2.22)$$

where $J_i(u)$ is given either by

$$J_i(u) = \frac{\int_{x_i}^{x_{i+1}} u'^2 - \lambda_n \int_{x_i}^{x_{i+1}} u^2}{u^2(x_{i+1})} \quad \text{if } u(x_i) = 0,$$

or by

$$J_i(u) = \frac{\int_{x_i}^{x_{i+1}} u'^2 - \lambda_n \int_{x_i}^{x_{i+1}} u^2}{u^2(x_i)} \quad \text{if } u(x_{i+1}) = 0.$$

Consequently, all inequalities in (2.22) transform into equalities. In particular, from Lemma 2.5 and the property (P) shown in Lemma 2.6 we obtain

$$m = n + 1, \quad x_{i+1} - x_i = \frac{L}{2(n+1)}, \quad 0 \leq i \leq 2n + 1.$$

Also, it follows that

$$J_i(u) = \frac{n\pi}{L} \cot \frac{n\pi}{L} \frac{L}{2(n+1)}, \quad 0 \leq i \leq 2n + 1.$$

From Lemma 2.3 we deduce that, up to some nonzero constants, in each interval $[x_i, x_{i+1}]$,

$$u(x) = \begin{cases} \frac{\sin \frac{n\pi}{L}(x - x_i)}{\sin \frac{n\pi}{L}(x_{i+1} - x_i)} & \text{if } i \text{ is odd,} \\ \frac{\sin \frac{n\pi}{L}(x - x_{i+1})}{\sin \frac{n\pi}{L}(x_i - x_{i+1})} & \text{if } i \text{ is even.} \end{cases}$$

In particular, in $[0, L/2(n+1)]$, u must be the function

$$u(x) = \frac{\sin \frac{n\pi}{L}(x - \frac{L}{2(n+1)})}{\sin \frac{n\pi}{L}(-\frac{L}{2(n+1)})},$$

which does not satisfy the condition $u'(0) = 0$. The conclusion is that $\beta_{1,n}$ is not attained. \square

Finally, as a trivial consequence of Lemmas 2.6, 2.7 and 2.8 we have the conclusion of Theorem 2.1. \square

Remark 1. Let us observe that if we consider $\beta_{1,n}$ as a function of $n \in (0, +\infty)$, then $\lim_{n \rightarrow 0^+} \beta_{1,n} = 4/L$, the constant of the classical L^1 Lyapunov inequality at the first eigenvalue ([5]).

Remark 2. The case where $L = 1$ and the function a satisfies the condition $A \leq a(x) \leq B$, a.e. in $(0, L)$ where $\lambda_k < A < \lambda_{k+1} \leq B$ for some $k \in \mathbb{N} \cup \{0\}$, has been considered in [6], where the authors use optimal control theory methods. They define $\Lambda_{A,B}$ to be the set of $a \in L^1(0, L)$ such that $A \leq a(x) \leq B$ a.e. in $(0, L)$ and (1.1) has nontrivial solutions. Then, by using Pontryagin's maximum principle, they prove that the number

$$\beta_{A,B} \equiv \inf_{a \in \Lambda_{A,B}} \|a\|_{L^1(0,L)}$$

is attained. In addition, they calculate $\lim_{B \rightarrow +\infty} \beta_{A,B}$.

Remark 3. In our opinion, the inequality $\int_0^1 b(t) dt \leq 2\sqrt{A} \cot(\sqrt{A}/2)$ in [6, Theorem 3] must be substituted by $\int_0^1 b(t) dt \leq A + 2(k+1)\sqrt{A} \cot \frac{\sqrt{A}}{2(k+1)}$. This may be easily derived from our method by modifying the definition of the set Λ_n (given in (2.1)) in a trivial way.

Remark 4. If $A \rightarrow \lambda_k^+$, it does not seem possible to deduce from [6] that the constant $\beta_{1,k}$ (defined in (2.2)) is not attained. In fact, to the best of our knowledge, this result is new. Moreover, our method, which combines a detailed analysis of the number and distribution of zeros of nontrivial solutions of (1.1) and their first derivatives with the use of suitable minimization problems, will be very useful to combine Lyapunov inequalities and disfocality. This will be seen in the next section.

Remark 5. We can use our methods to make an analogous study for other boundary conditions. In particular, with the help of Lemmas 2.2 and 2.3 we can consider the mixed linear problem

$$u''(x) + a(x)u(x) = 0, \quad x \in (0, L), \quad u'(0) = u(L) = 0 \quad (2.23)$$

where

$$a \in \Gamma_n = \{a \in L^1(0, L) : \mu_n < a \text{ and (2.23) has nontrivial solutions}\}.$$

Here μ_n is the n -th eigenvalue of the eigenvalue problem

$$u''(x) + \mu u(x) = 0, \quad x \in (0, L), \quad u'(0) = u(L) = 0. \quad (2.24)$$

The case where $L = 1$ and the function a satisfies the condition $A \leq a(x) \leq B$, a.e. in $(0, L)$ where $\mu_k < A < \mu_{k+1} \leq B$ has been considered in [7]. As in [6], the authors use optimal control theory methods. See also [12] for Dirichlet boundary conditions.

3. Lyapunov inequalities and disfocality

The L^∞ Lyapunov inequality is trivial from Dolph's result ([4]). In fact, by using Dolph's result, the constant

$$\beta_{\infty,n} \equiv \inf_{a \in \Lambda_n} \|a\|_{L^\infty(0,L)} \quad (3.1)$$

must be greater than or equal to λ_{n+1} . Since the constant function λ_{n+1} is an element of Λ_n , we deduce

$$\beta_{\infty,n} = \lambda_{n+1}. \quad (3.2)$$

Moreover, $\beta_{\infty,n}$ is attained at a unique element $a_\infty \in \Lambda_n$ given by the constant function $a_\infty \equiv \lambda_{n+1}$. On the other hand, under the restriction

$$a \in L^1(0, L), \quad \lambda_n < a, \quad (3.3)$$

the relation between Neumann boundary conditions and disfocality arises in a natural way. In fact, if $u \in H^1(0, L)$ is any nontrivial solution of (1.1) and the zeros of u are denoted by $x_1 < x_3 < \dots < x_{2m-1}$, and the zeros of u' are denoted by $0 = x_0 < x_2 < \dots < x_{2m} = L$, then for each i , $0 \leq i \leq 2m - 1$, the function u satisfies

$$u''(x) + a(x)u(x) = 0, \quad x \in (x_i, x_{i+1}), \quad u(x_i) = 0, \quad u'(x_{i+1}) = 0, \quad \text{if } i \text{ is odd,} \quad (3.4)$$

and

$$u''(x) + a(x)u(x) = 0, \quad x \in (x_i, x_{i+1}), \quad u'(x_i) = 0, \quad u(x_{i+1}) = 0, \quad \text{if } i \text{ is even.} \quad (3.5)$$

Consequently, each of the problems (3.4) and (3.5) with $0 \leq i \leq 2m - 1$ has a nontrivial solution. This simple observation can be used to deduce the following conclusion: if a is any function satisfying (3.3) such that for any $m \geq n + 1$ and any distribution of numbers $0 = x_0 < x_1 < x_2 < \dots < x_{2m-1} < x_{2m} = L$, either some problem of the type (3.4) or some problem of the type (3.5) has only the trivial solution, then problem (1.1) has only the trivial solution. Lastly, it has been established in [3] (Theorem 2.1 for the case $p = \infty$) that if $b \in L^\infty(c, d)$ satisfies

$$\|b\|_{L^\infty(c,d)} \leq \frac{\pi^2}{4(d-c)^2} \quad \text{and} \quad b \neq \frac{\pi^2}{4(d-c)^2} \quad \text{in } (c, d), \quad (3.6)$$

then the unique solution of the boundary value problems

$$u''(x) + b(x)u(x) = 0, \quad x \in (c, d), \quad u'(c) = u(d) = 0, \quad (3.7)$$

and

$$u''(x) + b(x)u(x) = 0, \quad x \in (c, d), \quad u(c) = u'(d) = 0 \quad (3.8)$$

is the trivial one.

We may use the previous reasonings to obtain the following result:

Theorem 3.1. *If $a \in L^\infty(0, L)$, $\lambda_n < a$ and there exists $0 = y_0 < y_1 < \dots < y_{2n+1} < y_{2n+2} = L$ such that*

$$\max_{0 \leq i \leq 2n+1} \{(y_{i+1} - y_i)^2 \|a\|_{L^\infty(y_i, y_{i+1})}\} \leq \pi^2/4, \quad (3.9)$$

and in addition a is not the constant $\pi^2/4(y_{i+1} - y_i)^2$ in at least one of the intervals $[y_i, y_{i+1}]$, $0 \leq i \leq 2n + 1$, then the boundary value problem (1.1) has only the trivial solution.

Proof. If $m \geq n + 1$ and $0 = x_0 < x_1 < x_2 < \dots < x_{2m-1} < x_{2m} = L$ is any distribution of numbers, then either

$$[x_j, x_{j+1}] \subset [y_i, y_{i+1}], \quad \text{strictly}, \quad (3.10)$$

for some $0 \leq i \leq 2n + 1$, $0 \leq j \leq 2m - 1$, or

$$m = n + 1 \quad \text{and} \quad x_i = y_i, \quad \forall 0 \leq i \leq 2n + 2. \quad (3.11)$$

If (3.10) is satisfied, then

$$\|a\|_{L^\infty(x_j, x_{j+1})} < \|a\|_{L^\infty(y_i, y_{i+1})} \leq \frac{\pi^2}{4(y_{i+1} - y_i)^2} < \frac{\pi^2}{4(x_{j+1} - x_j)^2} \quad (3.12)$$

and consequently we deduce from (3.4)–(3.6) that (1.1) has only the trivial solution.

If (3.11) is satisfied, we deduce from the hypotheses of the theorem that a is not the constant $\pi^2/4(x_{i+1} - x_i)^2$ in at least one of the intervals $[x_i, x_{i+1}]$, $0 \leq i \leq 2n + 1$. Therefore, again (3.4)–(3.6) imply that (1.1) has only the trivial solution. In any case, we have the desired conclusion. \square

Remark 6. If in the previous theorem we choose $y_i = iL/2(n + 1)$, $0 \leq i \leq 2n + 2$, then we have the so called nonuniform nonresonance conditions at higher eigenvalues ([4], [10]) but if for instance, $y_{j+1} - y_j < L/2(n + 1)$ for some j , $0 \leq j \leq 2n + 1$, the function a can satisfy $\|a\|_{L^\infty(y_j, y_{j+1})} = \pi^2/4(y_{j+1} - y_j)^2$ (which is a quantity greater than $\lambda_{n+1} = (n + 1)^2\pi^2/L^2$) as long as a satisfies (3.9) for each $i \neq j$.

Remark 7. The hypothesis of the previous theorem is optimal in the sense that if a is the constant $\pi^2/4(y_{i+1} - y_i)^2$ in each of the intervals (y_i, y_{i+1}) , $0 \leq i \leq 2n + 1$, then (1.1) has nontrivial solutions. In fact, if this is the case, it is easily checked that there exist appropriate constants k_i , $0 \leq i \leq 2n + 1$, such that the function

$$u(x) = \begin{cases} k_i \cos \frac{\pi(x - y_i)}{2(y_{i+1} - y_i)}, & x \in [y_i, y_{i+1}], \quad i \text{ even,} \\ k_i \cos \frac{\pi(y_{i+1} - x)}{2(y_{i+1} - y_i)}, & x \in [y_i, y_{i+1}], \quad i \text{ odd,} \end{cases}$$

is a nontrivial solution of (1.1).

Now we comment on some relations between the Lyapunov constant $\beta_{1,n}$ given in Theorem 2.1 and disfocality. It is clear from the definition of $\beta_{1,n}$ that if a function a satisfies

$$a \in L^1(0, L), \quad \lambda_n < a, \quad \|a - \lambda_n\|_1 < \beta_{1,n}, \quad (3.13)$$

then the unique solution of (1.1) is the trivial one. In the next theorem we prove that, with the use of disfocality, we can obtain a more general condition.

Theorem 3.2.

(1) If $a \in L^1(0, L)$, $\lambda_n < a$, and there exist $0 = y_0 < y_1 < \dots < y_{2n+1} < y_{2n+2} = L$ such that

$$y_{i+1} - y_i < \frac{L}{2n}; \quad \|a - \lambda_n\|_{L^1(y_i, y_{i+1})} < \frac{n\pi}{L} \cot \frac{n\pi(y_{i+1} - y_i)}{L}, \quad \forall 0 \leq i \leq 2n + 1, \quad (3.14)$$

then the unique solution of (1.1) is the trivial one.

(2) (3.13) implies (3.14).

(3) If $0 = y_0 < y_1 < \dots < y_{2n+1} < y_{2n+2} = L$ is any distribution of numbers such that $y_{k+1} - y_k < L/2n$ for all $0 \leq k \leq 2n + 1$ and $y_{i+1} - y_i \neq y_{j+1} - y_j$ for some $0 \leq i, j \leq 2n + 1$, then there exists $a \in L^1(0, L)$ with $\lambda_n < a$, satisfying (3.14) but not (3.13).

Proof. If a satisfies (3.14), then the unique solution of (1.1) is the trivial one. In fact, if this is not true, let u be a nontrivial solution of (1.1) and denote the zeros of u by $x_1 < x_3 < \dots < x_{2m-1}$ and the zeros of u' by $0 = x_0 < x_2 < \dots < x_{2m} = L$. Since $m \geq n + 1$,

$$[x_j, x_{j+1}] \subset [y_i, y_{i+1}] \quad (3.15)$$

for some $0 \leq i \leq 2n + 1$, $0 \leq j \leq 2m - 1$. Consequently,

$$\frac{\|a - \lambda_n\|_{L^1(x_j, x_{j+1})}}{\cot \frac{n\pi(x_{j+1} - x_j)}{L}} \leq \frac{\|a - \lambda_n\|_{L^1(y_i, y_{i+1})}}{\cot \frac{n\pi(y_{i+1} - y_i)}{L}} < \frac{n\pi}{L}.$$

From this we deduce

$$\|a - \lambda_n\|_{L^1(x_j, x_{j+1})} < \frac{n\pi}{L} \cot \frac{n\pi(x_{j+1} - x_j)}{L},$$

which is a contradiction with Lemmas 2.2 and 2.3.

Next we prove that (3.13) implies (3.14). We can certainly assume that $\inf a > \lambda_n$, for if not, we replace a by $a + \delta$ (for small $\delta > 0$) and the new function $a + \delta$ satisfies (3.13). Note that if condition (3.14) is satisfied for $a + \delta$ then it is also satisfied for a .

Now choose $\varepsilon > 0$ sufficiently small. Since the function

$$\frac{\|a - \lambda_n\|_{L^1(0, y)}}{\cot \frac{n\pi(y-0)}{L}}$$

is strictly increasing with respect to $y \in (0, L/2n)$ and

$$\lim_{y \rightarrow 0^+} \frac{\|a - \lambda_n\|_{L^1(0,y)}}{\cot \frac{n\pi(y-0)}{L}} = 0, \quad \lim_{y \rightarrow \frac{L}{2n}^-} \frac{\|a - \lambda_n\|_{L^1(0,y)}}{\cot \frac{n\pi(y-0)}{L}} = +\infty,$$

there is a unique y_1 , $0 = y_0 < y_1 < L/2n$, such that

$$\frac{\|a - \lambda_n\|_{L^1(0,y_1)}}{\cot \frac{n\pi(y_1-0)}{L}} = \frac{n\pi}{L} - \varepsilon. \quad (3.16)$$

With the help of a similar reasoning, it is possible to prove the existence of points $0 = y_0 < y_1 < \dots < y_{2n+1}$ such that

$$\frac{\|a - \lambda_n\|_{L^1(y_i, y_{i+1})}}{\cot \frac{n\pi(y_{i+1}-y_i)}{L}} = \frac{n\pi}{L} - \varepsilon, \quad y_{i+1} - y_i < \frac{L}{2n}, \quad 0 \leq i \leq 2n. \quad (3.17)$$

(If necessary, we can define $a(x) = \lambda_n$ for all $x > L$).

Since $y_{i+1} - y_i < L/2n$ for all $0 \leq i \leq 2n - 1$, we have $y_{2n} < L$.

If $y_{2n+1} \geq L$, then we replace y_{2n+1} with $y_{2n+1} = L - \mu$ (for small $\mu > 0$). Finally, choosing $y_{2n+2} = L$, we obtain (3.14).

If $y_{2n+1} < L$, take $y_{2n+2} = L$. We claim that

$$y_{2n+2} - y_{2n+1} < \frac{L}{2n} \quad \text{and} \quad \frac{\|a - \lambda_n\|_{L^1(y_{2n+1}, y_{2n+2})}}{\cot \frac{n\pi(y_{2n+2}-y_{2n+1})}{L}} < \frac{n\pi}{L} - \varepsilon. \quad (3.18)$$

In fact, if $y_{2n+2} - y_{2n+1} \geq L/2n$, then $y_{2n+1} \leq L(2n-1)/2n$. Then, from (3.17), Lemma 2.5 (with $r = 2n + 1$, $S = \frac{n\pi}{L}(y_{2n+1})$ and $z_i = \frac{n\pi}{L}(y_{i+1} - y_i)$) and using the monotonicity of \cot in $(0, \pi/2)$ we obtain

$$\begin{aligned} \frac{n\pi}{L} 2(n+1) \cot \frac{n\pi}{2(n+1)} &= \beta_{1,n} > \sum_{i=0}^{2n} \|a - \lambda_n\|_{L^1(y_i, y_{i+1})} \\ &= \left(\frac{n\pi}{L} - \varepsilon\right) \sum_{i=0}^{2n} \cot \frac{n\pi}{L} (y_{i+1} - y_i) \geq \left(\frac{n\pi}{L} - \varepsilon\right) (2n+1) \cot \frac{n\pi}{L(2n+1)} y_{2n+1} \\ &\geq \left(\frac{n\pi}{L} - \varepsilon\right) (2n+1) \cot \frac{\pi(2n-1)}{2(2n+1)}. \end{aligned}$$

If $\varepsilon \rightarrow 0^+$, we conclude

$$\beta_{1,n} \geq \frac{n\pi}{L} (2n+1) \cot \frac{\pi(2n-1)}{2(2n+1)}. \quad (3.19)$$

Now, since the function $x \mapsto \frac{2\pi \cot x}{\pi - 2x}$ is strictly decreasing in $(0, \pi/2)$ and $\frac{\pi(2n-1)}{2(2n+1)} < \frac{n\pi}{2(n+1)}$, we obtain

$$\beta_{1,n} \geq \frac{n\pi}{L} (2n+1) \cot \frac{\pi(2n-1)}{2(2n+1)} > \frac{n\pi}{L} 2(n+1) \cot \frac{n\pi}{2(n+1)} = \beta_{1,n},$$

which is a contradiction.

It remains to prove the second part of the claim (3.18). In fact, if it is not true, then from (3.17) and Lemma 2.5 (with $r = 2n + 2$, $S = n\pi$ and $z_i = \frac{n\pi}{L}(y_{i+1} - y_i)$) we have

$$\begin{aligned} \|a - \lambda_n\|_{L^1(0,L)} &= \sum_{i=0}^{2n+1} \|a - \lambda_n\|_{L^1(y_i, y_{i+1})} \\ &\geq \left(\frac{n\pi}{L} - \varepsilon\right) \sum_{i=0}^{2n+1} \cot \frac{n\pi(y_{i+1} - y_i)}{L} \geq \left(\frac{n\pi}{L} - \varepsilon\right) \frac{\beta_{1,n}}{n\pi/L} \end{aligned}$$

for $\varepsilon > 0$ sufficiently small. This is a contradiction with (3.13).

Finally, to prove (3), take $0 = y_0 < y_1 < \dots < y_{2n+1} < y_{2n+2} = L$ such that $y_{k+1} - y_k < L/2n$ for all $0 \leq k \leq 2n + 1$ and $y_{i+1} - y_i \neq y_{j+1} - y_j$ for some $0 \leq i, j \leq 2n + 1$. Then from Lemma 2.5 we obtain

$$\sum_{i=0}^{2n+1} \frac{n\pi}{L} \cot \frac{n\pi(y_{i+1} - y_i)}{L} > \frac{2\pi n(n+1)}{L} \cot \frac{n\pi}{2(n+1)} = \beta_{1,n}.$$

Now, choose $a \in L^1(0, L)$ with $\lambda_n < a$ satisfying

$$\|a - \lambda_n\|_{L^1(y_i, y_{i+1})} = \frac{n\pi}{L} \cot \frac{n\pi(y_{i+1} - y_i)}{L} - \varepsilon, \quad \forall 0 \leq i \leq 2n + 1.$$

It is trivial that if ε is sufficiently small, then a satisfies (3.14) whereas

$$\|a - \lambda_n\|_{L^1(0,L)} = \sum_{i=0}^{2n+1} \|a - \lambda_n\|_{L^1(y_i, y_{i+1})} > \beta_{1,n}. \quad \square$$

Final remark on nonlinear problems

We finish this paper by showing how to use previous reasonings to obtain new theorems on the existence and uniqueness of solutions of nonlinear b.v.p.

$$u''(x) + f(x, u(x)) = 0, \quad x \in (0, L), \quad u'(0) = u'(L) = 0. \quad (3.20)$$

For example, we have the following theorem related to Theorem 2.1 in [10]. This last theorem allows one to consider more general boundary value problems, but for ordinary problems with Neumann boundary conditions our hypotheses allow a more general behavior of the derivative $f_u(x, u)$. We omit the details of the proof (see [1] and [2] for similar results for the first two eigenvalues).

Theorem 3.3. *Consider (3.20) with the following requirements:*

- (1) f and f_u are Carathéodory functions on $[0, L] \times \mathbb{R}$ and $f(\cdot, 0) \in L^1(0, L)$.

(2) *There exist $\alpha, \beta \in L^\infty(0, L)$ satisfying*

$$\lambda_n \leq \alpha(x) \leq f_u(x, u) \leq \beta(x)$$

on $[0, L] \times \mathbb{R}$. Furthermore, α differs from λ_n on a set of positive measure and β satisfies either hypothesis (3.9) of Theorem 3.1 or hypothesis (3.14) of Theorem 3.2.

Then problem (3.20) has a unique solution.

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