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Gardens of Eden and amenability on cellular automata

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Abstract. We prove a converse to the "Garden-of-Eden" theorem by Ceccherini-Silberstein, Machì and Scarabotti, and to a theorem by Meyerovitch, yielding two new characterizations of amenable groups. The following are equivalent:

- the group G is amenable;
- all cellular automata living on G that admit mutually erasable patterns also admit gardens of Eden;
- all cellular automata living on G that do not preserve Bernoulli measure admit gardens of Eden.

This solves in particular Conjecture 6.2 (1) in [2].

1. Introduction

Von Neumann defined[1] *cellular automata* as creatures built out of infinitely many finite-state devices arranged on the nodes of $\mathbb{Z}^2$ or $\mathbb{Z}^3$, each device being capable of interaction with its immediate neighbours. We consider here the natural generalization to creatures living on a graph with simply transitive automorphism group, and show that some fundamental properties of the automaton are characterized by *amenability* of the underlying graph—a concept also due to von Neumann [15].

Definition 1.1. *Let G be a group. A finite* cellular automaton *on G is a G-equivariant continuous map $\Theta : Q^G \to Q^G$, where Q, the* state set*, is a finite set.*

Note that usually G is infinite; much of the theory holds trivially if G is finite. The map Θ computes the 1-step evolution of the automaton, and its continuity implies that the evolution of a site depends only on a finite neighbourhood.

For purposes of computation, it is convenient to express a cellular automaton by the following finite amount of data: a finite subset S of G, called the *memory set*, and the restriction $\theta : Q^S \to Q^{\{1\}}$ of Θ. The original cellular automaton is then recovered by setting

$$\Theta(\phi)(x) = \theta(s \mapsto \phi(xs))$$

for all $\phi : G \to Q$, which are called *configurations*.

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[1] It seems that von Neumann never published his work on cellular automata—see [1] for history of the subject.

Note that S may be supposed to generate G, although this is by no means a necessity. In general, if $\langle S\rangle = H \leq G$, then the evolution of the automaton is that of G/H parallel, independent cellular automata on H.

A cellular automaton should be thought of as a highly regular animal, composed of many cells labelled by G, each in a state $\in Q$. Each cell "sees" its neighbours as defined by S, and "evolves" according to its neighbours' states.

Two properties of cellular automata received special attention. A *pattern* is the restriction of a configuration to a finite subset $Y \subseteq G$. On the one hand, there can exist patterns that never appear in the image of Θ. These are called *Gardens of Eden* (GOE), the biblical metaphor expressing the notion of paradise lost forever.

On the other hand, Θ can be non-injective in a strong sense: there can exist patterns $\phi_1' \neq \phi_2' \in Q^Y$ such that, however one extends ϕ_1' to a configuration ϕ_1, if one extends ϕ_2' similarly (i.e. in such a way that ϕ_1 and ϕ_2 have the same restriction to $G \setminus Y$) then $\Theta(\phi_1) = \Theta(\phi_2)$. These patterns ϕ_1', ϕ_2' are called *Mutually Erasable Patterns* (MEP). Equivalently[2] there are two configurations ϕ_1, ϕ_2 which differ on a non-empty finite set, with $\Theta(\phi_1) = \Theta(\phi_2)$. The absence of MEP is sometimes called *pre-injectivity* [7, §8.G].

Cellular automata were initially considered on $G = \mathbb{Z}^n$. Celebrated theorems by Moore and Myhill [13, 14] prove that, in this context, a cellular automaton admits GOE if and only if it admits MEP; necessity is due to Myhill, and sufficiency to Moore. This result was generalized by Machì and Mignosi [10] to groups of subexponential growth, and by Ceccherini-Silberstein, Machì and Scarabotti [2] to amenable groups.

There is a natural measure, the *Bernoulli measure*, on the configuration space Q^G: it assigns measure $1/\#Q$ to each of the clopen sets $\mathcal{U}_{x,q} = \{\phi \in Q^G : \phi(x) = q\}$. Note that the action of G by translation preserves this measure. Hedlund proved (see [8, Theorem 5.4] or [4, Corollary 2.3]), for $G = \mathbb{Z}$, that a cellular automaton preserves Bernoulli measure if and only if it has no GOE. This result was generalized by Meyerovitch [11, Proposition 5.1] to amenable groups.

We prove that these last two results are essentially optimal, and yield new characterizations of amenable groups:

Theorem 1.2. *Let G be a group. Then the following are equivalent:*

(1) *the group G is amenable;*
(2) *all cellular automata on G that admit MEP also admit GOE;*
(3) *all cellular automata on G that do not preserve Bernoulli measure admit GOE.*

Schupp had already asked in [16, Question 1] in which precise class of groups the Moore–Myhill theorem holds. Ceccherini-Silberstein et al. write in [2]:[3]

Conjecture 1.3 ([2, Conjecture 6.2]). *Let G be a non-amenable finitely generated group. Then for any finite and symmetric generating set S of G there exist cellular automata* Θ_1, Θ_2 with that *S such that*

[2] In the non-trivial direction, let ϕ_1, ϕ_2 differ on a non-empty finite set F; set $Y = F(S \cup S^{-1})$ and let ϕ_1', ϕ_2' be the restrictions of ϕ_1, ϕ_2 to Y respectively.

[3] I changed their wording slightly to match this paper's.

(1) *in* Θ_1 *there are MEP but no GOE;*
(2) *in* Θ_2 *there are GOE but no MEP.*

As a first step, we will prove Theorem 1.2, in which we allow ourselves to choose an appropriate subset S of G. Next, we extend a little the construction to answer the first part of Conjecture 1.3:

Theorem 1.4. *Let* $G = \langle S \rangle$ *be a finitely generated, non-amenable group. Then there exists a cellular automaton* $\Theta : Q^G \to Q^G$ *with memory set* S *that has MEP but no GOE. Furthermore, this automaton does not preserve Bernoulli measure.*

We conclude that the property of "satisfying Moore's theorem", or "satisfying Hedlund's theorem", is independent of the memory set (provided that it generates a non-amenable subgroup), a fact which was not obvious *a priori*.

Note that Conjecture 1.3 was already known to hold for groups with a non-abelian free subgroup (see [2, Theorem 6.1]).

2. Proof of Theorem 1.2

The implication (1)⇒(2) has been proven by Ceccherini-Silberstein et al.; see also [7, §8] for a slicker proof. The implication (3)⇒(2) holds for all groups, because Bernoulli measure has full support. The implication (2)⇒(3) is [11, Proposition 5.1]. We need only prove (2)⇒(1).

Let us therefore be given a non-amenable group G. Let us also, as a first step, be given a large enough finite subset S of G. Then there exists a "bounded propagation $2:1$ compressing vector field" on G: a map $f : G \to G$ such that $f(x)^{-1}x \in S$ and $\#f^{-1}(x) = 2$ for all $x \in G$.

We construct the following automaton θ. Its state set is

$$Q = S \times \{0, 1\} \times S.$$

Order S in an arbitrary manner, and choose an arbitrary $q_0 \in Q$. Define $\theta : Q^S \to Q$ as follows:

$$\theta(\phi) = \begin{cases} (p, \alpha, q) & \text{if there exist unique } s < t \text{ in } S \text{ with } \begin{cases} \phi(s) = (s, \alpha, p), \\ \phi(t) = (t, \beta, q), \end{cases} \\ q_0 & \text{if no such } s, t \text{ exist, or if too many exist.} \end{cases} \tag{2.1}$$

2.1. Θ *is surjective*

That is, θ does not admit GOE. Let indeed ϕ be any configuration. We construct a configuration ψ with $\Theta(\psi) = \phi$.

Consider in turn all $x \in G$; write $\phi(x) = (p, \alpha, q)$, and $f^{-1}(x) = \{xs, xt\}$ for some $s, t \in S$ with $s < t$. Set then

$$\psi(xs) = (s, \alpha, p), \qquad \psi(xt) = (t, 0, q). \tag{2.2}$$

Note that $\psi(z) = (f(z)^{-1}z, *, *)$ for all $z \in G$. Since $\#f^{-1}(z) = 2$ for all $z \in G$, it is clear that, for every $x \in G$, there are exactly two $s \in S$ such that $\psi(xs) = (s, *, *)$; call them s, t, ordered so that $\psi(xs) = (s, \alpha, p)$ and $\psi(xt) = (t, 0, q)$. Then $\Theta(\psi)(x) = (p, \alpha, q)$, so $\Theta(\psi) = \phi$.

2.2. Θ is not pre-injective

That is, θ admits MEP. Let indeed $\phi : G \to Q$ be any configuration; then construct ψ following (2.2), and define ψ' as follows. Choose any $y \in G$, write $\phi(y) = (p, \alpha, q)$, and write $f^{-1}(y) = \{ys, yt\}$ for some $s, t \in S$ with $s < t$. Define $\psi' : G \to Q$ by

$$\psi'(x) = \begin{cases} \psi(x) & \text{if } x \neq yt, \\ (t, 1, q) & \text{if } x = yt. \end{cases}$$

Then ψ and ψ' differ only at yt; and $\Theta(\psi) = \Theta(\psi')$ because the value of β is unused in (2.1). We conclude that θ has MEP.

2.3. Θ does not preserve Bernoulli measure

Consider the open set

$$A = \{\phi \in Q^G : \phi(1) = q_0\}.$$

Let μ denote Bernoulli measure; then $\mu(A) = 1/\#Q$. Write $Q^G = X \sqcup X'$, where

$$X = \{\phi : \text{there are exactly two } s \in S \text{ such that } \phi(s) = (s, *, *)\}$$

and $X' = Q^G \setminus X$. Clearly $\mu(X), \mu(X') > 0$. Consider $B = \Theta^{-1}(A)$. Then $X' \subseteq B$, and $\mu(B \cap X)/\mu(X) = 1/\#Q$ because the restriction of the local rule to X is invariant under any permutation of Q. We get

$$\mu(B) = \mu(B \cap X) + \mu(B \cap X') = \mu(X)/\#Q + \mu(X') > 1/\#Q = \mu(A).$$

3. Proof of Theorem 1.4

We begin by a slightly extended formulation of amenability for finitely generated groups:

Lemma 3.1. *Let G be a finitely generated group. The following are equivalent:*

(1) *the group G is not amenable;*
(2) *for every generating set S of G, there exist $m > n \in \mathbb{N}$ and an "$m : n$ compressing correspondence on G with propagation S", i.e. a function $f : G \times G \to \mathbb{N}$ such that*

$$\forall y \in G : \quad \sum_{x \in G} f(x, y) = m, \tag{3.1}$$

$$\forall x \in G: \quad \sum_{y \in G} f(x, y) = n, \tag{3.2}$$

$$\forall x, y \in G: \quad f(x, y) \neq 0 \Rightarrow x \in yS. \tag{3.3}$$

Note that this definition generalizes the notion of "2 : 1 compressing vector field" introduced above. Indeed, f could be thought of as a multivalued function, which at x takes $f(x, y)$ times the value y; we write $f(x) = \{y : f(x, y) > 0\}$ and $f^{-1}(y) = \{x : f(x, y) > 0\}$.

Proof. For the forward direction, assuming that G is non-amenable, there exists a rational $m/n > 1$ such that every finite $F \subseteq G$ satisfies

$$\#(FS) \geq (m/n)\#F.$$

Construct the following bipartite oriented graph: its vertex set is $G \times \{1, \ldots, m\} \sqcup G \times \{-1, \ldots, -n\}$. There is an edge from (g, i) to $(gs, -j)$ for all $s \in S$ and all $i \in \{1, \ldots, m\}$, $j \in \{1, \ldots, n\}$. By hypothesis, every finite $F \subseteq G \times \{1, \ldots, m\}$ has at least $\#F$ neighbours. Since $m > n$ and multiplication by a generator is a bijection, every finite $F \subseteq G \times \{-1, \ldots, -n\}$ also has at least $\#F$ neighbours.

We now invoke the Hall–Rado theorem [12]: if a bipartite graph is such that every subset of any of the parts has as many neighbours as its cardinality, then there exists a "perfect matching"—a subset I of the edge set of the graph such that every vertex is contained in precisely one edge in I. Set then

$$f(x, y) = \#\{(i, j) \in \{1, \ldots, m\} \times \{1, \ldots, n\} : \\ I \text{ contains the edge between } (x, -j) \text{ and } (y, i)\}.$$

For the backward direction: assume that G is amenable, and let f be a bounded-propagation $m : n$ compressing correspondence. Let S be a finite set such that $y^{-1}x \in S$ whenever $f(x, y) \neq 0$, and let $F \subset G$ be a finite set such that $\#(FS) < (m/n)\#F$, a *Følner set*. Then $y \in F$ and $f(x, y) \neq 0$ imply $x \in FS$, so

$$m\#F = \sum_{y \in F} \sum_{x \in G} f(x, y) \leq \sum_{x \in FS} \sum_{y \in G} f(x, y) = n\#(FS),$$

a contradiction. □

Let now $G = \langle S \rangle$ be a non-amenable group, and apply Lemma 3.1 to $G = \langle S \rangle$, yielding $m > n \in \mathbb{N}$ and a contracting $m : n$ correspondence f. Consider the following cellular automaton θ with state set

$$Q = (S \times \{0, 1\} \times S^n)^n.$$

Choose $q_0 \in Q$, and give a total ordering to $S \times \{1, \ldots, n\}$.

Consider $\phi \in Q^S$. To define $\theta(\phi)$, seek whether there exists a unique sequence $(s_1, k_1) < \cdots < (s_m, k_m)$ in $(S \times \{1, \ldots, n\})^m$ such that

$$\phi(s_j)_{k_j} = (s_j, \alpha_j, t_{j,1}, \ldots, t_{j,n}) \in S \times \{0, 1\} \times S^n \quad \text{for } j = 1, \ldots, m.$$

If there are no, or too many, such $s_1, k_1, \dots, s_m, k_m$, set $\theta(\phi) = q_0$; otherwise, set

$$\theta(\phi) = ((t_{1,1}, \alpha_1, t_{2,1}, \dots, t_{n+1,1}), \dots, (t_{1,n}, \alpha_n, t_{2,n}, \dots, t_{n+1,n})) \in Q. \tag{3.4}$$

The same arguments as before apply. Given $\phi : G \to Q$, we construct $\psi : G \to Q$ such that $\Theta(\psi) = \phi$, as follows. We think of the coordinates $\psi(x)_k$ of $\psi(x)$ as n "slots", initially all "free", and will use the $m : n$ correspondence f to establish a correspondence between the slots of ϕ and those of ψ.

By definition, $\#f^{-1}(x) = m$ for all $x \in G$, while $\#f(x) = n$. Consider in turn all $x \in G$; write $f^{-1}(x) = \{xs_1, \dots, xs_m\}$, and let $k_1, \dots, k_m \in \{1, \dots, n\}$ be "free" slots in $\psi(xs_1), \dots, \psi(xs_m)$ respectively. By the definition of f, there always exist sufficiently many free slots.

Mark now these slots as "occupied". Reindex $s_1, k_1, \dots, s_m, k_m$ in such a way that $(s_1, k_1, \dots, s_m, k_m)$ is minimal among its $m!$ permutations. Set then

$$\psi(xs_j)_{k_j} = (s_j, \alpha_j, t_{j,1}, \dots, t_{j,n}) \quad \text{for } j = 1, \dots, m,$$

where $\alpha_{n+1}, \dots, \alpha_m$ are taken to be arbitrary values (say 0 for definiteness) and

$$\phi(x) = ((t_{1,1}, \alpha_1, t_{2,1}, \dots, t_{n+1,1}), \dots, (t_{1,n}, \alpha_n, t_{2,n}, \dots, t_{n+1,n})).$$

Finally, define ψ arbitrarily on slots that are still "free".

It is clear that $\Theta(\psi) = \phi$, so θ does not have GOE. On the other hand, θ has MEP as before, because the values of α_j in (3.4) are not used for $j \in \{n + 1, \dots, m\}$.

Similarly, setting $A = \{\phi \in Q^G : \phi(1) = q_0\}$, we have $\mu(\Theta^{-1}(A)) > \mu(A)$ as before.

4. Remarks

4.1. G-sets

A cellular automaton could more generally be defined on a right G-set X. There is a natural notion of amenability for G-sets, but it is not clear exactly to what extent Theorem 1.2 can be generalized to that setting—certainly not *verbatim*, since the G-set $G \sqcup \{\cdot\}$ is amenable for all G, but may support automata with MEP but without GOE. It is also unclear how to construct automata on graphs with a transitive, but not simply transitive, automorphism group (see e.g. [5]).

4.2. Myhill's theorem

It seems harder to produce counterexamples to Myhill's theorem ("GOE imply MEP") for arbitrary non-amenable groups, although there exists an example on $C = \mathbb{Z}/2*\mathbb{Z}/2*\mathbb{Z}/2$, due to Muller.[4] Let us make our task even harder, and restrict ourselves to linear automata over finite rings (so we assume Q is a module over a finite ring and the map $\Theta : Q^G \to Q^G$ is linear). The following approach seems promising.

[4] In his University of Illinois 1976 class notes, see [10, p. 55].

Conjecture 4.1 (Folklore? I learnt it from V. Guba). *Let G be a group. The following are equivalent:*

(1) *The group G is amenable.*
(2) *Let $\mathbb{K}$ be a field. Then $\mathbb{K}G$ admits right common multiples, i.e. for any $\alpha, \beta \in \mathbb{K}G$ there exist $\gamma, \delta \in \mathbb{K}G$ with $\alpha\gamma = \beta\delta$ and $(\gamma, \delta) \neq (0, 0)$.*

This last condition, if $\mathbb{K}G$ is a domain, is equivalent to Ore's condition, implying the existence of a classical ring of fractions—see [9] and [6]. The following direction is classical:

Proof of Conjecture 4.1 (1)⇒(2). Assume that G is amenable, and let $\alpha, \beta \in \mathbb{K}G$ be given. Let $S \subseteq G$ be a finite set containing the supports of α and β. By Følner's criterion, there exists $F \subseteq G$ finite such that $\#(SF) < 2\#F$. Consider $\gamma, \delta \in \mathbb{K}F$ as variables; then the equation system $\alpha\gamma = \beta\delta$ is linear, has $2\#F$ unknowns, and at most $\#(SF)$ equations, so has a non-trivial solution. □

Conjecture 4.2 (A possible converse to Myhill's Theorem). *Let $\mathbb{K}$ be a field. The following are equivalent:*

(1) *The group G is amenable.*
(2) *Any $\mathbb{K}$-linear cellular automaton which admits gardens of Eden also admits mutually erasable patterns.*

Proof, assuming Conjecture 4.1. Ceccherini-Silberstein and Coornaert proved the (1)⇒(2) direction in [3, Theorem 1.2].

Assume now the "hard" direction of Conjecture 4.1. Given G non-amenable, we may then find a finite field $\mathbb{K}$, and $\alpha, \beta \in \mathbb{K}G$ that do not have a common right multiple.

Set $Q = \mathbb{K}^2$ with basis (e_1, e_2), let S contain the inverses of the supports of α and β, and define the cellular automaton $\theta : Q^S \to Q$ by

$$\theta(\phi) = \sum_{x \in G} \big(\alpha(x^{-1})\langle\phi(x)|e_1\rangle - \beta(x^{-1})\langle\phi(x)|e_2\rangle, 0\big).$$

Then θ has GOE, indeed any configuration not in $(\mathbb{K} \times \{0\})^G$ is a GOE. On the other hand, assume for contradiction that θ had MEP; then by linearity we might as well assume $\Theta(\phi) = 0$ for some non-zero finitely-supported $\phi : G \to Q$. Write $\phi = (\gamma, \delta)$ in coordinates; then $\Theta(\phi) = 0$ would give $\alpha\gamma = \beta\delta$, showing that α, β actually did have a common right multiple. □

Muller's example is in fact a special case of this construction, with

$$G = \langle x, y, z \mid x^2, y^2, z^2 \rangle,$$

$\mathbb{K} = \mathbb{F}_2$, and $\alpha = x$, $\beta = y + z$.

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