



Anar Akhmedov · Scott Baldrige · R. İnanç Baykur  
· Paul Kirk · B. Doug Park

## Simply connected minimal symplectic 4-manifolds with signature less than $-1$

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**Abstract.** For each pair  $(e, \sigma)$  of integers satisfying  $2e + 3\sigma \geq 0$ ,  $\sigma \leq -2$ , and  $e + \sigma \equiv 0 \pmod{4}$ , with four exceptions, we construct a minimal, simply connected symplectic 4-manifold with Euler characteristic  $e$  and signature  $\sigma$ . We also produce simply connected, minimal symplectic 4-manifolds with signature zero (resp. signature  $-1$ ) with Euler characteristic  $4k$  (resp.  $4k + 1$ ) for all  $k \geq 46$  (resp.  $k \geq 49$ ).

**Keywords.** Symplectic topology, Luttinger surgery, fundamental group, 4-manifold

### 1. Introduction

In [6], a closed, simply connected, minimal symplectic 4-manifold with Euler characteristic 6 and signature  $-2$  is constructed. This manifold contains a symplectic genus 2 surface with trivial normal bundle and simply connected complement and also contains two Lagrangian tori with special properties. In this article we use this manifold and apply standard constructions to fill out the part of the symplectic geography plane corresponding to signature less than  $-1$ . Recall that Taubes proved ([35, 36, 37], also Li–Liu [21]) that minimal simply connected symplectic 4-manifolds satisfy  $2e + 3\sigma \geq 0$ , where  $e$  denotes the Euler characteristic and  $\sigma$  the signature. Moreover, every symplectic 4-manifold satisfies  $e + \sigma \equiv 0 \pmod{4}$ .

Our main result is the following.

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A. Akhmedov: School of Mathematics, University of Minnesota, Minneapolis, MN 55455, USA;  
e-mail: akhmedov@math.umn.edu

S. Baldrige: Department of Mathematics, Louisiana State University, Baton Rouge, LA 70817,  
USA; e-mail: sbaldrig@math.lsu.edu

R. İ. Baykur: Department of Mathematics, Brandeis University, Waltham, MA 02454, USA;  
e-mail: baykur@brandeis.edu

P. Kirk: Mathematics Department, Indiana University, Bloomington, IN 47405, USA;  
e-mail: pkirk@indiana.edu

B. D. Park: Department of Pure Mathematics, University of Waterloo, Waterloo, ON, N2L 3G1,  
Canada; e-mail: bdpark@math.uwaterloo.ca

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**Theorem A.** *Let  $\sigma$  and  $e$  denote integers satisfying  $2e + 3\sigma \geq 0$  and  $e + \sigma \equiv 0 \pmod{4}$ . If, in addition,*

$$\sigma \leq -2,$$

*then there exists a simply connected minimal symplectic 4-manifold with signature  $\sigma$  and Euler characteristic  $e$  and odd intersection form, except possibly for  $(\sigma, e)$  equal to  $(-3, 7)$ ,  $(-3, 11)$ ,  $(-5, 13)$ , or  $(-7, 15)$ .*

In terms of  $c_1^2$  and  $\chi_h$ , we construct symplectic manifolds realizing all pairs of integers satisfying  $0 \leq c_1^2 \leq 8\chi_h - 2$  except  $(c_1^2, \chi_h) = (5, 1)$ ,  $(13, 2)$ ,  $(11, 2)$ , and  $(9, 2)$ .

Using Freedman's theorem [13] and Taubes's results [36, 37] this theorem can be restated by saying that there exists a minimal symplectic manifold homeomorphic but not diffeomorphic to  $m\mathbb{C}\mathbb{P}^2 \# n\overline{\mathbb{C}\mathbb{P}^2}$  whenever  $m + 2 \leq n \leq 5m + 4$  and  $m$  is odd, except possibly for  $(m, n) = (1, 4)$ ,  $(3, 6)$ ,  $(3, 8)$ , or  $(3, 10)$ . The existence of minimal symplectic 4-manifolds homeomorphic to  $m\mathbb{C}\mathbb{P}^2 \# n\overline{\mathbb{C}\mathbb{P}^2}$  for these four pairs remains an open problem (as far as we know).

The *geography problem* refers to the problem of determining which pairs  $(\sigma, e)$  of integers arise as the signature and Euler characteristic of a 4-manifold in a certain class. The terminology was borrowed by topologists from algebraic geometers studying algebraic surfaces (see e.g. [7]). The motivation in 4-dimensional topology for studying the geography problem comes from Freedman's theorem [13] which shows that a simply connected smooth 4-manifold  $M$  (with odd intersection form) is determined up to homeomorphism by the pair  $(\sigma(M), e(M))$ . The smooth geography problem has a long history: see [9]. The monograph [17] and the recent survey [12] contain a comprehensive list of references.

The study of the geography problem for symplectic 4-manifolds has been an area of active study in recent years. In his seminal paper [15], Gompf constructed simply connected symplectic 4-manifolds filling in a large part of the geography plane, most of which can be proven to be minimal by more recent techniques. J. Park explored the topic in a series of articles [26, 27], addressing minimality and uniqueness questions using Seiberg–Witten invariants. The articles [11, 19, 28, 33] have focused attention on the problem of constructing small simply connected symplectic manifolds. Recently, the approach introduced in [2] has spurred the discovery of new constructions of small simply connected symplectic manifolds; see [3, 4, 5, 6, 8, 30].

The methods in this article are based on inductive constructions to produce simply connected manifolds starting with a few basic non-simply connected models. Although there are some formal similarities between some of the fundamental group calculations carried out in this article and those in the articles [2, 3, 4, 5, 6, 8], there is an important difference, as we now explain.

In those articles, the mechanism used to kill fundamental groups comes down to establishing precise enough control over certain group presentations to conclude that all generators die. This is a subtle process which depends critically on properly identifying words in fundamental groups, since e.g. in a group, a pair of elements  $x, y$  might commute but their conjugates  $gxg^{-1}, hyh^{-1}$  need not.

By contrast, the mechanism of the present paper is much softer. We use standard symplectic constructions pioneered by Gompf [15] and Luttinger [22] to kill a generator outright; subsequent generators are then killed by a simple argument. In particular, although we are explicit and careful in our fundamental group calculations in Theorem 1, Lemma 16, and elsewhere, the reader will quickly understand that our results follow as easily if one only knows the statements up to conjugacy.

To illustrate this point, in the statement of Lemma 16, the expressions for  $\mu_6, m_6, \ell_6$  are long, but it is straightforward to see that, *up to conjugacy*,  $\mu_6 = [a_1, x_2]$ ,  $m_6 = y_2$ ,  $\ell_6 = b_1^{-1}$ . This less precise information is quite sufficient to prove the results of this article.

The construction is also suitable for filling out a large region of the geography plane starting with any given symplectic 4-manifold with given characteristic numbers and containing a square zero symplectic torus. For example, Theorem 23 roughly says that given a symplectic 4-manifold  $X$ , one can construct a new symplectic manifold  $Y$  with the same fundamental group as  $X$  and satisfying  $c_1^2(Y) = c_1^2(X) + c$  and  $\chi_h(Y) = \chi_h(X) + \chi$ , for any  $(c, \chi)$  in the cone  $0 \leq c \leq 8\chi - 2$ .

Since it is known how to produce manifolds with positive signature ([32]) we apply this result to a positive signature symplectic 4-manifold and prove the following.

**Theorem B.** *For each integer  $k \geq 45$ , there exists a simply connected minimal symplectic 4-manifold  $X_{2k+1, 2k+1}$  with Euler characteristic  $e = 4k + 4$  and signature  $\sigma = 0$ . For each integer  $k \geq 49$ , there exists a simply connected minimal symplectic 4-manifold  $X_{2k-1, 2k}$  with Euler characteristic  $e = 4k + 1$  and signature  $\sigma = -1$ .*

All the manifolds we produce have odd intersection forms. Hence there remain four minimal simply connected symplectic odd 4-manifolds of signature less than or equal to  $-2$  and 97 minimal simply connected symplectic odd 4-manifolds of non-positive signature yet to be constructed.

We finish this introduction with a brief description of the proofs. We start with three models, the minimal symplectic 4-manifolds  $B, C, D$ . These manifolds have Euler characteristic 6, 8, and 10 and signatures  $-2, -4$ , and  $-6$  respectively. Each contains a disjoint pair of homologically independent Lagrangian tori  $T_1$  and  $T_2$  with nullhomotopic meridians and whose complement has fundamental group  $\mathbb{Z} \oplus \mathbb{Z}$ . Moreover,  $\pm 1$  Luttinger surgery (see Section 2) along certain curves on one or both of these tori yields a minimal symplectic 4-manifold.

We then produce a family  $B_g, g \in \mathbb{Z}$ , of minimal symplectic 4-manifolds with Euler characteristic  $6 + 4g$  and signature  $-2$  by taking a symplectic sum of  $B$  with a minimal manifold constructed from Luttinger surgeries on a product of surfaces. This family  $B_g$  again contains a pair of Lagrangian tori  $T_1, T_2$  with the same properties as those in  $B, C, D$ .

Taking the symplectic sum of many copies of  $B, B_g, C, D$  (and, if needed, the elliptic surfaces  $E(k)$ ) along their tori and performing a  $+1$  Luttinger surgery on each of the unused Lagrangian tori yields our even signature examples. Showing that the fundamental group vanishes is simple since the fundamental groups of  $B, B_g, C, D$  and the homomorphisms induced by the inclusions of the tori are known. Usher's theorem [38]

easily implies that the result is minimal. The manifolds  $B, B_g, C, D$  contain  $-1$  surfaces disjoint from the  $T_i$  which survive to  $-1$  surfaces in the symplectic sum and hence the result has an odd intersection form.

Producing odd signature manifolds follows the same general approach, but requires several small model manifolds with appropriate Lagrangian tori to use as seeds for the symplectic sums. The construction is not quite as clean as in the even signature case.

We construct a minimal symplectic 4-manifold  $P_{5,8}$  with fundamental group  $\mathbb{Z}$ , Euler characteristic 15, and signature  $-3$ . This and a few other known small manifolds with odd signature each contain a Lagrangian or symplectic torus appropriate for taking symplectic sums with many copies of  $B, B_g, C$ , and  $D$ . As in the even case, this produces minimal simply connected 4-manifolds of odd signature less than or equal to  $-5$ .

The signature  $-3$  examples are constructed by a separate argument, and a few small examples not covered by our general construction are culled from the literature (i.e.  $(\sigma, e) = (-7, 11), (-13, 21), (-11, 19), (-5, 9)$ ) or constructed explicitly  $((\sigma, e) = (-5, 17), (-7, 19), (-9, 21))$ .

## 2. Luttinger surgery

Given any Lagrangian torus  $T$  in a symplectic 4-manifold  $M$ , the Darboux–Weinstein theorem [23] implies that there is a parameterization of a tubular neighborhood  $T^2 \times D^2 \rightarrow \text{nb}(T) \subset M$  such that the image of  $T^2 \times \{d\}$  is Lagrangian for all  $d \in D^2$ . Choosing any point  $d \neq 0$  in  $D^2$  gives a push-off  $F_d : T \rightarrow T^2 \times \{d\} \subset M - T$  called the *Lagrangian push-off* or *Lagrangian framing*. Given any embedded curve  $\gamma \subset T$ , its image  $F_d(\gamma)$  is called the *Lagrangian push-off* of  $\gamma$ .

Any curve isotopic to  $\{t\} \times \partial D^2 \subset \partial(\text{nb}(T))$  will be called a *meridian* of  $T$  and typically denoted by  $\mu_T$ . In this article we will typically fix a pair of embedded curves on  $T$  intersecting transversally in one point and denote the two Lagrangian push-offs by  $m_T$  and  $\ell_T$ . The triple  $\mu_T, m_T, \ell_T$  generate  $H_1(\partial(\text{nb}(T)))$ . Since the 3-torus has abelian fundamental group we may choose a base point  $t$  on  $\partial(\text{nb}(T))$  and unambiguously refer to  $\mu_T, m_T, \ell_T \in \pi_1(\partial(\text{nb}(T)), t)$ .

The push-offs and meridian are used to specify coordinates for a *Luttinger surgery*. This is the process of removing a tubular neighborhood of  $T$  in  $M$  and regluing it so that the embedded curve representing  $\mu_T m_T^p \ell_T^q$  bounds a disk for some pair of integers  $p, q$ . The resulting 4-manifold admits a symplectic structure whose symplectic form is unchanged away from a neighborhood of  $T$  ([1, 22]).

When the base point  $x$  of  $M$  is chosen off the boundary of the tubular neighborhood of  $T$ , the based loops  $\mu_T, m_T$ , and  $\ell_T$  are to be joined to  $x$  by *the same path* in  $M - T$ . These curves then define elements of  $\pi_1(M - T, x)$ . With  $p, q$  as above, the 4-manifold resulting from Luttinger surgery on  $M$  has fundamental group

$$\pi_1(M - T, x) / N(\mu_T m_T^p \ell_T^q),$$

where  $N(\mu_T m_T^p \ell_T^q)$  denotes the normal subgroup generated by  $\mu_T m_T^p \ell_T^q$ .

We will only need the cases  $(p, q) = (\pm 1, 0)$  or  $(0, \pm 1)$  in this article, i.e.  $\pm 1$  *Luttinger surgery along  $m_T$  or  $\ell_T$* .

### 3. The fundamental group of the complement of tori in the product of surfaces

Let  $F$  be a genus  $f$  surface, with  $f \geq 2$ . Choose a base point  $h$  on  $F$  and pairs  $x_i, y_i$ ,  $i = 1, \dots, f$ , of circles forming a symplectic basis, with  $x_i, y_i$  intersecting at  $h_i \in F$ . Choose paths  $\alpha_i$  from  $h$  to  $h_i$ , so that the loops

$$\tilde{x}_i = \alpha_i x_i \alpha_i^{-1} \quad \text{and} \quad \tilde{y}_i = \alpha_i y_i \alpha_i^{-1}$$

generate  $\pi_1(F, h)$ . Let  $Y_i$  be a circle parallel to  $y_i$  which misses  $\alpha_i$ .

Let  $G$  be a genus  $g$  surface. Choose a base point  $k$  on  $G$ , and  $g$  pairs  $a_1, b_1, \dots, a_g, b_g$  of circles forming a symplectic basis, with  $a_i, b_i$  intersecting at  $k_i$ . Choose paths  $\beta_i$  from  $k$  to  $k_i$ , so that the loops

$$\tilde{a}_i = \beta_i a_i \beta_i^{-1} \quad \text{and} \quad \tilde{b}_i = \beta_i b_i \beta_i^{-1}$$

generate  $\pi_1(G, k)$ . Choose parallel copies  $A_i$  of  $a_i$  and  $B_i$  of  $b_i$  which miss the paths  $\beta_i$ . In Figure 1 we illustrate the notation when  $f = 2$  and  $g = 3$ .

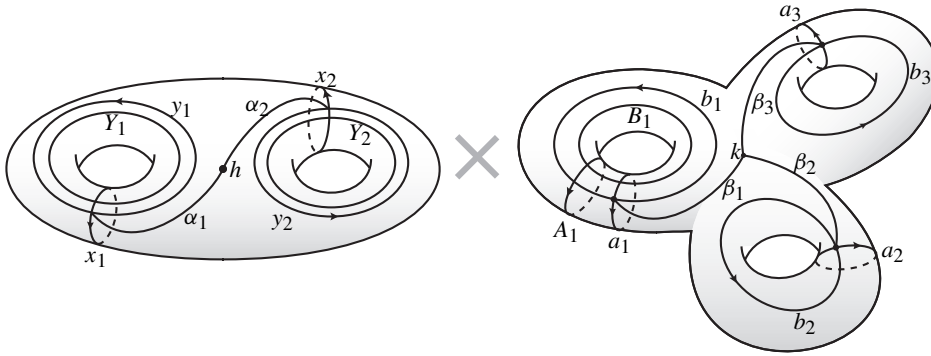


Fig. 1. The surface  $F \times G$ .

The product  $F \times G$  contains the union of the two symplectic surfaces  $F \times \{k\} \cup \{h\} \times G$  meeting at  $(h, k)$ . There is an identification  $\pi_1(F \times G, (h, k)) = \pi_1(F, h) \times \pi_1(G, k)$  which associates the loop  $\tilde{x}_i \times \{k\}$  to  $(\tilde{x}_i, 1)$ ,  $\tilde{y}_i \times \{k\}$  to  $(\tilde{y}_i, 1)$ ,  $\{h\} \times \tilde{a}_i$  to  $(1, \tilde{a}_i)$  and  $\{h\} \times \tilde{b}_i$  to  $(1, \tilde{b}_i)$ . In other words, the homomorphisms induced by the inclusions  $F \times \{k\} \subset F \times G$  and  $\{h\} \times G \subset F \times G$  present  $\pi_1(F \times G, (h, k))$  as the product of  $\pi_1(F, h)$  and  $\pi_1(G, k)$ .

When there is no chance of confusion we denote the  $2f + 2g$  loops  $\tilde{x}_i \times \{k\}$ ,  $\tilde{y}_i \times \{k\}$ ,  $\{h\} \times \tilde{a}_i$ ,  $\{h\} \times \tilde{b}_i$  simply by  $\tilde{x}_i, \tilde{y}_i, \tilde{a}_i, \tilde{b}_i$ . These are loops in  $F \times G$  based at  $(h, k)$ .

The product  $F \times G$  contains  $2g$  Lagrangian tori

$$Y_1 \times A_j, \quad Y_2 \times B_j, \quad j = 1, \dots, g.$$

These  $2g$  tori are pairwise disjoint and miss  $(F \times \{k\}) \cup (\{h\} \times G)$ .

Let  $N$  denote a tubular neighborhood of the union of these  $2g$  tori:

$$N = \text{nb}d\left(\left(\bigcup_j Y_1 \times A_j\right) \cup \left(\bigcup_j Y_2 \times B_j\right)\right) \subset F \times G.$$

The loops  $\tilde{x}_i, \tilde{y}_i, \tilde{a}_i, \tilde{b}_i$  are loops in  $F \times G - N$  based at  $(h, k)$ .

Typically, removing a surface from a 4-manifold increases the number of generators of the fundamental group, but since these tori respect the product structure one can prove the following theorem.

**Theorem 1.** *The  $2f + 2g$  loops  $\tilde{x}_1, \tilde{y}_1, \dots, \tilde{x}_f, \tilde{y}_f, \tilde{a}_1, \tilde{b}_1, \dots, \tilde{a}_g, \tilde{b}_g$  generate  $\pi_1(F \times G - N, (h, k))$ . There are paths  $d_j : [0, 1] \rightarrow F \times G - N$  from  $(h, k)$  to the boundary of the tubular neighborhood of  $Y_1 \times A_j$  and  $e_j : [0, 1] \rightarrow F \times G - N$  from  $(h, k)$  to the boundary of the tubular neighborhood of  $Y_2 \times B_j$  so that with respect to these paths, the meridian and two Lagrangian push-offs of  $Y_1 \times A_j$  are homotopic in  $F \times G - N$  rel endpoint to*

$$\mu_{Y_1 \times A_j} = [\tilde{x}_1, \tilde{b}_j], \quad m_{Y_1 \times A_j} = \tilde{y}_1, \quad \ell_{Y_1 \times A_j} = \tilde{a}_j,$$

and the meridian and two Lagrangian push-offs of  $Y_2 \times B_j$  are homotopic in  $F \times G - N$  rel endpoint to

$$\mu_{Y_2 \times B_j} = [\tilde{x}_2, \tilde{a}_j], \quad m_{Y_2 \times B_j} = \tilde{y}_2, \quad \ell_{Y_2 \times B_j} = \tilde{b}_j.$$

*Proof.* Before we start the proof, we give an indication of how it will proceed. Note that  $\bigcup_j (Y_1 \times A_j) = Y_1 \times \bigcup_j A_j$  lies on  $Y_1 \times G$  and that  $Y_2 \times \bigcup_j B_j$  lies on  $Y_2 \times G$ . Thus  $F \times G - N$  can be constructed by cutting  $F \times G$  along the hypersurface  $(Y_1 \cup Y_2) \times G$ , and then regluing the two copies of  $Y_1 \times G$  only along the complement of a neighborhood of the  $A_j$ , and regluing the two copies of  $Y_2 \times G$  only along the complement of a neighborhood of the  $B_j$ . However, in order to use the Seifert–Van Kampen theorem, the subsets and their intersection in a decomposition are required to be connected, and so we need to modify the decomposition slightly.

Let  $P_1$  be the annulus in  $F$  bounded by  $y_1$  and  $Y_1$ . Similarly let  $P_2$  denote the annulus in  $F$  bounded by  $y_2$  and  $Y_2$ . Let  $\alpha$  denote the arc  $(\alpha_1 \cup \alpha_2) \times \{k\}$ . Let  $\gamma_1$  denote the arc  $(x_1 \cap P_1) \times \{k\}$ ; it spans the two circles  $y_1$  and  $Y_1$ . Similarly let  $\gamma_2$  denote the arc  $(x_2 \cap P_2) \times \{k\}$ . See Figure 2. Set

$$S_1 = (P_1 \times G) \cup \alpha \cup (P_2 \times G), \quad S_2 = (\gamma_1 \cup \gamma_2) \cup ((F - \text{int}(P_1 \cup P_2)) \times G).$$

Then in  $F \times G$ ,  $S_1 \cap S_2$  is the union of four copies of  $S^1 \times G$  together with three arcs which connect the four components. In particular,  $S_1, S_2$  and  $S_1 \cap S_2$  are connected and contain the base point  $(h, k)$ .

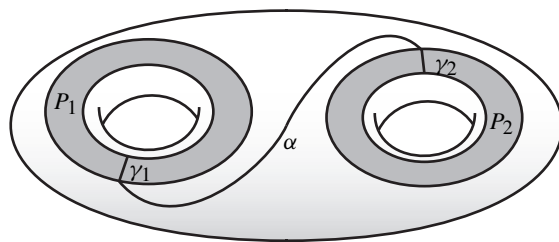


Fig. 2

Let  $G_A = G - \text{nb}d(\bigcup_j A_j)$  denote the complement of an open tubular neighborhood of the  $A_j$  in  $G$ . Since the  $A_j$  do not disconnect  $G$ ,  $G_A$  is path connected. Similarly let  $G_B = G - \text{nb}d(\bigcup_j B_j)$  denote the complement of an open tubular neighborhood of the  $B_j$  in  $G$ .

To construct  $F \times G - N$  we form the identification space

$$F \times G - N = S_1 \sqcup S_2 / \sim$$

by identifying  $(f, s) \in S_1$  with its corresponding point  $(f', s')$  in  $S_2$  *except if*  $f \in Y_1$  and  $s \in \text{nb}d(A_j)$  or  $f \in Y_2$  and  $s \in \text{nb}d(B_j)$ . In other words, along  $Y_1 \times G$  we identify only the two copies of  $Y_1 \times G_A$  and along  $Y_2 \times G$  we identify only the two copies of  $Y_2 \times G_B$ .

Hence we have exhibited  $F \times G - N$  as the union of  $S_1$  and  $S_2$  with connected intersection

$$S_1 \cap S_2 = (Y_1 \times G_A) \cup \gamma_1 \cup (y_1 \times G) \cup \alpha \cup (y_2 \times G) \cup \gamma_2 \cup (Y_2 \times G_B).$$

It is easy to see that  $\pi_1(S_1 \cap S_2, (h, k)) \rightarrow \pi_1(S_1, (h, k))$  is surjective. Indeed, one can use the product parameter in the annuli  $P_1$  and  $P_2$  to define a deformation retraction (fixing  $\alpha$  and hence also  $(h, k)$ ) of  $S_1$  to the subset  $(y_1 \times G) \cup \alpha \cup (y_2 \times G)$  of  $S_1 \cap S_2$ .

The Seifert–Van Kampen theorem applies and implies that there is a surjection

$$\pi_1(S_2, (h, k)) \rightarrow \pi_1(F \times G - N, (h, k))$$

induced by inclusion.

We will show that the image of  $\pi_1(S_2, (h, k)) \rightarrow \pi_1(F \times G - N, (h, k))$  is generated by the loops  $\tilde{x}_1, \tilde{y}_1, \tilde{x}_2, \tilde{y}_2, \dots, \tilde{x}_f, \tilde{y}_f, \tilde{a}_1, \tilde{b}_1, \dots, \tilde{a}_g, \tilde{b}_g$ . Notice that all these loops are contained in  $S_2$ .

We find generators for  $\pi_1(S_2, (h, k))$ . This is again a straightforward application of the Seifert–Van Kampen theorem, as we will now show.

Since the arcs  $\gamma_i$  are just segments that lie on  $\tilde{x}_i$  (and the rest of the loops  $\tilde{x}_i$  lie in  $S_2$ ), we can decompose  $S_2$  as

$$S_2 = (\tilde{x}_1 \cup \tilde{x}_2) \cup ((F - \text{int}(P_1 \cup P_2)) \times G).$$

The intersection of the two pieces in this decomposition is the (contractible) set  $\tilde{x}_1 \cup \tilde{x}_2 - (\gamma_1 \cup \gamma_2)$ .

Hence  $\pi_1(S_2, (h, k))$  is generated by  $\tilde{x}_1, \tilde{x}_2$  and any set of generators of

$$\pi_1((F - \text{int}(P_1 \cup P_2)) \times G, (h, k)) = \pi_1(F - \text{int}(P_1 \cup P_2), h) \times \pi_1(G, k).$$

The loops  $\tilde{a}_i, \tilde{b}_i$  generate  $\pi_1(G, k)$ . The space  $F - \text{int}(P_1 \cup P_2)$  is a 4-punctured genus  $f - 2$  surface. Its fundamental group is generated by  $\tilde{y}_1, \tilde{y}_2, \tilde{x}_3, \tilde{y}_3, \dots, \tilde{x}_f, \tilde{y}_f$  and one other loop  $\tau$  based at  $h$  which is obtained by traveling from the base point to a point on the boundary component  $Y_1$ , following  $Y_1$ , then returning to the base point.

We have shown that the loops  $\tilde{x}_1, \tilde{y}_1, \tilde{x}_2, \tilde{y}_2, \tilde{x}_3, \tilde{y}_3, \dots, \tilde{x}_f, \tilde{y}_f, \tilde{a}_1, \tilde{b}_1, \dots, \tilde{a}_g, \tilde{b}_g$ , and  $\tau \times \{k\}$  generate  $\pi_1(S_2, (h, k))$ . Hence, considered as loops in  $F \times G - N$ , they generate  $\pi_1(F \times G - N, (h, k))$ . We need only show that the generator  $\tau \times \{k\}$  is not

needed. But this is obvious since  $\tilde{x}_1, \tilde{y}_1, \dots, \tilde{x}_f, \tilde{y}_f$  and  $\tau \times \{k\}$  all lie on the surface  $F \times \{k\} \subset F \times G - N$ , and  $\tilde{x}_1, \tilde{y}_1, \dots, \tilde{x}_f, \tilde{y}_f$  generate  $\pi_1(F \times \{k\}, (h, k))$ .

We next turn to the problem of expressing the meridians and Lagrangian push-offs of the generators of the Lagrangian tori  $Y_1 \times A_j, Y_2 \times B_j$  in terms of the loops  $\tilde{x}_1, \tilde{y}_1, \dots, \tilde{x}_f, \tilde{y}_f, \tilde{a}_1, \tilde{b}_1, \dots, \tilde{a}_g, \tilde{b}_g$ . We do this for  $Y_1 \times A_1$ . Symmetric arguments provide the analogous calculations for the rest.

In Figure 1, denote by  $h_1$  the intersection of  $x_1$  and  $y_1$  (i.e. the endpoint of  $\alpha_1$ ), and by  $k_1$  the intersection of  $a_1$  and  $b_1$ . Then the point  $(h_1, k_1)$  lies on the boundary of a tubular neighborhood of  $Y_1 \times A_1$ .

Since we take the product symplectic form on  $F \times G$ , referring to Figure 1 one sees that the loops  $y_1 \times \{k_1\}$  and  $\{h_1\} \times a_1$  are Lagrangian push-offs of two generators of  $\pi_1(Y_1 \times A_1)$  to the boundary of the tubular neighborhood of  $Y_1 \times A_1$ .

There is a map of a square into  $F \times G - N$  given by  $\alpha_1 \times \beta_1$ :

$$\alpha_1 \times \beta_1 : [0, 1] \times [0, 1] \rightarrow F \times G - N.$$

The point  $(0, 0)$  is mapped to the base point  $(h, k)$  of  $F \times G - N$ , and  $(1, 1)$  is mapped to  $(h_1, k_1)$ . Thus the diagonal path  $d(t) = (\alpha_1(t), \beta_1(t))$  connects the base point to the boundary of the tubular neighborhood of  $Y_1 \times A_1$ .

Conjugating by  $d$  expresses the Lagrangian push-offs as based curves in  $F \times G - N$ . So

$$m_{Y_1 \times A_1} = d * (y_1 \times \{k_1\}) * d^{-1} \quad \text{and} \quad \ell_{Y_1 \times A_1} = d * (\{h_1\} \times a_1) * d^{-1}.$$

But  $m_{Y_1 \times A_1}$  is homotopic rel basepoint in  $F \times G - N$  to  $\tilde{y}_1$ . An explicit homotopy is given by the formula

$$(s, t) \mapsto \begin{cases} (\alpha_1(3t), \beta_1((1-s)3t)) & \text{if } 0 \leq t \leq 1/3, \\ (y_1(3t-1), \beta_1(1-s)) & \text{if } 1/3 \leq t \leq 2/3, \\ (\alpha_1(3-3t), \beta_1((1-s)(3-3t))) & \text{if } 2/3 \leq t \leq 1. \end{cases}$$

A similar homotopy, but exchanging the roles of  $\alpha_1$  and  $\beta_1$ , establishes that  $\ell_{Y_1 \times A_1}$  is homotopic rel basepoint in  $F \times G - N$  to  $\tilde{a}_1$ . (These homotopies clearly miss all the other  $Y_1 \times A_j$  and  $Y_2 \times B_j$ .)

It remains to calculate the meridian of  $Y_1 \times A_1$ . For this, consider the map  $x_1 \times b_1 : [0, 1] \times [0, 1] \rightarrow F \times G$ . This has image a torus intersecting  $Y_1 \times A_1$  transversally in one point (near the point  $(x_1(.9), b_1(.9))$ , as one sees from Figure 1). Since

$$(h_1, k_1) = (x_1 \times b_1)(0, 0) = (x_1 \times b_1)(0, 1) = (x_1 \times b_1)(1, 0) = (x_1 \times b_1)(1, 1),$$

by conjugating the path that follows the boundary of this square by the path  $d$  from the base point  $(h, k)$  to  $(h_1, k_1)$ , we see that the meridian  $\mu_{Y_1 \times A_1}$  is given by the composite

$$\mu_{Y_1 \times A_1} = d * (x_1 \times \{k_1\}) * (\{h_1\} \times b_1) * (x_1 \times \{k_1\})^{-1} * (\{h_1\} \times b_1)^{-1} * d^{-1}.$$



Now  $d * (x_1 \times \{k_1\}) * d^{-1}$  is homotopic rel basepoint to  $\tilde{x}_1$  in  $F \times G - N$  by the same argument given above. The key observation is that  $\beta_1$  misses  $A_j$  and  $B_j$  for all  $j$ . Similarly  $d * (\{h_1\} \times b_1) * d^{-1}$  is homotopic rel basepoint to  $\tilde{b}_1$  in  $F \times G - N$ . Thus

$$\mu_{Y_1 \times A_1} \sim \tilde{x}_1 * \tilde{b}_1 * \tilde{x}_1^{-1} * \tilde{b}_1^{-1} = [\tilde{x}_1, \tilde{b}_1].$$

Similar calculations establish all other assertions.  $\square$

#### 4. Telescoping triples and symplectic sums

Our construction of symplectic 4-manifolds which fill large regions in the geography plane is based on using telescoping symplectic sums along symplectic tori as well as Luttinger surgeries. The basic models in our constructions have a convenient property preserved under appropriate symplectic sum, and so we formalize the property in the following definition.

**Definition 2.** An ordered triple  $(X, T_1, T_2)$  where  $X$  is a symplectic 4-manifold and  $T_1, T_2$  are disjointly embedded Lagrangian tori is called a telescoping triple if

- (1) The tori  $T_1, T_2$  span a 2-dimensional subspace of  $H_2(X; \mathbb{R})$ .
- (2)  $\pi_1(X) \cong \mathbb{Z}^2$  and the inclusion induces an isomorphism  $\pi_1(X - (T_1 \cup T_2)) \rightarrow \pi_1(X)$  (in particular the meridians of the  $T_i$  are trivial in  $\pi_1(X - (T_1 \cup T_2))$ ).
- (3) The image of the homomorphism induced by inclusion  $\pi_1(T_1) \rightarrow \pi_1(X)$  is a summand  $\mathbb{Z} \subset \pi_1(X)$ .
- (4) The homomorphism induced by inclusion  $\pi_1(T_2) \rightarrow \pi_1(X)$  is an isomorphism.

If  $X$  is minimal we call  $(X, T_1, T_2)$  a minimal telescoping triple.

Note that the order of  $(T_1, T_2)$  matters in this definition. Notice also that since the meridians  $\mu_{T_1}, \mu_{T_2} \in \pi_1(X - (T_1 \cup T_2))$  are trivial and the relevant fundamental groups are abelian, the push-off of an oriented loop  $\gamma \subset T_i$  into  $X - (T_1 \cup T_2)$  with respect to any framing of the normal bundle of  $T_i$  (e.g. the Lagrangian framing) represents a well defined element of  $\pi_1(X - (T_1 \cup T_2))$ , independent of the choice of framing (and basing).

The definition of a telescoping triple includes the hypothesis that the Lagrangian tori  $T_1$  and  $T_2$  are linearly independent in  $H_2(X; \mathbb{R})$ . This implies ([15]) that the symplectic form on  $X$  can be slightly perturbed so that one of the  $T_i$  remains Lagrangian while the other becomes symplectic. It can also be perturbed so that both become symplectic. Moreover, if  $F$  is a symplectic surface in  $X$  disjoint from  $T_1$  and  $T_2$ , the perturbed symplectic form can be chosen so that  $F$  remains symplectic.

Recall that the symplectic sum ([15]) of two symplectic 4-manifolds  $X$  and  $X'$  along genus  $g$  symplectic surfaces  $F \subset X$  and  $F' \subset X'$  of opposite self-intersection is a symplectic 4-manifold described topologically as the union

$$X \#_{F, F'} X' = (X - \text{nbdd}(F)) \cup (X' - \text{nbdd}(F'))$$

where the boundaries of the tubular neighborhoods are identified by a fiber-preserving diffeomorphism of the corresponding circle bundles. When the surfaces are clear from context we write  $X \#_s X'$ .

**Proposition 3.** *Let  $(X, T_1, T_2)$  and  $(X', T'_1, T'_2)$  be two telescoping triples. Then for an appropriate gluing map the triple*

$$(X \#_{T_2, T'_1} X', T_1, T'_2)$$

*is again a telescoping triple. The Euler characteristic and signature of  $X \#_{T_2, T'_1} X'$  are given by  $e(X) + e(X')$  and  $\sigma(X) + \sigma(X')$ .*

*Proof.* Let  $i_j : \pi_1(T_j) \rightarrow \pi_1(X)$  be the homomorphisms induced by inclusion for  $j = 1, 2$ . Choose  $x_1, y_1 \in \pi_1(T_1)$  so that  $x_1$  spans the kernel of  $i_1$  and  $i_1(y_1)$  spans the image of  $i_1$ . Denote  $i_1(y_1)$  by  $t$  and choose  $s \in \pi_1(X)$  so that  $s, t$  forms a basis of  $\pi_1(X)$ . Then choose generators  $x_2, y_2$  for  $\pi_1(T_2)$  so that  $i_2(x_2) = s$  and  $i_2(y_2) = t$ . Thus the inclusions induce

$$x_1 \mapsto 1, \quad y_1 \mapsto t, \quad x_2 \mapsto s, \quad y_2 \mapsto t.$$

Similarly, construct generators  $x'_1, y'_1$  for  $\pi_1(T'_1)$ ,  $x'_2, y'_2$  for  $\pi_1(T'_2)$  and  $s', t'$  for  $\pi_1(X')$ .

The inclusion induces an isomorphism  $\pi_1(X - (T_1 \cup T_2)) \rightarrow \pi_1(X)$  and the boundary of the tubular neighborhood of  $T_1$  (resp.  $T_2$ ) is a 3-torus whose fundamental group is spanned by  $x_1, y_1, \mu_{T_1}$  (resp.  $x_2, y_2, \mu_{T_2}$ ) (for definiteness use the Lagrangian framing to push the  $x_i, y_i$  into the boundary of the tubular neighborhood). Similar assertions hold for  $(X', T'_1, T'_2)$ . The symplectic sum of  $X$  and  $X'$  along the surfaces  $T_2 \subset X$  and  $T'_1 \subset X'$  can be formed so that the ordered triple  $(x_2, y_2, \mu_2)$  is sent to  $(x'_1, y'_1, \mu'_1)$  by the gluing diffeomorphism (perhaps after a change of orientation on some of the loops to ensure that the gluing diffeomorphism is orientation preserving).

The Seifert–Van Kampen theorem and the fact that all meridians are trivial imply that

$$\pi_1(X \#_{T_2, T'_1} X') = \langle s, t, s', t' \mid [s, t], [s', t'], s, t(t')^{-1} \rangle = \mathbb{Z}s' \oplus \mathbb{Z}t'.$$

The inclusion  $T_1 \subset X \#_{T_2, T'_1} X'$  induces  $x_1 \mapsto 1, y_1 \mapsto t'$ . The inclusion  $T'_2 \subset X \#_{T_2, T'_1} X'$  induces  $x'_2 \mapsto s', y'_2 \mapsto t'$ . Hence  $(X \#_{T_2, T'_1} X', T_1, T'_2)$  is indeed a telescoping triple.

The assertions about the Euler characteristic and signature are clear.  $\square$

Since the meridians of the Lagrangian tori are trivial in a telescoping triple, one immediately deduces the following.

**Proposition 4.** *Let  $(X, T_1, T_2)$  be a telescoping triple. Let  $\ell_{T_1}$  be a Lagrangian push-off of a curve on  $T_1$  and  $m_{T_2}$  the Lagrangian push-off of a curve on  $T_2$  so that  $\ell_{T_1}$  and  $m_{T_2}$  generate  $\pi_1(X)$ . Then the symplectic 4-manifold obtained by performing  $+1$  Luttinger surgery on  $T_1$  along  $\ell_{T_1}$  and  $+1$  surgery on  $T_2$  along  $m_{T_2}$  is simply connected.  $\square$*

We will make frequent use of the following two results. The first is a criterion given by Usher [38] to determine when a symplectic sum is minimal. The second is a useful result of T.-J. Li which we will use to verify that the hypotheses in Usher's theorem hold in certain contexts.

**Theorem 5** (Usher). *Let  $Z = X_1 \#_{F_1, F_2} X_2$  denote the symplectic sum of  $X_1$  and  $X_2$  along symplectic surfaces  $F_i$  of positive genus  $g$ . Then:*

- (i) If either  $X_1 - F_1$  or  $X_2 - F_2$  contains an embedded symplectic sphere of square  $-1$ , then  $Z$  is not minimal.
- (ii) If one of the summands  $X_i$  (for definiteness, say  $X_1$ ) admits the structure of an  $S^2$ -bundle over a surface of genus  $g$  such that  $F_1$  is a section of this fiber bundle, then  $Z$  is minimal if and only if  $X_2$  is minimal.
- (iii) In all other cases,  $Z$  is minimal.

Corollary 3 of T.-J. Li's article [20] provides a useful method to eliminate the first two cases of Usher's theorem in some contexts.

**Theorem 6** (Li). *Let  $M$  be a symplectic 4-manifold which is neither rational nor ruled. Then every smoothly embedded  $-1$  sphere is homologous to a symplectic  $-1$  curve up to sign. If  $M$  is the blowup of a minimal symplectic 4-manifold with  $E_1, \dots, E_n$  represented by exceptional curves, then the  $E_i$  are the only classes represented by a smoothly embedded  $-1$  sphere, hence any orientation preserving diffeomorphism maps  $E_i$  to some  $\pm E_j$ .*

## 5. The model even signature manifolds

We will set up an inductive argument by constructing telescoping symplectic sums starting with several basic telescoping triples. Proposition 4 then applies to produce simply connected 4-manifolds.

To begin with, in [6, Theorem 20], a minimal telescoping triple  $(B, T_1, T_2)$  is constructed ( $B$  is denoted  $B_1$  in that article) so that  $B$  contains a genus 2 surface  $F$  with trivial normal bundle, and a geometrically dual symplectic  $-1$  torus  $H_1$ . The tori  $T_1, T_2$  miss  $F \cup H_1$ . Moreover,  $(B - F, T_1, T_2)$  is also a telescoping triple. These facts follow immediately from the following theorem, which summarizes the assertions established in [6].

**Theorem 7.** *There exists a minimal symplectic 4-manifold  $B$  containing a pair of homologically essential Lagrangian tori  $T_1$  and  $T_2$  and a square zero symplectic genus 2 surface  $F$  so that  $T_1, T_2$ , and  $F$  are pairwise disjoint,  $e(B) = 6$  and  $\sigma(B) = -2$ , and:*

- (1)  $\pi_1(B - (F \cup T_1 \cup T_2)) = \mathbb{Z}^2$ , generated by  $t_1$  and  $t_2$ .
- (2) The inclusion  $B - (F \cup T_1 \cup T_2) \subset B$  induces an isomorphism on fundamental groups. In particular, the meridians  $\mu_F, \mu_{T_1}, \mu_{T_2}$  all vanish in  $\pi_1(B - (F \cup T_1 \cup T_2))$ .
- (3) The Lagrangian push-offs  $m_{T_1}, \ell_{T_1}$  of  $\pi_1(T_1)$  are sent to 1 and  $t_2$  respectively in the fundamental group of  $B - (F \cup T_1 \cup T_2)$ .
- (4) The Lagrangian push-offs  $m_{T_2}, \ell_{T_2}$  of  $\pi_1(T_2)$  are sent to  $t_1$  and  $t_2$  respectively in the fundamental group of  $B - (F \cup T_1 \cup T_2)$ .
- (5) The push-off  $F \subset B - (F \cup T_1 \cup T_2)$  takes the first three generators of a standard symplectic generating set  $\{a_1, b_1, a_2, b_2\}$  for  $\pi_1(F)$  to 1 and the last element to  $t_2$ .
- (6) There exists a symplectic torus  $H_1 \subset B$  which intersects  $F$  transversally once, which has square  $-1$ , and the homomorphism  $\pi_1(H_1) \rightarrow \pi_1(B)$  takes the first generator to 1 and the second to  $t_1$ . Moreover,  $H_1$  is disjoint from  $T_1$  and  $T_2$  (see [6, Proposition 12, Theorem 20]). □

The following is a restatement of [6, Theorem 13]. We state it formally since we will have frequent need of it.

**Corollary 8.** *The symplectic 4-manifold  $X_{1,3}$  obtained from  $B$  by +1 Luttinger surgery on  $T_1$  along  $\ell_{T_1}$  and +1 Luttinger surgery on  $T_2$  along  $m_{T_2}$  is a minimal symplectic 4-manifold homeomorphic to  $\mathbb{C}\mathbb{P}^2 \# 3\overline{\mathbb{C}\mathbb{P}^2}$ . It contains a genus 2 symplectic surface of square zero with simply connected complement and a symplectic torus  $H_1$  of square  $-1$  intersecting  $F$  transversally and positively in one point.  $\square$*

**Corollary 9.** *For each  $g \geq 0$  there exists a minimal telescoping triple  $(B_g, T_1, T_2)$  satisfying  $e(B_g) = 6 + 4g$  and  $\sigma(B_g) = -2$  and containing a square  $-1$  genus  $g + 1$  surface disjoint from  $T_1 \cup T_2$ .*

*Proof.* To avoid confusing notation, during this proof we denote the symplectic genus 2 surface in  $B$  of Theorem 7 by  $F_B$ .

Take the product  $F \times G$  of a genus 2 surface  $F$  and a genus  $g$  surface  $G$ , as in Section 3. Let  $Z_g$  denote the 4-manifold obtained from  $F \times G$  by performing  $-1$  Luttinger surgeries on the  $2g$  disjoint Lagrangian tori  $Y_1 \times A_i$  and  $Y_2 \times B_i$  along the curves  $\ell_{Y_1 \times A_i} = \tilde{a}_i$  and  $\ell_{Y_2 \times B_i} = \tilde{b}_i$ . Then by Theorem 1 the fundamental group of  $Z_g$  is generated by loops  $\tilde{x}_1, \tilde{y}_1, \tilde{x}_2, \tilde{y}_2, \tilde{a}_1, \tilde{b}_1, \dots, \tilde{a}_g, \tilde{b}_g$  and the relations

$$[\tilde{x}_1, \tilde{b}_i] = \tilde{a}_i, \quad [\tilde{x}_2, \tilde{a}_i] = \tilde{b}_i$$

hold in  $\pi_1(Z_g)$ . Moreover, the standard symplectic generators for  $\pi_1(F)$  are sent to  $\tilde{x}_1, \tilde{y}_1, \tilde{x}_2, \tilde{y}_2$  in  $\pi_1(Z_g)$ .

Since the meridian  $\mu_{F_B}$  of  $F_B \subset B$  is trivial, the symplectic sum of  $B$  with  $Z_g$  along their genus 2 symplectic surfaces  $F_B \subset B$  and  $F = F \times \{k\} \subset F \times G$ ,

$$B_g = B \#_{F_B, F} Z_g,$$

has fundamental group a quotient of  $(\mathbb{Z}t_1 \oplus \mathbb{Z}t_2) * \pi_1(Z_g)$ . We choose this symplectic sum so that the generators  $a_1, b_1, a_2, b_2$  for  $\pi_1(F_B)$  are identified (in order) with the generators  $\tilde{x}_1, \tilde{y}_1, \tilde{x}_2, \tilde{y}_2$ .

The fifth assertion of Theorem 7 shows that  $\tilde{x}_1, \tilde{y}_1$ , and  $\tilde{x}_2$  are trivial in  $\pi_1(B_g)$ . The relations coming from the Luttinger surgeries then show that  $\tilde{a}_i = 1 = \tilde{b}_i$ . Since  $b_2 = \tilde{y}_2$  is identified with  $t_2$ ,  $\pi_1(B_g)$  is generated by  $t_1$  and  $t_2$ . A calculation using the Mayer–Vietoris sequence shows that  $H_1(B_g) = \mathbb{Z}^2$ , and so  $\pi_1(B_g) = \mathbb{Z}t_1 \oplus \mathbb{Z}t_2$ . Hence  $(B_g, T_1, T_2)$  is a telescoping triple, as desired.

The Euler characteristic of  $B_g$  is calculated as

$$e(B_g) = e(B) + e(F \times G) + 4 = 6 + 4g - 4 + 4 = 6 + 4g,$$

and the signature is computed by Novikov additivity:  $\sigma(B_g) = \sigma(B) = -2$ .

The torus  $H_1$  in  $B$  geometrically dual to  $F_B$  can be lined up with one of the parallel copies  $\{z\} \times G$  in  $F \times G$  (i.e. take a *relative symplectic sum*, [15]) to produce a square  $-1$  genus  $g + 1$  surface in  $B_g$ .

Minimality follows from [6, Lemma 2], which shows that  $Z_g$  is minimal (its universal cover is contractible, so  $\pi_2(Z_g) = 0$ ), and from Usher’s theorem (Theorem 5).  $\square$

We can also produce telescoping triples with odd signature starting with  $B$ . Recall that a symplectic 4-manifold  $X$  containing a symplectic surface  $F$  is called *relatively minimal* if every  $-1$  sphere in  $X$  intersects  $F$ .

**Lemma 10.** *The blowup  $A = B \# \overline{\mathbb{C}\mathbb{P}^2}$  contains a genus 3 symplectic surface  $F_3$  with trivial normal bundle and two Lagrangian tori  $T_1$  and  $T_2$  so that the surfaces  $F_3, T_1, T_2$  are pairwise disjoint,  $(A, F_3)$  is relatively minimal, and:*

- (1)  $\pi_1(A - (F_3 \cup T_1 \cup T_2)) = \mathbb{Z}^2$ , generated by  $t_1$  and  $t_2$ .
- (2) The inclusion  $A - (F_3 \cup T_1 \cup T_2) \subset A$  induces an isomorphism on fundamental groups. In particular, the meridians  $\mu_{F_3}, \mu_{T_1}, \mu_{T_2}$  all vanish in  $\pi_1(A - (F_3 \cup T_1 \cup T_2))$ .
- (3) The Lagrangian push-offs  $m_{T_1}, \ell_{T_1}$  of  $\pi_1(T_1)$  are sent to 1 and  $t_2$  respectively in the fundamental group of  $A - (F_3 \cup T_1 \cup T_2)$ .
- (4) The Lagrangian push-offs  $m_{T_2}, \ell_{T_2}$  of  $\pi_1(T_2)$  are sent to  $t_1$  and  $t_2$  respectively in the fundamental group of  $A - (F_3 \cup T_1 \cup T_2)$ .
- (5) There is a standard symplectic generating set  $\{a_1, b_1, a_2, b_2, a_3, b_3\}$  for  $\pi_1(F_3)$  so that the push-off  $F_3 \subset A - (F_3 \cup T_1 \cup T_2)$  takes  $b_2$  to  $t_2$ ,  $b_3$  to  $t_1$ , and all other generators to 1.

In particular,  $(A, T_1, T_2)$  is a telescoping triple.

*Proof.* The 4-manifold  $B$  of Theorem 7 contains a symplectic genus 2 surface  $F$  of square zero and a geometrically dual symplectic torus  $H_1$  of square  $-1$ . Symplectically resolve the union  $F \cup H_1$  to get  $F'_3$ , a genus 3 symplectic surface in  $B$  which misses  $T_1$  and  $T_2$ . The surface  $F'_3$  has square  $(F + H_2)^2 = 1$ . Blow up  $B$  at one point on  $F'_3$  to construct  $A$  and denote the proper transform of  $F'_3$  by  $F_3$ .

Since  $F_3$  has a geometrically dual 2-sphere (the exceptional sphere), the meridian of  $F_3$  in  $A - F_3 \subset F_3$  is nullhomotopic. The rest of the fundamental group assertions follow from Theorem 7.

Although  $A$  is not minimal, T.-J. Li's theorem (Theorem 6) implies that every  $-1$  sphere in  $A$  intersects  $F_3$ , since  $B$  is minimal, and neither rational nor ruled.  $\square$

Note that Luttinger surgery on  $T_1$  and  $T_2$  in  $A$  produces a symplectic 4-manifold homeomorphic to  $\mathbb{C}\mathbb{P}^2 \# 4\overline{\mathbb{C}\mathbb{P}^2}$ , but this manifold is not minimal; it is just the blowup  $X_{1,3} \# \overline{\mathbb{C}\mathbb{P}^2}$ . We do not know how to produce a minimal symplectic 4-manifold with this homeomorphism type.

We next produce a 4-manifold  $C$  with  $e = 8$  and  $\sigma = -4$  by stopping the construction of a minimal symplectic 4-manifold homeomorphic to  $\mathbb{C}\mathbb{P}^2 \# 5\overline{\mathbb{C}\mathbb{P}^2}$  in the proof of [6, Theorem 10] before the last two Luttinger surgeries to obtain the following.

**Theorem 11.** *There exists a minimal telescoping triple  $(C, T_1, T_2)$  with  $e(C) = 8$  and  $\sigma(C) = -4$ . Moreover,  $C$  contains a square  $-1$  torus disjoint from  $T_1 \cup T_2$ .*

*Proof.* We follow the notation and proof of [6, Theorem 10]. By *not* performing the Luttinger surgeries on the tori  $T_3$  and  $T_4$ , one obtains a minimal symplectic 4-manifold  $C$  such that  $\pi_1(C - (T_3 \cup T_4))$  is generated by the two commuting elements  $y$  and  $a_2$ . The Mayer-Vietoris sequence shows that  $H_1(C - (T_3 \cup T_4); \mathbb{Z}) = \mathbb{Z}^2$ , and so  $\pi_1(C - (T_1 \cup T_2)) =$

$\mathbb{Z}y \oplus \mathbb{Z}a_2$ . The meridians and Lagrangian push-offs of  $T_3$  and  $T_4$  are given by  $\mu_{T_3} = 1$ ,  $m_{T_3} = 1$ ,  $\ell_{T_3} = a_2$  and  $\mu_{T_4} = 1$ ,  $m_{T_4} = y$ ,  $\ell_{T_4} = a_2$ . Thus  $(C, T_3, T_4)$  is a telescoping triple. We relabel  $T_3$  as  $T_1$  and  $T_4$  as  $T_2$ .

The  $-1$  torus comes about from the construction. Briefly,  $C$  is obtained by performing Luttinger surgeries on the symplectic sum  $(T^2 \times F_2) \#_s ((T^2 \times S^2) \# 4\overline{\mathbb{C}\mathbb{P}^2})$  along the genus 2 surface  $\{x\} \times F_2$  in  $T^2 \times F_2$  and the genus 2 surface  $F'_2 \subset (T^2 \times S^2) \# 4\overline{\mathbb{C}\mathbb{P}^2}$  obtained by resolving the singularities of  $(T^2 \times \{p_1\}) \cup (\{q\} \times S^2) \cup (T^2 \times \{p_2\})$  and blowing up four times at points on this genus 2 surface. One can choose a square zero torus of the form  $T^2 \times \{y\} \subset T^2 \times F_2$  which matches up (i.e. take a relative symplectic sum) with one of the four exceptional curves to provide a  $-1$  symplectic torus disjoint from the Lagrangian tori where the Luttinger surgeries are performed.  $\square$

The symplectic 4-manifold  $X_{1,5}$  obtained from  $C$  by  $+1$  Luttinger surgeries on  $T_1$  and  $T_2$  as in Proposition 4 is minimal and homeomorphic to  $\mathbb{C}\mathbb{P}^2 \# 5\overline{\mathbb{C}\mathbb{P}^2}$ .

Our next small model is a minimal telescoping triple built in the process of constructing a minimal symplectic 4-manifold homeomorphic to  $\mathbb{C}\mathbb{P}^2 \# 7\overline{\mathbb{C}\mathbb{P}^2}$  in [6, Theorem 8]. One stops the construction before performing the two Luttinger surgeries, and these unused tori provide the desired  $T_1$  and  $T_2$ .

**Theorem 12.** *There exists a minimal telescoping triple  $(D, T_1, T_2)$  with  $e(D) = 10$  and  $\sigma(D) = -6$ . Moreover,  $D$  contains a square  $-1$  torus disjoint from  $T_1 \cup T_2$ .*  $\square$

*Proof.* The proof is similar to that of Theorem 11. We follow the notation and proof of [6, Theorem 8]. The 4-manifold  $S$  contains two Lagrangian tori  $T_1, T_2$  such that  $\pi_1(S - (T_1 \cup T_2))$  is generated by the two commuting elements  $s_1, t_1$ . The Mayer–Vietoris sequence computes  $H_1(S - (T_1 \cup T_2)) = \mathbb{Z}^2$  so that  $\pi_1(S - (T_1 \cup T_2)) = \mathbb{Z}s_1 \oplus \mathbb{Z}t_1$ .

The meridians and Lagrangian push-offs of  $T_1$  and  $T_2$  are given by  $\mu_{T_1} = 1$ ,  $m_{T_1} = s_1$ ,  $\ell_{T_1} = s_1^{-1}$  and  $\mu_{T_2} = 1$ ,  $m_{T_2} = t_1$ ,  $\ell_{T_2} = s_1$ . Thus  $(S, T_1, T_2)$  is a telescoping triple. It is shown to be minimal in the proof of [6, Theorem 8]. The existence of a square  $-1$  torus follows exactly as in the proof of Theorem 11, since the manifold  $S$  is obtained by Luttinger surgeries on the symplectic sum of  $(T^2 \times T^2) \# 2\overline{\mathbb{C}\mathbb{P}^2}$  and  $(T^2 \times S^2) \# 4\overline{\mathbb{C}\mathbb{P}^2}$  along a genus 2 surface. Relabel  $S$  as  $D$ .  $\square$

The symplectic 4-manifold  $X_{1,7}$  obtained from  $D$  by  $+1$  Luttinger surgeries on  $T_1$  and  $T_2$  as in Proposition 4 is minimal and homeomorphic to  $\mathbb{C}\mathbb{P}^2 \# 7\overline{\mathbb{C}\mathbb{P}^2}$  ([6]). More generally, the following proposition is true.

**Proposition 13.** *Let  $X$  be one of the manifolds  $B, B_g, C, D$  and  $T_1, T_2$  the corresponding Lagrangian tori as described in Theorems 7, 11, 12, with Lagrangian push-offs  $m_{T_i}$  and  $\ell_{T_i}$  (and trivial meridians). Then the symplectic 4-manifolds obtained from  $\pm 1$  Luttinger surgery on one or both of  $T_1, T_2$  along  $m_{T_i}$  or  $\ell_{T_i}$  are all minimal.*  $\square$

We omit the proof, which is based on Usher’s theorem and a repeated use of [6, Lemma 2]. The reader may look at the proofs of Theorems 8, 10, and 13 of [6].

Since our emphasis in this article is on 4-manifolds with odd intersection forms, we recall the following result [14, Theorem VII.3.2 for minimality].

**Theorem 14.** *The symplectic manifold  $E'(k) = E(k)_{2,3}$  obtained from the elliptic surface  $E(k)$  by performing two log transforms of order 2 and 3 is simply connected and minimal. It has Euler characteristic  $e(E'(k)) = 12k$ , signature  $\sigma(E'(k)) = -8k$ , and an odd intersection form.  $\square$*

## 6. Minimal symplectic 4-manifolds with $\sigma = -3$ and $e \geq 15$

The most complicated examples we construct are simply connected minimal symplectic 4-manifolds with signature  $-3$ . Putting these in the context of telescoping triples is more trouble than constructing them directly. Moreover, with the exception of the  $\sigma = -3$  manifolds, our inductive scheme for filling out the entire geography for  $\sigma \leq -2$  only requires at most one copy of the manifold  $A$  of Lemma 10. Hence in this section we prove the following theorem.

**Theorem 15.** *For each integer  $k \geq 2$ , there exists a simply connected minimal symplectic 4-manifold  $X_{1+2k,4+2k}$  with  $e(X_{1+2k,4+2k}) = 7 + 4k$  and  $\sigma(X_{1+2k,4+2k}) = -3$ .*

The construction of 4-manifolds with signature  $-3$  for  $e = 7 + 8g$  is easier than for  $e = 11 + 8g$ . Roughly speaking, to produce a 4-manifold with  $e = 7 + 8g$ , we take the symplectic sum along a genus 3 surface of the 4-manifold  $A$  of Lemma 10 with  $F \times G$ , where  $F$  is a genus 3 surface and  $G$  is a genus  $g$  surface, and perform Luttinger surgeries on the Lagrangian tori in  $G$ . To produce a 4-manifold with  $e = 11 + 8g$  requires producing a substitute  $A'$  for  $A$  which has signature  $-3$  and  $e = 11$ , and which satisfies the conclusions of Lemma 10. To do this, we take the symplectic sum of  $A$  with the product  $F \times G$  of two genus 2 surfaces along a symplectic torus.

**Lemma 16.** *There exists a minimal symplectic 4-manifold  $Z$  with  $e(Z) = 4$  and  $\sigma(Z) = 0$  which contains eight homologically essential Lagrangian tori  $S_1, \dots, S_8$  (in fact each  $S_i$  has a geometrically dual torus  $S_i^d$  so that all other intersections are zero) so that  $\pi_1(Z - \bigcup_i S_i)$  is generated by  $x_1, y_1, x_2, y_2$  and  $a_1, b_1, a_2, b_2$ , and so that the meridians and Lagrangian push-offs are given by*

- $S_1 : \mu_1 = [b_1^{-1}, y_1^{-1}], m_1 = x_1, \ell_1 = a_1,$
- $S_2 : \mu_2 = [x_1^{-1}, b_1], m_2 = y_1, \ell_2 = b_1 a_1 b_1^{-1},$
- $S_3 : \mu_3 = [b_2^{-1}, y_1^{-1}], m_3 = x_1, \ell_3 = a_2,$
- $S_4 : \mu_4 = [x_1^{-1}, b_2], m_4 = y_1, \ell_4 = b_2 a_2 b_2^{-1},$
- $S_5 : \mu_5 = [b_1 a_1^{-1} b_1^{-1}, y_2^{-1}], m_5 = x_2, \ell_5 = b_1^{-1},$
- $S_6 : \mu_6 = [x_2^{-1}, b_1 a_1 b_1^{-1}], m_6 = y_2, \ell_6 = b_1 a_1 b_1^{-1} a_1^{-1} b_1^{-1},$
- $S_7 : \mu_7 = [b_2 a_2^{-1} b_2^{-1}, y_2^{-1}], m_7 = x_2, \ell_7 = b_2^{-1},$
- $S_8 : \mu_8 = [x_2^{-1}, b_2 a_2 b_2^{-1}], m_8 = y_2, \ell_8 = b_2 a_2 b_2^{-1} a_2^{-1} b_2^{-1}.$

*Proof.* Proposition 7 of [6] (see also the construction of the manifold  $P$  in [5]) computes the fundamental group of the complement of four Lagrangian tori  $S_1, S_2, S_3, S_4$  in the product  $F_1 \times G$  of a punctured torus  $F_1$  with a genus 2 surface  $G$ . This group is generated

by loops  $x_1, y_1, a_1, b_1, a_2, b_2$  (called  $\tilde{x}, \tilde{y}, \tilde{a}_1, \tilde{b}_1, \tilde{a}_2, \tilde{b}_2$  there) where  $a_1, b_1, a_2, b_2$  are a standard generating set for  $\pi_1(G)$ , and  $x_1, y_1$  are a standard generating set for  $\pi_1(F_1)$  based at a point  $h$  on the boundary. In particular, the copy  $\{h\} \times G$  in the boundary of  $F_1 \times G$  carries the loops  $a_1, b_1, a_2, b_2$ .

We take two copies of this manifold, calling the second  $F_2 \times G$ , its tori  $S_5, S_6, S_7, S_8$ , and its generators  $x_2, y_2, a'_1, b'_1, a'_2, b'_2$ . Glue the two copies together using a diffeomorphism of their boundary of the form  $\text{Id} \times \phi : \partial F_2 \times G \rightarrow \partial F_1 \times G$ , where  $\phi : G \rightarrow G$  is the basepoint preserving diffeomorphism inducing the map

$$(a'_1, b'_1, a'_2, b'_2) \mapsto (b_1^{-1}, b_1 a_1 b_1^{-1}, b_2^{-1}, b_2 a_2 b_2^{-1})$$

(a composite of six Dehn twists: see [6, Lemma 9]).

The resulting manifold  $Z$  can also be described as the symplectic sum of two copies of a product of a genus 1 and a genus 2 surface. Thus the result is symplectic and the 8 tori are Lagrangian. The tori  $S_1, S_2, S_3, S_4$  in  $F_1 \times G$  have geometrically dual tori  $S_1^d, S_2^d, S_3^d, S_4^d$  which form a direct sum (geometrically) of four hyperbolic pairs, and similarly for  $S_5, S_6, S_7, S_8$ . Clearly  $e(Z) = 4$  and  $\sigma(Z) = 0$ . Applying the Seifert–Van Kampen theorem to the formulae of Proposition 7 of [6] yields the fundamental group assertions. Since the diffeomorphism  $\text{Id} \times \phi : \partial F_2 \times G \rightarrow \partial F_1 \times G$  extends to  $F_2 \times G \rightarrow F_1 \times G$ , the manifold  $Z$  is just the product of two genus 2 surfaces. In particular  $Z$  is minimal.  $\square$

Let  $Y$  be the symplectic 4-manifold obtained from the manifold  $Z$  of Lemma 16 by performing the following seven Luttinger surgeries on  $S_1, \dots, S_7$ :

- (1)  $S_1$  : +1 surgery along  $m_1$ .
- (2)  $S_2$  : +1 surgery along  $\ell_2$ .
- (3)  $S_3$  : +1 surgery along  $\ell_3$ .
- (4)  $S_4$  : +1 surgery along  $m_4$ .
- (5)  $S_5$  : +1 surgery along  $\ell_5$ .
- (6)  $S_6$  : +1 surgery along  $m_6$ .
- (7)  $S_7$  : +1 surgery along  $m_7$ .

Since the torus  $S_8$  has not been surgered, it remains as a Lagrangian torus in  $Y$ . Since  $S_8$  is homologically essential, the symplectic form can be perturbed so that  $S_8$  becomes symplectic. The symplectic 4-manifold  $Y$  is minimal, since it is a symplectic sum of manifolds with contractible universal cover (see [6, Lemma 2]).

Let  $(B, T_1, T_2)$  be the telescoping triple of Theorem 7, with  $B$  containing the genus 2 symplectic surface  $F$  and geometrically dual  $-1$  torus  $H_1$ . Perform +1 Luttinger surgery on  $T_2$  along  $m_{T_2}$  to kill  $t_1$ , yielding a minimal (Proposition 13) symplectic 4-manifold  $\hat{B}$ . Note that  $\hat{B}$  still contains the three surfaces  $T_1, F, H_1$  and  $\pi_1(\hat{B} - (T_1 \cup F \cup H_1)) = \mathbb{Z}t_2$ . The torus  $T_1$  is disjoint from the geometrically dual symplectic surfaces  $F$  and  $H_1$ , and its Lagrangian push-offs are  $m_{T_1} = 1$  and  $\ell_{T_1} = t_2$  by Theorem 7.

**Lemma 17.** *The symplectic sum  $X_{3,5} = \hat{B} \#_{T_1, S_8} Y$  is simply connected, minimal, and contains a symplectic genus 2 surface of square 0 and a geometrically dual symplectic torus of square  $-1$ . Moreover,  $e(X_{3,5}) = 10$  and  $\sigma(X_{3,5}) = -2$ , so that  $X_{3,5}$  is homeomorphic but not diffeomorphic to  $3\mathbb{C}\mathbb{P}^2 \# 5\overline{\mathbb{C}\mathbb{P}^2}$ .*



*Proof.* We refer to the notation in the statement of Theorem 7. The fundamental group of  $X_{3,5} - F$  is generated by  $t_2, x_1, y_1, x_2, y_2, a_1, b_1, a_2, b_2$  by the Seifert–Van Kampen theorem (recall that  $t_1$  is killed by Luttinger surgery on  $T_2$ ).

Since the meridian of  $T_1$  in  $\pi_1(B - (F \cup T_1 \cup T_2))$  is trivial,  $\mu_8$  is trivial in  $\pi_1(X_{3,5} - F)$ . Choose the gluing map  $S_8 \rightarrow T_1$  so that  $\ell_8$  is killed and  $m_8$  is sent to  $t_2$  (i.e.  $m_8 \mapsto \ell_{T_1}, \ell_8 \mapsto m_{T_1}^{-1}$ ).

Since  $\ell_8$  is a conjugate of  $b_2^{-1}$  and  $m_8 = y_2$ , it follows that  $b_2 = 1$  and  $y_2 = t_2$ . This implies that  $\mu_3$  and  $\mu_4$  are trivial, and hence the third and fourth Luttinger surgeries listed above show that  $a_2 = 1$  and  $y_1 = 1$ . Thus  $\mu_1$  and  $\mu_7$  are killed. The first and seventh Luttinger surgeries now show that  $x_1 = 1$  and  $x_2 = 1$ . Continuing, we see that  $\mu_2$  and  $\mu_6$  are killed so that the corresponding surgeries give  $a_1 = 1$  and  $y_2 = 1$ . This implies  $\mu_5 = 1$  and so  $b_1 = 1$ . Hence  $\pi_1(X_{3,5}) = 1$ .

The minimality of  $X_{3,5}$  follows from Usher’s theorem. The genus 2 surface  $F$  and torus  $H_1$  in  $B$  survive to give the required surfaces in  $X_{3,5}$ .  $\square$

*Proof of Theorem 15.* We define two minimal simply connected symplectic 4-manifolds: let  $X_- = X_{3,5}$  and let  $X_+ = X_{1,3}$  (thus  $X_+$  is obtained from the manifold  $\hat{B}$  defined above by performing  $+1$  Luttinger surgery on  $T_1$  along  $\ell_{T_1}$ ; see Corollary 8). Then  $X_-$  and  $X_+$  each contain a symplectic genus 3 surface  $F_3$  of square 1 obtained by resolving the union  $H_1 \cup F$ . Moreover,  $e(X_-) = 10$ ,  $\sigma(X_-) = -2$ ,  $e(X_+) = 6$ , and  $\sigma(X_+) = -2$ .

Blow up  $X_{\pm}$  once at a point on  $F_3$  and take the proper transform. Call the result  $\tilde{X}_{\pm}$  and denote by  $\tilde{F}_3$  the proper transform of  $F_3$ . Thus  $\tilde{F}_3$  is a genus 3, square zero symplectic surface with simply connected complement, which meets every  $-1$  sphere in  $\tilde{X}_{\pm}$  since  $X_{\pm}$  is minimal.

We now mimic the proof of Corollary 9. Take the product  $F_3 \times G$  of a genus 3 surface with a genus  $g$  surface. Perform Luttinger surgeries on the  $2g$  disjoint Lagrangian tori  $Y_1 \times A_j$  and  $Y_2 \times B_j$  along the curves  $\ell_{Y_1 \times A_j} = a_j$  and  $\ell_{Y_2 \times B_j} = b_j$  to obtain a manifold  $Z_g$ .

Then by Theorem 1 the fundamental group of  $Z_g$  is generated by the  $6 + 2g$  loops  $\tilde{x}_1, \tilde{y}_1, \tilde{x}_2, \tilde{y}_2, \tilde{x}_3, \tilde{y}_3, \tilde{a}_1, \tilde{b}_1, \dots, \tilde{a}_g, \tilde{b}_g$ , and the relations

$$[\tilde{x}_1, \tilde{b}_j] = \tilde{a}_j, \quad [\tilde{x}_2, \tilde{a}_j] = \tilde{b}_j, \quad j = 1, \dots, g,$$

hold in  $\pi_1(Z_g)$ . Moreover, the standard symplectic generators for  $\pi_1(F)$  are sent to  $\tilde{x}_1, \tilde{y}_1, \tilde{x}_2, \tilde{y}_2, \tilde{x}_3, \tilde{y}_3$  in  $\pi_1(Z_g)$ .

Since  $\pi_1(\tilde{X}_{\pm} - \tilde{F}_3) = 1$ , the fundamental group of the symplectic sum

$$Q_{\pm, g} = \tilde{X}_{\pm} \#_{\tilde{F}_3, F} Z_g$$

is trivial. Indeed, the  $\tilde{x}_i, \tilde{y}_i$  are killed by taking the symplectic sum, and the relations coming from the Luttinger surgeries show the  $\tilde{a}_j$  and  $\tilde{b}_j$  are killed also.

Now  $Q_{\pm, g}$  is minimal provided  $g \geq 1$  by Usher’s theorem since  $\tilde{X}_{\pm}$  is relatively minimal by Li’s theorem (Theorem 6).

One computes:

$$e(Q_{-,g}) = e(\tilde{X}_-) + e(Z_g) + 8 = 11 + 8g - 8 + 8 = 11 + 8g,$$

$$\sigma(Q_{-,g}) = \sigma(\tilde{X}_-) + \sigma(Z_g) = -3,$$

and

$$e(Q_{+,g}) = e(\tilde{X}_+) + e(Z_g) + 8 = 7 + 8g - 8 + 8 = 7 + 8g,$$

$$\sigma(Q_{+,g}) = \sigma(\tilde{X}_+) + \sigma(Z_g) = -3.$$

Thus we set  $X_{1+2k,4+2k} = Q_{+,k/2}$  if  $k$  is even and  $X_{1+2k,4+2k} = Q_{-, (k-1)/2}$  if  $k$  is odd. This completes the proof of Theorem 15.  $\square$

**Remark 1.** In the construction of the manifold  $X_{5,8}$ , the first step (see the paragraph preceding Lemma 17) involves Luttinger surgery on the torus  $T_2$  to kill  $t_1$ . If one constructs the manifold  $P_{5,8}$  following the same construction as for  $X_{5,8}$  without performing this surgery, then  $P_{5,8}$  is a minimal symplectic 4-manifold with  $\pi_1(P_{5,8}) = \mathbb{Z}t_1$  containing an essential Lagrangian (or if desired symplectic) torus  $T = T_2$  such that the inclusion map  $\pi_1(T) \rightarrow \pi_1(P_{5,8})$  is a surjection and the inclusion map  $\pi_1(P_{5,8} - T) \rightarrow \pi_1(P_{5,8})$  an isomorphism. Moreover,  $e(P_{5,8}) = 15$  and  $\sigma(P_{5,8}) = -3$ .

More generally, for any  $k \geq 2$  the same construction yields a minimal symplectic 4-manifold  $P_{1+2k,4+2k}$  containing a Lagrangian or symplectic torus  $T$  with these properties and such that  $e(P_{1+2k,4+2k}) = 7 + 4k$ ,  $\sigma(P_{1+2k,4+2k}) = -3$ .

## 7. Small examples with odd signature

In this section, we remind the reader of some known examples of small manifolds with odd signature, and construct a few new ones.

Kotschick showed in [19] that the Barlow surface is smoothly irreducible and hence it is a minimal symplectic 4-manifold homeomorphic to  $\mathbb{C}\mathbb{P}^2 \# 8\overline{\mathbb{C}\mathbb{P}^2}$ . This manifold realizes the pair  $e = 11$ ,  $\sigma = -7$ .

In [15], Gompf constructs small minimal symplectic 4-manifolds which contain appropriate tori. For example, the manifold Gompf calls  $S_{1,1}$  is minimal, has  $e = 23$  and  $\sigma = -15$ , and contains a symplectic torus of square zero with simply connected complement ([15, Lemma 5.5]). The minimality of  $S_{1,1}$  was proved by Stipsicz [31].

Gompf also constructs other minimal symplectic 4-manifolds: the manifold  $R_{2,1}$  has  $e = 21$  and  $\sigma = -13$  and  $R_{2,2}$  has  $e = 19$  and  $\sigma = -11$ . The minimality of  $R_{2,1}$  was proved by J. Park [27], and  $R_{2,2}$  was proved to be minimal by Szabó [34].

In [33], Stipsicz and Szabó construct a minimal symplectic 4-manifold homeomorphic to  $\mathbb{C}\mathbb{P}^2 \# 6\overline{\mathbb{C}\mathbb{P}^2}$ , realizing  $e = 9$ ,  $\sigma = -5$ .

In [25], the fifth author constructs a minimal simply connected symplectic 4-manifold homeomorphic to  $3\mathbb{C}\mathbb{P}^2 \# 12\overline{\mathbb{C}\mathbb{P}^2}$ , hence with  $e = 17$  and  $\sigma = -9$ , containing a symplectic torus  $T_{2,4}$  with simply connected complement. This manifold is called  $X_{12}$  in that article; we will use the notation  $X_{3,12}$  here to avoid confusion.

We produce a few more small examples.

**Proposition 18.** *There exists a minimal simply connected symplectic 4-manifold  $X_{5,10}$  homeomorphic to  $5\mathbb{C}P^2 \# 10\overline{\mathbb{C}P^2}$ , hence with  $e = 17$  and  $\sigma = -5$ .*

*Proof.* The manifold  $X_{1,3}$  of Corollary 8 contains a symplectic genus 2 surface  $F$  of square zero, and a geometrically dual symplectic torus  $H_1$  with square  $-1$ . Symplectically resolve  $F \cup H_1$  to produce a square 1 symplectic genus 3 surface  $F_3 \subset X_{1,3}$ .

Blow up  $X_{1,3}$  at a point on  $F_3$  to obtain  $\tilde{X}_{1,3}$  and take  $\tilde{F}_3$  to be the proper transform of  $F_3$ . Then  $\tilde{F}_3$  is a square zero symplectic surface that meets every  $-1$  sphere in  $\tilde{X}_{1,3}$  by Li's theorem (Theorem 6). Moreover, since  $X_{1,3}$  is simply connected and  $\tilde{F}_3$  meets the exceptional sphere,  $\tilde{X}_{1,3} - \tilde{F}_3$  is simply connected.

Take  $Y = T \times F_2$ , the product of a torus with a genus 2 surface. Then  $Y$  contains the geometrically dual symplectic surfaces  $T \times \{p\}$  and  $\{q\} \times F_2$ . Symplectically resolve their union to obtain a genus 3, square 2 symplectic surface  $F'_3 \subset Y$ . Note that the homomorphism induced by inclusion  $\pi_1(F'_3) \rightarrow \pi_1(Y)$  is surjective. Blow up  $Y$  twice at points on  $F'_3$  to obtain  $\tilde{Y}$  and the proper transform  $\tilde{F}'_3$ , a square zero genus 3 symplectic surface. Then the symplectic sum

$$X_{5,10} = \tilde{X}_{1,3} \#_{\tilde{F}_3, \tilde{F}'_3} \tilde{Y}$$

is simply connected. It is minimal by Usher's theorem.

Its characteristic numbers are

$$e(X_{5,10}) = e(\tilde{X}_{1,3}) + e(\tilde{Y}) + 8 = 7 + 2 + 8 = 17,$$

$$\sigma(X_{5,10}) = \sigma(\tilde{X}_{1,3}) + \sigma(\tilde{Y}) = -3 - 2 = -5.$$

The proposition follows.  $\square$

**Proposition 19.** *There exists a minimal simply connected symplectic 4-manifold  $X_{5,12}$  homeomorphic to  $5\mathbb{C}P^2 \# 12\overline{\mathbb{C}P^2}$ , hence with  $e = 19$  and  $\sigma = -7$ .*

*Proof.* The proof is very similar to the proof of Proposition 18. Construct  $\tilde{X}_{1,3}$  and  $\tilde{F}_3$  as in that proof.

Take  $Z = T \times T$ , the product of two tori. Pick three distinct points  $p_1, p_2, q$  in  $T$ . Then  $Z$  contains the three symplectic surfaces  $T \times \{p_1\}$ ,  $T \times \{p_2\}$  and  $\{q\} \times T$ . Symplectically resolve their union to obtain a genus 3, square 4 symplectic surface  $F'_3 \subset Z$ . Note that the homomorphism induced by inclusion  $\pi_1(F'_3) \rightarrow \pi_1(Z)$  is surjective. Blow up  $Z$  four times at points on  $F'_3$  to obtain  $\tilde{Z}$  and the proper transform  $\tilde{F}'_3$ , a square zero genus 3 symplectic surface.

Then the symplectic sum

$$X_{5,12} = \tilde{X}_{1,3} \#_{\tilde{F}_3, \tilde{F}'_3} \tilde{Z}$$

is simply connected. It is minimal by Usher's theorem.

Its characteristic numbers are

$$e(X_{5,12}) = e(\tilde{X}_{1,3}) + e(\tilde{Z}) + 8 = 7 + 4 + 8 = 19,$$

$$\sigma(X_{5,12}) = \sigma(\tilde{X}_{1,3}) + \sigma(\tilde{Z}) = -3 - 4 = -7.$$

The proposition follows.  $\square$

**Proposition 20.** *There exists a minimal simply connected symplectic 4-manifold  $X_{5,14}$  homeomorphic to  $5\mathbb{C}\mathbb{P}^2 \# 14\overline{\mathbb{C}\mathbb{P}^2}$ , hence with  $e = 21$  and  $\sigma = -9$ .*

*Proof.* The proof is very similar to the proof of Proposition 19. Construct  $\tilde{X}_{1,3}$  and  $\tilde{F}_3$  as in that proof.

Take  $Z = T \times S^2$ , the product of a torus and a sphere. Pick three distinct points  $p_1, p_2, p_3$  in  $S^2$  and  $q \in T$ . Then  $Z$  contains the four symplectic surfaces  $T \times \{p_1\}$ ,  $T \times \{p_2\}$ ,  $T \times \{p_3\}$  and  $\{q\} \times S^2$ . Symplectically resolve their union to obtain a genus 3, square 6 symplectic surface  $F'_3 \subset Z$ . Note that the homomorphism induced by inclusion  $\pi_1(F'_3) \rightarrow \pi_1(Z)$  is surjective. Blow up  $Z$  six times at points on  $F'_3$  to obtain  $\tilde{Z}$  and the proper transform  $\tilde{F}'_3$ , a square zero genus 3 symplectic surface.

Then the symplectic sum

$$X_{5,14} = \tilde{X}_{1,3} \#_{\tilde{F}_3, \tilde{F}'_3} \tilde{Z}$$

is simply connected. It is minimal by Usher's theorem.

Its characteristic classes are

$$\begin{aligned} e(X_{5,14}) &= e(\tilde{X}_{1,3}) + e(\tilde{Z}) + 8 = 7 + 6 + 8 = 21, \\ \sigma(X_{5,14}) &= \sigma(\tilde{X}_{1,3}) + \sigma(\tilde{Z}) = -3 - 6 = -9. \end{aligned}$$

The proposition follows.  $\square$

## 8. The main theorem

In this section we prove Theorem A stated in the introduction. We begin with an arithmetic lemma. The purpose of this lemma is to calculate the number of each of the model manifolds  $B, B_g, C, D, E(k)$  needed to construct a 4-manifold with specified signature and Euler characteristic. The proof includes an algorithm for finding these numbers.

**Lemma 21.** *Given any pair of non-negative integers  $(m, n)$  such that*

$$0 \leq m \leq 4n - 1$$

*there exist non-negative integers  $b, c, d, g$ , and  $k$  such that*

$$m = d + 2c + 3b + 4g \quad \text{and} \quad n = b + c + d + g + k$$

*and such that  $b \geq 1$  if  $g > 0$ .*

*Proof.* If  $m = 0$ , set  $k = n$  and  $b = c = d = g = 0$ .

Assume then that  $m > 0$ . Choose a non-negative integer  $\ell$  so that  $(m + 1)/4 \leq n - \ell \leq m$ . Let

$$s = \max\{z \in \mathbb{Z} \mid 3z \leq 4n - 4\ell - m - 1\} \quad \text{and} \quad \Delta = 4n - 4\ell - m - 1 - 3s.$$

Then  $\Delta = 0, 1$  or  $2$ , and  $s \geq 0$ . Moreover,

$$n - \ell - s - 1 = \frac{1}{3}(m - (n - \ell)) - \frac{2}{3} + \frac{\Delta}{3} \geq \frac{\Delta}{3} - \frac{2}{3}.$$

If  $\Delta = 0$ , then set  $b = 1$ ,  $c = 0$ ,  $d = s$ ,  $g = n - \ell - s - 1$ , and  $k = \ell$ . Since  $\Delta = 0$ , and since  $g$  is an integer,  $g \geq 0$ .

If  $\Delta = 1$  and  $s \geq 1$ , then set  $b = 1$ ,  $c = 0$ ,  $d = s - 1$ ,  $g = n - \ell - s - 1$ , and  $k = \ell + 1$ . Note that  $g \geq -1/3$  so that  $g \geq 0$ .

If  $\Delta = 1$  and  $s = 0$ , then either  $n - \ell - 2 \geq 0$  in which case we set  $b = 2$ ,  $c = 0$ ,  $d = 0$ ,  $k = \ell$ , and  $g = n - \ell - 2$ , or else  $n - \ell - 2 = -1$  in which case we take  $b = 0$ ,  $c = 1$ ,  $d = 0$ ,  $k = \ell$ , and  $g = 0$ .

If  $\Delta = 2$  and  $s \geq 2$ , set  $b = 1$ ,  $c = 0$ ,  $d = s - 2$ ,  $g = n - \ell - s - 1$ , and  $k = \ell + 2$ .

If  $\Delta = 2$ ,  $s = 1$ , and  $n - \ell \geq 3$  then set  $b = 2$ ,  $c = 0$ ,  $d = 0$ ,  $g = n - \ell - 3$ , and  $k = \ell + 1$ . If  $\Delta = 2$ ,  $s = 1$ , and  $n - \ell < 3$ , then necessarily  $n - \ell = 2$ , and so  $(m, n) = (2, \ell + 2)$  and we set  $b = 0$ ,  $c = 1$ ,  $d = 0$ ,  $k = \ell + 1$ , and  $g = 0$ .

This leaves the cases when  $\Delta = 2$  and  $s = 0$ . If  $n - \ell \geq 2$ , set  $b = 1$ ,  $c = 1$ ,  $d = 0$ ,  $g = n - \ell - 2$ , and  $k = \ell$ . Finally, if  $n - \ell = 1$ , then  $(m, n) = (1, 1 + \ell)$ , so we take  $b = 0$ ,  $c = 0$ ,  $d = 1$ ,  $g = 0$  and  $k = \ell$ .  $\square$

We can now prove our main result. We state it in terms of  $c_1^2 = 2e + 3\sigma$  and  $\chi_h = \frac{1}{4}(e + \sigma)$  because it is simpler to work with these numbers than with pairs  $(e, \sigma)$  where  $e + \sigma \equiv 0 \pmod{4}$ . Note that in this notation, a 4-manifold with  $c_1^2 = 8\chi_h + k$  has signature  $k$ , so the line  $c_1^2 = 8\chi_h - 2$  corresponds to manifolds with signature  $-2$ .

**Theorem 22.** *For any pair  $(c, \chi)$  of non-negative integers satisfying*

$$0 \leq c \leq 8\chi - 2$$

*with the possible exceptions of  $(c, \chi) = (5, 1), (9, 2), (11, 2),$  or  $(13, 2)$ , there exists a minimal simply connected symplectic 4-manifold  $Y = X_{2\chi-1, 10\chi-c-1}$  with odd intersection form and*

$$c = c_1^2(Y) \quad \text{and} \quad \chi = \chi_h(Y).$$

*Hence  $Y$  is homeomorphic but not diffeomorphic to  $(2\chi - 1)\mathbb{C}\mathbb{P}^2 \# (10\chi - c - 1)\overline{\mathbb{C}\mathbb{P}^2}$ .*

*Proof.* We make extensive use of the manifolds  $A, B, B_g, C, D, E'(k)$  of (respectively) Lemma 10, Theorem 7, Corollary 9, Theorem 11, Theorem 12, and Theorem 14. We will also use the sporadic examples of Section 7.

We first realize all pairs with  $c$  even. Let  $(m, n) = (\frac{1}{2}c, \chi)$ . Lemma 21 produces integers  $b, c, d, g$ , and  $k$  such that  $m = d + 2c + 3b + 4g$  and  $n = b + c + d + g + k$  and with  $b \geq 1$  if  $g > 0$ .

Construct the symplectic sum along tori of

- (1)  $b$  copies of  $B$  if  $g = 0$ , or one copy of  $B_g$  and  $b - 1$  copies of  $B$  if  $g \geq 1$ ,
- (2)  $c$  copies of  $C$ ,
- (3)  $d$  copies of  $D$ .

More precisely, each of the manifolds  $B, C, D$  contains two essential Lagrangian tori. Construct the symplectic sum  $Z$  of these manifolds by chaining them together, using Proposition 3 to ensure that at each stage one has a telescoping triple.

Specifically, if  $g = 0$  take

$$Z = \underbrace{B \#_s \cdots \#_s B}_b \#_s \underbrace{C \#_s \cdots \#_s C}_c \#_s \underbrace{D \#_s \cdots \#_s D}_d$$

and if  $g \geq 1$  take

$$Z = B_g \#_s \underbrace{B \#_s \cdots \#_s B}_{b-1} \#_s \underbrace{C \#_s \cdots \#_s C}_c \#_s \underbrace{D \#_s \cdots \#_s D}_d$$

where  $\#_s$  denotes the symplectic sum along the appropriate tori (perturbing the symplectic forms so that they become symplectic) according to the recipe of Proposition 3, so that the two unused Lagrangian tori (which we relabel  $T_1$  and  $T_2$ ) make  $(Z, T_1, T_2)$  a telescoping triple.

If  $k = 0$ , then perform +1 Luttinger surgery on  $T_1$  and  $T_2$  to obtain a simply connected (according to Proposition 4) symplectic 4-manifold  $Y$ .

If  $k \geq 1$  and one of  $b, c, d$  is positive, perform +1 Luttinger surgery on  $T_2$  in  $Z$  and take the symplectic sum of the result with the elliptic surface  $E(k)$  along  $T_1$  and a generic fiber  $T$  of  $E(k)$  to obtain the manifold  $Y$ . Since  $E(k) - T$  is simply connected, so is  $Y$ , by the same reasoning as in the proof of Theorem 15. Since  $B, C$ , and  $D$  contain  $-1$  tori disjoint from the Lagrangian tori  $T_1, T_2$ , the manifold  $Y$  has odd intersection form.

If  $k \geq 1$  and  $b, c, d$  are zero, take  $Y = E'(k)$  (see Theorem 14), which has an odd intersection form.

Thus  $Y$  is a simply connected symplectic manifold realizing the pair  $(c, \chi)$ . Since each of the manifolds  $B, B_g, C$ , and  $D$  contains a surface of odd square which misses the tori used in forming the symplectic sums, and since  $E'(k)$  has an odd intersection form, it follows that  $Y$  has an odd intersection form.

Since the 4-manifold  $Y$  has indefinite, odd intersection form, Freedman's theorem [13] implies that  $Y$  is homeomorphic to an appropriate connected sum of  $\mathbb{C}\mathbb{P}^2$ s and  $\overline{\mathbb{C}\mathbb{P}^2}$ s.

Now we turn to the case when  $c$  is odd. Suppose first that  $1 \leq c \leq 8\chi - 17$ . Let  $(c', \chi') = (c - 1, \chi - 2)$ . Thus  $0 \leq c' \leq 8\chi' - 2$ , and  $c'$  is even. Construct the manifold  $Z$  corresponding to the pair  $(c', \chi')$  and either perform +1 Luttinger surgery on  $T_1$  or take the symplectic sum with  $E(k)$  if  $k \geq 1$ . But rather than performing +1 Luttinger surgery on  $T_2$  as we did above, perturb the symplectic form to make  $T_2$  symplectic, and then take the symplectic sum with Gompf's manifold  $S_{1,1}$  (see Section 7) along the symplectic torus in  $S_{1,1}$  with simply connected complement. Since  $S_{1,1}$  has  $c_1^2 = 1$  and  $\chi_h = 2$  the resulting symplectic manifold  $Y$  has  $(c_1^2, \chi_h) = (c, \chi)$ .

Next suppose that  $c$  is odd and  $7 \leq c \leq 8\chi - 11$ . Set  $(c', \chi') = (c - 7, \chi - 2)$ . Thus  $0 \leq c' \leq 8\chi' - 2$  and  $c'$  is even. Construct the manifold  $Z$  corresponding to the pair  $(c', \chi')$ . We repeat the argument of the previous paragraph, replacing Gompf's manifold  $S_{1,1}$  with the manifold  $X_{3,12}$  of Section 7. Take the symplectic sum of  $Z$  with  $X_{3,12}$  along  $T_2$  and  $T_{2,4}$ . Since  $c_1^2(X_{3,12}) = 7$  and  $\chi_h(X_{3,12}) = 2$ , the resulting manifold  $Y$  realizes the pair  $(c, \chi)$ .

To realize all pairs  $(c, \chi)$  with  $c$  odd and  $21 \leq c \leq 8\chi - 5$ , repeat the argument once more, this time using the manifold  $P_{5,8}$  described in Remark 1 at the end of the proof of

Theorem 15, which has  $c_1^2 = 21$  and  $\chi_h = 3$ . A bit of care must be taken to ensure that the result is simply connected since  $\pi_1(P_{5,8}) = \mathbb{Z}$ . This is accomplished by making sure that the generator of  $\pi_1(T)$  sent to the generator of  $\pi_1(P_{5,8} - T)$  is identified with an element in the kernel of  $\pi_1(T_2) \rightarrow \pi_1(Z - T_2)$  when forming the symplectic sum  $Y = Z \#_s P_{5,8}$ .

The manifold  $Y = X_{1+2k,4+2k}$  of Theorem 15 provides an example realizing the pair  $(c, \chi) = (5 + 8k, 1 + k)$  for any  $k \geq 2$ , i.e.  $21 \leq c = 8\chi - 3$ .

Since  $c_1^2 \equiv \sigma \pmod{2}$ , and simply connected 4-manifolds with odd signature have an odd intersection form, it follows that the manifolds constructed for  $c$  odd also have an odd intersection form.

It remains to show that  $Y$  is minimal. Since  $E'(k)$  is minimal, we assume that  $c_1^2 > 0$ . By Proposition 13, the 4-manifold obtained by performing one or two  $\pm 1$  Luttinger surgeries on  $T_1$  or  $T_2$  along  $\ell_{T_i}$  or  $m_{T_i}$  in  $B$ ,  $C$ , or  $D$  is minimal. The  $E(k)$  are minimal for  $k \geq 2$ . Although  $E(1)$  is not minimal, every  $-1$  sphere intersects the generic torus fiber. Thus  $Y$  is the symplectic sum of minimal (or, if  $k = 1$ , relatively minimal) symplectic 4-manifolds and therefore is minimal by Usher's theorem.

It is easy to check that the only pairs  $(c, \chi)$  with  $0 \leq c \leq 8\chi - 2$  which are omitted by these cases are

$$\begin{aligned} &(1, 1), (3, 1), (5, 1), \\ &(1, 2), (3, 2), (5, 2), (7, 2), (9, 2), (11, 2), (13, 2), \\ &(15, 3), (17, 3), (19, 3). \end{aligned}$$

The examples listed in Section 7 realize most of these pairs. The only ones left unrealized are  $(5, 1)$ ,  $(9, 2)$ ,  $(11, 2)$ , and  $(13, 2)$ .  $\square$

The four unrealized pairs do correspond to (non-minimal) symplectic 4-manifolds; e.g. blowups of  $X_{1,3}$  or  $X_{3,5}$ . It is conjectured that one of the irreducible smooth 4-manifolds homeomorphic to  $3\mathbb{C}\mathbb{P}^2 \# 10\overline{\mathbb{C}\mathbb{P}^2}$  constructed in [25] and one of the irreducible smooth 4-manifolds homeomorphic to  $3\mathbb{C}\mathbb{P}^2 \# 8\overline{\mathbb{C}\mathbb{P}^2}$  constructed in [29] are symplectic (and hence minimal): their Seiberg–Witten invariants have the right form to be the invariants of a minimal symplectic manifold.

There exist small simply connected minimal symplectic 4-manifolds with non-negative signatures (e.g.  $\mathbb{C}\mathbb{P}^2$ ,  $S^2 \times S^2$ ). To date, no small examples are known that contain a suitable Lagrangian torus for which we can extend the construction of Theorem 22. Some moderately large examples are known and we will briefly explore the consequences for the geography problem in the next section.

**Remark 2.** Each of the manifolds constructed in Theorem 22, with the possible exception of those corresponding  $c_1^2 = 0$  and some of the small manifolds with  $c$  odd, contains a nullhomologous torus suitable for altering the differentiable structure as explained in [8], using [24] to compute the change in Seiberg–Witten invariants. Those with  $c_1^2 = 0$  are  $E'(k)$ , for which the methods of [9, 10, 16] show how to alter the differentiable structure. Hence the manifolds of Theorem 22 admit infinitely many smooth structures.

The proofs of Lemma 21 and Theorem 22 provide an algorithm for constructing simply connected minimal 4-manifolds with desired characteristic numbers, using the model manifolds  $A, B, B_g, C, D,$  and  $E(k)$ .

For example, to construct a minimal symplectic manifold homeomorphic but not diffeomorphic to  $3\mathbb{C}\mathbb{P}^2 \# 17\overline{\mathbb{C}\mathbb{P}^2}$ , one sees that such a manifold would have  $(c_1^2, \chi_h) = (2, 2)$ . This corresponds to  $(m, n) = (1, 2)$  in Lemma 21. In the notation of the proof of Lemma 21, we see that in this case  $\ell = 1, s = 0$  and  $\Delta = 2$ , so that  $b = 0, c = 0, d = 1, g = 0$ , and  $k = 1$ . Thus the desired manifold is obtained by taking the symplectic sum

$$D \#_s E'(1)$$

and performing  $+1$  Luttinger surgery on the remaining Lagrangian torus in  $D$ .

As another example, we construct a minimal symplectic manifold homeomorphic but not diffeomorphic to  $21\mathbb{C}\mathbb{P}^2 \# 31\overline{\mathbb{C}\mathbb{P}^2}$ , i.e.  $\chi_h = 11$  and  $c_1^2 = 78$ . Thus  $(m, n) = (39, 11)$ . The proof of Lemma 21 provides  $\ell = 0, s = 1$ , and  $\Delta = 1$ , and so  $b = 1, c = 0, d = 0, g = 9$ , and  $k = 1$ . Thus the desired manifold is obtained by taking the symplectic sum

$$B_9 \#_s E(1)$$

and performing  $+1$  Luttinger surgery on the remaining Lagrangian torus.

The integers produced by the algorithm in the proof of Lemma 21 are not unique. For example, the choice  $b = 2, c = 0, d = 1, g = 8$ , and  $k = 0$  yields a manifold

$$B_8 \#_s B \#_s D.$$

Performing two  $+1$  Luttinger surgeries on this manifold produces a (possibly different) minimal symplectic manifold homeomorphic to but not diffeomorphic to  $21\mathbb{C}\mathbb{P}^2 \# 31\overline{\mathbb{C}\mathbb{P}^2}$ .

## 9. Signature greater than $-2$

Finding small minimal symplectic 4-manifolds with signature greater than  $-2$  poses a special challenge. Stipsicz [32] shows how to produce simply connected minimal symplectic 4-manifolds with positive signature. The following theorem provides a method for producing many examples, given one. It is also useful in studying the geography problem for non-simply connected 4-manifolds.

To avoid an overly technical statement and proof, we separate the cases of  $c$  odd and even, but a more complete statement would have similar hypotheses on  $(c, \chi)$  as in Theorem 22.

**Theorem 23.** *Let  $X$  be a symplectic 4-manifold and suppose that  $X$  contains a symplectic torus  $T$  such that the homomorphism  $\pi_1(T) \rightarrow \pi_1(X)$  induced by inclusion is trivial. Then, for any pair  $(c, \chi)$  of non-negative integers satisfying*

$$\begin{aligned} 0 \leq c \leq 8\chi - 2 & \quad \text{if } c \text{ is even,} \\ 1 \leq c \leq 8\chi - 17 & \quad \text{if } c \text{ is odd,} \end{aligned}$$



there exists a symplectic 4-manifold  $Y$  with  $\pi_1(Y) = \pi_1(X)$ ,

$$c_1^2(Y) = c_1^2(X) + c \quad \text{and} \quad \chi_h(Y) = \chi_h(X) + \chi.$$

Moreover, if  $X$  is minimal (or more generally if  $(X, T)$  is relatively minimal) then the manifold  $Y$  is minimal and has an odd, indefinite intersection form.

*Proof.* The argument is similar to the proof of Theorem 22, save for the last step. Consider first the case  $c$  even. Let  $(Z, T_1, T_2)$  be the telescoping triple corresponding to  $(c, \chi)$  as in the first part of the proof of Theorem 22. If  $k = 0$ , then do +1 Luttinger surgery on  $T_1$  to get a minimal (by Proposition 13 and Usher's theorem) manifold  $Z_1$  with  $\pi_1(Z_1) \cong \mathbb{Z}$  containing a symplectic torus  $T_2$  (after perturbing the symplectic structure) so that the induced map  $\pi_1(T_2) \rightarrow \pi_1(Z_1)$  is a split surjection. If  $k \geq 1$ , then take a fiber sum of  $Z$  with  $E(k)$  to again get a manifold  $Z_1$  with  $\pi_1(Z_1) \cong \mathbb{Z}$  containing a symplectic torus  $T_2$  so that the induced map  $\pi_1(T_2) \rightarrow \pi_1(Z_1)$  is a surjection.

Next consider the case of  $c$  odd. Following the proof of Theorem 22, let  $(c', \chi') = (c - 1, \chi - 2)$  and construct the telescoping triple  $(Z, T_1, T_2)$  corresponding to the pair  $(c', \chi')$ . If  $k = 0$ , form the symplectic sum of  $Z$  along  $T_1$  with  $S_{1,1}$  and perturb the symplectic form so that  $T_2$  is symplectic. If  $k \geq 1$ , form the symplectic sum of  $Z$  along  $T_1$  with  $E(k)$  along an elliptic fiber, then form a further symplectic sum along a different elliptic fiber of  $E(k)$  with  $S_{1,1}$ . Then perturb the symplectic form so that  $T_2$  becomes symplectic. In either case, this results in a minimal symplectic manifold  $Z_1$  with  $\pi_1(Z_1) \cong \mathbb{Z}$  containing a symplectic torus  $T_2$  so that the induced map  $\pi_1(T_2) \rightarrow \pi_1(Z_1)$  is a surjection and  $(c_1^2, \chi_h) = (c, \chi)$ .

Since the meridian of  $T_2$  is nullhomotopic in  $Z_1$ , the symplectic sum,  $Y$ , of  $Z_1$  and  $X$  has fundamental group isomorphic to that of  $X$ , since the homomorphism  $\pi_1(T) \rightarrow \pi_1(X)$  is trivial. Minimality follows as in the proof of Theorem 22 using Usher's theorem. Since  $c_1^2$  and  $\chi_h$  are both additive with respect to symplectic sums along tori, the result follows.  $\square$

Before we can prove Theorem B stated in the introduction, we will require one more useful fact about  $B$  and  $X_{1,3}$  not mentioned in Theorem 7 or Corollary 8, namely, the existence of a genus 2 square zero symplectic surface  $G$  geometrically dual to  $F$ . We indicate how to find  $G$ :  $X_{1,3}$  is obtained by Luttinger surgeries on eight Lagrangian tori in the symplectic sum of the twice blown up 4-torus  $(T^2 \times T^2) \# 2\overline{\mathbb{C}\mathbb{P}^2}$  and the product  $T \times F_2$  of a torus and a genus 2 surface.

This symplectic sum is taken along the genus 2 surface in  $(T^2 \times T^2) \# 2\overline{\mathbb{C}\mathbb{P}^2}$  obtained by resolving  $(T^2 \times \{p\}) \cup (\{q\} \times T^2)$  and blowing up twice (for definiteness at points on  $T^2 \times \{p\}$ ). In  $T \times F_2$  one takes the surface  $\{x\} \times F_2$ .

The square  $-1$  torus  $H_1$  of Theorem 7 and Corollary 8 was obtained by taking the torus of the form  $T \times \{z\}$  which matches up with one of the exceptional spheres in the symplectic sum. To find the surface  $G$ , take another nearby torus of the form  $T \times \{z'\}$  in  $T \times F_2$  and match it up with a torus of the form  $\{q'\} \times T^2$ . This is the required surface  $G$ . (The surface  $F$  is a parallel copy of  $\{x\} \times F_2$ .)

*Proof of Theorem B.* Start with the telescoping triple  $(B, T_1, T_2)$  of Theorem 7. It contains a genus 2 square zero symplectic surface  $F$  and a geometrically dual square zero symplectic genus 2 surface  $G$ . The union  $F \cup G$  is disjoint from  $T_1 \cup T_2$ .

Perform +1 Luttinger surgery on  $T_1$  along  $\ell_{T_1}$  to kill  $t_2$ . Call the result  $R$ . Perturb the symplectic form on  $R$  slightly so that  $T_2$  becomes symplectic. Note that  $\pi_1(R - T_2) = \pi_1(R) = \mathbb{Z}t_1$ ,  $\pi_1(T_2) \rightarrow \pi_1(R)$  is surjective, and  $R$  is minimal (Proposition 13).

In [6, Theorem 18], a minimal symplectic 4-manifold  $\tilde{X}_{3,5}$  homeomorphic to  $3\mathbb{C}\mathbb{P}^2 \# 5\overline{\mathbb{C}\mathbb{P}^2}$  and containing a pair of symplectic tori  $T_3, T_4$  with simply connected complement is constructed. The symplectic sum  $Q = R \#_{T_2, T_3} \tilde{X}_{3,5}$  is minimal by Usher's theorem. Moreover,  $Q$  is simply connected, since  $T_2 \subset R$  induces a surjection on fundamental groups. The surfaces  $F$  and  $G$  persist as square zero, symplectic, geometrically dual surfaces. Since  $e(Q) = 16$  and  $\sigma(Q) = -4$ ,  $Q$  is neither rational nor ruled. Notice that the symplectic torus  $T_4$  in  $Q$  has simply connected complement.

In  $Q$ , take eight parallel copies of the genus 2 surface  $F$  and one copy of  $G$  and symplectically resolve to obtain a genus 18 surface  $\Sigma \subset Q$  of square 16. Blow up  $Q$  16 times, yielding a genus 18 square zero surface  $\tilde{\Sigma} \subset \tilde{Q} = Q \# 16\overline{\mathbb{C}\mathbb{P}^2}$ . By Li's theorem, every  $-1$  sphere in  $\tilde{Q}$  intersects  $\tilde{\Sigma}$ . Moreover,  $\pi_1(\tilde{Q} - \tilde{\Sigma}) = 1$ .

In [32, Lemma 2.1], a Lefschetz fibration  $H \rightarrow K$  over a surface  $K$  of genus 2 is constructed which has  $e = 75$  and  $\sigma = 25$ . This fibration admits a symplectic section of square  $-1$  and has fiber genus 16. The 4-manifold  $H$  is an algebraic surface, and by the Bogomolov–Miyaoaka–Yau inequality [7] is holomorphically minimal. By [18], it is also symplectically minimal. Moreover,  $H$  is neither rational nor ruled since it lies on the BMY line.

Let  $\Sigma' \subset H$  denote the symplectic surface obtained by symplectically resolving the union of a fiber and a section. Then  $\Sigma'$  has square 1, and the exact sequence of fundamental groups for a Lefschetz fibration shows that  $\pi_1(\Sigma') \rightarrow \pi_1(H)$  is surjective. Blow up  $H$  once along  $\Sigma'$  and take the proper transform to obtain a square zero, genus 18 surface  $\tilde{\Sigma}' \subset \tilde{H} = H \# \overline{\mathbb{C}\mathbb{P}^2}$  so that  $\pi_1(\tilde{H} - \tilde{\Sigma}') \rightarrow \pi_1(\tilde{H})$  is an isomorphism and  $\pi_1(\tilde{\Sigma}') \rightarrow \pi_1(\tilde{H})$  is surjective. By Li's theorem (Theorem 6), every  $-1$  sphere in  $\tilde{H}$  intersects  $\tilde{\Sigma}'$ , since  $H$  is neither rational nor ruled.

Hence the symplectic sum  $S = \tilde{Q} \#_{\tilde{\Sigma}, \tilde{\Sigma}'} \tilde{H}$  is minimal. It is simply connected since  $\pi_1(\tilde{Q} - \tilde{\Sigma}) = 1$  and  $\pi_1(\tilde{\Sigma}') \rightarrow \pi_1(\tilde{H})$  is surjective. Moreover, the symplectic torus  $T_4 \subset S$  has simply connected complement.

Since  $S$  is the symplectic sum along genus 18 surfaces,

$$\begin{aligned} e(S) &= e(\tilde{Q}) + e(\tilde{H}) + 4(18 - 1) = 176, \\ \sigma(S) &= \sigma(\tilde{Q}) + \sigma(\tilde{H}) = 24 - 20 = 4. \end{aligned}$$

Thus  $c_1^2(S) = 364$  and  $\chi_h(S) = 45$ . It contains the symplectic torus  $T_4$  with simply connected complement. Hence Theorem 23 establishes the existence of minimal, simply connected symplectic 4-manifolds

$$X_{89+2\chi, 85+10\chi-c}$$

with  $c_1^2 = 364 + c$  and  $\chi_h = 45 + \chi$  for any  $(c, \chi)$  satisfying  $0 \leq c \leq 8\chi - 2$  when  $c$  is even.

Taking  $c = 8\chi - 4$  for any  $\chi \geq 1$  yields  $X_{89+2\chi, 89+2\chi}$ , a minimal simply connected symplectic 4-manifold with signature zero. The intersection form is odd since, as one can check from Lemma 21, either  $\chi = 1$  in which case the model manifold  $C$  (with its  $-1$  torus) is used in the construction of  $X_{91, 91}$ , or else  $\chi > 1$ , in which case the model manifold  $B$  (with its  $-1$  torus) is used in the construction of  $X_{89+2\chi, 89+2\chi}$ .

To get minimal symplectic 4-manifolds with signature  $-1$ , consider the symplectic sum

$$Y = B \#_{T_1, T} P_{1+2k, 4+2k}$$

of the manifold  $B$  of Theorem 7 with the manifold  $P_{1+2k, 4+2k}$  of Remark 1 (at the end of Section 6) along  $T_1$  in  $B$  and  $T$  in  $P_{1+2k, 4+2k}$ . Since  $\pi_1(T) \rightarrow \pi_1(P_{1+2k, 4+2k}) = \mathbb{Z}$  is surjective,  $\pi_1(P_{1+2k, 4+2k} - T) \rightarrow \pi_1(P_{1+2k, 4+2k})$  is an isomorphism, and  $\pi_1(T_1) \rightarrow \pi_1(B)$  has image a cyclic summand, the gluing map for the symplectic sum can be chosen so that  $B - \text{nbnd}(T_1) \subset Y$  induces an isomorphism on fundamental groups. Hence  $\pi_1(T_2) \rightarrow \pi_1(Y)$  is an isomorphism.

The symplectic sum

$$X_{93+2k, 94+2k} = Y \#_{T_2, T_4} S$$

is a simply connected minimal symplectic 4-manifold with  $e = 189 + 4k$  and  $\sigma = -1$ , for any  $k \geq 2$ .  $\square$

Since any symplectic signature zero 4-manifold has  $e$  a multiple of 4, there remain 45 signature zero minimal symplectic 4-manifolds with odd intersection form to be constructed. Also missing are 48 signature  $-1$  minimal symplectic 4-manifolds. Hence to complete the geography problem for minimal simply connected symplectic 4-manifolds of non-positive signature and odd intersection form, there remain 97 manifolds to discover.

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