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Probability theory. — *On the martingale problem associated to the* 2*D and* 3*D stochastic Navier–Stokes equations*, by GIUSEPPE DA PRATO and ARNAUD DEBUSSCHE.

ABSTRACT. — We consider a Markov semigroup $(P_t)_{t\geq 0}$ associated to the 2D and 3D Navier–Stokes equations. In the two-dimensional case P_t is unique, whereas in the three-dimensional case (where uniqueness is not known) it is constructed as in [4] and [7].

For $d = 2$, we specify a core, identify the abstract generator of $(P_t)_{t\geq 0}$ with the differential Kolmogorov operator L on this core and prove existence and uniqueness for the corresponding martingale problem. In dimension 3, we are not able to prove a similar result and we explain the difficulties encountered. Nonetheless, we specify a core for the generator of the transformed semigroup $(S_t)_{t\geq 0}$, obtained by adding a suitable potential and then using the Feynman–Kac formula. Then we identify the abstract generator $(S_t)_{t\geq 0}$ with a differential operator N on this core and prove uniqueness for the stopped martingale problem.

KEY WORDS: Stochastic Navier–Stokes; Kolmogorov equations; martingale problems; weak uniqueness.

MATHEMATICS SUBJECT CLASSIFICATION (2000): 76D05, 60H15, 37A25.

1. INTRODUCTION

We consider the stochastic Navier–Stokes equations on a bounded domain $\mathscr{O}$ of $\mathbb{R}^d$, $d = 2$ or 3, with Dirichlet boundary conditions, the unknowns being the velocity $X(t,\xi)$ and the pressure $p(t,\xi)$ defined for $t > 0$ and $\xi \in \overline{\mathscr{O}}$:

$$\begin{cases} dX(t,\xi) = [\Delta X(t,\xi) - (X(t,\xi)\cdot\nabla)X(t,\xi)]\,dt \\ \qquad\qquad - \nabla p(t,\xi)\,dt + f(\xi)\,dt + \sqrt{Q}\,dW, \\ \operatorname{div} X(t,\xi) = 0, \end{cases} \tag{1.1}$$

with Dirichlet boundary conditions

$$X(t,\xi) = 0, \qquad t > 0,\ \xi \in \partial\mathscr{O},$$

and supplemented with the initial condition

$$X(0,\xi) = x(\xi), \qquad \xi \in \mathscr{O}.$$

We have taken the viscosity equal to 1 since it plays no particular role in this work.

The understanding of the stochastic Navier–Stokes equations has recently progressed considerably. In dimension two, impressive progress has been obtained and difficult ergodic properties have been proved (see [1], [8], [10], [13], [14]–[19]). In dimension three, the theory is not so advanced. Uniqueness is still an open problem. However, Markov solutions have been constructed and ergodic properties have been proved (see [2]–[4], [7], [9], [11], [12], [22], [25], [26]). In [20], [21], a general form of the stochastic Navier–Stokes equations is derived from the assumptions that the fluid particles are subject to turbulent diffusion. These equations are studied theoretically in the martingale and strong sense. Also, in dimension two, it is shown that the equations are equivalent to an infinite system of deterministic PDEs obtained by Wiener chaos decomposition.

In this article, our aim is to try to improve the understanding of the martingale problems associated to these equations. Let us first set some notations. Let

$$H = \{x \in (L^2(\mathscr{O}))^d : \operatorname{div} x = 0 \text{ in } \mathscr{O},\ x \cdot n = 0 \text{ on } \partial\mathscr{O}\},$$

where n is the outward normal to $\partial\mathscr{O}$, and $V = (H_0^1(\mathscr{O}))^d \cap H$. The norm and inner product in H will be denoted by $|\cdot|$ and $(\cdot,\cdot)$ respectively. Moreover, W is a cylindrical Wiener process on H and the covariance of the noise Q is trace class and nondegenerate (see (1.3) and (1.4) below for more precise assumptions).

We also denote by A the Stokes operator in H:

$$A = P\Delta, \quad D(A) = (H^2(\mathscr{O}))^d \cap V,$$

where P is the orthogonal projection of $(L^2(\mathscr{O}))^3$ onto H, and by b the operator

$$b(x,y) = -P((x\cdot\nabla)y), \qquad b(x) = b(x,x), \qquad x, y \in V.$$

With these notations we rewrite the equations as

$$\begin{cases} dX = (AX + b(X))\,dt + \sqrt{Q}\,dW, \\ X(0) = x. \end{cases} \tag{1.2}$$

We assume that

$$\operatorname{Tr}((-A)^{1+g}Q) < \infty \qquad \text{for some } g > 0 \tag{1.3}$$

and

$$|Q^{-1/2}x| \le c|(-A)^r x| \qquad \text{for some } r \in (1, 3/2). \tag{1.4}$$

In dimension $d = 3$, it is well known that there exists a solution to the martingale problem, but weak or strong uniqueness is an open problem (see [9] for a survey). However, it has been proved in [4], [7] (see also [11]) that the above assumptions allow constructing a transition semigroup $(P_t)_{t\ge0}$ associated to a Markov family of solutions

$$((X(t,x))_{t\ge0}, \Omega_x, \mathscr{F}_x, \mathbb{P}_x)$$

for $x \in D(A)$. Moreover, for sufficiently regular φ defined on $D(A)$, $P_t\varphi$ is a solution of the Kolmogorov equation associated to (1.2),

$$\begin{cases} \dfrac{du}{dt} = Lu, \quad t > 0,\ x \in D(A), \\ u(0, x) = \varphi(x), \quad x \in D(A), \end{cases} \tag{1.5}$$

where the Kolmogorov operator L is defined by

$$L\varphi(x) = \frac{1}{2}\operatorname{Tr}\{QD^2\varphi(x)\} + (Ax + b(x), D\varphi(x))$$

for sufficiently smooth functions φ on $D(A)$.

In all the article, we choose $((X(t, x))_{t\geq 0}, \Omega_x, \mathscr{F}_x, \mathbb{P}_x)$ to be the Markov family constructed in [7].

The fundamental idea in [4] is to introduce a modified semigroup $(S_t)_{t\geq 0}$ defined by

$$S_t\varphi(x) = \mathbb{E}\big(e^{-K\int_0^t |AX(s,x)|^2\,ds}\varphi(X(t, x))\big). \tag{1.6}$$

It can be seen that for K large enough, this semigroup has very nice smoothing properties and various estimates can be proved. Note that, thanks to the Feynman–Kac formula, this semigroup is formally associated to the following equation:

$$\begin{cases} \dfrac{dv}{dt} = Nv, \quad t > 0,\ x \in D(A), \\ v(0, x) = \varphi(x), \quad x \in D(A), \end{cases} \tag{1.7}$$

where N is defined by

$$N\varphi(x) = \frac{1}{2}\operatorname{Tr}\{QD^2\varphi(x)\} + (Ax + b(x), D\varphi(x)) - K|Ax|^2\varphi(x),$$

for sufficiently smooth functions φ on $D(A)$.

In [4], [7], this semigroup is defined only on the Galerkin approximations of (1.2). Let P_m denote the projector associated to the first m eigenvalues of A. We consider the following equation in P_mH:

$$\begin{cases} dX_m = (AX_m + b_m(X_m))\,dt + \sqrt{Q}_m\,dW, \\ X_m(0) = P_m x, \end{cases} \tag{1.8}$$

where $b_m(x) = P_m b(P_m x)$ and $Q_m = P_m Q P_m$. This defines, with obvious notations, $(P_t^m)_{t\geq 0}$ and $(S_t^m)_{t\geq 0}$. The following formula holds by a standard argument:

$$P_t^m\varphi = S_t^m\varphi + K\int_0^t S_{t-s}^m(|A\cdot|^2 P_s^m\varphi)\,ds, \quad \varphi \in C_b(P_m H).$$

Various estimates are proved on $(S_t^m)_{t\geq 0}$ and transferred to $(P_t^m)_{t\geq 0}$ thanks to this identity. A compactness argument allows one to construct $(P_t)_{t\geq 0}$. Moreover, a subsequence m_k can be constructed so that for any $x \in D(A)$, $(X_{m_k}(\cdot, x))_{t\geq 0}$ converges in law to $(X(\cdot, x))_{t\geq 0}$.

Note also that similar arguments to those in [4] may be used to prove that for smooth φ, $(S_t\varphi)_{t\geq 0}$ is a strict solution to (1.7).

In dimension 2 this result also holds with exactly the same proofs since all arguments for $d = 3$ are still valid. Note that it is well known that for $d = 2$ conditions (1.3)–(1.4) imply that, for $x \in H$, there exists a unique strong solution to (1.2), and the proof of the above facts can be simplified.

In the following, we give some properties of the generator of $(P_t)_{t\geq 0}$ and $(S_t)_{t\geq 0}$. For $d = 2$, we specify a core, identify the abstract generator with the differential operator L on this core and prove existence and uniqueness for the corresponding martingale problem. (See [24] for a similar result.) Again, this follows from strong uniqueness but we think that it is interesting to have a direct proof. Moreover, it can be very useful to have a better knowledge of the Kolmogorov generator and we think that this work is a contribution in this direction. In dimension 3, we are not able to prove a similar result. We explain the difficulties encountered. We hope that this article will help the reader to get a better insight into the problem of weak uniqueness for the three-dimensional Navier–Stokes equations. Nonetheless, we specify a core for the generator of the transformed semigroup $(S_t)_{t\geq 0}$, identify it with the differential operator N on this core and prove uniqueness for the stopped martingale problem. In other words, we prove weak uniqueness up to the time when solutions are smooth. Again, this could be proved directly thanks to local strong uniqueness.

2. THE GENERATORS

The space of continuous functions on $D(A)$ is denoted by $C_b(D(A))$. Its norm is denoted by $\|\cdot\|_0$. For $k \in \mathbb{N}$, $C^k(D(A))$ is the space of C^k functions on $D(A)$. We need several other function spaces on $D(A)$.

Let us introduce the set $\mathscr{E}_1 \subset C_b(D(A))$ of C^3 functions on $D(A)$ such that there exists a constant c satisfying

$$\begin{aligned}
&|(-A)^{-1}Df(x)|_H \leq c(|Ax|^2 + 1),\\
&|(-A)^{-1}D^2f(x)(-A)^{-1}|_{\mathscr{L}(H)} \leq c(|Ax|^4 + 1),\\
&|(-A)^{-1/2}D^2f(x)(-A)^{-1/2}|_{\mathscr{L}(H)} \leq c(|Ax|^6 + 1),\\
&\|D^3f(x)((-A)^{-1}\cdot, (-A)^{-1}\cdot, (-A)^{-1}\cdot)\| \leq c(|Ax|^6 + 1),\\
&\|D^3f(x)((-A)^{-\gamma}\cdot, (-A)^{-\gamma}\cdot, (-A)^{-\gamma}\cdot)\| \leq c(|Ax|^8 + 1),\\
&|Df(x)|_H \leq c(|Ax|^4 + 1),
\end{aligned}$$

for all $\gamma \in (1/2, 1]$, and

$$\mathscr{E}_2 = \left\{ f \in C_b(D(A)) : \sup_{x,y\in D(A)} \frac{|f(x) - f(y)|}{|A(x-y)|(1 + |Ax|^2 + |Ay|^2)} < +\infty \right\}.$$

Note that we identify the gradient and the differential of a real-valued function. Also, the second differential is identified with a function with values in $\mathscr{L}(H)$. The third differential is a trilinear operator on $D(A)$, and the norm $\|\cdot\|$ above is the norm of such operators.

Slightly improving the arguments in [4], it can be proved[†] that P_t maps $\mathscr{E}_i$ into itself and that there exists a constant $c > 0$ such that

$$\|P_t f\|_{\mathscr{E}_i} \le c\|f\|_{\mathscr{E}_i}. \tag{2.1}$$

Moreover, for $f \in \mathscr{E}_1$, $P_t f$ is a strict solution of (1.5) in the sense that it is satisfied for any $x \in D(A)$ and $t \ge 0$. Again, the result of [4] has to be slightly improved to get this result. In fact, using an interpolation argument, Proposition 5.9 and various other estimates in [4], it is easy to deduce that, for any $x \in D(A)$, $LP_t f(x)$ is continuous on $[0, T]$.

For $f \in \mathscr{E}_2$, $P_t f$ is still a solution of (1.5), but in the mild sense. We define the Ornstein–Uhlenbeck semigroup associated to the linear equation by

$$R_t\varphi(x) = \varphi\left(e^{tA}x + \int_0^t e^{A(t-s)}\sqrt{Q}\, dW(s)\right), \qquad t \ge 0,\ \varphi \in C_b(D(A)).$$

Then it is shown in [4] that

$$P_t f(x) = R_t f(x) + \int_0^t R_{t-s}(b, DP_s f)\, ds, \qquad t \ge 0,\ f \in \mathscr{E}_2. \tag{2.2}$$

For any $\lambda > 0$ we set

$$F_\lambda f = \int_0^\infty e^{-\lambda t} P_t f\, dt, \qquad f \in C_b(D(A)).$$

Then since $\|P_t f\|_0 \le \|f\|_0$, we have

$$\|F_\lambda f\|_0 \le \frac{1}{\lambda}\|f\|_0.$$

Moreover, since P_t is Feller, by dominated convergence we have

$$F_\lambda f \in C_b(D(A)).$$

It can be easily deduced that

$$F_\lambda f - F_\mu f = (\mu - \lambda)F_\lambda F_\mu f, \qquad \mu, \lambda > 0,$$

and

$$\lim_{\lambda\to\infty} \lambda F_\lambda f(x) = \lim_{\lambda\to\infty} \int_0^\infty e^{-\tau} P_{\tau/\lambda}(x)\, d\tau = f(x), \qquad x \in D(A).$$

[†] In fact, only Lemma 5.3 there has to be improved. In that lemma, the term L_1 can in fact be estimated in a single step by using Proposition 3.5 of [7] instead of Proposition 5.1 of [4].

It follows classically (see for instance [23]) that there exists a unique maximal dissipative operator $\bar{L}$ on $C_b(D(A))$ with domain $D(\bar{L})$ such that

$$F_\lambda f = (\lambda - \bar{L})^{-1} f.$$

We recall the following well known characterization of $D(\bar{L})$: $f \in D(\bar{L})$ if and only if

(i) $f \in C_b(D(A))$,
(ii) $t^{-1}\|P_t f - f\|_0$ is bounded for $t \in [0, 1]$,
(iii) $t^{-1}(P_t f(x) - f(x))$ has a limit as $t \to 0$ for any $x \in D(A)$.

Moreover, in this case we have

$$\bar{L} f(x) = \lim_{t\to 0} \frac{1}{t}(P_t f(x) - f(x)).$$

Recall also that

$$(\lambda - \bar{L})^{-1} f = \int_0^\infty e^{-\lambda t} P_t f\, dt, \qquad f \in C_b(D(A)).$$

By (2.1) we deduce that

$$\|(\lambda - \bar{L})^{-1} f\|_{\mathscr{E}_i} \le \frac{c}{\lambda}\|f\|_{\mathscr{E}_i}. \tag{2.3}$$

Similarly, we may define, for $k \ge 0$, $\mathscr{E}_3^k$ as the space of C^3 functions on $D(A)$ such that there exists a constant c satisfying

$$\begin{aligned}
&|(-A)^{-1} Df(x)|_H \le c(|Ax|^k + 1),\\
&|(-A)^{-1} D^2 f(x)(-A)^{-1}|_{\mathscr{L}(H)} \le c(|Ax|^k + 1),\\
&|(-A)^{-1/2} D^2 f(x)(-A)^{-1/2}|_{\mathscr{L}(H)} \le c(|Ax|^k + 1),\\
&\|D^3 f(x)((-A)^{-1}\cdot, (-A)^{-1}\cdot, (-A)^{-1}\cdot)\| \le c(|Ax|^k + 1),\\
&\|D^3 f(x)((-A)^{-\gamma}\cdot, (-A)^{-\gamma}\cdot, (-A)^{-\gamma}\cdot)\| \le c(|Ax|^k + 1),\\
&|Df(x)|_H \le c(|Ax|^k + 1),
\end{aligned}$$

for all $\gamma \in (1/2, 1]$. By various estimates given in [4], it is easy to check that, provided K is chosen large enough, S_t maps $\mathscr{E}_3^k$ into itself and there exists a constant $c > 0$ such that

$$\|S_t f\|_{\mathscr{E}_3^k} \le c\|f\|_{\mathscr{E}_3^k}. \tag{2.4}$$

Moreover, for $f \in \mathscr{E}_3^k$, $S_t f$ is a strict solution of (1.7) in the sense that it satisfies (1.7) for any $x \in D(A)$ and $t \ge 0$.

For any $\lambda > 0$ we set

$$\widetilde{F}_\lambda f = \int_0^\infty e^{-\lambda t} S_t f\, dt, \qquad f \in C_b(D(A)),$$

and prove that there exists a unique maximal dissipative operator $\bar{N}$ on $C_b(D(A))$ with domain $D(\bar{N})$ such that

$$\widetilde{F}_\lambda f = (\lambda - \bar{N})^{-1} f,$$

and $f \in D(\bar{N})$ if and only if

(i) $f \in C_b(D(A))$,
(ii) $t^{-1}\|S_t f - f\|_0$ is bounded for $t \in [0, 1]$,
(iii) $t^{-1}(S_t f(x) - f(x))$ has a limit as $t \to 0$ for any $x \in D(A)$.

Finally, by (2.4), we see that

$$\|(\lambda - \bar{N})^{-1} f\|_{\mathscr{E}_3^k} \le \frac{c}{\lambda}\|f\|_{\mathscr{E}_3^k}. \tag{2.5}$$

3. Construction of cores and identification of the generators

In this section, we analyse the generators defined in the preceding section. We start with the following definition.

DEFINITION 3.1. *Let K be an operator with domain $D(K)$. A set $\mathscr{D} \subset D(K)$ is a π-*core *for K if for any $\varphi \in D(K)$, there exists a sequence $(\varphi_n)_{n\in\mathbb{N}}$ in $\mathscr{D}$ which π-converges*‡ *to φ and such that $(K\varphi_n)_{n\in\mathbb{N}}$ π-converges to $K\varphi$.*

Let us set $\mathscr{G}_1 = (\lambda - \bar{L})^{-1}\mathscr{E}_1$ for some $\lambda > 0$. Clearly, for any $\varphi \in \mathscr{G}_1$ we have $\varphi \in D(\bar{L})$, and by (2.3), $\varphi \in \mathscr{E}_1$. Moreover,

$$P_t\varphi(x) - \varphi(x) = \int_0^t LP_s\varphi(x)\,ds,$$

since $(P_t\varphi)_{t\ge0}$ is a strict solution of the Kolmogorov equation. By (2.1) and the definition of $\mathscr{E}_1$, for any $x \in D(A)$ we have

$$|LP_s\varphi(x)| \le c(1 + |Ax|^6)\|P_s\varphi\|_{\mathscr{E}_1} \le c(1 + |Ax|^6)\|\varphi\|_{\mathscr{E}_1}.$$

Moreover, since $t \mapsto LP_t\varphi(x)$ is continuous, we have

$$\lim_{t\to0}\frac{1}{t}(P_t\varphi(x) - \varphi(x)) = L\varphi(x).$$

We deduce that

$$\bar{L}\varphi(x) = L\varphi(x), \qquad x \in D(A).$$

Since $\mathscr{E}_1$ is π-dense in $C_b(D(A))$, we deduce that $\mathscr{G}_1$ is a π-core for $\bar{L}$.

Also $\mathscr{E}_1 \subset \mathscr{E}_2$ so that

$$\mathscr{G}_1 \subset \mathscr{G}_2 = (\lambda - \bar{L})^{-1}\mathscr{E}_2$$

and $\mathscr{G}_2$ is also a π-core for $\bar{L}$.

‡ Recall that the π-convergence (also called b.p. convergence) is defined by: $(f_n)_{n\in\mathbb{N}}$ π-converges to f iff $f_n(x) \to f(x)$ for any $x \in D(A)$ and $\sup_{n\in\mathbb{N}}\|f_n\|_0 < \infty$.

These results hold both in dimension 2 and 3. The problem is that these cores are abstract and strongly depend on the semigroup $(P_t)_{t\geq 0}$. In dimension 3, this is a real problem since we do not know if the transition semigroup is unique. If we were able to construct a core in terms of the differential operator L, this would certainly imply uniqueness of this transition semigroup.

In dimension 2, we are able to construct such a core. Of course, in this case, uniqueness is well known. However, we think that it is important to have explicit cores, as they give much information on the transition semigroup $(P_t)_{t\geq 0}$.

THEOREM 3.2. *Set*

$$\mathscr{H} = \{f \in \mathscr{E}_1 : Lf \in \mathscr{E}_1\}.$$

Then, in dimension $d = 2$, $\mathscr{H} \subset D(\bar{L})$ and it is a π-core for $\bar{L}$. Moreover, for any $f \in \mathscr{H}$, we have

$$\bar{L}f = Lf.$$

The crucial point is to prove the following result.

PROPOSITION 3.3. *Let $d = 2$. For any $f \in \mathscr{H}$ we have*

$$P_{t_1}f - P_{t_2}f = \int_{t_1}^{t_2} P_s Lf\, ds, \qquad 0 \leq t_1 \leq t_2.$$

PROOF. Let $f \in \mathscr{H}$. By the Itô formula applied to the Galerkin equation (1.8), we have, for $\epsilon > 0$,

$$\begin{aligned}
&d(e^{-\epsilon \int_0^t |(-A)^{1/2}X_m(s,x)|^6\,ds} f(X_m(t,x)))\\
&= (-\epsilon|(-A)^{1/2}X_m(t,x)|^6 f(X_m(t,x)) + L_m f_m(X(t,x)))e^{-\epsilon\int_0^t |(-A)^{1/2}X_m(s,x)|^6\,ds}\,dt\\
&\quad + e^{-\epsilon\int_0^t |(-A)^{1/2}X_m(s,x)|^6\,ds}(Df_m(X_m(t,x)), \sqrt{Q_m}\,dW)
\end{aligned}$$

and

$$\begin{aligned}
(3.1)\qquad &\mathbb{E}(e^{-\epsilon\int_0^{t_2} |(-A)^{1/2}X_m(s,x)|^6\,ds} f(X_m(t_2,x)))\\
&\qquad - \mathbb{E}(e^{-\epsilon\int_0^{t_1} |(-A)^{1/2}X_m(s,x)|^6\,ds} f(X_m(t_1,x)))\\
&= \mathbb{E}\bigg(\int_{t_1}^{t_2} \big(-\epsilon|(-A)^{1/2}X_m(s,x)|^6 f(X_m(s,x))\\
&\qquad + L_m f(X_m(s,x))\big)e^{-\epsilon\int_0^s |(-A)^{1/2}X_m(\sigma,x)|^6\,d\sigma}\,ds\bigg).
\end{aligned}$$

We have denoted by L_m the Kolmogorov operator associated to (1.8). Since $f \in \mathscr{H}$, we have

$$|Lf_m(x)| \leq c(1 + |Ax|^6).$$

By Proposition 5.4 and Lemma 5.3, the right hand side of (3.1) is uniformly integrable on $\Omega \times [t_1, t_2]$ with respect to m. Thus, we can take the limit as $m \to \infty$ in (3.1) to obtain

$$
\begin{aligned}
(3.2)\qquad \mathbb{E}_x(e^{-\epsilon\int_0^{t_2}|(-A)^{1/2}X(s,x)|^6\,ds}f(X(t_2,x)))& \\
-\mathbb{E}_x(e^{-\epsilon\int_0^{t_1}|(-A)^{1/2}X(s,x)|^6\,ds}f(X(t_1,x)))& \\
=\mathbb{E}_x\biggl(\int_{t_1}^{t_2}\bigl(-\epsilon|(-A)^{1/2}X(s,x)|^6 f(X(s,x))& \\
+Lf(X(s,x))\bigr)e^{-\epsilon\int_0^{s}|(-A)^{1/2}X(\sigma,x)|^6\,d\sigma}\,ds\biggr)&.
\end{aligned}
$$

It is easy to prove by dominated convergence that

$$
\begin{aligned}
&\mathbb{E}_x(e^{-\epsilon\int_0^{t_i}|X(s,x)|_1^6\,ds}f(X(t_i,x)))\to P_{t_i}f(x),\\
&\mathbb{E}_x\int_{t_1}^{t_2}Lf(X(s,x))e^{-\epsilon\int_0^{s}|X(\sigma,x)|_1^6\,d\sigma}\,ds\to\mathbb{E}_x\int_{t_1}^{t_2}P_sLf(x)\,ds,
\end{aligned}
$$

as $\epsilon\to 0$. Indeed, by Lemma 5.3 below, we have

$$
\int_0^{t_i}|X(s,x)|_1^6\,ds<\infty\qquad \mathbb{P}\text{-a.s.}
$$

Moreover,

$$
\begin{aligned}
&\left|\mathbb{E}_x\int_{t_1}^{t_2}\bigl(\epsilon|X(s,x)|_1^6 f(X(s,x))e^{-\epsilon\int_0^{s}|X(\sigma,x)|_1^6\,d\sigma}\bigr)\,ds\right|\\
&\qquad\le\|f\|_0\mathbb{E}_x(e^{-\epsilon\int_0^{t_1}|X(\sigma,x)|_1^6\,d\sigma}-e^{-\epsilon\int_0^{t_2}|X(\sigma,x)|_1^6\,d\sigma})\to 0
\end{aligned}
$$

as $\epsilon\to 0$. The result follows. □

It is now easy to conclude the proof of Theorem 3.2. Indeed, by Proposition 3.3, for $f\in\mathscr{H}$ we have, since $\|P_sLf\|_0\le\|Lf\|_0$,

$$
\|P_tf-f\|_0\le t\|Lf\|_0.
$$

Moreover, since $s\mapsto P_sLf(x)$ is continuous, for any $x\in D(A)$ we have

$$
\frac{1}{t}(P_tf(x)-f(x))\to Lf(x)\qquad\text{as } t\to 0.
$$

It follows that $f\in D(\bar{L})$ and $\bar{L}f=Lf$. Finally,

$$
\mathscr{G}_1\subset\mathscr{H},
$$

and since $\mathscr{G}_1$ is a π-core we deduce that $\mathscr{H}$ is also a π-core.

REMARK 3.4. We do not use the fact that P_tf is a strict solution of the Kolmogorov equation to prove that $\mathscr{H}\subset D(\bar{L})$ and $\bar{L}f=Lf$. But we do not know if there is a direct proof of $\mathscr{H}$ being a π-core. We have used the inclusion $\mathscr{G}_1\subset\mathscr{H}$ and the fact that $\mathscr{G}_1$ is a π-core. The proof of $\mathscr{G}_1\subset\mathscr{H}$ requires (2.1), which is almost as strong as the construction of a strict solution.

REMARK 3.5. For $d = 3$, using Lemma 3.1 in [4], it is easy to prove a formula similar to (3.2) with $|(-A)^{1/2}X(s,x)|^6$ replaced by $|AX(s,x)|^4$ in the exponential terms. The problem is that

$$\lim_{\epsilon\to 0} e^{-\epsilon\int_0^t |AX(s,x)|^4\,ds} = \mathbb{1}_{[0,\tau^*(x))}(t),$$

where $\tau^*(x)$ is the life time of the solution in $D(A)$. Thus we are not able to prove Proposition 3.3 in this case.

We have the following result on the operator $\bar{N}$.

THEOREM 3.6. *Let $d = 2$ or 3 and $k \in \mathbb{N}$, and define*

$$\widetilde{\mathscr{H}_k} = \{f \in \mathscr{E}_3^k : Nf \in \mathscr{E}_3^k\}.$$

Then $\widetilde{\mathscr{H}_k} \subset D(\bar{N})$ and it is a π-core for $\bar{N}$. Moreover, for any $f \in \widetilde{\mathscr{H}_k}$ we have

$$\bar{N}f = Nf.$$

The proof follows the one above. Indeed, it is easy to use similar arguments to those in [4] and prove that for $f \in \mathscr{E}_3^k$, $(S_t f)_{t\ge 0}$ is a strict solution to (1.7). Arguing as above, we deduce that $(\lambda - \bar{N})^{-1}\mathscr{E}_3^k$ is a π-core for $\bar{N}$.

Moreover, applying the Itô formula to the Galerkin approximations and letting $m \to \infty$ along a subsequence m_k (to get uniform integrability by Lemma 3.1 of [4]) we prove, for $f \in \widetilde{\mathscr{H}_k}$,

$$\begin{aligned}&\mathbb{E}_x(e^{-K\int_0^{t_2}|AX(s,x)|^2\,ds} f(X(t_2,x))) - \mathbb{E}_x(e^{-K\int_0^{t_1}|AX(s,x)|^2\,ds} f(X(t_1,x)))\\ &\quad= \mathbb{E}_x\left(\int_{t_1}^{t_2}\big(-K|AX(s,x)|^2 f(X(s,x)) + Lf(X(s,x))\big)e^{-K\int_0^s |AX(\sigma,x)|^2\,d\sigma}\,ds\right).\end{aligned}$$

We rewrite this as

$$S_{t_2}f(x) - S_{t_1}f(x) = \int_{t_1}^{t_2} S_s Nf(x)\,ds,$$

and deduce as above that $f \in D(\bar{N})$ and $\bar{N}f = Nf$. Finally, since $(\lambda - \bar{N})^{-1}\mathscr{E}_3^k \subset \widetilde{H}_k$, we conclude that $\widetilde{\mathscr{H}_k}$ is also a π-core.

4. UNIQUENESS FOR THE MARTINGALE PROBLEM

Let us study the following martingale problem.

DEFINITION 4.1. *We say that a probability measure $\mathbb{P}_x$ on $C([0,T]; D((-A)^{-\epsilon}))$, $\epsilon > 0$, is a* solution of the martingale problem *associated to* (1.2) *if*

$$\mathbb{P}_x(\eta(t) \in D(A)) = 1, \quad t \ge 0, \quad \mathbb{P}_x(\eta(0) = x) = 1$$

and for any $f \in \mathscr{H}$,

$$f(\eta(t)) - \int_0^t Lf(\eta(s))\,ds$$

is a martingale with respect to the natural filtration.

REMARK 4.2. In general, the existence of a solution is only proved for a different martingale problem, where f is required to be in a smaller class. In particular, it is required that $f \in C_b(D((-A)^{-\epsilon}))$ for some $\epsilon > 0$. However, in all concrete constructions of solutions, it can be shown that a solution of our martingale problem is in fact obtained.

THEOREM 4.3. *Let $d = 2$. Then for any $x \in D(A)$, there exists a unique solution to the martingale problem.*

PROOF. By a similar proof to that of Proposition 3.3, we know that there exists a solution to the martingale problem.

Uniqueness follows from a classical argument. Let $f \in \mathscr{E}_1$ and, for $\lambda > 0$, set $\varphi = (\lambda - \bar{L})^{-1} f$. Then $\varphi \in \mathscr{G}_1 \subset \mathscr{H}$ and

$$\varphi(\eta(t)) - \varphi(x) - \int_0^t L\varphi(\eta(s))\,ds$$

is a martingale. Thus, for any solution $\tilde{P}_x$ of the martingale problem,

$$\tilde{\mathbb{E}}_x\left(\varphi(\eta(t)) - \varphi(x) - \int_0^t L\varphi(\eta(s))\,ds\right) = \varphi(x).$$

We multiply by $\lambda e^{-\lambda t}$ and integrate over $[0, \infty)$ to obtain, since $\bar{L}\varphi = L\varphi$,

$$\tilde{\mathbb{E}}_x \int_0^\infty e^{-\lambda t} f(\eta(t))\,dt = \varphi(x) = (\lambda - \bar{L})^{-1} f(x) = \int_0^\infty e^{-\lambda t} P_t f(x)\,dt.$$

By inversion of the Laplace transform we deduce that

$$\tilde{\mathbb{E}}_x(f(\eta(t))) = P_t f(x).$$

Thus the law at a fixed time t is uniquely defined. A standard argument proves that this implies uniqueness for the martingale problem. □

For $d = 3$ the proof of uniqueness still works. The problem is that we cannot prove existence of a solution of the martingale problem. More precisely, we cannot prove Proposition 3.3.

We can prove existence and uniqueness in $d = 3$ for the martingale problem with $\mathscr{H}$ replaced by $\mathscr{G}_1$, but since the definition of $\mathscr{G}_1$ depends on the semigroup, this does not give any real information.

We have the following weaker result on a stopped martingale problem.

DEFINITION 4.4. *We say that a probability measure* $\mathbb{P}_x$ *on* $C([0,T];D(A))$ *is a* solution of the stopped martingale problem *associated to* (1.2) *if*

$$\mathbb{P}_x(\eta(0)=1)=1,$$

and for any $f\in\widetilde{\mathscr{H}_k}$,

$$f(\eta(t\wedge\tau^*))-\int_0^{t\wedge\tau^*}Lf(\eta(s))\,ds$$

is a martingale with respect to the natural filtration and

$$\eta(t)=\eta(\tau^*),\qquad t\geq\tau^*.$$

The stopping time τ^* *is defined by*

$$\tau^*=\lim_{R\to\infty}\tau_R,\qquad \tau_R=\inf\{t\in[0,T]:|A\eta(t)|\geq R\}.$$

THEOREM 4.5. *For any* $x\in D(A)$, *there exists a unique solution to the stopped martingale problem.*

PROOF. Existence of a solution is classical. The proof follows the proofs of Proposition 3.3 and Theorem 3.6 (see also Remark 3.5, and [9] for more details). In fact, we may choose the Markov family $((X(t,x))_{t\geq 0},\Omega_x,\mathscr{F}_x,\mathbb{P}_x)$ constructed in [7]. It is easy to see that $X(t,x)$ is continuous up to τ^*. We slightly change notation and set $X(t,x)=X(t\wedge\tau^*,x)$.

Uniqueness follows from a similar argument to that for Theorem 4.3. For $\epsilon>0$, we define $(S^\epsilon(t))_{t\geq 0}$ just as $(S_t)_{t\geq 0}$ but we replace $e^{-K\int_0^t|A\eta(s)|^2\,ds}$ by $e^{-\epsilon\int_0^t|A\eta(s)|^4\,ds}$ in (1.6). Proceeding as above, we then define N_ϵ, $\bar{N}_\epsilon$, $\widetilde{\mathscr{H}_k^\epsilon}$, and prove that $\widetilde{\mathscr{H}_k^\epsilon}$ is a π-core for $\bar{N}_\epsilon$ and $N_\epsilon\varphi=\bar{N}_\epsilon\varphi$ for $\varphi\in\widetilde{\mathscr{H}_k^\epsilon}$.

Let $\tilde{\mathbb{P}}_x$ be a solution to the martingale problem and $f\in\mathscr{E}_3^k$. For $\lambda,\epsilon>0$, we set $\varphi=(\lambda-\bar{N}_\epsilon)^{-1}$; then $\varphi\in\widetilde{\mathscr{H}_k^\epsilon}$.

By the Itô formula (note that in Definition 4.4 it is required that the measure is supported by $C([0,T];D(A))$) we prove that

$$\begin{aligned}e^{-\epsilon\int_0^t|A\eta(s)|^4\,ds}\varphi(\eta(t))-\int_0^t\big(-\epsilon|A\eta(s)|^4\varphi(\eta(s))+L\varphi(\eta(s))\big)e^{-\epsilon\int_0^s|A\eta(\sigma)|^4\,d\sigma}\,ds\\=e^{-\epsilon\int_0^t|A\eta(s)|^4\,ds}\varphi(\eta(t))-\int_0^t N_\epsilon\varphi(\eta(s))e^{-\epsilon\int_0^s|A\eta(\sigma)|^4\,d\sigma}\,ds\end{aligned}$$

is also a martingale. We have used the fact that

$$e^{-\epsilon\int_0^t|A\eta(s)|^4\,ds}=0,\qquad t\geq\tau^*.$$

Thus

$$\tilde{\mathbb{E}}_x\left(e^{-\epsilon\int_0^t|A\eta(s)|^4\,ds}\varphi(\eta(t))-\int_0^t N_\epsilon\varphi(\eta(s))e^{-\epsilon\int_0^s|A\eta(\sigma)|^4\,d\sigma}\,ds\right)=\varphi(x).$$

We multiply by $e^{-\lambda t}$ and integrate over $[0,\infty)$ to obtain, since $\bar{N}_\epsilon\varphi = N_\epsilon\varphi$,

$$\tilde{\mathbb{E}}_x\left(\int_0^\infty e^{-\lambda t-\epsilon\int_0^t |A\eta(s)|^4\,ds} f(\eta(t))\,dt\right) = \varphi(x) = (\lambda-\bar{N}_\epsilon)^{-1}f(x)$$
$$= \int_0^\infty e^{-\lambda t} S_t^\epsilon f(x)\,dt.$$

By dominated convergence, we may let $\epsilon\to 0$ to obtain

$$\tilde{\mathbb{E}}_x\left(\int_0^\infty e^{-\lambda t}\mathbb{I}_{t\le\tau^*} f(\eta(t))\,dt\right) = \int_0^\infty e^{-\lambda t} S_t^0 f(x)\,dt,$$

where $S_t^0 f(x) = \lim_{\epsilon\to 0} S_t^\epsilon f(x) = \mathbb{E}_x(\mathbb{I}_{t\le\tau^*} f(X(t,x)))$. The conclusion follows. $\square$

5. TECHNICAL RESULTS

Throughout this section, we assume that $d=2$. Also, for $s\in\mathbb{R}$, we set $|\cdot|_s = |(-A)^s\cdot|$.

LEMMA 5.1. *There exists c depending on T, Q, A such that*

$$\mathbb{E}\left(\sup_{t\in[0,T]}|X(t,x)|^2 + \int_0^T |X(s,x)|_1^2\,ds\right) \le c(1+|x|^2),$$
$$\mathbb{E}\left(\sup_{t\in[0,T]}|X(t,x)|^4 + \int_0^T |X(s,x)|^2|X(s,x)|_1^2\,ds\right) \le c(1+|x|^4).$$

PROOF. We first apply Itô's formula to $\frac{1}{2}|x|^2$ (as usual the computation is formal and it should be justified by Galerkin approximations):

$$\frac{1}{2}\,d|X(t,x)|^2 + |X(t,x)|_1^2\,dt = (X(t,x),\sqrt{Q}\,dW) + \frac{1}{2}\operatorname{Tr}Q\,dt.$$

We deduce, thanks to a classical martingale inequality,

$$\mathbb{E}\left(\frac{1}{2}\sup_{t\in[0,T]}|X(t,x)|^2 + \int_0^T |X(s,x)|_1^2\,ds\right)$$
$$\le \mathbb{E}\left(\sup_{t\in[0,T]}\left|\int_0^t (X(s,x),\sqrt{Q}\,dW(s))\right|\right) + \frac{1}{2}(|x|^2 + (\operatorname{Tr}Q)T)$$
$$\le 2\mathbb{E}\left(\left(\int_0^T |\sqrt{Q}X(s,x)|^2\,ds\right)^{1/2}\right) + \frac{1}{2}(|x|^2 + (\operatorname{Tr}Q)T)$$
$$\le \frac{1}{2}\mathbb{E}\int_0^T |X(s,x)|^2\,ds + C + \frac{1}{2}|x|^2,$$

where C depends on T, Q, A. It follows that

$$\mathbb{E}\left(\sup_{t\in[0,T]}|X(t,x)|^2+\int_0^T|X(s,x)|_1^2\,ds\right)\le C+|x|^2. \tag{5.1}$$

We now apply Itô's formula to $\frac{1}{4}|x|^4$:

$$\begin{aligned}&\frac{1}{4}d|X(t,x)|^4+|X(t,x)|^2|X(t,x)|_1^2\,dt\\&=|X(t,x)|^2(X(t,x),\sqrt{Q}\,dW)+\left(\frac{1}{2}(\mathrm{Tr}\,Q)|X(t,x)|^2+|\sqrt{Q}\,X(t,x)|^2\right)dt\\&\le|X(t,x)|^2(X(t,x),\sqrt{Q}\,dW)+c|X(t,x)|^2\,dt.\end{aligned}$$

We deduce

$$\begin{aligned}&\mathbb{E}\left(\frac{1}{4}\sup_{t\in[0,T]}|X(t,x)|^2+\int_0^T|X(s,x)|^2|X(s,x)|_1^2\,ds\right)\\&\le\mathbb{E}\left(\sup_{t\in[0,T]}\left|\int_0^t|X(s,x)|^2(X(s,x),\sqrt{Q}\,dW(s))\right|\right)+c\mathbb{E}\int_0^T|X(s,x)|^2\,ds+\frac{1}{4}|x|^4\\&\le2\mathbb{E}\left(\left(\int_0^T|X(s,x)|^4|\sqrt{Q}\,X(s,x)|^2\,ds\right)^{1/2}\right)+c(1+|x|^4)\\&\le2\mathbb{E}\left(\sup_{t\in[0,T]}|X(s,x)|^2\left(\int_0^T|\sqrt{Q}\,X(s,x)|^2\,ds\right)^{1/2}\right)+c(1+|x|^4)\\&\le\frac{1}{8}\mathbb{E}\left(\sup_{t\in[0,T]}|X(t,x)|^4\right)+c\mathbb{E}\left(\int_0^T|\sqrt{Q}\,X(s,x)|^2\,ds\right)+c(1+|x|^4).\end{aligned}$$

Since $\sqrt{Q}$ is a bounded operator, using (5.1) we deduce that

$$\mathbb{E}\left(\sup_{t\in[0,T]}|X(t,x)|^4+\int_0^T|X(s,x)|^2|X(s,x)|_1^2\,ds\right)\le(1+|x|^4). \qquad \square$$

LEMMA 5.2. *There exists c depending on T, Q, A such that*

$$\begin{aligned}&\mathbb{E}\left(\sup_{t\in[0,T]}e^{-c\int_0^t|X(s,x)|^2|X(s,x)|_1^2\,ds}|X(t,x)|_1^2\right)\\&\qquad+\mathbb{E}\left(\int_0^T e^{-c\int_0^s|X(\sigma,x)|^2|X(\sigma,x)|_1^2\,d\sigma}|X(s,x)|_2^2\,ds\right)\le c(1+|x|_1^2).\end{aligned}$$

PROOF. We apply Itô's formula to

$$e^{-c\int_0^t|X(s,x)|^2|X(s,x)|_1^2\,ds}|X(t,x)|_1^2,$$

and obtain

$$\frac{1}{2}d(e^{-c\int_0^t |X(s,x)|^2|X(s,x)|_1^2\,ds}|X(t,x)|_1^2) + e^{-c\int_0^t |X(s,x)|^2|X(s,x)|_1^2\,ds}|X(t,x)|_2^2\,dt$$
$$= e^{-c\int_0^t |X(s,x)|^2|X(s,x)|_1^2\,ds}\big(-c|X(t,x)|^2|X(t,x)|_1^4 + (b(X(t,x)), AX(t,x))\big)\,dt$$
$$+ (Ax, \sqrt{Q}\,dW) - \frac{1}{2}\operatorname{Tr}(AQ)\,dt.$$

We have

$$(b(x), Ax) \le |b(x)|\,|Ax| \le \tilde{c}|x|_{L^4}|\nabla x|_{L^4}|Ax| \le \tilde{c}|x|^{1/2}|x|_1|x|_2^{3/2}$$
$$\le \frac{1}{2}|x|_2^2 + \tilde{c}|x|^2|x|_1^4.$$

We deduce that if $c \ge \tilde{c}$, then

$$\frac{1}{2}d(e^{-c\int_0^t |X(s,x)|^2|X(s,x)|_1^2\,ds}|X(t,x)|_1^2) + \frac{1}{2}e^{-c\int_0^t |X(s,x)|^2|X(s,x)|_1^2\,ds}|X(t,x)|_2^2\,dt$$
$$\le e^{-c\int_0^t |X(s,x)|^2|X(s,x)|_1^2\,ds}(AX(t,x), \sqrt{Q}\,dW) + c\,dt$$

and

$$\mathbb{E}\Big(\sup_{t\in[0,T]} e^{-c\int_0^t |X(s,x)|^2|X(s,x)|_1^2\,ds}|X(t,x)|_1^2\Big)$$
$$+ \mathbb{E}\bigg(\int_0^T e^{-c\int_0^s |X(\sigma,x)|^2|X(\sigma,x)|_1^2\,d\sigma}|AX(s,x)|^2\,ds\bigg)$$
$$\le 2\mathbb{E}\bigg(\bigg(\int_0^T e^{-2c\int_0^s |X(\sigma,x)|^2|X(\sigma,x)|_1^2\,d\sigma}|\sqrt{Q}\,X(s,x)|^2\,ds\bigg)^{1/2}\bigg) + cT + |x|_1^2.$$

Since $\operatorname{Tr}(QA) < \infty$, we know that QA is a bounded operator and

$$\mathbb{E}\bigg(\bigg(\int_0^T e^{-2c\int_0^s |X(\sigma,x)|^2|X(\sigma,x)|_1^2\,d\sigma}|\sqrt{Q}\,X(s,x)|^2\,ds\bigg)^{1/2}\bigg)$$
$$\le c\mathbb{E}\bigg(\bigg(\int_0^T |X(s,x)|_1^2\,ds\bigg)^{1/2}\bigg) \le |x| + 1,$$

by Lemma 5.1. The result follows. □

LEMMA 5.3. *For any $k \in \mathbb{N}$, there exists c depending on k, T, Q, A such that*

$$\mathbb{E}\Big(\sup_{t\in[0,T]} e^{-c\int_0^t |X(s,x)|^2|X(s,x)|_1^2\,ds}|X(t,x)|_1^k\Big) \le c(1 + |x|_1^k).$$

The proof of this lemma follows the same argument as above. It is left to the reader.

PROPOSITION 5.4. *For any $k \in \mathbb{N}$ and $\epsilon > 0$, there exists $C(\epsilon, k, T, Q, A)$ such that for any $m \in \mathbb{N}$, $x \in D(A)$, $t \in [0, T]$,*

$$\mathbb{E}(e^{-\epsilon\int_0^t |(-A)^{1/2}X_m(s,x)|^6\,ds}|AX_m(t,x)|^k) \le C(\epsilon, k, T, Q, A)(1 + |Ax|^k).$$

PROOF. Let us set

$$Z(t) = \int_0^t e^{(t-s)A}\sqrt{Q}\, dW(s), \qquad Y(t) = X(t,x) - z(t).$$

Then, by the factorization method (see [5]),

$$\mathbb{E}\Big(\sup_{t\in[0,T]} |Z(t)|_{2+\epsilon}^h\Big) \le C \tag{5.2}$$

for any $\epsilon < g$, and we have

$$\frac{dY}{dt} = AY + b(Y+Z).$$

We take the scalar product with A^2Y:

$$\frac{1}{2}\frac{d}{dt}|Y|_2^2 + |Y|_3^2 = (b(Y+Z), A^2Y) = ((-A)^{1/2}b(Y+Z), (-A)^{3/2}Y).$$

We have

$$|(-A)^{1/2}b(Y+Z)| = |\nabla b(Y+Z)| \le c(|Y+Z|_{W^{1,4}}^2 + |Y+Z|_{L^p}|Y+Z|_{W^{2,q}}),$$

where $1/p + 1/q = 1/2$.

By the Gagliardo–Nirenberg inequality

$$|Y+Z|_{W^{1,4}}^2 \le c|Y+Z|_1|Y+Z|_2.$$

Setting $1/p = 1/2 - s/2$ we have, by Sobolev's embedding,

$$|Y+Z|_{L^p}|Y+Z|_{W^{2,q}} \le c|Y+Z|_{s/2}|Y+Z|_{3-s/2}.$$

Therefore

$$\begin{aligned}((-A)^{1/2}b(Y+Z), (-A)^{3/2}Y) &\le c|Y+Z|_1|Y+Z|_2|Y|_3\\ &\le c|Y+Z|_{s/2}|Y+Z|_{3-s/2}|Y|_3\\ &\le \frac{1}{4}|Y|_3^2 + c|Y+Z|_1^2|Z|_2^2 + c|Y+Z|_1^2|Y|_2^2\\ &\quad + c|Y+Z|_{s/2}|Y|_{3-s/2}|Y|_3 + c|Y+Z|_{s/2}^2|Z|_{3-s/2}^2.\end{aligned}$$

Since

$$|Y+Z|_{s/2}|Y|_{3-s/2}|Y|_3 \le |Y+Z|_{s/2}|Y|_1^{s/4}|Y|_3^{2-s/4} \le c|Y+Z|_{s/2}^{8/s}|Y|_1^2 + \frac{1}{4}|Y|_3^2,$$

we finally get

$$\frac{d}{dt}|Y|_2^2 \le c|Y+Z|_1^2|Y|_2^2 + c(|Y+Z|_1^2|Z|_2^2 + |Y+Z|_{s/2}^2|Z|_{3-s/2}^2 + |Y+Z|_{s/2}^{8/s}|Y|_1^2)$$

and

$$|Y(t)|_2^2 \le e^{c\int_0^t |Y+Z|_1^2\,ds} \times \left(|x|_2^2 + c\int_0^t (|Y+Z|_1^2|Z|_2^2 + |Y+Z|_{s/2}^2|Z|_{3-s/2}^2 + |Y+Z|_1^{8/s}|Y|_{1/2}^2)\,ds \right).$$

We then obtain, by the Hölder and Poincaré inequalities,

$$e^{-\epsilon\int_0^t |Y+Z|_1^6\,ds}|Y(t)|_2^k \le c_k e^{-\epsilon\int_0^t |Y+Z|_1^6\,ds + c_k\int_0^t |Y+Z|_1^2\,ds} \times \left(|x|_2^k + \int_0^t (|Y+Z|_1^4 + |Z|_{2+g}^4 + |Y+Z|_1^{16/s})\,ds \right)$$

(we choose $3 - s/2 < 2 + g$ and set $\epsilon = 1 - s/2$).

The conclusion follows from Lemma 5.3 and the boundedness of the function $x \mapsto -\epsilon x^6 + cx^4$. □

REFERENCES

[1] J. BRICMONT - A. KUPIAINEN - R. LEFEVERE, *Exponential mixing for the 2D Navier–Stokes dynamics*, Comm. Math. Phys. 230 (2002), 87–132.

[2] I. CHUESHOV - S. B. KUKSIN, *Random kick-forced 3D Navier–Stokes equations in a thin domain*. Arch. Ration. Mech. Anal. 188 (2008), 117–153.

[3] I. CHUESHOV - S. B. KUKSIN, *Stochastic 3D Navier–Stokes equations in a thin domain and its α-approximation*. Preprint (2007).

[4] G. DA PRATO - A. DEBUSSCHE, *Ergodicity for the 3D stochastic Navier–Stokes equations*. J. Math. Pures Appl. 82 (2003), 877–947.

[5] G. DA PRATO - J. ZABCZYK, *Stochastic Equations in Infinite Dimensions*. Encyclopedia Math. Appl. 44, Cambridge Univ. Press, 1992.

[6] G. DA PRATO - J. ZABCZYK, *Ergodicity for Infinite Dimensional Systems*. London Math. Soc. Lecture Note Ser. 229, Cambridge Univ. Press, 1996.

[7] A. DEBUSSCHE - C. ODASSO, *Markov solutions for the 3D stochastic Navier–Stokes equations with state dependent noise*. J. Evol. Equations 6 (2006), 305–324.

[8] W. E - J. C. MATTINGLY - Y. G. SINAI, *Gibbsian dynamics and ergodicity for the stochastically forced Navier–Stokes equation*. Comm. Math. Phys. 224 (2001), 83–106.

[9] F. FLANDOLI, *An Introduction to 3D Stochastic Fluid Dynamics*. CIME lecture notes, 2005.

[10] F. FLANDOLI - B. MASLOWSKI, *Ergodicity of the 2D Navier–Stokes equations under random perturbations*. Comm. Math. Phys. 172 (1995), 119–141.

[11] F. FLANDOLI - M. ROMITO, *Markov selections for the 3D stochastic Navier–Stokes equation*. Probab. Theory Related Fields 140 (2008), 407–458.

[12] B. GOLDYS - M. RÖCKNER - X. ZHANG, *Martingale solutions and Markov selections for stochastic partial differential equations*. Preprint, Bielefeld, 2008.

[13] M. HAIRER - J. C. MATTINGLY, *Ergodicity of the 2D Navier–Stokes equations with degenerate stochastic forcing*. Ann. of Math. 164 (2006), 993–1032.

[14] M. HAIRER - J. C. MATTINGLY, *Spectral gaps in Wasserstein distances and the 2D stochastic Navier–Stokes equations*. Preprint, 2006.

[15] S. B. KUKSIN, *Randomly Forced Nonlinear PDEs and Statistical Hydrodynamics in 2 Space Dimensions*. Eur. Math. Soc., 2006.

[16] S. KUKSIN - A. SHIRIKYAN, *Ergodicity for the radomly forced 2D Navier–Stokes equations*. Math. Phys. Anal. Geom. 4 (2001), 147–195.

[17] S. KUKSIN - A. SHIRIKYAN, *A coupling approach to randomly forced nonlinear PDEs. I.* Comm. Math. Phys. 221 (2001), 351–366.

[18] J. C. MATTINGLY, *Exponential convergence for the stochastically forced Navier–Stokes equations and other partially dissipative dynamics*. Comm. Math. Phys. 230 (2002), 421–462.

[19] J. C. MATTINGLY - E. PARDOUX, *Malliavin calculus for the stochastic 2D Navier–Stokes equation*. Comm. Pure Appl. Math. 59 (2006), 1742–1790.

[20] R. MIKULEVICIUS - B. L. ROZOVSKII, *Stochastic Navier–Stokes equations for turbulent flows*. SIAM J. Math. Anal. 35, 1250–1310, 2005.

[21] R. MIKULEVICIUS - B. L. ROZOVSKII, *Global L_2-solutions of stochastic Navier–Stokes equations*. Ann. Probab. 33 (2004), 137–176.

[22] C. ODASSO, *Exponential mixing for the 3D Navier–Stokes equation*. Comm. Math. Phys. 270 (2007), 109–139.

[23] E. PRIOLA, *On a class of Markov type semigroups in spaces of uniformly continuous and bounded functions*. Studia Math. 136 (1999), 271–295.

[24] M. RÖCKNER - Z. SOBOL, *A new approach to Kolmogorov equations in infinite dimensions and applications to the stochastic 2D Navier–Stokes equation*. C. R. Math. Acad. Sci. Paris 345 (2007), 289–292.

[25] M. ROMITO, *Analysis of equilibrium states of Markov solutions to the 3D Navier–Stokes equations driven by additive noise*. J. Statist. Phys. 131 (2008), 415–444.

[26] A. SHIRIKYAN, *Qualitative properties of stationary measures for three-dimensional Navier–Stokes equations*. J. Funct. Anal. 249 (2007), 284–306.

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