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Asymptotic behavior of a stochastic combustion growth process

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Abstract. We study a continuous time growth process on the d-dimensional hypercubic lattice $\mathbb{Z}^d$, which admits a phenomenological interpretation as the combustion reaction $A + B \to 2A$, where A represents heat particles and B inert particles. This process can be described as an interacting particle system in the following way: at time 0 a simple symmetric continuous time random walk of total jump rate one begins to move from the origin of the hypercubic lattice; then, as soon as any random walk visits a site previously unvisited by any other random walk, it creates a new independent simple symmetric random walk starting from that site. Let P_d be the law of such a process and $S_d^0(t)$ the set of sites visited at time t. We prove that there exists a bounded, non-empty, convex set $C_d \subset \mathbb{R}^d$ such that for every $\epsilon > 0$, P_d-a.s. eventually in t, the set $S_d^0(t)$ is within an ϵt distance of the set $[C_d t]$, where for $A \subset \mathbb{R}^d$ we define $[A] := A \cap \mathbb{Z}^d$. Furthermore, answering questions posed by M. Bramson and R. Durrett, we prove that the empirical density of particles converges weakly to a product Poisson measure of parameter one, and moreover, for d large enough, we establish that the set C_d is not a ball under the Euclidean norm.

Keywords. Random walk, Green function, subadditivity

1. Introduction

In this article we consider a stochastic growth process associated with a system of interacting random walks on the lattice $\mathbb{Z}^d$. At time $t = 0$ a continuous time simple symmetric random walk of total jump rate one begins to move from the origin 0. Then, as soon as any random walk visits a site previously unvisited by any other random walk, it creates a new independent simple symmetric random walk. Thus, the set $S_d^0(t)$ of visited sites at time t, for the d-dimensional process starting with one random walk at the origin, is a random connected cluster which is growing in time.

A natural interpretation for this process can be given in terms of a chemical reaction associated to the steady-state burning of a homogeneous solid. Here heat is conducted into the solid from a reaction region raising its temperature and deflagrating it. This is the situ-

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ation for example of combustion in solid propellant rocket motors (see Chapter 9 of [W]). A very simplified description of this phenomenon can be given by a system composed of two types of particles: active particles A representing diffusing heat, and passive particles B representing inert combustible molecules (see Section 1.3 and Chapter 4 of [BE] for a description of this phenomenon in terms of partial differential equations).

The system starts with one heat particle A at the origin and one passive at any other site. Then, whenever an active particle A reaches a passive particle B, the passive particle is "burned" becoming active, and the following chemical reaction takes place:

$$A + B \to A + A.$$

This process can be viewed as well as a dependent long-range version of first-passage percolation. In fact, here to each site $x \in \mathbb{Z}^d$ we can associate the countable collection $\{t_{x,y} : y \in \mathbb{Z}^d\}$ of passage times, where $t_{x,y}$ represents the first hitting time of site y by a continuous time d-dimensional simple symmetric random walk starting from site x and of total jump rate 1. Then, given points $x_i \in \mathbb{Z}^d$, $1 \leq i \leq n$, and the corresponding path $r = (x_1, \dots, x_n)$ we can define its passage time as $T(r) = \sum_{i=1}^{n-1} t_{x_i,x_{i+1}}$. Next, this gives us, as in first-passage percolation, the travel time $T(0, x) := \inf\{T(r) : r$ a path from 0 to $x\}$, from site 0 to site x. Thus, the set of sites visited at time t is represented as $S_d^0(t) = \{y \in \mathbb{Z}^d : T(0, y) \leq t\}$. This representation will not be explicitly used in this paper, though first-passage percolation type techniques will recurrently appear in the proofs. On the other hand, our model presents some similarities to problems of random walks in random potentials and it can be viewed as an opposite of the Internal DLA (see [LBG] and [BR]), where particles are killed when visiting an unvisited site. For this reason, some aspects of the analysis that will be presented show similarities with the study of Internal DLA, where elementary potential theory and some spectral estimates are used.

The first problem which we address about this process, which we call the *combustion growth process*, is the asymptotic behavior in time of the set $S_d^0(t)$ of sites visited at time t and its geometric properties. The first result of this paper gives a partial answer to this question, stating that to leading order $S_d^0(t)$ approaches a linearly growing deterministic shape. Moreover, for large enough dimensions d, the set $S_d^0(t)$ is not a ball under the Euclidean norm. This is the content of the following theorem, where P_d is the probability measure associated to the d-dimensional combustion growth process, and for any subset $A \subset \mathbb{R}^d$, we define $[A] = A \cap \mathbb{Z}^d$.

Theorem 1.1. *There is a closed convex bounded subset $C_d \subset \mathbb{R}^d$, symmetric under permutations of the coordinate axes and with non-empty interior, such that for every $\epsilon > 0$, P_d-a.s. eventually in t one has*

$$[C_d t(1 - \epsilon)] \subset S_d^0(t) \subset [C_d t(1 + \epsilon)]. \tag{1.1}$$

Furthermore, for d large enough, C_d is not a ball under the Euclidean norm.

Remark 1.1. The fact that for large enough dimensions d, the set C_d is not a ball, is a corollary of Theorem 7.1 of Section 7. There it is actually shown that for large enough dimensions, the asymptotic speed of growth of $S_d^0(t)$ in the axial direction is larger than $Cd^{-1/3-\epsilon}$ for every $\epsilon > 0$ and for some constant $C(\epsilon)$, while in the diagonal direction it is smaller than $d^{-1/2}$.

In spite of the fact that the proof of the linear growth of (1.1) has a technical character, it is possible to give an informal heuristic argument providing some insight into the mechanism taking place. Indeed, by definition the number of particles in the set $S := S_d^0(t)$ is equal to its cardinality. If we assume that this distribution is close to uniform, then the number of particles in the interior boundary ∂S of S should be close to something proportional to the cardinality of this interior boundary. This can be expressed as

$$\frac{dS}{dt} = \text{const} \cdot \partial S.$$

Thus, if S is approximately proportional to some bounded set $C_d \subset \mathbb{R}^d$, the above equation shows that this scaling has to be linear, so that $S \sim \text{const} \cdot C_d t$.

The non-isotropy statement of Theorem 1.1 is an effect of the random walk structure on the discrete lattice. This can be seen from the fact that in high dimensions of the lattice $\mathbb{Z}^d$, the number of steps necessary to move a fixed Euclidean distance in a diagonal direction is much larger than of those needed to move in an axial direction. We could consider for instance a model where there are particles represented by balls of a fixed radius in $\mathbb{R}^d$ and centers having a Poisson distribution. Initially a single particle moves so that its center travels like a Brownian motion. Then, every time a particle hits the boundary of another inert ball, the latter is activated and begins to follow a Brownian motion trajectory. We expect that in such a model there will be no non-isotropy effects.

The second result of this paper answers a question posed by M. Bramson and R. Durrett several years ago concerning the behavior of the empirical density of random walks within the set $S_d^0(t)$. We denote by $\eta_x(t)$ the total number of random walks at site $x \in \mathbb{Z}^d$ at time t and refer to the quantity $\eta(t) = \{\eta_x(t) : x \in \mathbb{Z}^d\}$ as the *occupation field* of random walks at time t. Then let $\mu(t)$ be the distribution of the empirical density $\eta(t)$ under P_d. Theorem 1.2 below shows that the occupation field distribution of particles inside $S_d^0(t)$ converges to a product Poisson measure of parameter 1. To state this result let us endow $\mathcal{M} := \mathbb{N}^{\mathbb{Z}^d}$ with the product topology and the Borel σ-algebra $\mathcal{C}$.

Theorem 1.2. *Let ν be the product Poisson measure of parameter 1 on $(\mathcal{M}, \mathcal{C})$. Then*

$$\lim_{t\to\infty} \mu(t) = \nu,$$

where the convergence is in the sense of the weak topology on $\mathcal{M}$.

Remark 1.2. Given a probability measure α on $(\mathcal{M}, \mathcal{C})$, and some subset $\Lambda \subset \mathbb{Z}^d$, denote by α_Λ the restriction of α to $\mathcal{M}_\Lambda := \mathbb{N}^\Lambda$ endowed with its Borel σ-algebra. With a modification of the method used to prove the above theorem, it can be proved that for every finite $\Lambda \subset \mathbb{Z}^d$ and $0 < \alpha < 1$ one has $\lim_{t\to\infty} \|\mu(t)_{t^\alpha\Lambda} - \nu_{t^\alpha\Lambda}\| = 0$, where $\|\cdot\|$ denotes the total variation norm on $\mathcal{M}_\Lambda$.

Theorem 1.2 is a corollary of Theorem 1.1. In fact, once one knows the presence of a growing connected cluster of activated sites growing linearly in time, it is possible to reduce the problem to the study of the occupation field of particles of a set of independent simple random walks on the group $\mathbb{Z}^d_{2N+1}$ defined as the direct product of d copies of the group of integers modulo $2N+1$, with $N = [t^{2/3}]$, and having as initial condition one random walk per site. The reduction is achieved by first observing that random walks at a distance larger than $O(\sqrt{t})$ (as is the case for $t^{2/3}$) do not affect what happens within some finite set Λ at time t. Thus, the behavior of $\mu_\Lambda(t)$ can be approximated by the behavior of the corresponding marginal of a version of the combustion growth process defined on $\mathbb{Z}^d_{2N+1}$ with $N = [t^{2/3}]$. Next, one observes that particles within some small enough central region of $S^0_d(t)$ are born at times which are small enough compared to t so that it is irrelevant if one approximates their birth time by 0. This is a consequence of the fact that the linear growth of $S^0_d(t)$ is much larger than the typical distance $\sqrt{t}$ traveled by a random walk at time t. Once this is proved, the convergence follows via standard methods of approximation by Poisson product measures. Here we rely on Laplace transform techniques.

The proof of Theorem 1.1 is more involved. The basic idea is to apply the subadditive ergodic theorem (see Kesten [K] or Liggett [Li]). Its use is not new, and one of its first applications was to prove shape theorems in the context of first-passage percolation ([R], [CD], [K], [BG]). Here we apply it to show linear growth in n for the set of first passage times $T(0, nz)$, where $z \in \mathbb{Z}^d$ is arbitrary. This proves that the set $S^0_d(t)$ grows linearly in the direction defined by z. An appropriate pasting and continuity argument enables us to finish the proof of Theorem 1.1. The most serious difficulty is the verification of the hypothesis $E_d(T(0, z)) < \infty$ needed to apply the subadditive ergodic theorem, where E_d denotes expectation with respect to P_d. If one remarks that the hitting time of a site by a simple random walk is not integrable, it becomes clear that to prove $E_d(T(0, z)) < \infty$ it is necessary to control both the number of active random walks and their distance to z. One wants to show that at time t there are "many" random walks "close" to z.

An important idea of this paper is the use of induction on dimension to obtain lower bounds on the number of random walks at some given time. This captures the natural intuition that since the amount of space somehow grows with dimension, it is conceivable that if we consider random walks whose total jump rate in dimension d is d, then $S^{d+1}_0(t)$ is larger than $S^d_0(t)$ in a certain sense. More precisely, we construct a coupling between the d-dimensional process and the $(d+1)$-dimensional process which shows that if the total jump rate of the walks in dimension d is d, then with probability one, $S^{d+1}_0(t) \geq S^d(t)$. Hence, once a linear shape theorem is proved in dimension d with a good enough control on the slowdown deviations from this linear growth, we know modulo this slowdown deviation probability that the number of random walks at time t in dimension $d+1$ must be at least of the order of t^d, which corresponds to the order of the volume of $S^0_d(t)$. A separate argument shows that the distance of these random walks to z at time t cannot be larger than t. Therefore, with a probability tending to one as $t \to \infty$, we know that in dimension $d+1$, at time $t^{1/4}$ we have at least $t^{d/4}$ random walks. Then using the fact that $t^{1/4} = o(\sqrt{t})$, classical random walk hitting time probability estimates show that in dimension $d+1$, the probability that at time t site z has not been visited by any random walk is smaller than $(1 - \text{const}/t^{(d-1)/4})^{t^{d/4}} \approx \exp(-t^{1/4})$, which is integrable.

It is interesting to note some similarities of this problem with upper bound computations for the slowdown deviations of the so called marginal nestling and plain nestling random walks in random environment, defined in Sznitman [Sz2], which are subexponential. When optimal upper bounds of this kind have to be computed, usually it is helpful to use renormalization methods (see Piztora, Povel and Zeitouni [PPZ] for the so called positive and zero drift case in dimension $d = 1$). In our case, crude upper bounds are enough, so that it is not necessary to introduce any sophisticated machinery and a simple block argument does the job. We would like to remark that independently, and by different means, recently Alves, Machado and Popov [AMP1] proved a result analogous to the first part of Theorem 1.1 (existence of the asymptotic shape) obtained in the context of a discrete time dynamics, which is slightly easier to handle than the original continuous time model, even though the structure of the process is essentially the same in both cases (see also [AMP2] for other related work of these authors). In contrast to the method of [AMP1], our method is based on, and makes precise, some ideas related to the dependence of growth on dimension, and to the knowledge of the authors it has not been applied in this form previously in growth problems originating from first-passage percolation.

The proof of the non-isotropy result of Theorem 1.1, stating that the convex set C_d is not a ball for large enough dimensions d, is influenced by the approach of Hara and Slade [HS] for self-avoiding random walks. Similar ingredients can also be found in the non-isotropy proof for large dimensions in first-passage percolation [K]. Namely, at a heuristic level, as the dimension increases the hypercubic lattice becomes richer in terms of connections and locally it has a structure similar to that of a tree. The proof of the non-isotropy result is contained in Theorem 7.1, where it is proved that the asymptotic axial speed is larger than $Cd^{-1/3-\epsilon}$ for large dimensions and every $\epsilon > 0$ and some constant $C(\epsilon)$, and the asymptotic diagonal speed smaller than $d^{-1/2}$. For the upper bound in the diagonal direction, we essentially use the fact that asymptotically, the maximum number of sites that a rate one random walk can visit at time t is bounded by $t(1+\epsilon)$, for some $\epsilon > 0$. Then, the time it takes for the combustion growth process to visit the site $z_1 := (1, \ldots, 1)$ of the d-dimensional hypercubic lattice is at least of order d. Since the Euclidean distance of z_1 to the origin is $\sqrt{d}$, our process moves at most at a speed of $d^{-1/2}$ Euclidean units per unit time. The lower bound in the axial direction uses the large space in terms of connections which is available for large dimensions. This is contained in Lemma 7.2, where it is proved that in dimension d, at time $d^{1/3}$ there are at least $\sim d^{2/3+\epsilon}$ moving random walks, for some $\epsilon > 0$, at a unit distance from a hyperplane orthogonal to one of the axes. Since the probability for each of these random walks to hit this hyperplane by time t is of order t/d when $t \ll d$, it follows that the probability of not hitting it by time $d^{1/3-\epsilon/2}$ is of order $(1 - d^{1/3-\epsilon/2}/d)^{d^{2/3+\epsilon}} \sim \exp\{-d^{\epsilon/2}\}$, which goes to zero as $d \to \infty$. In Section 7, these ideas are developed in order to obtain the corresponding bounds for the expectations of the passage and passage to line times.

Before closing this introduction, we would like to say some words in relation to the boundary fluctuations of the set $S_d^0(t)$ of visited sites. This problem will not be touched here and remains beyond the scope of this article. A challenging question would be to settle whether or not the combustion growth process falls in the same universality class of growth models described by the KPZ theory, where in dimension $d = 1$ the boundary fluctuations are normal, and in dimension $d = 2$ the longitudinal fluctuations of the boundary follow a power law with exponent $1/3$ (see Krug–Spohn [KS] for a detailed discussion about these issues). Presumably standard first-passage percolation processes belong to this class ([K], [NP]). We do not, however, have any strong indication to believe that the combustion growth process should fall in the KPZ universality class.

Let us describe the organization of this paper. Sections 2 to 7 contain the proof of Theorem 1.1. In Section 2 we construct the process and show in Lemma 2.1 that the family $\{T(x, y) : x, y \in \mathbb{Z}^d\}$ of travel times has a subadditivity property. In Section 3 we obtain, under an assumption on the tail probabilities for the travel times, slowdown deviation estimates on the growth of the cluster $S_d^0(t)$ in a fixed direction. This is needed to implement the induction argument explained previously. In Section 4 a coupling between dimension d and dimension $d+1$ is constructed. The properties we need about this coupling are stated in Lemma 4.1. This is then used in Lemma 4.2 to prove that if a "weak" shape theorem is satisfied in dimension d then the travel times in dimension $d+1$ have small enough tails. In Section 5 it is shown in Lemma 5.1 how to obtain the first estimates for the induction argument in dimension 1. In Section 6, Theorem 1.1 is proved using the previous results. The convex set C_d of Theorem 1.1 is defined and its properties verified. Here, a pasting technique requiring a continuity property of the growth is used to complete the proof. This is the content of Lemma 6.3. In Section 7, the non-isotropy result is proven. Section 8 contains the proof of the second result of this paper, Theorem 1.2, showing how to obtain it as a corollary from Theorem 1.1. One appendix has been added, with results of a more technical character. In Theorem A.1 of the appendix some precise estimates on the hitting probabilities of random walks are obtained. These estimates are stronger than what is needed for the proof of Theorem 1.1, but have been kept in their present form for completeness.

2. Construction of the process and its properties

We will construct the combustion growth process in the hypercubic lattice $\mathbb{Z}^d$, where $d \geq 1$ is the space dimensionality. Let $\mathbf{X} = \{X^x : x \in \mathbb{Z}^d\}$ be a family of independent random walks, each being a d-dimensional continuous time simple symmetric random walk starting at site x of total jump rate 1, and R will stand for the corresponding probability measure on the space $\Omega_d = \mathbf{D}([0, \infty); (\mathbb{Z}^d)^{\mathbb{N}})$ of right-continuous functions with left limits, endowed with the Skorokhod topology and with its Borel σ-algebra $\mathcal{B}_d$.

The combustion growth process starting at the site $x \in \mathbb{Z}^d$ will be defined as a function of $\mathbf{X}$. It will be convenient for our purposes to represent this process as a kind of branching process: we start with a single random walk from site x, and once it jumps to an unvisited site of $\mathbb{Z}^d$ it creates another random walk, which moves independently; next, once one of the two independent random walks jumps to a site previously unvisited by any of them it will create a third independent random walk, etc. Our first concern is the asymptotic growth of the set of visited sites.

Now we will make the above construct formal. Fix $x \in \mathbb{Z}^d$ and let $Z_1^x := X^x$. In particle terminology Z_1^x represents the first moving particle of the process starting at x.

Set $\tau_1 = 0$ and define $S_d^x(1) := \{Z_1^x(\tau_1)\}$ and

$$\tau_2 := \inf_{t \geq 0}\{t : Z_1^x(t) \neq x\},$$

which represents the first time when the particle Z_1^x leaves the site x. The superscript d in S_d^x refers to the dimension and will later be helpful when constructing a coupling between dimensions. Next we define

$$Z_2^x(t) := \begin{cases} Z_1^x(\tau_2) & \text{if } 0 \leq t \leq \tau_2, \\ X^{Z_1^x(\tau_2)}(t - \tau_2) & \text{if } t > \tau_2, \end{cases}$$

which represents the second moving particle of the process created at the site $Z_1^x(\tau_2)$ at time τ_2. Here, for technical reasons, we have defined the dynamics of this second particle from time $t = 0$ and set it coinciding with the dynamics of the first created particle up to time τ_2. We define

$$S_d^x(2) := \{Z_1^x(\tau_1), Z_2^x(\tau_2)\}.$$

Now we proceed inductively on $n \geq 3$ and recursively define the successive creation times

$$\tau_n := \min_{1 \leq k \leq n-1} \inf_{t \geq \tau_{n-1}} \{t : Z_k^x(t) \notin S_d^x(n-1)\}, \tag{2.1}$$

which represent the first time when some of the particles $Z_1^x, \ldots, Z_{n-1}^x$ leaves the set $S_d^x(n-1)$, and the indices

$$\kappa_n := \{k \leq n-1 : \inf_{t \geq \tau_{n-1}} \{Z_k^x(t) \notin S_d^x(n-1)\} = \tau_n\}, \tag{2.2}$$

which for each n are R-a.s. unique, and show which one among the particles $Z_1^x, \ldots, Z_{n-1}^x$ leaves the set $S_d^x(n-1)$ at time τ_n. Next we define the corresponding *created* particle,

$$Z_n^x := \begin{cases} Z_{\kappa_n}^x(\tau_n) & \text{if } 0 \leq t \leq \tau_n, \\ X^{Z_{\kappa_n}^x(\tau_n)}(t - \tau_n) & \text{if } t > \tau_n, \end{cases} \tag{2.3}$$

and set

$$S_d^x(n) := \{Z_1^x(\tau_1), Z_2^x(\tau_2), \ldots, Z_n^x(\tau_n)\}.$$

Finally, we define

$$S_d^x(t) := S_d^x(n) \quad \text{if } \tau_n \leq t < \tau_{n+1}, \tag{2.4}$$

which is the set of sites of $\mathbb{Z}^d$ which have been visited by time t at least by one moving particle. The family $\mathbf{Z}^x = \{Z_n^x : n \in \mathbb{N}\}$ of random walks with a distribution constructed from the law R of the random walks $\mathbf{X}$ will be called the *combustion growth process* starting at site x. We will denote by P_d^x the canonical probability measure on $(\Omega_d, \mathcal{B}_d)$ corresponding to the process Z^x and refer to P_d as the probability measure defined on $\Omega_d^{\mathbb{Z}^d}$ endowed with its Borel σ-algebra and corresponding to the family $\mathbf{Z} := \{\mathbf{Z}^x : x \in \mathbb{Z}^d\}$ of combustion growth processes. The construction above (eqs. (2.1)–(2.3)) defines a version of a coupling between the families $\mathbf{X}$ and $\mathbf{Z}$.

In order to construct a reasonable filtration for the combustion growth process, so that for example the random variables $\{\tau_n : n \in \mathbb{N}\}$ become stopping times, it is necessary to introduce a branching process associated to $\mathbf{Z}$. This branching process will also be useful to prove rough estimates on the growth of the set S_d^0 in Lemma 3.2. Fix $x \in \mathbf{Z}^d$ and let $Y_1^x := Z_1^x$. Define $Y_2^x(t) := Z_1^x(t)$ if $t \leq \tau_2$, while $Y_2^x(t) := Z_2^x(t)$ if $t > \tau_2$. Then define recursively, for $n \geq 3$,

$$Y_n^x(t) := \begin{cases} Y_{\kappa_n}^x(t) & \text{if } 0 \leq t \leq \tau_n, \\ Z_n^x(t) & \text{if } t > \tau_n. \end{cases} \tag{2.5}$$

We will call the family $\mathbf{Y}^x := \{Y_n^x : n \in \mathbb{N}\}$ of random walks the *combustion branching process*. Note that due to right-continuity of $\mathbf{Y}^x$, the random variables $\{\tau_n : n \in \mathbb{N}\}$ are stopping times with respect to the minimal filtration $\{\mathcal{F}_t^x : t \in [0, \infty)\}$ in $(\Omega_d, \mathcal{B}_d)$, where $\mathcal{F}_t^x := \sigma(\mathbf{Y}^x(s) : s \in [0, t])$ is the σ-algebra generated by the process $\mathbf{Y}^x$ between times $s = 0$ and $s = t$.

In the rest of the section we will prove two important and useful properties of the process. The first is the subadditivity of travel times.

Definition 2.1. *For any pair of sites $x, y \in \mathbb{Z}^d$, we define the* travel time *by*

$$T(x, y) := \inf_{t \geq 0}\{t : y \in S_d^x(t)\}. \tag{2.6}$$

In other words, $T(x, y)$ is the first time when the site $y \in \mathbb{Z}^d$ is visited by some particle of the process beginning at the site x.

It follows from the definition and from the processes Z being right-continuous that $T(x, y),\ x, y \in \mathbb{Z}^d$, are stopping times. Moreover, the family $\{T(x, y) : x, y \in \mathbb{Z}^d\}$ has the following subadditivity property:

Lemma 2.1. *For any $x, y, z \in \mathbb{Z}^d$,*

$$T(x, y) \leq T(x, z) + T(z, y) \qquad P_d\text{-a.s.} \tag{2.7}$$

Proof. For any $x, y \in \mathbb{Z}^d$ define $t_{x,y} := \inf\{s : X^x(s) = y\}$, which is the hitting time of y by the random walk X^x starting at x. A sequence $\{x_0, \ldots, x_L\}$, $L = 1, 2, \ldots,$ of distinct sites with $x_0 = x$ and $x_L = y$ will be called a *chain* (of length L) connecting x to y. Then one can check that

$$T(x, y) = \inf \sum_{i=1}^{L} t_{x_{i-1}, x_i}, \tag{2.8}$$

where the infimum is taken over all chains of length L, $L = 1, 2, \ldots,$ connecting x to y. The inequality (2.7) follows immediately from (2.8). □

3. Slowdown deviations from linear growth

In this section we will derive the first step of the inductive procedure explained in the introduction: we will show that if the tails of the distribution of the travel times $T(0, z)$ for some fixed $z \in \mathbb{Z}^d$ decay polynomially fast, then the probability that the set $S_d^0(t)$ of sites visited at time t does not contain a ball of radius rt, where r is small enough, also decays polynomially. We let $\{e_i : 1 \le i \le d\}$ be the canonical basis of $\mathbb{Z}^d$, $w_d := 2\pi^{d/2}/(d\Gamma(d/2))$ the volume of a d-dimensional ball of radius one, and E_d the expectation with respect to the measure P_d. Also, given $x \in \mathbb{Z}^d$ we denote by $|x|$ its Euclidean norm and define for $r > 0$ the Euclidean ball centered at x of radius r as $B(x, r) := \{y \in \mathbb{Z}^d : |y - x| \le r\}$.

Proposition 3.1. *Let $n \in \mathbb{N}$. Assume that there is a constant $c_1(n, d)$ such that for every $t > 0$,*

$$P_d\,(T(0, e_1) \ge t) \le \frac{c_1}{t^{4(n+d+1)}}. \tag{3.1}$$

Then, for every r such that $r < 1/(dE_d[T(0, e_1)])$ and $t > 0$,

$$P_d(B(0, rt) \subset S_d^0(t)) \ge 1 - \frac{c_2}{t^n}, \tag{3.2}$$

where

$$c_2(n, d) := dw_d \frac{c_1(n, d)}{(\frac{1}{dr} - E_d(T(0, e_1)))^{4(n+d)} r^n}.$$

The proof of Proposition 3.1 relies on the following lemma.

Lemma 3.1. *Let $n \in \mathbb{N}$. Assume that there is a constant $c_3(n, d)$ such that for every $t > 0$,*

$$P_d(T(0, e_1) \ge t) \le \frac{c_3}{t^{4(n+1)}}. \tag{3.3}$$

Then, for every $\delta > 0$ and $t > 0$,

$$P_d(T(0, \lfloor t \rfloor \cdot e_1) \ge t(E_d(T(0, e_1)) + \delta)) \le \frac{c_4}{\delta^{4n} t^n}, \tag{3.4}$$

where $c_4(n, d) := 6c_3(n, d)4^n + 2^d(d + n)^{8(d+n)}$.

Before proving Lemma 3.1 we will establish two rather standard estimates which will be used at various steps of the proof. We define the L_1 norm for points $x \in \mathbb{Z}^d$ by $|x|_1 := |x_1| + \cdots + |x_d|$, where x_i, $1 \le i \le d$, denote the coordinates of x. For $x \in \mathbb{Z}^d$ and $r > 0$, we let $B_1(x, r) := \{x \in \mathbb{Z}^d : |x|_1 < r\}$ be the open ball in the L_1 metric. Moreover, given any subset A of $\mathbb{Z}^d$, we will denote by $|A|$ its cardinality.

Lemma 3.2. *For every $\epsilon > 0$, $d \ge 1$ and $t \ge 0$ we have*

$$P_d(S_d^0(t) \subset B_1(0, (1 + \epsilon)t)) \ge 1 - \frac{[(1 + \epsilon)t]^d}{e^{tI(\epsilon)}}, \tag{3.5}$$

where $I : \mathbb{R} \to [0, \infty)$ is defined by $I(x) := (1 + x)\log(1 + x)/e + 1$.

Proof. Here it will be useful to express the set $S_d^0(t)$ in terms of the combustion branching process $\mathbf{Y}^0 := \{Y_n^0 : n \in \mathbb{N}\}$ defined in (2.5). In fact note that $S_d^0(t) = \bigcup_{n : \tau_n \le t} \{Y_n^0(\tau_n)\}$, where τ_n as defined in Section 2 represents the birth time of the n-th born random walk Z_n^0. We can then clearly write

$$S_d^0(t) = \bigcup_{n : \tau_n \le t} \{Y_n^0(\tau_n)\}.$$

Let $\gamma = 1 + \epsilon$. Then

$$P_d(S_d^0(t) \subset B_1(0, \gamma t)) = P_d\Big(\bigcap_{n : \tau_n \le t} \{Y_n^0(\tau_n) \in B_1(0, \gamma t)\} \Big) \tag{3.6}$$

Now let $\mathbf{M} := \{M_x : x \in \mathbb{Z}^d\}$ be a set of independent rate one Poisson processes indexed by the hypercubic lattice. Furthermore assume that $\mathbf{M}$ is independent of P_d. Now for each $x \in \mathbb{Z}^d$ define the processes $N_x(\cdot)$ as follows. Given $t \ge 0$, if there is an $n \ge 1$ such that $\tau_n \le t$ and $x = Y_n^0(\tau_n)$ we let $N_x(t) :=$ total number of jumps up to time t of $Y_n(t)$. Otherwise we define $N_x(t) := M_x(t)$. Note that for every $x \in \mathbb{Z}^d$ and $t \ge 0$, the random variable $N_x(t)$ has a Poisson distribution of parameter t. Lower bounding the right hand side of (3.6), we now conclude that

$$\begin{aligned} P_d(S_d^0(t) \subset B_1(0, \gamma t)) &\ge P_d\Big(\bigcap_{x \in B_1(0,\gamma t)} \{N_x(t) \le \gamma t\} \Big) \\ &= 1 - P_d\Big(\bigcup_{x \in B_1(0,\gamma t)} \{N_x(t) > \gamma t\} \Big) \\ &\ge 1 - |B_1(0, \gamma t)| P_d(N_0(t) > \gamma t). \end{aligned}$$

By an elementary large deviation bound we find that $P_d\,(N_0(t) \ge \gamma t) \le \exp\{-tI(\epsilon)\}$. Furthermore, since for $r > 0$ we have $|B_1(0, r)| = \binom{[r]+d-1}{[r]}$, where $[r]$ is the integer part of r, and $\binom{n+m}{m} \le n^m$ for $n, m \in \mathbb{N}$, it follows that $|B_1(0, \gamma t)| \le (\gamma t)^d$. Thus, $P_d(S_d^0(t) \subset B_1(0, \gamma t)) \ge 1 - (\gamma t)^d \exp\{-tI(\epsilon)\}$. □

At the next step we will introduce criteria characterizing "weak dependence" of the evolution in far apart space-time regions. Informally, having estimate (3.5) on the growth of $S_d^0(t)$, it becomes natural to expect that conditioned on the whole evolution up to time t, the random variables $T(x_1, y_1)$ and $T(x_2, y_2)$ are "basically independent" whenever $\min\{|x_1 - x_2|, |y_1 - y_2|\}$ is much larger than $t^{1+\epsilon}$ for some $\epsilon > 0$, in a sense to be made precise.

To make this precise, we will define independent families of independent symmetric random walks. For each $y \in \mathbb{Z}^d$ define the family $\mathbf{X}_y := \{X_y^x : x \in \mathbb{Z}^d\}$ of independent random walks, where each X_y^x is a random walk starting from site x distributed like X^x. The families $\{\mathbf{X}_y : y \in \mathbb{Z}^d\}$ and $\mathbf{X} = \{X^x : x \in \mathbb{Z}^d\}$ are taken independent of each other, and their joint distribution will be denoted by $\widehat{R}$.

With each given pair (y, r), $y \in \mathbb{Z}^d$, $r > 0$, and $x \in \mathbb{Z}^d$, we associate the random variable defined as follows:

$$X^x_{(y,r)} := \begin{cases} X^x & \text{if } |x-y| < r, \\ X^x_y & \text{if } |x-y| \geq r, \end{cases} \tag{3.7}$$

and the new family $\mathbf{X}_{(y,r)} := \{X^x_{(y,r)} : x \in \mathbb{Z}^d\}$. From (3.7) it follows that under $\widehat{R}$ the family $\mathbf{X}_{(y,r)}$ has the same law as $\mathbf{X}$. Moreover, with the random walks $\mathbf{X}$ we associate the corresponding combustion growth process $\mathbf{Z}$, and with each family $\mathbf{X}_{(y,r)}$ we associate another combustion growth process $\mathbf{Z}_{(y,r)}$ constructed using the procedure of Section 2, taking the family $\mathbf{X}_{(y,r)}$ as the underlying free random walks. We will denote by $\widehat{P}_d$ the joint law of the processes $\{\mathbf{Z}_{(y,r)} : y \in \mathbb{Z}^d\}$ and $\mathbf{Z}$ under $\widehat{R}$, when taken as the canonical coordinate process. As before, we define $T_r(y, z)$ as the time of the first visit to the site z by a moving particle of this process. From (3.7) it follows that $T_r(x_1, y_1)$ is independent of $T_r(x_2, y_2)$ if $|x_1 - x_2| \geq 2r$, since the families $\mathbf{X}_{(x_1,r)}$ and $\mathbf{X}_{(x_2,r)}$ are then independent of each other. This is so since $\mathbf{X}_{(x_1,r)}$ is defined using random walks from two different classes: random walks $\{X^x : |x_1 - x| < r\}$ from the family $\mathbf{X}$ and random walks from $\mathbf{X}_{x_1}$. To define $\mathbf{X}_{(x_2,r)}$ we use the random walks $\{X^x : |x_2 - x| < r\}$, again from $\mathbf{X}$, and random walks from $\mathbf{X}_{x_2}$. Since $|x_1 - x_2| \geq 2r$, the sets of indices $\{x : |x_1 - x| < r\}$ and $\{x : |x_2 - x| < r\}$ do not intersect, which implies that all random walks involved in the above construction are mutually independent, and thus we get independence of $\mathbf{X}_{(x_1,r)}$ and $\mathbf{X}_{(x_2,r)}$.

Lemma 3.3. *For any $\gamma \geq 1$, $x, y \in \mathbb{Z}^d$, and $t \geq 0$, we have*

$$\widehat{P}_d(T(x, y) \neq T_{\gamma t}(x, y),\ T(x, y) < t) \leq (\gamma t)^d \exp(-tI(\gamma - 1)). \tag{3.8}$$

Proof. Observe that the occurrence of the event $\{T(x, y) \neq T_{\gamma t}(x, y)\} \cap \{T(x, y) < t\}$ implies that before time t at least one random walk of the process which is constructed using the family $\mathbf{X} = \{X^x : x \in \mathbb{Z}^d\}$ visits the complement of the ball $B(x, \gamma t)$. Thus, the event $\{T(x, y) \neq T_{\gamma t}(x, y)\} \cap \{T(x, y) < t\}$ is contained in $\{S^x_d(t) \subset B(x, \gamma t)\}^c$. The result now follows from (3.5). □

Proof of Lemma 3.1. To simplify notation we set $\bar{E} := E_d(T(0, e_1))$ and $T(k) := T(ke_1, (k+1)e_1)$, $T_1(k) := T_{t^{1/4}}(ke_1, (k+1)e_1)$ for $k \in \mathbb{N}$. Assume that (3.3) is satisfied. Define the events $G_j = \{T(j) = T_1(j)\}$. Using subadditivity we get

$$P_d(T(0, \lfloor t \rfloor e_1) > t(\bar{E} + \delta)) \leq P_d\Big(\sum_{k=0}^{\lfloor t \rfloor - 1} T(k) > \lfloor t \rfloor (\bar{E} + \delta)\Big), \tag{3.9}$$

and decomposing according to the occurrence or not of $\bigcap_{j=0}^{\lfloor t \rfloor - 1} G_j$ and using translation invariance, we bound the right hand side of (3.9) by $t\widehat{P}_d(T(0, e_1) \neq T_{t^{1/4}}(0, e_1)) + \widehat{P}_d(\sum_{k=0}^{\lfloor t \rfloor - 1} T_1(k) > \lfloor t \rfloor (\bar{E} + \delta))$, which, due to the estimate (3.8) with $\gamma = 1$, and the assumed validity of (3.3), is smaller than

$$\widehat{P}_d\Big(\sum_{k=0}^{\lfloor t \rfloor - 1} (T_1(k) - \bar{E}) > \delta \lfloor t \rfloor\Big) + t(2t^{1/4})^d \exp(-t^{1/4}) + \frac{c_3(n, d)}{t^{n+1}}. \tag{3.10}$$

Next we will estimate the first term of the last expression. It follows from the Chebyshev inequality that

$$\widehat{P}_d\Big(\sum_{k=0}^{\lfloor t\rfloor-1}(T_1(k)-\bar{E})>\delta\lfloor t\rfloor\Big)\le\frac{\widehat{E}_d((\sum_{k=0}^{\lfloor t\rfloor-1}(T_1(k)-\bar{E}))^{2m})}{(\delta\lfloor t\rfloor)^{2m}},\tag{3.11}$$

with m to be chosen equal to $2n$, and where $\widehat{E}_d$ denotes the expectation with respect to the measure $\widehat{P}_d$. Now,

$$\widehat{E}_d\Big(\Big(\sum_{k=0}^{\lfloor t\rfloor-1}(T_1(k)-\bar{E})\Big)^{2m}\Big)=\sum_{k_1,\dots,k_{2m}=0}^{\lfloor t\rfloor-1}\widehat{E}_d\Big(\prod_{j=1}^{2m}(T_1(k_j)-\bar{E})\Big).\tag{3.12}$$

Observe that if k_je_1 is not in $B(k_ie_1,2t^{1/4})$ then the random variables $T_{t^{1/4}}(k_ie_1,y_1)$ and $T_{t^{1/4}}(k_je_1,y_2)$ are independent for all $y_1,y_2\in\mathbb{Z}^d$, and moreover $\widehat{E}_d(T_{t^{1/4}}(0,e_1))-\bar{E}=0$. This immediately implies that all terms on the right hand side of the last equality which contain at least one index k_i such that $k_je_1\notin B(k_ie_1,2t^{1/4})$ for all $j\ne i$ will vanish. On the other hand, the number of non-zero terms in the above sum is bounded from above by the number of ways of selecting m pairs of indices $k_i,k_j\in\{0,1,\dots,\lfloor t\rfloor-1\}$, with the property that $|e_1k_i-e_1k_j|\le 2t^{1/4}$. This immediately implies that on the right hand side of (3.12) there are at most $(2t^{1/4}\lfloor t\rfloor)^m$ terms different from zero. Thus continuing (3.12), we have

$$\widehat{E}_d\Big(\Big(\sum_{k=0}^{\lfloor t\rfloor-1}(T_1(k)-\bar{E})\Big)^{2m}\Big)\le 2^mt^{5m/4}\sup_{0\le k_1,\dots,k_{2m}\le\lfloor t\rfloor}\widehat{E}_d\Big(\prod_{j=1}^{2m}(T_1(k_j)-\bar{E})\Big).$$

By the Hölder inequality, the right hand side is bounded by $(\prod_{j=1}^{2m}\widehat{E}_d(T_1(k_j)-\bar{E})^{2m})^{1/2m}=E_d((T(0,e_1)-\bar{E})^{2m})$. From the inequality $(a-b)^n\le a^n+b^n$, valid for a and b positive and $n\in\mathbb{N}$, it follows that $E_d((T(0,e_1)-\bar{E})^{2m})\le 2E_d(T(0,e_1)^{2m})$. This estimate together with the assumption (3.3) shows that $E_d((T(0,e_1)-\bar{E})^{2m})<3c_3$, where we have chosen $m=2n$. Substituting these estimates in (3.11) we obtain

$$\widehat{P}_d\Big(\sum_{k=0}^{\lfloor t\rfloor-1}(T_1(k)-\bar{E})>\delta\lfloor t\rfloor\Big)\le\frac{3c_34^n}{\delta^{4n}t^n}.$$

Inserting this into the bound (3.10), we see that $P_d(T(0,\lfloor t\rfloor\cdot e_1)\ge t(E_d(T(0,e_1))+\delta))\le t(2t^{1/4})^de^{-t^{1/4}}+c_3/t^n+3c_34^n/(\delta^{4n}t^n)$. Since $e^{-x}\le(\frac{n}{ex})^n$ for $x\in\mathbb{R}$ and $n\in\mathbb{N}$, the first term of this bound can be estimated by $2^d(\frac{4+4n+d}{4e})^{4+4n+d}\frac{1}{t^n}$. This proves Lemma 3.1. □

Proof of Proposition 3.1. Observe that

$$P_d(B(0,rt)\subset S_d^0(t))\ge P_d\Big(\bigcap_{z\in B(0,rt)}\{T(0,z)<t\}\Big).$$

Now, by subadditivity, this last quantity can be lower bounded by

$$1 - \sum_{z \in B(0,rt)} P_d\Big(\sum_{i=1}^{d} T(0, z_i e_i) \geq t\Big),$$

where z_i denote the coordinates of z. Therefore, using the fact that $\sum_{i=1}^{d} T(0, z_i e_i) \leq d \max_{1 \leq i \leq d} T(0, z_i e_i)$, and

$$P_d(d \max_{1 \leq i \leq d} T(0, z_i e_i) \geq t) \geq d P_d(T(0, z_1 e_1) \geq t/d),$$

we see that $P_d(B(0, rt) \subset S_d^0(t)) \geq 1 - d \sum_{z \in B(0,rt)} P_d(T(0, z_1 e_1) \geq t/d)$. But then, from the hypothesis $r < 1/(d E_d(T(0, e_1)))$ and (3.1), we get the lower bound

$$1 - d w_d \frac{c_1(n, d)}{(\frac{1}{dr} - E_d(T(0, e_1)))^{4(n+d)} r^n t^n},$$

which ends up the proof. □

4. Tail probabilities for infection times

In this section we will show that an inverse polynomial bound on the probability that $S_d^0(t)$ does not contain a ball of radius rt for r small enough implies an inverse polynomial bound on the decay of the tails of the random variables $\{T(x, y) : x, y \in \mathbb{Z}^d\}$ in dimension $d + 1$. The main tool needed for this is the construction of a coupling between the combustion growth processes in dimension d and $d + 1$ which will enable us to control the number of live particles in the $d + 1$-dimensional process in terms of the number of live particles in the d-dimensional process. That is the content of Lemma 4.1 below. This lets us conclude that if there is a reasonable shape theorem in dimension d, then the number of live particles in dimension $d + 1$ is at least the number of live particles in dimension d (modulo the probability of deviating from the d-dimensional shape). In what follows, given a subset $C \subset \mathbb{Z}^{d+1}$, we will denote by $\pi^d C$ its projection on the first d coordinates, so that $\pi^d(x_1, \ldots, x_d, x_{d+1}) = (x_1, \ldots, x_d)$. Recall that P_d^0 denotes the law of a d-dimensional combustion growth process starting from the origin.

Lemma 4.1. *For each $d \geq 1$, there exists a probability measure Q_d defined on the Cartesian product $\Omega_d \times \Omega_{d+1}$ endowed with its Borel σ-algebra, such that*

(i) $Q_d(A \times \Omega_{d+1}) = P_d^0(A)$ *for every* $A \in \mathcal{B}_d$,
(ii) $Q_d(\Omega_{d+1} \times A) = P_{d+1}^0(A)$ *for every* $A \in \mathcal{B}_{d+1}$,
(iii) *for any* $t \geq 0$,

$$S_d^0\left(\frac{d}{d+1} t\right) \subset \pi^d S_{d+1}^0(t) \qquad Q_d\text{-a.s.} \tag{4.1}$$

Proof. In the processes to be considered, it will be helpful to exhibit explicitly the dimension of the underlying hypercubic lattice. We will thus consider a d-dimensional combustion growth process starting at the origin, $\mathbf{Z}^{0,d} = \{Z_n^{0,d} : n \geq 1\}$, and the family $\mathbf{X}^d = \{X^{x,d} : x \in \mathbb{Z}^d\}$ of underlying d-dimensional independent random walks. Moreover consider a family $\mathbf{X}^{d+1} = \{X^{x,d+1} : x \in \mathbb{Z}^{d+1}\}$ of $d+1$-dimensional independent random walks, which are also independent of the family $\{X^{x,d}\}$. All these random walks are simple symmetric of total jump rate one.

Now we will construct a combustion process $\mathbf{Z}^{0,d+1} = \{Z_n^{0,d+1} : n \geq 1\}$ on Ω_{d+1}, as a function of $\mathbf{X}^d$, $\mathbf{Z}^{0,d}$ and $\mathbf{X}^{d+1}$, in such a way that the joint distribution of $\mathbf{Z}^{0,d} = \{Z_n^{0,d} : n \in \mathbb{N}\}$ and $\mathbf{Z}^{0,d+1} = \{Z_n^{0,d+1} : n \in \mathbb{N}\}$, denoted by Q_d, has properties (i)–(iii) of the lemma.

First we will define another family $\widetilde{\mathbf{X}}^{d+1} = \{\widetilde{X}^{x,d+1} : x \in \mathbb{Z}^d\}$ of random walks as a function of $\mathbf{X}^d$ and $\mathbf{X}^{d+1}$, by setting

$$\widetilde{X}^{x,d+1}(\cdot) = \left(X^{x,d}\left(\cdot \frac{d}{d+1}\right), X_{d+1}^{x,d+1}(\cdot)\right), \tag{4.2}$$

where $X_{d+1}^{x,d+1}$ denotes the $d+1$-coordinate of the random walk $X^{x,d+1}$, and $X^{x,d}(\cdot \frac{d+1}{d})$ denotes the d-dimensional random walk $X^{x,d}$ with time reduced by a factor of $d/(d+1)$. In a way similar to what was done in Section 2, we define $Z_1^{0,d+1} = \widetilde{X}^{0,d+1}$. Observe that the first d coordinates of this random walk coincide with $Z_1^{0,d}$. We then set $\tau_1 = 0$ and $\tau_2 = \inf_{t \geq 0}\{t : Z_1^{0,d+1}(t) \neq 0\}$. Moreover, we will need to introduce an extra sequence of random variables, whose first two terms will be given by $\sigma_1 = 0$ and $\sigma_2 = \inf_{t\geq 0}\{t : \pi^d Z_1^{0,d+1}(t) \neq 0\}$, where σ_2 represents the first time when the random walk $Z_1^{0,d+1}$ moves in a direction orthogonal to the $d+1$-th coordinate axis. Now, define

$$Y^{Z_1^{0,d+1}(\tau_2),d+1} := \begin{cases} X^{Z_1^{0,d+1}(\tau_2),d+1} & \text{if } \tau_2 < \sigma_2, \\ \widetilde{X}^{Z_1^{0,d+1}(\tau_2),d+1} & \text{if } \tau_2 = \sigma_2, \end{cases}$$

and finally define the second random walk of the infection process,

$$Z_2^{0,d+1}(t) := \begin{cases} Z_1^{0,d+1}(\tau_2) & \text{if } t \leq \tau_2, \\ Y^{Z_1^{0,d+1}(\tau_2),d+1}(t-\tau_2) & \text{if } t > \tau_2. \end{cases}$$

In other words, the first activated particle $Z_2^{0,d+1}$ evolves coupled to the d-dimensional process if the first jump of $Z_1^{0,d+1}$ is orthogonal to the $d+1$-th direction, and it evolves independently of this process otherwise. Now, let $S_{d+1}^0(2) := \{Z_1^{0,d+1}(\tau_1), Z_2^{0,d+1}(\tau_2)\}$ and let $\pi^d S_{d+1}^0(2)$ be its projection. Proceeding inductively, for $n \geq 3$ we recursively define

$$\tau_n := \min_{1\leq k\leq n-1} \inf_{t\geq\tau_{n-1}} \{t : Z_k^{0,d+1}(t) \notin S_{d+1}^0(n-1)\},$$

$$\sigma_n := \min_{1\leq k\leq n-1} \inf_{t\geq\tau_{n-1}} \{t : \pi^d Z_k^{0,d+1}(t) \notin \pi^d S_{d+1}^0(n-1)\}.$$

Moreover, we let $\kappa_n := \{k \le n-1 : \inf_{t \ge \tau_k}\{Z_k^x(t) \notin S_d^x(n-1)\} = \tau_n\}$, which a.s. has a unique element. So we define the random walk

$$Y^{Z^{0,d+1}_{\kappa_n(\tau_n)},d+1} := \begin{cases} X^{Z^{0,d+1}_{\kappa_n}(\tau_n),d+1} & \text{if } \tau_n < \sigma_n, \\ \widetilde{X}^{Z^{0,d+1}_{\kappa_n}(\tau_n),d+1} & \text{if } \tau_n = \sigma_n, \end{cases} \tag{4.3}$$

and the n-th particle of the process

$$Z_n^{d+1}(t) = \begin{cases} Z_{\kappa_n}^{d+1}(\tau_n) & \text{if } t \le \tau_n, \\ Y^{Z^{d+1}_{\kappa_n}(\tau_n),d+1}(t-\tau_n) & \text{if } t > \tau_2. \end{cases} \tag{4.4}$$

Finally, we let $S^0_{d+1}(n) := \{Z_1^{0,d+1}(\tau_1), Z_2^{0,d+1}(\tau_2), \dots, Z_n^{0,d+1}(\tau_n)\}$, and $\pi^d S^0_{d+1}(n)$ the corresponding projection.

Speaking informally, the above construction couples the d-dimensional process with the $d+1$-dimensional process in the following way: we begin in dimension $d+1$ with a particle whose first d coordinates evolve as those of the first particle in the d-dimensional process with time reduced by the factor $d/(d+1)$, and the $d+1$-th coordinate evolves according to an independent one-dimensional random walk of total jump rate $1/(d+1)$. Now, if some active particle activates a new one by performing a jump parallel to the $d+1$-th coordinate axis, this new activated particle will evolve according to an independent $d+1$-dimensional random walk, initially associated with a given site. If the new particle at site $x \in \mathbb{Z}^{d+1}$ is activated by some already active particle, which performs a jump in a direction orthogonal to the $d+1$-th axis, but the projection of the first d coordinates of x are in the set already visited by the d-dimensional process (with time reduced by the factor $d/(d+1)$), then this new particle will evolve according to an independent $d+1$-dimensional random walk, initially associated with a given site. And only if a new particle at site $y \in \mathbb{Z}^{d+1}$ is activated by some already active particle, which performs a jump in a direction orthogonal to the $d+1$-th axis, and the projection of the first d coordinates of y are not in the set already visited by the d-dimensional process then the first d coordinates of this new particle will evolve as those of the particle in the d-dimensional process, associated to the site $\pi^d y$, and the $d+1$-th coordinate will evolve according to an independent one-dimensional random walk associated with the site y.

Thus properties (i) and (iii) are immediate consequences of the construction. On the other hand, all "underlying" random walks involved in the construction of the family $\mathbf{Z}^{0,d+1} = \{Z_n^{0,d+1} : n \in \mathbb{N}\}$ are independent of each other. Moreover a given random walk is used to represent the motion of one and only one particle. Finally, since the choice of the random walk which will be used to describe the evolution of the particle activated at time t depends only on the state of the infection process *up to time t*, we conclude that the constructed family $\mathbf{Z}^{0,d+1} = \{Z_n^{0,d+1} : n \in \mathbb{N}\}$ has the distribution of the infection process starting at the origin, which proves (ii). □

We are now ready to state the main result of this section.

Lemma 4.2. *Let $n \geq 1$ be a natural number. Assume that there is a constant $r > 0$ independent of n and the dimension d and a constant $c_5(n, d)$ depending on n and d such that for every $t > 0$,*

$$P_d^0(B(0, rt) \subset S_d^0(t)) \geq 1 - \frac{c_5}{t^n}. \tag{4.5}$$

Then for every $z \in \mathbb{Z}^{d+1}$ there is a constant $c_6(n, z, d+1)$, depending on n, the site z and the dimension d, such that for every $t > 0$,

$$P_{d+1}^0(T(0, z) \geq t) \leq \frac{c_6}{t^{n/4}}. \tag{4.6}$$

Proof. Let

$$v_d(r) := w_d r^d \left(\frac{d}{d+1}\right)^d.$$

Note that by Lemma 4.1,

$$P_{d+1}^0(|S_{d+1}^0(t)| \leq v_d(r)t^d) \leq P_d^0(|S_d^0(dt/(d+1))| \leq v_d(r)t^d).$$

A combination of this fact and the hypothesis (4.5) gives

$$P_{d+1}^0(|S_{d+1}^0(t)| \leq v_d(r)t^d) \leq \left(\frac{d+1}{d}\right)^n \frac{c_5(n, d)}{t^n}. \tag{4.7}$$

Now let $\alpha < 1/2$. Then

$$\begin{aligned} P_{d+1}^0(T(0, z) > t) &\leq P_{d+1}^0(|S_{d+1}^0(t^\alpha)| < v_d(r)t^{d\alpha}) \\ &\quad + P_{d+1}^0(T(0, z) > t, |S_{d+1}^0(t^\alpha)| \geq v_d(r)t^{d\alpha}) \\ &\leq \frac{c_5(d+1)^n}{d^n t^{n\alpha}} + ((e-1)t^\alpha)^{d+1} \exp(-t^\alpha) \\ &\quad + P_{d+1}^0(T(0, z) > t, |S_{d+1}^0(t^\alpha)| \geq v_d(r)t^{d\alpha}, S_{d+1}^0(t^\alpha) \subset B(0, (e-1)t^\alpha)). \end{aligned} \tag{4.8}$$

Here, in the second inequality we used inequality (4.7) and Lemma 3.2 with $\epsilon = e - 1$ together with the fact that $B_1(0, r) \subset B(0, r)$ for $r > 0$. The last term on the right hand side of (4.8) is smaller than the probability that $v_d(r)t^{d\alpha}$ independent random walks born at times $\leq t^\alpha$ and at sites within Euclidean distance $(e-1)t^\alpha$ from the origin do not hit site z at time t. Then

$$\begin{aligned} P_{d+1}^0(T(0, z) > t,\ |S_{d+1}^0(t^\alpha)| \geq v_d(r)t^{d\alpha},\ S_{d+1}^0(t^\alpha) \subset B(0, (e-1)t^\alpha)) \\ \leq \Big(\sup_{x:|x-z| \leq (e-1)t^\alpha} P_x(\tau > t)\Big)^{v_d(r)t^{d\alpha}}, \end{aligned} \tag{4.9}$$

where τ is the first hitting time of the origin by a $d+1$-dimensional simple symmetric random walk of total jump rate one starting from site x and P_x is its law. Now, for $d + 1 = 2$ and t large enough, we can use Theorem A.1(iii) of the appendix, the fact that $P_x(\tau > t) \leq P_y(\tau > t)$ if $|x|_1 \leq |y|_1$ and that $\alpha < 1/2$ to bound the right hand side of (4.9) by $(2\alpha')^{v_1 t^{\alpha'}} = \exp(-v_1 |\log(2\alpha')| t^{\alpha'})$, where α' is some number such that

$\alpha \le \alpha' < 1/2$. We then conclude that for every $n \ge 1$ there is a constant $c_7(n, z, 2)$ (depending on n, the site z, and where the 2 indicates that it corresponds to a bound for $d+1 = 2$ and that it will also be defined for $d+1 \ge 3$) such that $c_7(n, z, 2)/t^{n/4}$ is a bound for the right hand side of (4.9). Finally, if $d+1 \ge 3$, as in the case $d+1 = 2$, we can use Theorem A.1(iv) to get for t large enough the bound

$$\left(1 - \frac{c_8(z,d)}{(2t)^{(d-1)\alpha}}\right)^{v_d(r)t^{d\alpha}} \le \exp\left(-\frac{v_d c_8 t^\alpha}{2^{(d-1)\alpha}}\right),$$

where $c_8(z, d)$ is a constant depending only on z and d. For every $n \ge 1$, this is in any case at most $c_7(n, z, d+1)/t^{n/4}$ for an appropriate constant c_7. Combining these estimates with (4.8) we see that

$$P^0_{d+1}(T(0,z) > t) \le c_5 \frac{(d+1)^n}{d^n} \frac{1}{t^{n\alpha}} + 2^{d+1} t^{(d+1)\alpha} e^{-t^\alpha} + \frac{c_7(n, z, d+1)}{t^{n/4}}.$$

Choosing $\alpha = 1/4$, we deduce the existence of a constant $c_6(n, z, d+1)$ depending on n, z and d such that (4.6) is satisfied. □

5. One-dimensional estimates

The objective of this section is to obtain good enough estimates on the tail probabilities of the stopping times $\{T(x, y) : x, y \in \mathbb{Z}\}$ in dimension $d = 1$. This will enable us to apply Proposition 3.1 and then Lemma 4.2 to begin the induction argument. To obtain the tail estimates we will first need the following lemma which gives us control on the number of live particles at time t.

Lemma 5.1. *There is a constant c_9 such that for every $0 < \alpha < 1/3$,*

$$P_1(|S_1^0(t)| < t^\alpha) \le \exp\left(-\frac{1}{2}t^{1-\alpha}\right) \quad \text{whenever } t > c_9^{1/(1-3\alpha)}.$$

Proof. Let $\Delta\tau_n = \tau_{n+1} - \tau_n$, where $\{\tau_n : n \in \mathbb{N}\}$ are the stopping times corresponding to the birth times of the successive particles of the combustion growth process defined in (2.1). First note that for $0 < \alpha < 1/2$ and $r > 0$,

$$P_1(|S_1^0(t)| < t^\alpha) \le P_1^0(\tau_{\lfloor t^\alpha \rfloor + 1} > t) \le e^{-rt} E_1^0\Big(\prod_{k=1}^{\lfloor t^\alpha \rfloor} e^{r\Delta\tau_k}\Big), \tag{5.1}$$

where in the last inequality we have used the Chebyshev inequality; recall that E_1 is the expectation with respect to P_1. Now, let $\mathcal{F}_n$ be the σ-algebra of events prior to the stopping time τ_n. Then, by the strong Markov property,

$$E_1^0\Big(\prod_{k=1}^{\lfloor t^\alpha \rfloor} e^{r\Delta\tau_k}\Big) \le E_1^0(e^{r\tau_1} E_1^0(e^{r\tau_2} \cdots E_1^0(e^{r\tau_{\lfloor t^\alpha+1 \rfloor}} \mid \mathcal{F}^0_{\lfloor t^\alpha \rfloor}) \cdots \mid \mathcal{F}^0_1)), \tag{5.2}$$

where $\{\mathcal{F}^0_t : t \in [0,\infty)\}$ as defined in Section 2 is the filtration generated by the associated one-dimensional combustion branching process $\mathbf{Y}^0$. We now claim that there is a constant $c_{10} > 1$ such that for every natural k and real r such that $r \le 3/(2k)$,

$$E^0_1(e^{r\tau_k} \mid \mathcal{F}^0_{k-1}) \le c^k_{10}. \tag{5.3}$$

This is enough to prove the lemma. Indeed, using the fact that $\sum_{k=1}^{[t^\alpha]} k = \frac{1}{2}[t^\alpha](1+[t^\alpha]) \le [t^{2\alpha}]$ and substituting (5.3) in (5.2), and (5.2) in (5.1), we obtain $P^0_1(|S^0_1(t)| < t^\alpha) \le e^{-rt}e^{t^{2\alpha}\ln c_{10}}$ whenever $r \le 3/(2([t^\alpha]+1))$. Hence, choosing $r = 3/(2t^\alpha)$ we conclude that

$$P^0_1(|S^0_1(t)| < t^\alpha) \le e^{-t^{1-\alpha}+(\ln c_{10})t^{2\alpha}} e^{-\frac{1}{2}t^{1-\alpha}}.$$

Therefore, the statement of the lemma follows if we choose $c_9 \ge \ln c_{10}$.

Thus, it remains to prove (5.3). Recall that $\tau_1 = 0$ while we have $P^0_1(\tau_{k+1} > t \mid \mathcal{F}^0_k) \le P_0(T > t)^k$ for $k \ge 1$, where T is the first-exit time of a random walk from the set $S_k := [-k, k]$, and P_0 is its law. Then from standard estimates (see [A] or [BR]) we obtain

$$P^0_1(\tau_{k+1} > t \mid \mathcal{F}^0_k) \le (c_{11}(\lambda(S_k)t+1)^{1/2}e^{-\lambda(S_k)t})^k$$

for $k \ge 1$, where $c_{11} > 1$ is a constant. Here, for any interval $I \subset \mathbb{Z}$ with $l = |I|$, $\lambda(I) = 1 - \cos(\frac{\pi}{l+1})$ is the principal Dirichlet eigenvalue of the normalized discrete Laplacian on I defined as $\Delta f(y) := \frac{1}{2}\sum_{e:|e|=1}(f(x+e) - f(x))$ for functions $f : \mathbb{Z} \to \mathbb{R}$ which vanish outside I. Thus,

$$E^0_1(e^{r\tau_{k+1}} \mid \mathcal{F}^0_k) \le 1 + c^k_{11} r \int_0^\infty e^{rt}((\lambda(S_k)t+1)^{1/2}e^{-\lambda(S_k)t})^k\, dt \tag{5.4}$$

for all $k \ge 1$. Now, for $|x| \le 1$ we have

$$x^2 \ge 1 - \cos x \ge \frac{x^2}{2}\left(1 - \frac{x^2}{12}\right) \ge \frac{1}{3}x^2.$$

Hence, since $|S_k| = 2k+1$ we have

$$\frac{\pi^2}{8k^2} \ge \lambda(S_k) \ge \frac{\pi^2}{8}\frac{1}{k^2}\frac{1}{(1+\frac{1}{2k})^2}\left(1 - \frac{\pi^2}{48k^2}\right).$$

But $\pi^2 > 9.8$ while $\frac{1}{(1+\frac{1}{2k})^2} > \frac{1}{1.11}$ and $1 - \frac{\pi^2}{48k^2} > 0.99$ if $k \ge 10$, so

$$\frac{\pi^2}{8}\frac{1}{k^2}\frac{1}{(1+\frac{1}{2k})^2}\left(1 - \frac{\pi^2}{48k^2}\right) > \frac{1}{k^2} \quad \text{if } k \ge 10.$$

It follows that $2/k^2 \ge \lambda(S_k) \ge 1/k^2$ if $k \ge 10$. Making the substitution $x = t/k^2$ in the integral of (5.4) we get

$$E^0_1(e^{r\tau_{k+1}} \mid \mathcal{F}^0_k) \le 1 + c^k_{11} r k^2 \int_0^\infty (2x+1)^{k/2} e^{-xk(1-kr)}\, dx \tag{5.5}$$

whenever $k \geq 10$. Looking at inequality (5.4) directly again, it is easy to see that the conditional expectation $E_1^0(e^{r\tau_{k+1}} \mid \mathcal{F}_k^0)$ is bounded by some constant independent of k if $1 \leq k \leq 9$. Therefore, provided we change the constant c_{11} if necessary, inequality (5.5) continues to be valid for every $k \geq 1$. Now, if $r \leq 1/(2k)$, we have

$$\begin{aligned}
&\int_0^\infty (2x+1)^{k/2} e^{-xk(1-kr)}\,dx \\
&\qquad \leq (2e)^{k/2} \int_0^\infty x^{k/2} e^{-xk/2}\,dx = \frac{2^{k/2}}{(k/2)^{k/2+1}} \int_0^\infty x^{k/2} e^{-x}\,dx \\
&\qquad = \frac{(2e)^{k/2}}{(k/2)^{k/2+1}} \Gamma\left(\frac{k}{2}+1\right) \leq 3 \frac{(2e)^{k/2}}{(k/2)^{k/2+1}} \left(\frac{k}{2e}\right)^{k/2} \sqrt{k} \leq 2^{k/2} \frac{8}{\sqrt{k}} \\
&\qquad \leq 8 \times 2^{k/2}.
\end{aligned}$$

Here $\Gamma(x)$ is the gamma function. Substituting this estimate into (5.5) we conclude that

$$E(e^{r\tau_{k+1}} \mid \mathcal{F}_k^0) \leq 1 + 8rk^2 (c_{11}\sqrt{2})^k \leq 1 + 4k(c_{11}\sqrt{2})^k.$$

Finally, from the inequality $k \leq e^k$ we see that the claim (5.3) is true whenever $c_{10} \geq 4ec_{11}\sqrt{2}$. □

Now we state the main result of this section.

Lemma 5.2. *There is a constant $c_{12} > 0$ such that*

$$P_1(T(0,1) > t) \leq 2\left(\frac{5}{t^{1/8}}\right)^{t^{1/4}} \qquad \textit{whenever } t \geq c_{12}. \tag{5.6}$$

Proof. Let $0 < \alpha < 1/3$. From Lemma 5.1 we conclude that there is a constant $c_9 > 0$ such that whenever $t > c_9^{1/(1-3\alpha)}$,

$$\begin{aligned}
P_1(T(0,1) > t) &\leq e^{-t^{1-\alpha}/2^{2-\alpha}} + P_1(T(0,1) > t,\ |S(t/2)| \geq t^\alpha/2^\alpha) \\
&\leq e^{-t^{1-\alpha}/4} + \Big(\sup_{z\,:\,|z| \leq 1 + t^\alpha/2^\alpha} P_z(\tau_1 \geq t/2)\Big)^{t^\alpha/2^\alpha} \\
&\leq e^{-t^{1-\alpha}/4} + \Big(\sup_{z\,:\,|z| \leq 2 + t^\alpha/2^\alpha} P_z(\tau_0 \geq t/2)\Big)^{t^\alpha/2^\alpha},
\end{aligned}$$

where for every site y, P_y is the law of a random walk starting from y, and τ_y is the hitting time of y. Now, by Theorem A.1(ii), the right hand side above is bounded by $(20/t^{1/2-\alpha})^{t^\alpha/2^\alpha}$ for $t \geq 2^{1+1/\alpha}$, so that choosing $\alpha = 1/4$ and $c_{12} \geq \max\{c_9, 2^{5/4}\}$ we get (5.6). □

6. Proof of the shape theorem

First note that by Lemma 5.2, for $d = 1$, the hypothesis (3.1) of Proposition 3.1 is satisfied for every n. In other words, for every $n \in \mathbb{N}$, there is a constant $c_1(n, 1)$ (where the 1 stands for dimension one) such that $P_1(T(0, e_1) \geq t) \leq c_1(n, 1)/t^{4(n+2)}$ for every $t > 0$. Hence, an application of Proposition 3.1 shows that for every $r < 1/(E_1[T(0, e_1)])$ and $n \in \mathbb{N}$, there is a constant $c_2(n, 1)$ such that $P_1(B(0, rt) \subset S_1^0(t)) \geq 1 - c_2(n, 1)/t^n$ for all $t > 0$. Therefore, the hypothesis (4.5) of Lemma 4.2 with $d = 1$ is satisfied, and we deduce that in dimension $d = 2$, for every $n \in \mathbb{N}$ and $z \in \mathbb{Z}^2$ there is a constant $c_1(n, z, 2)$ such that $P_2(T(0, z) \geq t) \leq c_1/t^{n/4}$ for every $t > 0$. We continue in this way by induction on d, applying Proposition 3.1 and Lemma 4.2 alternately, to conclude that for every $d \geq 1$, $r < 1/(dE_d(T(0, e_1)))$ and $n \in \mathbb{N}$ there is a constant $c_2(n, d)$ such that

$$P_d(B(0, rt) \subset S_d^0(t)) \geq 1 - \frac{c_2(n, d)}{t^n}, \tag{6.1}$$

and that for every $d \geq 1$, $n \in \mathbb{N}$ and $z \in \mathbb{Z}^d$ there is a constant $c_1(n, z, d)$ such that

$$P_d(T(0, z) \geq t) \leq \frac{c_1}{t^n}. \tag{6.2}$$

In particular, for every $d \geq 1$ and $z \in \mathbb{Z}^d$,

$$E_d(T(0, z)) < \infty. \tag{6.3}$$

We will now proceed to prove some ergodic properties of the collection of stopping times $\{T(x, y) : x, y \in \mathbb{Z}^d\}$ that will enable us to apply the subadditive ergodic theorem.

Lemma 6.1. *Consider the collection* $\{T(x, y) : x, y \in \mathbb{Z}^d\}$ *of travel times.*

(i) *For each* $z \in \mathbb{Z}^d - \{0\}$, $\{T((k-1)z, kz) : k \geq 1\}$ *is a stationary ergodic process.*
(ii) *For each* $z \in \mathbb{Z}^d - \{0\}$ *and* $j \in \mathbb{N}$, $\{T(jz, (j+k)z) : k \geq 0\} = \{T((j+1)z, (j+1+k)z) : k \geq 0\}$ *in distribution.*

Proof. Let $z \neq 0$. Part (ii) is a consequence of translation invariance. Note that the stationarity of part (i) is a consequence of translation invariance. To prove ergodicity, note that it is enough to show that for any pair of Borel subsets A, B of $[0, \infty)$,

$$\lim_{k \to \infty} P_d(T((k-1)z, kz) \in A,\ T(0, z) \in B) = P_d(T(0, z) \in A) P_d(T(0, z) \in B).$$

Now let us remark that for any $0 < R \leq 19$, the event

$$\begin{aligned} A_k = &\left\{ S_d^{(k-1)z}\left(\frac{k}{40}\right) \subset B\left((k-1)z, R\frac{k-1}{40}\right) \right\} \cap \left\{ S_d^0\left(\frac{k}{40}\right) \subset B\left(0, R\frac{k-1}{40}\right) \right\} \\ &\cap \left\{ T((k-1)z, zk) < \frac{k}{40} \right\} \cap \left\{ T(0, z) < \frac{k}{40} \right\} \end{aligned}$$

decouples the random variables $T((k-1)z, kz)$ and $T(0, z)$ for k large enough. Therefore,

$$P_d(T((k-1)z, kz) \in A,\ T(0,z) \in B) = P_d(T((k-1)z, kz) \in A,\ T(0,z) \in B,\ A_k) + P_d(T((k-1)z, kz) \in A,\ T(0,z) \in B,\ A_k^c).$$

Now, by decoupling the events A_k, the first term of the right hand side above can be expanded as

$$\begin{aligned} P_d(T((k-1)z, kz) \in A,\ A_k) P_d(T(0,z) \in B,\ A_k) &= P_d(T(0,z) \in A) P_d(T(0,z) \in B) \\ &\quad - P_d(T(0,z) \in A,\ A_k^c) P(T(0,z) \in B) \\ &\quad - P_d(T((k-1)z, kz) \in B,\ A_k^c) P_d(T(0,z) \in A). \end{aligned}$$

Thus, if $19 \geq R \geq 2$, by the estimate (3.5) of Lemma 3.2, which establishes that the set of visited sites is contained in some ball, and estimate (6.2), for any $n \in \mathbb{N}$ and k large enough,

$$\begin{aligned} |P_d(T((k-1)z, kz) \in A,\ T(0,z) \in B) - P_d(T(0,z) \in A) P_d(T(0,z) \in B)| &\leq 3 P_d(A_k^c) \\ &\leq 6\left((2k)^d e^{-kI(R-1)/40} + \frac{c_1}{k^n} \right), \end{aligned}$$

which completes the proof of part (i). □

We are now in a position to apply the subadditive ergodic theorem (see Liggett [Li]). In fact, fix $z \in \mathbb{Z}^d$, $z \neq 0$, and consider the family of random variables $\{T(nz, mz) : n, m \in \mathbb{N}\}$. By Lemma 2.1 this family is subadditive. Then by the stationarity and ergodic properties proved in Lemma 6.1 and the finite expectation of each of them expressed in the bound (6.3), we conclude that P_d-a.s.,

$$\lim_{n \to \infty} \frac{T(0, nz)}{n} = \mu_d(z), \tag{6.4}$$

where

$$\mu_d(z) := \inf_{n \geq 1} \frac{E_d(T(0, nz))}{n} = \lim_{n \to \infty} \frac{E_d(T(0, nz))}{n}. \tag{6.5}$$

Note that $\mu_d(nz) = n\mu_d(z)$ for every $z \in \mathbb{Z}^d$ and $n \in \mathbb{N}$, and $\mu_d(0) = 0$. This leads to the following definition:

Definition 6.1. *For each $q \in \mathbb{Q}^d - \{0\}$, let n_q be the smallest positive natural number n such that $qn \in \mathbb{Z}^d$, and define $z_q := qn_q$. Now for any $q \in \mathbb{Q}^d$ let*

$$\mu_d(q) := \begin{cases} |q| \dfrac{\mu_d(z_q)}{|z_q|} & \text{if } q \neq 0, \\ 0 & \text{if } q = 0, \end{cases} \tag{6.6}$$

where $\mu_d(z)$ is defined in (6.5). *We refer to the set $\{\mu_d(q) : q \in \mathbb{Q}^d\}$ as the* time constants *associated to the combustion growth process.*

A similar quantity can be defined in the context of first-passage percolation (see Kesten [K]) and is analogous to the Lyapunov exponents introduced by Sznitman [Sz1] to study some large deviation principles of Brownian motion among Poissonian obstacles. In our context, the quantity $\mu_d(q)$ represents the time needed for the set $S_d^0(t)$ to reach the point q. We continue with the following linearity and subadditivity properties of the time constants.

Lemma 6.2. *For any $q, r, s \in \mathbb{Q}^d$,*

$$\text{(i)} \quad \mu_d(sq) = s\mu_d(q), \tag{6.7}$$

$$\text{(ii)} \quad \mu_d(q+r) \leq \mu_d(q) + \mu_d(r). \tag{6.8}$$

This is a simple consequence of the definition (6.6) of the time constants, and of the subadditivity of the family $\{T(x, y) : x, y \in \mathbb{Z}^d\}$.

Let us now begin with the proof of Theorem 1.1, defining the set that will correspond to the limiting shape of the set of visited sites in the combustion growth process.

Corollary 6.1. *Consider the following subset of $\mathbb{R}^d$:*

$$C_d^o = \{q \in \mathbb{Q}^d : \mu_d(q) \leq 1\},$$

where $\{\mu_d(q) : q \in \mathbb{Q}^d\}$ are the time constants associated to the combustion growth process. Let C_d be the closure of C_d^o in $\mathbb{R}^d$. Then C_d is a closed convex bounded subset of $\mathbb{R}^d$, symmetric under permutations of the coordinate axes and with a non-empty interior.

Proof. By definition, C_d is closed. To prove it is convex, it is enough to show that C_d^o is convex as a subset of $\mathbb{Q}^d$. But this is a trivial consequence of the linearity and subadditivity properties expressed in Lemma 6.2. Next note that $\mu_d(e_i) > 0$ by Lemma 3.2. This shows that C_d is bounded along each coordinate axis. Combined with the convexity of C_d, this implies its boundedness. Since $E_d(T(0, e_i))$ is independent of $1 \leq i \leq d$, it follows that C_d is symmetric under permutations of the axes. Finally, the fact that $E_d(T(0, e_i)) < \infty$ and convexity imply that C_d has a non-empty interior. □

We now proceed to complete the proof of Theorem 1.1. As a preliminary step, which will be used as a fill-up technique, we prove the following.

Lemma 6.3. *Let s be such that $s < 1/(3dE_d(T(0, e_1)))$. Then for every $a > 0$, P_d-a.s. eventually in t, we have*

$$(S_d^0(t))_{ast} \subset S_d^0(t + at),$$

where for $A \subset \mathbb{Z}^d$ and $b > 0$, we define $A_b := \{x \in \mathbb{Z}^d : \inf_{y \in A} |x - y| \leq b\}$, the b-neighborhood of A in the Euclidean norm.

Proof. We will essentially use three elements: the subadditivity of $\{T(x, y) : x, y \in \mathbb{Z}^d\}$ (Lemma 2.1); the internal balls assured by inequality (6.1) with n large enough; and the external balls, giving an upper bound on the speed of growth (Lemma 3.2). We first need to consider the event that the boundary of the set $S_d^0(t)$ of sites visited at time t is contained in a ring whose size is proportional to t. So let $r > 0$ be such that $r < 1/(dE_d(T(0,e_1)))$ and

$$E_t := \{B(0,rt) \subset S_d^0(t) \subset B(0,2t)\}. \tag{6.9}$$

By Lemma 3.2 and (6.1) we know that for every n there is a constant $c_2(n,d)$ such that

$$P_d(E_t^c) \le \frac{c_2}{t^n} + \frac{(2t)^d}{e^{tI(2)}}.$$

Define $R_t := \{x \in \mathbb{Z}^d : x \in B(0, 2t + ast),\ x \notin B(0,rt)\}$. Let $a' > 0$. Now cover R_t by a finite number N (independent of t) of balls of radius $a't$. We can suppose that these balls are $\{B(tx_i, a't) : x_i \in R_1$ for $1 \le i \le N\}$. For every ball from this collection which intersects $S_0^d(t)$, we choose one point $y_i \in S_0^d(t) \cap B(tx_i, a't)$. Define $F_t := E_t \cap \bigcup_{i=1}^N \{B(ty_i, 3a't) \subset S_0^d(3a't/r)\}$. Then, by (6.1), (6.9) and translation invariance, for every n there is a constant $c_2(n,d)$ such that

$$P_d(F_t^c) \le \frac{c_2}{t^n} + N\frac{c_2 r^n}{(3a't)^n} + \frac{(2t)^d}{e^{tI(2)}}. \tag{6.10}$$

On the other hand, if $y \in S_0^d(t)_{a't}$, then there is a y_i such that $|y - y_i| \le 3a'$. Therefore, since $y_i \in S_0^d(t)$, whenever F_t occurs we have $T(0,y) \le T(0,y_i) + T(y_i,y) \le t + 3a't/r$. In other words, $S_0^d(t)_{a't} \subset S_0^d(t + 3a't/r) \subset F_t$. If we choose $a' = as$, an application of Borel-Cantelli together with the estimate (6.10) proves the lemma. □

Now, the following lemma together with Corollary 6.1 finishes the proof of Theorem 1.1.

Lemma 6.4. *For every $\epsilon > 0$, P_d-a.s. there is a $t_0 > 0$ such that*

$$[C_d t(1-\epsilon)] \subset S_d^0(t) \subset [C_d t(1+\epsilon)] \quad \text{whenever } t \ge t_0.$$

Proof. For $r = (r_1, \ldots, r_d) \in \mathbb{R}^d$, we define $[r] := ([r_1], \ldots, [r_d]) \in \mathbb{Z}^d$. Let $\epsilon > 0$. First we will show that P_d-a.s. eventually in t we have

$$[C_d t(1-\epsilon)] \subset S_d(t). \tag{6.11}$$

We remark that for any $s > 0$, if $q \in \mathbb{Q}^d - \{0\}$ and $z_q \in \mathbb{Z}^d$ is given by Definition 6.1, then $[sz_q/\mu_d(z_q)] = [sq/\mu_d(q)]$. Also, $(z_q)_i[s/\mu_d(z_q)] \le [(z_q)_i s/\mu_d(z_q)] \le (z_q)_i([s/\mu_d(z_q)] + 1)$ for $1 \le i \le d$. Finally, if $y \in \mathbb{Z}^d$ differs from z_q at most by one in each coordinate, then reformulating (6.6) we find that $\lim_{t\to\infty} T(0, y[t/\mu_d(y)])/t = 1$ P_d-a.s. Combining these three facts we conclude that for every $q \in \mathbb{Q}^d$ and $\epsilon' > 0$, P_d-a.s. eventually in t,

$$T(0, [qt(1-2\epsilon')/\mu_d(q)]) \le t(1-\epsilon'). \tag{6.12}$$

Thus, for every $q \in \mathbb{Q}^d$ and $\epsilon' > 0$, P_d-a.s. eventually in t, the point $[qt(1-2\epsilon')]$ belongs to $S_d^0(t(1-\epsilon'))$. Since $[x]_a = B(x,a)$ for $x \in \mathbb{Z}^d$ and $a > 0$, choosing $\epsilon' = \epsilon/2$, by Lemma 6.3 we conclude that there is an $r > 0$ such that for every $q \in \mathbb{Z}^d$, P_d-a.s. eventually in t,

$$B([qt(1-\epsilon)/\mu_d(q)], rt\epsilon/2) \subset S_d^0(t). \tag{6.13}$$

Now, choose a number $M = M(\epsilon)$, depending on ϵ, of points $q_1, \ldots, q_M \in \mathbb{Q}^d$ such that given any point $p \in \mathbb{Q}^d$, we have

$$\inf_{1\le i\le M}\left|\frac{p}{\mu_d(p)} - \frac{q_i}{\mu_d(q_i)}\right| < \frac{r\epsilon}{3}. \tag{6.14}$$

In other words, $r\epsilon$-neighborhoods of the points q_i, normalized to be on the boundary of C_d, cover the boundary of C_d. By (6.13) we conclude that P_d-a.s. eventually in t one has $\bigcup_{1\le i\le M} B([q_i t(1-\epsilon)/\mu_d(q_i)], rt\epsilon/2) \subset S_d^0(t)$, where we have used the fact that the set of points $q_1, \ldots, q_M$ is finite. Now, again by Lemma 6.3 (or also directly from (6.1)) we conclude that if $v = \max_{1\le i\le M} |q_i|/\mu_d(q_i)$, then P_d-a.s. there is a t_0 such that

$$B(0, t_0 v) \cup \bigcup_{t_0\le s\le t}\ \bigcup_{1\le i\le M} B([q_i s(1-\epsilon)/\mu_d(q_i)], rs\epsilon/2) \subset S_d^0(t) \tag{6.15}$$

whenever $t \ge t_0$. But by the definition of v, and by (6.14),

$$[C_d t(1-\epsilon)] \subset B(0, t_0 v) \cup \bigcup_{t_0\le s\le t}\ \bigcup_{1\le i\le M} B([q_i t(1-\epsilon)/\mu_d(q_i)], rt\epsilon/2).$$

Hence, by (6.15) we obtain the lower bound (6.11).

To finish the proof one has to show that P_d-a.s. eventually in t,

$$S_d(t) \subset [C_d t(1+\epsilon)]. \tag{6.16}$$

First, let us remark that as in (6.12), one can show that for every $q \in \mathbb{Q}^d$ and $\epsilon' > 0$, P_d-a.s. eventually in t,

$$t(1+\epsilon') \le T(0, [qt(1+2\epsilon')/\mu_d(q)]). \tag{6.17}$$

Let us now choose $M(\epsilon)$ and $q_1, \ldots, q_M$ as in (6.14), and define the rays $R_{q_i}(t) := \{x \notin C_d t(1+\epsilon) : x = vq_i \text{ for some } v > 0\}$. Then, choosing $\epsilon' = \epsilon/2$ in (6.17), we see that P_d-a.s. eventually in t, $(\bigcup_{1\le i\le M}[R_{q_i}(t(1+\epsilon))]) \cap S_d^0(t(1+\epsilon/2)) = \emptyset$. Now, by Lemma 6.3 there is an $r > 0$ such that P_d-a.s. eventually in t one has $(S_d^0(t))_{r\epsilon t/2} \subset S_d^0(t(1+\epsilon/2))$. Therefore P_d-a.s. eventually in t,

$$(S_d^0(t))_{r\epsilon t/2} \cap \bigcup_{1\le i\le M} [R_{q_i}(t(1+\epsilon))] = \emptyset.$$

By the above and (6.14), there can be no point in $S_d^0(t)$ at a distance smaller than $r\epsilon t/2$ to the rays R_{q_i}, $1 \le i \le M$. It follows that P_d-a.s. eventually in t,

$$S_d^0(t) \subset \mathbb{Z}^d - \left(\bigcup_{1\le i\le M} [R_{q_i}(t(1+\epsilon))]\right)_{r\epsilon t/2}.$$

But again by (6.14), the right hand side above is contained in $[C_d t(1+\epsilon)]$. This proves (6.16). □

7. Non-isotropy under the Euclidean norm

We proceed to prove that for high enough dimensions d, the limiting shape C_d is not a ball under the Euclidean norm. It is enough to show that to leading order in time, for high enough dimensions the growth of $S_d(t)$ is faster in the axial direction than in the diagonal one. We state this more precisely in terms of the time constants in the following theorem.

Theorem 7.1. *Consider a combustion process $\mathbf{Z}^0$ starting from the origin in the d-dimensional lattice, and the corresponding time constants $\{\mu_d(z) : z \in \mathbb{Z}^d\}$ defined in* (6.5). *Let $z_1 = (1, \dots, 1) \in \mathbb{Z}^d$ and $e_1 = (1, 0, \dots, 0) \in \mathbb{Z}^d$.*

(i) *For every $\epsilon > 0$ there is a constant $C(\epsilon) > 0$ such that*

$$\mu_d(e_1) \le C d^{1/3+\epsilon} \quad \textit{whenever } d \ge C(\epsilon).$$

(ii) *For $d \ge 1$, we have*

$$\frac{1}{|z_1|}\mu_d(z_1) \ge d^{1/2}.$$

To prove (i) we will need to introduce some quantities and prove a couple of lemmas. First, it will be necessary to define the so called *point-to-plane passage times*. Given $n \in \mathbb{N}$, we define

$$U(0, n) := \inf_{z\in\mathbb{Z}^d} \{T(0, z) : \pi_1 z = m\},$$

where for $1 \le i \le d$, $\pi_i z$ denotes the i-th coordinate of z. $U(0, n)$ represents the first time the hyperplane $\pi_1 z = n$ is visited. The following property of the point-to-plane passage times will be useful.

Lemma 7.1.

$$\mu(e_1) = \inf_{n\ge1} \frac{E_d(U(0, n))}{n} = \lim_{n\to\infty} \frac{E_d(U(0, n))}{n}.$$

Proof of Lemma 7.1. Note that for any natural n we have $U(0, n) \le T(0, z)$ if $\pi_1 z = n$. It follows from the integrability of the travel times that $U(0, n)$ is integrable. Now, it is easy to verify that the family of functions $f(n) := E_d(U(0, n))$, indexed by natural n, is subadditive. The existence of $\lim_{n\to\infty} E_d(U(0, n))/n$ and the second equality are consequences of this property. The first equality follows easily from the definition of the time constant $\mu(e_1)$ together with the convexity of the limiting set C_d. □

Denote by $H_n = \{x \in \mathbb{Z}^d : \pi_1 x = n\}$ the hyperplane orthogonal to the first coordinate axis and passing through $z = (n, 0, \dots, 0)$. Our second lemma is a key step in the proof of Theorem 7.1(i). It gives an estimate on the number of visited sites in the hyperplane H_0 for times which are short compared with the dimension.

Lemma 7.2. *For every $1/5 < \beta < 2/5$ and $0 < \epsilon < \beta - 1/5$ there is a constant $C(\epsilon)$ such that*

$$P_d(|H_0 \cap S_d^0(d^\beta)| \ge d^{2\beta-\epsilon}) \ge 1 - 10^{12}\frac{d^\beta}{d} - \frac{d^{4\beta-2\epsilon}}{d^2} \quad \textit{whenever } d \ge C(\epsilon).$$

Proof of Lemma 7.2. The proof consists basically of two steps. First, we will show that with a high enough probability which decreases with d, any single random walk in dimension d large enough visits t sites by time t. Hence, by some time d^α with $\alpha < \beta$, there are d^α random walks. In the second step we will show that if we wait an additional time of order d^β, then each one of these random walks will in turn visit d^β new sites, producing a total of $d^{\beta+\alpha}$ random walks by time $d^\alpha + d^\beta$. A key feature of this proof will be the fact that the dimension d is much larger than the order d^β of the times involved.

First recall that the family $\mathbf{Z}$ of combustion growth processes can be constructed from the family of underlying free random walks $\mathbf{X} := \{X^x : x \in \mathbb{Z}^d\}$ (see Section 2). We denote by $C_x(t)$ the set of visited sites of X^x up to time t. Our first step will be to obtain a uniform estimate on the cardinality of $C_0(t)$ intersected with a collection of sets of high enough cardinality. More precisely, we will prove that

$$\inf_{A\subset\mathbb{Z}^d : |A|\le d^{4/5}} P_d\left(|C_0(t) - A| \ge \frac{t}{100}\right) \ge 1 - \exp\left\{-\frac{t}{100}\right\} \tag{7.1}$$

whenever $d \ge 4$ and $t \le d/2$. So for the moment fix $A \subset \mathbb{Z}^d$ such that $|A| \le d^{4/5}$. Set

$$\mathcal{A} := \{i : 1 \le i \le d \text{ and } |\pi_i z| = 1 \text{ for some } z \in A\}.$$

Note that since $|A| \le d^{4/5}$, also $|\mathcal{A}| \le d^{4/5}$. Now, we can estimate the cardinality of $C_0(t) - A$ in terms of the number of coordinates of the random walk X^0 that have changed up to time t and do not belong to $\mathcal{A}$. In fact,

$$|C_0(t) - A| \ge \sum_{i\in\mathcal{A}^c} (1 - \theta_i(t)),$$

where $\mathcal{A}^c$ denotes the complement of $\mathcal{A}$ in $\{1, \ldots, d\}$ and $\theta_i(t)$ denotes the random variable which has the value 0 if X^0 has performed a jump in the i-th coordinate, and 1 otherwise. Therefore,

$$P_d\left(|C_0(t) - A| \ge \frac{t}{100}\right) \ge P_d\left(\sum_{i\in\mathcal{A}^c} \theta_i(t) \le d(\mathcal{A}^c) - \frac{t}{100}\right), \tag{7.2}$$

where $d(\mathcal{A}^c) := |\mathcal{A}^c|$. Now, by the Chebyshev inequality for every $\lambda > 0$,

$$\begin{aligned} P_d\left(\sum_{i\in\mathcal{A}^c} \theta_i(t) \ge d(\mathcal{A}^c) - \frac{t}{100}\right) &\le E_d(e^{\lambda\sum_{i\in\mathcal{A}^c}\theta_i(t)})e^{-\lambda(d(\mathcal{A}^c)-t/100)} \\ &= ((1 - e^{-t/d}) + e^{-t/d}e^\lambda)^{d(\mathcal{A}^c)}e^{-\lambda(d(\mathcal{A}^c)-t/100)}, \end{aligned} \tag{7.3}$$

where we have used the independence of the random variables θ_i and the fact they have a Bernoulli distribution of parameter $e^{-t/d}$. Now, using the bounds $e^{-x} \le 1 - x + x^2$ and $1 - e^{-x} \le x$, valid for $x \ge 0$, we can conclude that $(1 - e^{-t/d}) + e^{-t/d}e^\lambda \le t/d + (1 - t/d + (t/d)^2)e^\lambda$. Hence, we can upper bound the left hand side of (7.3) by

$$\left(1-\left(1-\frac{t}{d}-e^{-\lambda}\right)\frac{t}{d}\right)^{d(\mathcal{A}^c)} e^{\lambda t/100} \le \exp\left\{-\left(1-\frac{t}{d}-e^{-\lambda}\right)\frac{d(\mathcal{A}^c)}{d}t+\lambda\frac{t}{100}\right\}. \tag{7.4}$$

Now, since $d \geq 4$, we have $1 - 1/d^{1/5} \geq 0.2$. On the other hand, since $t \leq d/2$ we have $1 - t/d - e^{-1} \geq 1/2 - 2/5 = 1/10$. Hence, using the fact that $d(\mathcal{A}^c) \geq d - d^{4/5}$ we conclude that

$$\frac{d(\mathcal{A}^c)}{d}\left(1 - \frac{t}{d} - e^{-1}\right) \geq \left(1 - \frac{1}{d^{1/5}}\right)\left(\frac{1}{2} - e^{-1}\right) \geq \frac{1}{50}.$$

Therefore, for $\lambda = 1$, the left hand side of (7.4) can be upper bounded by $\exp\{-t/100\}$. Substituting this back in (7.4) and (7.2), we obtain (7.1).

We are now ready for the second step of the proof. Let $0 < \alpha < \beta < 2/5$ and $d \geq 4$. By (7.1), we know that with a very high probability, at time d^α, the first random walk Z_0^0 of the combustion process, which starts from the origin, has visited $d^\alpha/100$ sites. If d is large enough, then $d^\beta \gg d^\alpha$ in the sense that $\lim_{d\to\infty} d^\alpha/d^\beta = 0$. We want to argue that each of the random walks Z_n^0, $n \geq 1$, created by the first one Z_0^0 between times 0 and d^β, will visit about $d^\beta/100$ different sites, making a total of $d^{\alpha+\beta}/10000$ new random walkers. First, let us linearly order the sites of $\mathbb{Z}^d$. Then each subset of $\mathbb{Z}^d$ inherits this order and we can write $C_0(d^\alpha) = \{x_1, \ldots, x_n\}$, where $n := |C_0(d^\alpha)|$ and $x_1 = 0 < x_2 < \cdots < x_n$. Now, given any subset $A \subset \mathbb{Z}^d$ and $m \in \mathbb{N}$ we denote by $(A)_m$ the first m sites of A according to this order if $m \leq |A|$, while $(A)_m = A$ if $m \geq |A|$. We then make the following recursive definition. First, let $A_1 := C_0(d^\alpha)$ and

$$A_2 := C_{x_n}(d^\beta) - (A_1)_{d^\beta}.$$

Note that A_2 is a subset of the set of sites visited by the random walk X^{x_n} up to time d^β and which do not belong to $(A_1)_{d^\beta}$. We then define recursively, for $2 \leq k \leq n$,

$$A_k := C_{x_{n+2-k}}(d^\beta) - \bigcup_{i=1}^{k-1}(A_i)_{d^\beta}.$$

Now let E_1 be the event that $|C_0(d^\alpha)| \geq d^\alpha/100$, and for $2 \leq i \leq n$, E_i the event that $|A_i| \geq d^\beta/100$. Furthermore, define F_1 as the event that the random walk X^0 has not exited the hyperplane H_0 before time d^α, and F_2 as the event that at least $n \wedge d^\alpha - 2$ random walks from the set of $n \wedge d^\alpha - 1$ random walks $\{X^{x_2}, X^{x_3}, \ldots, X^{x_{n\wedge d^\alpha}}\}$ have not exited the hyperplane H_0 before time d^β. Note that

$$\bigcup_{i=1}^{n\wedge d^\alpha}(A_i)_{d^\beta} \subset S_d^0(d^\alpha + d^\beta).$$

Since by definition, the collection $\{(A_i)_{d^\beta} : i \geq 1\}$ of sets is disjoint, and $n \wedge d^\alpha - 2 \geq d^\alpha/100 - 2$, we have the lower bound

$$P_d\left(|H_0 \cap S_d^0(d^\alpha + d^\beta)| \geq \left(\frac{d^\alpha}{100} - 2\right)\frac{d^\beta}{100}\right) \geq P_d\left(\{F_1 \cap E_1\} \cap \bigcap_{i=2}^{n\wedge d^\alpha} E_i \cap F_2\right). \quad (7.5)$$

Using the estimate $P_d(A \cap B) \geq P_d(A) + P_d(B) - 1$ for any events A, B, we can lower bound the right hand side of the above expression by

$$P_d\left(F_1 \cap E_1 \cap F_2\right) + P_d\Big(\bigcap_{i=1}^{n \wedge d^\alpha} E_i\Big) - 1. \tag{7.6}$$

Now, note that $P_d(F_1 \cap E_1 \cap F_2) = P_d(F_2 \mid F_1 \cap E_1) P_d(F_1 \cap E_1)$. But the probability of F_1 is the probability that a one-dimensional random walk of total jump rate $1/d$ does not exit the origin by time d^α. Thus, $P_d(F_1) = \exp\{-d^\alpha/d\}$, and $P_d(F_1 \cap E_1) \geq \exp\{-d^\alpha/d\} + P_d(E_1) - 1$. Similarly, if we define $\mathcal{X}$ as the σ-algebra generated by the random walk X^0, and $p := n \wedge d^\alpha - 1$, we have

$$\begin{aligned} P_d(F_2 \mid F_1 \cap E_1) &= E_d(E_d(F_2 \mid \mathcal{X}) \mid F_1 \cap E_1) \\ &= E_d(E_d((e^{-d^\beta/d})^p + p(e^{-d^\beta/d})^{p-1}(1 - e^{-d^\beta/d}) \mid F_1 \cap E_1)). \end{aligned}$$

But since $1 - x \leq e^{-x} \leq 1 - x + x^2/2$ for $0 \leq x \leq 1$, the argument of the expectation above is lower bounded by

$$1 - p(p-1)\frac{d^{2\beta}}{d^2} + p(p-1)\frac{d^{3\beta}}{d^3} - p\frac{d^{2\beta}}{d^2} \geq 1 - p^2\frac{d^{2\beta}}{d^2}$$

whenever $p \geq 1$. But on F_1 we have $d^\alpha/100 - 1 \leq p \leq d^\alpha$. Thus, if $d \geq 200^{1/\alpha}$, we obtain the lower bound

$$P_d(F_2 \mid F_1 \cap E_1) \geq 1 - \frac{d^{2\alpha+2\beta}}{d^2}.$$

Using again the estimate $e^{-x} \geq 1 - x$ for $0 \leq x \leq 1$, and inequality (7.1), we see that

$$P_d(F_1 \cap E_1 \cap F_2) \geq \left(1 - \frac{d^{2\alpha+2\beta}}{d^2}\right)\left(1 - e^{-\frac{d^\alpha}{100}} - \frac{d^\alpha}{d}\right) \geq 1 - (1 + 400^4)\frac{d^\beta}{d} - \frac{d^{2\alpha+2\beta}}{d^2}$$

whenever $d \geq \max\{200^{1/\alpha}, (100(1-\alpha)/\alpha)^{1/\alpha}\}$, where we used the fact that $e^{-d^\alpha/100} \leq 400^4 d^\alpha/d$ for $d \geq (100(1-\alpha)/\alpha)^{1/\alpha}$ and that $(1-\alpha)/\alpha \leq 4$ for $1/5 \leq \alpha \leq 2/5$.

To estimate the second term $P_d(\bigcap_{i=1}^{n \wedge d^\beta} E_i)$ of (7.6), since for any $m \leq d^\alpha$, we have $|\bigcup_{i=1}^m (A_i)_{d^\beta}| \leq d^{\alpha+\beta} \leq d^{4/5}$ (because $0 < \alpha < \beta < 2/5$), we can apply recursively, via conditional expectation, inequality (7.1) to conclude that

$$P_d\Big(\bigcap_{i=1}^{n \wedge d^\beta} E_i\Big) \geq (1 - e^{-d^\alpha/100})(1 - e^{-d^\beta/100})^{d^\beta}. \tag{7.7}$$

Now, for any $y > 0$ we have $1 - 1/y \leq e^{-1/y} \leq 1 - 1/(1+y)$. Therefore,

$$(1 - e^{-d^\beta/100})^{d^\beta} \geq \exp\left\{-\frac{d^\beta}{e^{d^\beta/100} - 1}\right\} \geq 1 - \frac{d^\beta}{e^{d^\beta/100} - 1}.$$

But since $e^x \geq x$ for $x \geq 0$, we have $e^{d^\delta/100} \geq d(\delta/100)^{1/\delta}$ for any $\delta > 0$. Therefore, $e^{d^\delta/100} - 1 \geq \frac{1}{2}d(\delta/100)^{1/\delta}$ whenever $d \geq 2(100/\delta)^{1/\delta}$. Hence, from (7.7) and the supposition that $\beta \geq \alpha \geq 1/5$, we see that

$$P_d\Big(\bigcap_{i=1}^{n\wedge d^\beta} E_i\Big) \geq \left(1 - \frac{1}{d}\left(\frac{100}{\alpha}\right)^{1/\alpha}\right)\left(1 - \frac{d^\beta}{d}\left(\frac{100}{\beta}\right)^{1/\beta}\right) \geq 1 - 3\frac{d^\beta}{d}500^5$$

whenever $d \geq 2 \times 500^5$. Putting together (7.5)–(7.7) we obtain

$$P_d\left(|H_0 \cap S_d^0(d^\alpha + d^\beta)| \geq \left(\frac{d^\alpha}{100} - 2\right)\frac{d^\beta}{100}\right) \geq 1 - 5 \times 500^5\frac{d^\beta}{d} - \frac{d^{2\alpha+2\beta}}{d^2},$$

whenever $d \geq 10^{11}$, where we used the fact that $2 \times 500^5 \leq 10^{11}$. Finally, choosing $\alpha = \beta - \epsilon$ we obtain the desired conclusion. □

Proof of Theorem 7.1. Proof of part (i). We will make use of Lemma 7.1, so that good enough lower bounds on the tail probabilities $P_d(U(0,1) > t)$ of the point-to-plane passage times will prove the theorem. First fix $2/5 > \gamma > \beta > 1/5$ and $0 < \epsilon < \beta - 1/5$. Note that

$$\begin{aligned} &P_d(U(0,1) \leq d^\gamma) \\ &\quad = P_d(U(0,1) \leq d^\gamma \,|\, |H_0 \cap S_d^0(d^\beta)| \geq d^{2\beta-\epsilon})P_d(|H_0 \cap S_d^0(d^\beta)| \geq d^{2\beta-\epsilon}). \qquad (7.8)\end{aligned}$$

The probability $P_d(U(0,1) > d^\gamma \,|\, |H_0 \cap S_d^0(d^\beta)| \geq d^{2\beta-\epsilon})$ can be bounded above by the probability that $d^{2\beta-\epsilon}$ independent random walks in the hyperplane H_0 at time d^β do not hit the hyperplane H_1 by time d^γ. If $t_d := d^\gamma - d^\beta$, this bound is given by $P(\tau > t_d)^{d^{2\beta-\epsilon}}$, where P is the law of a one-dimensional simple symmetric random walk x_t of total jump rate $1/d$ starting from the origin 0, and τ is the first hitting time to 1. Now, $P(\tau > t_d) \leq P(x_{t_d} \leq 0)$. On the other hand, $P(x_{t_d} \leq 0) = Q(N_{t_d} - M_{t_d} \leq 0) = \sum_{m=0}^\infty Q(N_{t_d} \leq m)Q(M_{t_d} = m)$, where N_s and M_s, $s \geq 0$, are independent Poisson processes of rate $1/(2d)$, and Q is their joint law. We therefore have

$$\begin{aligned} P(\tau > t_d) &\leq e^{-t_d/d} + t_d/(2d)e^{-t_d/(2d)}\sum_{m=0}^\infty Q(N_{t_d} \leq m+1)\frac{1}{(m+1)!}\left(\frac{t_d}{2d}\right)^m \\ &\leq e^{-t_d/d} + \frac{t_d}{2d} \leq 1 - \left(\frac{t_d}{t_d+d} - \frac{t_d}{2d}\right), \end{aligned}$$

where in the last inequality we used the fact that $e^{-1/y} \leq 1 - 1/(1+y)$ for $y > 0$. But for $d \geq 2^{1/(1-\gamma)}$, we have $t_d \leq d^\gamma \leq d/2$ and hence $t_d/(t_d+d) \geq \frac{2}{3}t_d/d$, so that

$$\frac{t_d}{t_d+d} - \frac{t_d}{2d} \geq \frac{1}{6}\frac{t_d}{d}.$$

It follows that whenever $d \geq 2^{1/(1-\gamma)}$ then

$$P(\tau > t_d) \leq 1 - \frac{1}{6}\frac{t_d}{d}.$$

Thus, whenever $d \geq 2^{1/(1-\gamma)}$ we have

$$P_d(U(0,1) > d^\gamma \mid |H_0 \cap S_d^0(d^\beta)| \leq d^{2\beta-\epsilon}) \leq \left(1 - \frac{1}{6}\frac{t_d}{d}\right)^{d^{2\beta-\epsilon}}.$$

And since $t_d \geq d^\gamma/2$ when $d \geq 2^{1/(\gamma-\beta)}$, we see that if $d \geq \max\{2^{1/(\gamma-\beta)}, 2^{1/(1-\gamma)}\}$, then

$$P_d(U(0,1) > d^\gamma \mid |H_0 \cap S_d^0(d^\beta)| \leq d^{2\beta-\epsilon}) \leq \left(1 - \frac{1}{12}\frac{1}{d^{1-\gamma}}\right)^{d^{2\beta-\epsilon}}.$$

Plugging this estimate into inequality (7.8) and using Lemma 7.2 with $0 < \beta < 2/5$ to bound the second factor of the right hand side of (7.8), we get

$$\begin{aligned} P_d(U(0,1) \leq d^\gamma) &\geq \left(1 - \left(1 - \frac{1}{12}\frac{1}{d^{1-\gamma}}\right)^{d^{2\beta-\epsilon}}\right) \cdot \left(1 - 10^{12}\frac{d^\beta}{d} - \frac{d^{4\beta-2\epsilon}}{d^2}\right) \\ &\geq \left(1 - \exp\left\{-\frac{1}{12}\frac{d^{2\beta-\epsilon}}{d^{1-\gamma}}\right\}\right) \cdot \left(1 - 10^{12}\frac{d^\beta}{d} - \frac{d^{4\beta-2\epsilon}}{d^2}\right) \end{aligned}$$

whenever $d \geq C'(\epsilon, \beta, \gamma)$, where $C'(\epsilon, \beta, \gamma)$ is some constant depending only on ϵ, β and γ. Choosing $\beta = 1/3 + \epsilon$, $\gamma = 1/3 + 2\epsilon$, and $\epsilon > 0$ small enough, we see that the exponential in the above inequality decreases to zero like $\exp\{-2ed^{3\epsilon}\}$. Hence, for $d \geq C''(\epsilon)$, where $C''(\epsilon)$ is a constant depending only on ϵ, we have

$$P_d(U(0,1) > d^{1/3+2\epsilon}) \leq 2\frac{d^{1/3+2\epsilon}}{d}. \tag{7.9}$$

On the other hand, it is not difficult to see, using a similar argument and Theorem A.1(ii), that there is a constant $C > 0$ such that

$$P_d(U(0,1) \geq t) \leq \left(C\frac{d^{1/2}}{t^{1/2}}\right)^{d^{\beta-\epsilon}} \tag{7.10}$$

whenever $t \geq d^{1+\epsilon}$. From estimates (7.9) and (7.10) we easily conclude that there is a constant $C'''(\epsilon)$ such that

$$\begin{aligned} E_d(U(0,1)) &= E_d(U(0,1), U(0,1) \leq d^{1/3+2\epsilon}) \\ &\quad + E_d(U(0,1), d^{1/3+2\epsilon} < U(0,1) \leq d^{1+\epsilon}) \\ &\quad + E_d(U(0,1), U(0,1) \geq d^{1+\epsilon}) \\ &\leq d^{1/3+2\epsilon} + 2d^{1/3+3\epsilon} + o(d^{1/3}), \end{aligned}$$

whenever $d \geq C'''(\epsilon)$. Now Lemma 7.1 completes the proof of part (i).

Proof of part (ii). Let $\epsilon > 0$ and $n \in \mathbb{N}$. Note that $P_d(T(0, [(1+2\epsilon)n] \cdot z_1) \leq nd) \leq 1 - P_d(S_d^0(nd) \subset B_1(0, (1+\epsilon)nd))$. Hence, by (3.5),

$$P_d(T(0, [(1+2\epsilon)n] \cdot z_1) \leq nd) \leq \frac{((1+\epsilon)nd)^d}{e^{ndI(\epsilon)}}.$$

We then have

$$E_d(T(0,[(1+2\epsilon)n]\cdot z_1)) \geq nd\cdot(1-P_d(T(0,(1+\epsilon)n\cdot z_1)\leq nd))$$
$$\geq nd\cdot\left(1-\frac{((1+\epsilon)nd)^d}{e^{ndI(\epsilon)}}\right).$$

Dividing this inequality by n, and letting $n\to\infty$ and $\epsilon\to 0$, we conclude that $\mu_d(z_1)\geq d$ for $d\geq 1$. □

8. Proof of the density theorem

Let us define the empirical measure of particles at time t, associated to the combustion growth process starting from site 0, as $\eta(t) := \{\eta_x(t) : x\in\mathbb{Z}^d\}$, where

$$\eta_x(t) := \sum_{n=0}^{\infty} \mathbf{1}_x(Z_n^0(t))$$

represents the total number of particles at site x at time t and $\mathbf{1}_x(y)$ is the indicator function of the site $x\in\mathbb{Z}^d$. The empirical measure is a measure on the space $\mathcal{M}=\mathbb{N}^{\mathbb{Z}^d}$, endowed with its Borel σ-algebra $\mathcal{C}$. Let us recall the following notation introduced in Remark 1.2: given a measure α defined on $(\mathcal{M},\mathcal{C})$ and some subset $\Lambda\subset\mathbb{Z}^d$, we denote by α_Λ the restriction of α to $\mathcal{M}_\Lambda := \mathbb{N}^\Lambda$ endowed with its Borel σ-algebra. In this section we will prove Theorem 1.2. Namely,

Theorem 1.2. *Let ν be the product Poisson measure of parameter* 1 *on $(\mathcal{M},\mathcal{C})$. Then*

$$\lim_{t\to\infty}\mu(t)=\nu,$$

where the convergence is in the sense of the weak topology on $\mathcal{M}$.

The first step in the proof is a comparison result between the combustion growth process and a periodic combustion growth process defined below. For a natural N, consider the group $\mathbb{Z}_{2N+1}$ of integers modulo $2N+1$. We denote by $\mathbb{Z}^d_{2N+1}$ the direct product of d copies of this group. Consider the homomorphism $h:\mathbb{Z}^d\to\mathbb{Z}^d_{2N+1}$ which maps $z=(z_1,\dots,z_d)\in\mathbb{Z}^d$ to $h(z) := (\langle z_1\rangle,\dots,\langle z_d\rangle)\in\mathbb{Z}^d_{2N+1}$, where for $1\leq i\leq d$, $\langle z_i\rangle$ is the equivalence class of z_i modulo $2N+1$. Note that under h the set $\Lambda_N := [-N,N]^d\subset\mathbb{Z}^d$ can be identified with $\mathbb{Z}^d_{2N+1}$ and both are isomorphic if a proper addition operation is defined on Λ_N. We recall the notation $\{e_i : 1\leq i\leq d\}$ for the canonical basis of $\mathbb{Z}^d$. We will similarly denote by $\{\bar{e}_i : 1\leq i\leq d\}$ the canonical generators of the group $\mathbb{Z}^d_{2N+1}$.

In analogy with the construction of Section 2, it is possible to define a periodic combustion growth process on the d-dimensional group $\mathbb{Z}^d_{2N+1}$, starting from the origin. Thus, initially there is a simple symmetric continuous time total jump rate one random walk at $0\in\mathbb{Z}^d_{2N+1}$, and thereafter, each time a random walk visits an unvisited site, it branches. As in Section 2, this can be formalized by defining first a family $X_N := \{X_N^x : x\in\mathbb{Z}^d\}$

of independent random walks X_N^x on $\mathbb{Z}_{2N+1}^d$, each being simple, symmetric of total jump rate one and such that X_N^x starts from site $x \in \mathbb{Z}_{2N+1}^d$. We let $R_{d,N}$ be the corresponding probability measure defined on the Skorokhod space $\Omega_d^N := \mathbf{D}([0,\infty); \mathbb{Z}_{2N+1}^d)$, endowed with its Borel σ-algebra $\mathcal{B}_d^N$. We denote by $E_{d,N}$ the expectation associated to $R_{d,N}$. Then we define the successive random walks $Z_{n,N}$, $n \geq 1$, so that the initial one $Z_{0,N} := X_N^0$ starts from the origin, and the next ones have the dynamics given by the random walk from the family X_N starting from the same site. Furthermore, following (2.1), we let $\tau_{n,N}$, $n \geq 1$, denote the successive creation times of the random walks $Z_{n,N}$; as in (2.2) we let $\kappa_{n,N}$, $n \geq 1$, denote the index of the random walk within the set $\{Z_{1,N}, \dots, Z_{n-1,N}\}$, which created $Z_{n,N}$; and as in (2.4) we let $S_{d,N}^0(t)$ denote the set of sites of $\mathbb{Z}_{2N+1}^d$ visited by time t. Finally, we call the set $\mathbf{Z}_N := \{Z_{n,N} : n \in \mathbb{N}\}$ of random walks the *periodic combustion growth process at scale N* starting from the origin. Define the *occupation field* at time t for this process as $\eta^N(t) := \{\eta_x^N(t) : x \in \mathbb{Z}_{2N+1}^d\}$, where

$$\eta_x^N(t) := \sum_{n=0}^{(2N+1)^d} \mathbf{1}_x(Z_{n,N}(t)).$$

The above sum stops at $(2N+1)^d$ because no more than $(2N+1)^d$ random walks can be created. Due to the identification of $\mathbb{Z}_{2N+1}^d$ with Λ_N, given any local function f on $\mathbb{N}^{\mathbb{Z}^d}$ with support $A \subset \mathbb{Z}^d$, for N such that $A \subset \Lambda_N$ the quantity $f(\eta^N)$ is well defined. The next lemma tells us that at a certain scale, the occupation field of the combustion growth process is close to that of the periodic combustion growth process.

Lemma 8.1. *Let f be some local function on $\mathcal{M} = \mathbb{N}^{\mathbb{Z}^d}$. Let $N : [0,\infty) \to [0,\infty)$ be an increasing function. If there is an $\epsilon > 0$ such that $N(t) \gg t^{1/2+\epsilon}$, then*

$$\lim_{t\to\infty} |E_d(f(\eta(t)) - E_{d,N(t)}(f(\eta^{N(t)}(t))| = 0.$$

Proof. To simplify notation we will shorten $N(t)$ to N. Note that for each $x \in \mathbb{Z}_{2N+1}^d$, the quantity $h(X^x) \in \mathbb{Z}_{2N+1}^d$, where h is the homomorphism defined above, has the same law as the random walk $X_N^{\bar{x}}$, where $\langle \bar{x} \rangle = x$. Now, let A be the support of f. Then clearly, $\lim_{t\to\infty} |E_d(f(\eta(t)) - E_{d,N(t)}(f(\eta^{N(t)}(t))|$ is smaller than $\sup_{x\in A} |f(x)|$, which is finite because A is finite (f being local), times the probability that some random walk X^x has traveled from some site $y \notin \Lambda_N$ to the support A of f in a time smaller than t. Since $N \gg t^{1/2+\epsilon}$ for some $\epsilon > 0$, from Theorem A.1(i) controlling the hitting probabilities of random walks, we see that this probability goes to 0. □

In the second step of our proof we will establish a relationship involving the multi-parametric Laplace transform of the occupation field of the periodic combustion growth process at scale N on a subset $\Lambda \subset \mathbb{Z}_{2N+1}^d$, which will enable us to decouple those particles which contribute in the computation of the Laplace transform (those born in a $\sqrt{t}$-neighborhood of Λ) from those which do not (born in a $\sqrt{t}$-neighborhood of Λ).

Lemma 8.2. *Let $N \in \mathbb{N}$ and $\Lambda \subset \mathbb{Z}^d_{2N+1}$. Consider a family $\lambda := \{\lambda_x : x \in \Lambda\}$ of parameters indexed by Λ. Then*

$$E_{d,N}\Big(\exp\Big\{ -\sum_{x\in\Lambda} \lambda_x \eta^N_x(t)\Big\}\Big) = \mathbf{T}_{(2N+1)^d-1} \circ \mathbf{T}_{(2N+1)^d-2} \circ \cdots \circ \mathbf{T}_1 \circ \mathbf{T}_0(\bar{1}). \quad (8.1)$$

Here, for each $0 \le k \le (2N+1)^d - 1$, the operator $\mathbf{T}_k : C^N_B \to C^N_B$, where C^N_B is the space of real-valued functions on $(\Omega^N_d)^{\mathbb{N}}$, is defined by

$$\mathbf{T}_k(g)(w) := E_{d,N}(g(\mathbf{Z}_N)\exp\{-\lambda^\Lambda_{Z_{(2N+1)^d-k,N}(t)}\} \mid Z_{1,N}, \ldots, Z_{(2N+1)^d-k-1,N})(w) \quad (8.2)$$

$w \in (\Omega_d)^{\mathbb{N}}$, with $\lambda^\Lambda_x := \lambda_x$ for $x \in \Lambda$ and 0 otherwise, and $\bar{1} \in (\Omega^N_d)^{\mathbb{N}}$ is the constant function equal to 1.

Proof. First observe that $\sum_{x\in\Lambda} \lambda_x \eta^N_x(t) = \sum_{n=1}^{(2N+1)^d} \lambda^\Lambda_{Z_{n,N}(t)}$. Therefore,

$$E_{d,N}\Big(\exp\Big\{ -\sum_{x\in\Lambda} \lambda_x \eta^N_x(t)\Big\}\Big) = E_{d,N}(E_{d,N}(e^{-\sum_{n=2}^{(2N+1)^d} \lambda^\Lambda_{Z_{n,N}(t)}} \mid Z_{1,N})e^{-\lambda^\Lambda_{Z_{1,N}(t)}}).$$

Iterating this conditioning $N^d - 1$ times, we obtain (8.1). □

At this point we need the following version of Theorem 1.1, in the context of the periodic combustion growth process. Here, for any subset $A \subset \mathbb{R}^d$, we define $[A]_N := A \cap [-N, N]^d \cap \mathbb{Z}^d$.

Proposition 8.1. *There is a closed convex bounded subset $C_d \subset \mathbb{R}^d$, symmetric under permutations of the coordinate axes and with non-empty interior, such that for every $\epsilon > 0$, and every function $f : [0,\infty) \to [0,\infty)$ with $\lim_{t\to\infty} f(t) = \infty$ and $f(t) \ll t$, one has*

$$\lim_{t\to\infty} R_{d,[t]}([C_d f(t)(1-\epsilon)]_{[t]} \subset S^0_{d,[t]}(f(t)) \subset [C_d f(t)(1+\epsilon)]_{[t]}) = 1. \quad (8.3)$$

Furthermore, for d large enough, C_d is not a ball under the Euclidean norm.

Remark 8.1. The set C_d is the same subset of Theorem 1.1.

Proof. Let $x \in \mathbb{Z}^d$. Note that whenever $x \in \Lambda_N$ and X^x is a random walk which has not exited Λ_N in the time interval $[0,t]$, then the random walk $h(X^x) = X^{h^{-1}(x)}$ in $\mathbb{Z}^d_{2N+1}$ is in the same position as X^x in the sense that given integers $\{n_i : 1 \le i \le d\}$ we have $X^x = \sum_{i=1}^d n_i e_i$ if and only if $X^{h^{-1}(x)} = \sum_{i=1}^d n_i \bar{e}_i$. It follows that whenever $f(t) \times \mathrm{diam}(C_d) < t$, where $\mathrm{diam}(C_d)$ is the diameter of C_d, then the event of the right hand side of (1.1) occurs if and only if the event $[C_d f(t)(1-\epsilon)] \subset S^0_d(f(t)) \subset [C_d f(t)(1+\epsilon)]$ occurs in the standard combustion growth process. But the probability P_d of such occurrence tends to one by Theorem 1.1. □

We continue with the proof of Theorem 1.2 via another lemma, which shows that in the recursion formula (8.1) of Lemma 8.2 we can eliminate those particles which are born very far away from the subset Λ (at a distance larger than $\sqrt{t}$). This time whenever we have a subset $\Lambda \subset \mathbb{Z}^d$ and a natural N such that $\Lambda \subset [-N, N]^d$ we will identify Λ with the corresponding subset of $\mathbb{Z}^d_{2N+1}$ and denote it by the same symbol. In analogy with the definition prior to Proposition 3.1, we define in $\mathbb{Z}^d_{2N+1}$ a ball centered at $x \in \mathbb{Z}^d_{2N+1}$ of radius r as $B(x,r)_N := \{y \in \mathbb{Z}^d_{2N+1} : |y - x| \leq r\}$, where $|\cdot|$ is the natural Euclidean metric in $\mathbb{Z}^d_{2N+1}$, defined by $|z| = \sqrt{\sum_{i=1}^d (\bar{e}_i)^2}$ for $z = \sum_{i=1}^d \bar{e}_i$. For the following lemmas, we need to introduce two important events. Let r, R and s be positive real numbers. We first define the event that at time s, the set of visited sites of the periodic combustion process at scale N contains a Euclidean ball of radius r,

$$I_{N,r,s} := \{B(0,r)_N \subset S^0_{d,N}(s)\}. \tag{8.4}$$

Secondly, we define the event that at time s, the set of visited sites of the periodic combustion process at scale N is contained in a Euclidean ball of radius R,

$$O_{N,R,s} := \{S^0_{d,N}(s) \subset B(0,R)_N\}. \tag{8.5}$$

Finally, in what follows we will denote by $R_d := \sup\{|x| : x \in C_d\}$ and $r_d := \inf\{|x| : x \notin C_d\}$ the outer and inner radius, respectively, of the convex set C_d.

Lemma 8.3. *Let $\Lambda \subset \mathbb{Z}^d$ be some fixed subset. Consider a family of parameters $\lambda := \{\lambda_x : x \in \Lambda\}$. Let $N := [t]$. Then*

$$\lim_{t\to\infty} |E_{d,N}(e^{-\sum_{x\in\Lambda}\lambda(x)\eta^N_x(t)}) - \mathbf{T}_{(2N+1)^d-1} \circ \mathbf{T}_{(2N+1)^d-2} \circ \cdots \circ \mathbf{T}_{N^d-M}(\bar{1})| = 0, \tag{8.6}$$

where $M := [ct^{2d/3}]$, c is an arbitrary constant and for $0 \leq k \leq (2N+1)^d - 1$, $\mathbf{T}_k$ and $\bar{1} \in (\Omega^N_d)^{\mathbb{N}}$ are defined in Lemma 8.2.

Proof. Fix k so that $M \leq k \leq (2N+1)^d$. If $p^N_t(x,y)$ is the probability that a simple symmetric random walk of total jump rate one on $\mathbb{Z}^d_{2N+1}$ starting from $x \in \mathbb{Z}^d_{2N+1}$ is at site $y \in \mathbb{Z}^d_{2N+1}$ at time t, we have

$$\begin{aligned} E_{d,N}(e^{-\lambda^\Lambda_{Z_{k,N}(t)}} \mid Z_{1,N}, \ldots, Z_{k-1,N}) &= \sum_{x\in\mathbb{Z}^d_{2N+1}} p^N_{(t-\tau_{k,N})_+}(Z_{k,N}(\tau_{k,N}), x) e^{-\lambda^\Lambda_x} \\ &= 1 + \sum_{x\in\Lambda} p^N_{(t-\tau_{k,N})_+}(Z_{k,N}(\tau_{k,N}), x)(e^{-\lambda^\Lambda_x} - 1). \end{aligned} \tag{8.7}$$

Now fix four positive reals s_1, s_2, r_1 and r_2. Note that if the event O_{N,r_1,s_1} occurs then every random walk of the periodic combustion growth process which starts from some site outside the ball $B(0,r_1)_N$ is born at a time larger than or equal to s_1. On the other hand, if $I_{N,r_1,s_2} \cap O_{N,r_2,s_2}$ occurs, then every random walk $Z_{k,N}$ such that $k \geq |B(0,r_2)_N|$ is born outside $B(0,r_1)_N$. Thus, if $O_{N,r_1,s_1} \cap I_{N,r_1,s_2} \cap O_{N,r_2,s_2}$ occurs, then every random walk $Z_{k,N}$ such that $k \geq |B(0, r_2)_N|$ is born at a distance larger than r_1 from the origin. In other words,

$$O_{N,r_1,s_1} \cap I_{N,r_1,s_2} \cap O_{N,r_2,s_2} \subset \bigcap_{k=[w_d r_2^d]}^{(2N+1)^d} \{|Z_{k,N}(\tau_{k,N})| \geq r_1\}.$$

Let us now choose r_2 so that $[w_d r_2^d] = M = [ct^{2d/3}]$ and $r_1 = r_2/2$. Also we let $s_1 = r_2/(4R_d)$. Thus, we choose $r_1 = c^{1/d} t^{2/3}/w_d^{1/d}$. With these choices of r_1, r_2 and s_1 and an appropriate choice of s_2, Proposition 8.1 yields $\lim_{t\to\infty} R_{d,N}(O_{N,r_1,s_1} \cap I_{N,r_1,s_2} \cap O_{N,r_2,s_2}) = 1$ and hence

$$\lim_{t\to\infty} R_{d,N}(A_M) = 1 \tag{8.8}$$

for $A_M := \bigcap_{k=M}^{(2N+1)^d} \{|Z_{k,N}(\tau_{k,N})| \geq a_1 t^{2/3}\}$ and $a_1 = c^{1/d}/w_d^{1/d}$.

Now, a simple estimate (for example a local central limit theorem for a periodic random walk like Theorem A.1(i)) shows that on A_M, for $x \in \Lambda$ and k such that $M \leq k \leq (2N+1)^d$, we have

$$p^N_{(t-\tau_k)_+}(Z_{k,N}(\tau_k), x) \leq K e^{-Kt^{1/4}}$$

for some constant K. Substituting this back into (8.7) we conclude that on A_M,

$$\begin{aligned} E_{d,N}(e^{-\lambda^{\Lambda}_{Z^N_k(t)}} \mid Z^N_1, \ldots, Z^N_{k-1}) &= \sum_{x\in T^d_N} p^N_{(t-\tau_k)_+}(Z^N_k(\tau_k), x) e^{-\lambda^{\Lambda}_x} \\ &= 1 + o_2(t), \end{aligned} \tag{8.9}$$

where $|o_2(t)| \leq K e^{-Kt^{1/4}}$. Thus, iterating (8.9) from $k = (2N+1)^d$ to $k = M$ and using Lemma 8.2 together with the fact that the probability of A_M tends to one (see (8.8)), we finish the proof of (8.6). □

The following lemma enables us to decouple the dynamics of those particles born before times $\sim t^{2/3}$. In order to have a reasonable filtration, we will also need to introduce a branching process analogous to the process $\mathbf{Y}^0$ defined in (2.5). Thus for each fixed natural N, we perform a construction analogous to $\mathbf{Y}^0$ and define a *periodic branching combustion process* $\mathbf{Y}_N := \{Y_{n,N} : n \in \mathbb{N}\}$ so that each $Y_{n,N}$ is a random walk on $\mathbb{Z}^d_{2N+1}$ starting from the origin. We now let $\mathcal{F}_N := \{\mathcal{F}_{t,N} : t \in [0,\infty)\}$, where $\mathcal{F}_{t,N} := \sigma(\mathbf{Y}_N(s) : s \in [0,t])$ is the σ-field generated by $\mathbf{Y}_N$ between times $s = 0$ and $s = t$. In what follows, given an event E of the Borel σ-algebra of Ω^N_d, we denote by $\mathbf{1}_E$ its indicator function.

Lemma 8.4. *Let $\Lambda \subset \mathbb{Z}^d$ be some fixed subset. Consider a family of parameters $\lambda := \{\lambda_x : x \in \Lambda\}$. Let r be a positive constant such that $r < \inf\{|x| : x \notin C_d\} = r_d$. Then*

$$\lim_{t\to\infty} \left| E_{d,[t]}(e^{-\sum_{x\in\Lambda} \lambda(x)\eta^{[t]}_x(t)}) - E_{d,[t]}\Big(\mathbf{1}_{A_t} \cdot \prod_{k=1}^{[w_d r^d t^{2d/3}]} E_{d,[t]}(e^{-\lambda^{\Lambda}_{Z_{k,[t]}(t)}} \mid \mathcal{F}_{t^{2/3},[t]})\Big)\right| = 0, \tag{8.10}$$

where $A_t := I_{[t],rt^{2/3},t^{2/3}}$*, and as defined in* (8.4)*,* $I_{[t],rt^{2/3},t^{2/3}}$ *is the event that the set of sites visited at time* $t^{2/3}$ *contains a Euclidean ball of radius* $rt^{2/3}$*.*

Proof. Note that Lemma 8.3 implies that

$$E_{d,[t]}\Big(\exp\Big\{-\sum_{x\in\Lambda}\lambda(x)\eta_x^{[t]}(t)\Big\}\Big) = E_{d,[t]}\Big(\exp\Big\{-\sum_{k=1}^{[w_d r^d t^{2d/3}]}\lambda^{\Lambda}_{Z_{k,[t]}(t)}\Big\}\Big) + o(t).$$

Now, by Proposition 8.1 we know that $R_{d,[t]}(A_t^c) = o(t)$. Hence,

$$\begin{aligned} &E_{d,[t]}\Big(\exp\Big\{-\sum_{x\in\Lambda}\lambda(x)\eta_x^{[t]}(t)\Big\}\Big) \\ &\qquad = E_{d,[t]}(\mathbf{1}_{A_t}\cdot E_{d,[t]}(e^{-\sum_{k=1}^{[w_d r^d t^{2d/3}]}\lambda^{\Lambda}_{Z_{k,[t]}(t)}} \mid \mathcal{F}_{t^{2/3},[t]})) + o(t). \qquad (8.11)\end{aligned}$$

Now, in the event A_t we have $\tau_{k,[t]} \le t^{2/3}$ for $k \le w_d r^d t^{2d/3}$. Thus, at time $t^{2/3}$ in the event A_t, all particles $Z_{k,[t]}$ with $k \le w_d r^d t^{2d/3}$ have been born, and hence their dynamics become independent of A_t when conditioned on the σ-field $\mathcal{F}_{t^{2/3},[t]}$. Hence, the conditioning on the right hand side of (8.11) decouples the sum in the exponential, so that

$$\begin{aligned} &E_{d,[t]}(\mathbf{1}_{A_t}\cdot E_{d,[t]}(e^{-\sum_{k=1}^{[w_d r^d t^{2d/3}]}\lambda_\Lambda(Z_{k,[t]}(t))} \mid \mathcal{F}_{t^{2/3},[t]})) \\ &\qquad = E_{d,[t]}\Big(\mathbf{1}_{A_t}\cdot\prod_{k=1}^{[w_d r^d t^{2d/3}]} E_{d,[t]}(e^{-\lambda_\Lambda(Z_{k,[t]}(t))} \mid \mathcal{F}_{t^{2/3},[t]})\Big). \qquad (8.12)\end{aligned}$$

This completes the proof of the lemma. □

In order to prove Theorem 1.2 we will need to define two events. For this purpose let r' be any positive constant chosen so that $4R_d r' < \inf\{|x| : x \notin C_d\}$. The first event is

$$\begin{aligned} B_t &:= I_{[t],r_d r' t^{2/3},2r' t^{2/3}} \cap O_{[t],4R_d r' t^{2/3},2r' t^{2/3}} \\ &= \{B(0, r_d r' t^{2/3})_{[t]} \subset S^0_{d,[t]}(2r' t^{2/3}) \subset B(0, 4R_d r' t^{2/3})_{[t]}\}. \qquad (8.13)\end{aligned}$$

Note that Proposition 8.1 implies that

$$\lim_{t\to\infty} R_{d,[t]}(B_t^c) = 0. \qquad (8.14)$$

The event B_t defined in (8.13), ensures that if r is chosen so that $4R_d r' < r < \inf\{|x| : x \notin C_d\}$, then the random walks $Z_{k,[t]}$ of the right hand side of (8.10) are born from a set of sites containing a Euclidean ball of radius $r_d r' t^{2/3}$. Furthermore, (8.14) ensures that this happens with a probability converging to one. Next, choosing r as above, we define the second event as

$$C_t := \bigcap_{k=1}^{[w_d r^d t^{2/3}]} \{|Z_{k,[t]}(t^{2/3}) - Z_{k,[t]}(\tau_{k,[t]})| \le t^{4/9}\}.$$

In other words, C_t is the event that at time $t^{2/3}$, none of the first $[w_d r^d t^{2/3}]$ born random walks has moved a distance larger than $t^{4/9}$ from its point of departure. This time note that the probability of C_t^c is upper bounded by the probability that there is some random walk from a set of $[w_d r^d t^{2/3}]$ independent simple symmetric random walks on $\mathbb{Z}^d$, which is at a distance larger than $t^{4/9}$ from its point of departure at time $t^{2/3}$. Now, from Theorem A.1(i), we know that the probability that a single random walk is at a distance larger than $t^{4/9}$ from its point of departure at time $t^{2/3}$ is bounded above by $k_1 e^{-k_2 t^{1/9}}$, for appropriate constants k_1 and k_2. Here we have used the fact that the quantity $tI(t^{4/9}/t)$ behaves like $t^{8/9-1} = t^{2/9}$ times a slowly varying function when $t \to \infty$. We thus conclude that

$$R_{d,[t]}(C_t^c) \le k_1 [w_d r^d t^{2/3}] e^{-k_2 t^{1/9}},$$

and hence that

$$\lim_{t\to\infty} R_{d,[t]}(C_t^c) = 0. \tag{8.15}$$

Let us now remark that in order to prove Theorem 1.2 it is enough to show that for every finite $\Lambda \subset \mathbb{Z}^d$ and a family of parameters $\{\lambda_x : x \in \Lambda\}$,

$$\lim_{t\to\infty} E_d\Big(\exp\Big\{-\sum_{x\in\Lambda} \lambda_x \eta_x(t)\Big\}\Big) = \exp\Big\{-\prod_{x\in\Lambda}(e^{-\lambda_x} - 1)\Big\}. \tag{8.16}$$

Indeed, the right hand side is the multi-parametric Laplace transform of the product Poisson measure of parameter 1 on $\mathbb{N}^\Lambda$. Hence (8.16) implies that $\lim_{t\to\infty} \mu(t) = \nu$ weakly. But by Lemmas 8.1 and 8.4 and by (8.14) and (8.15), it is enough to prove that

$$\lim_{t\to\infty} E_{d,[t]}\Big(\mathbf{1}_{A_t} \cdot \mathbf{1}_{B_t} \cdot \mathbf{1}_{C_t} \cdot \prod_{k=1}^{[w_d r^d t^{2d/3}]} E_{d,[t]}(e^{-\lambda^\Lambda_{Z_{k,[t]}(t)}} \mid \mathcal{F}_{t^{2/3},[t]})\Big) = e^{-\prod_{x\in\Lambda}(e^{-\lambda_x}-1)}, \tag{8.17}$$

which in turn we will do by showing that

$$\lim_{t\to\infty} \sup_{w\in A_t\cap B_t\cap C_t} \Big| \prod_{k=1}^{[w_d r^d t^{2d/3}]} E_{d,[t]}(e^{-\lambda^\Lambda_{Z_{k,[t]}(t)}} \mid \mathcal{F}_{t^{2/3},[t]})(w) - e^{-\prod_{x\in\Lambda}(e^{-\lambda_x}-1)} \Big| = 0. \tag{8.18}$$

In fact, (8.17) follows from (8.18), (8.14), (8.15) and the fact that $\lim_{t\to\infty} R_{d,[t]}(A_t^c) = 0$. So let us now turn to (8.18). First note that for $w \in A_t$ and $1 \le k \le [w_d r^d t^{2d/3}]$ we can write

$$E_{d,[t]}(e^{-\lambda^\Lambda_{Z_{k,[t]}(t)}} \mid \mathcal{F}_{t^{2/3},[t]})(w) = \sum_{x\in\mathbb{Z}^d_{2N+1}} p^{[t]}_{t-t^{2/3}}(Z_{k,[t]}(t^{2/3}), x) e^{-\lambda^\Lambda_x}.$$

Combining this with the fact that on B_t, the random walks $Z_{k,[t]}$ with $1 \le k \le [w_d r^d t^{2d/3}]$ are born from a set of sites containing a ball of radius $r_d r' t^{2/3}$, and the fact that on C_t

none of those random walks is at a distance larger than $t^{4/9}$ from its point of departure at time $t^{2/3}$, we conclude that for $w \in A_t \cap B_t \cap C_t$,

$$\prod_{k=1}^{[w_d r^d t^{2/3}]} E_{d,[t]}(e^{-\lambda^{\Lambda}_{Z_{k,[t]}(t)}} \mid \mathcal{F}_{t^{2/3},[t]})(w) = \prod_{y\in B(0,r_d r' t^{2/3})} \Big[\sum_{x\in\mathbb{Z}^d_{2N+1}} p^{[t]}_{t-t^{2/3}}(X^y_N(t^{2/3}) + x_y, x) \Big] \times \prod_{k\in S} \Big[\sum_{x\in\mathbb{Z}^d_{2N+1}} p^{[t]}_{t-t^{2/3}}(Z_{k,[t]}(t^{2/3}), x) \Big], \tag{8.19}$$

where $x_y(w) \in \mathbb{Z}^d_{2N+1}$ are random sites such that $|x_y| \le t^{4/9}$ and $S(w)$ is a random subset of the set of indices $1 \le k \le [w_d r^d t^{2d/3}]$ such that the corresponding random walks $Z_{k,[t]}$ are born from sites outside a ball of radius $r_d r' t^{2/3}$. A computation similar to the proof of Lemma 8.4 shows that the second factor of the right hand side of (8.19) converges to 1 uniformly in w. On the other hand, using the local central limit theorem, it is easy to compute that the first factor converges uniformly to $e^{-\prod_{x\in\Lambda}(e^{-\lambda_x}-1)}$. This proves (8.18).

9. Appendix

This section has a technical character; we derive some asymptotic estimates for the probability that a random walk starting from a site x_t, depending on time t, hits the origin by time t. The estimates are more precise than what is needed for the proof of Theorem 1.1, but are included here for completeness. We do not claim any originality about them whatsoever, but considering that they are elementary to derive and we were unable to find proper references up to this precision, we have decided to include them here (see Lemma 2 of Bramson, Cox and Le Gall [BCL] for similar estimates).

Theorem A.1. *Let τ be the first hitting time of the origin by a continuous time symmetric simple random walk $Y(t)$ of total jump rate one, starting from $x \in \mathbb{Z}^d$, and let P_x be the corresponding probability measure.*

(i) *If $d \ge 1$ and $\min_{1\le i\le d} |x_i| \ge C\sqrt{(t/d)\log((t/d)^2+1)}$ for some $C > 0$, then*

$$P_x(\tau < t) \le 2 \min_{1\le i\le d} \frac{\exp\{-(t/d) I(|x_i|/t)\}}{\sqrt{2\pi}\, a_{t/d,x_i}(1 - e^{-I'(|x_i|/t)})} [1 + R_0(t, x_i)],$$

where for $u \ge 0$, $I(u) = u \sinh^{-1}(u) - \sqrt{1+u^2} + 1$, $I'(u) = \sinh^{-1}(u)$, $a_{t,u} = (t^2+u^2)^{1/4}$ and $|R_0(t,u)| \le 30C^{-1}(4 \log a_{t/d,u})^{-1/6}$.

(ii) *If $d = 1$ and $|x| \le t^{1/2-\epsilon}$ for some $\epsilon > 0$, then*

$$P_x(\tau > t) = \frac{2|x|}{(2\pi t)^{1/2}} [1 + R_1(t)], \tag{9.1}$$

where $|R_1(t)| \le 8t^{-\epsilon/4}$.

(iii) *If* $d = 2$ *and* $|x| \leq t^{1/2-\epsilon}$ *for some* $\epsilon > 0$*, then*

$$P_x(\tau > t) = 2\frac{\log|x|}{\log t}[1 + R_2(x)], \tag{9.2}$$

where $\lim_{|x|\to\infty} R_2(x) = 0$.

(iv) *If* $d \geq 3$ *and* $|x| \leq (t/d)^{1/2-\epsilon}$ *for some* $\epsilon > 0$*, then*

$$P_x(\tau > t) = 1 - \frac{a_d}{|x|^{d-2}}[1 + R_3(x, d)], \tag{9.3}$$

where $a_d = (d/2)\Gamma(d/2-1)\pi^{-d/2}$ *and* $R_3(x, d)$ *is an error depending on* d *and* $|x|$ *which satisfies* $\lim_{|x|\to\infty} R_3(x, d) = 0$.

Proof. The following estimate will prove useful. Let $y, z \in \mathbb{Z}$ be sites which may eventually depend on t, and $a_{t,z} = (t^2 + z^2)^{1/4}$. Then

$$P_0(Y(t) = z + y) = \frac{1}{a_{t,z}} e^{-yI'(z/t)-tI(z/t)} \int_{-\pi a_{t,z}}^{\pi a_{t,z}} e^{-iuy/a_{t,z}-\frac{1}{2}u^2+R_2(u)}\,\frac{du}{2\pi}, \tag{9.4}$$

where $R_2(u)$ is an error term satisfying $R_2(u) = \frac{1}{6}iu^3 z a_{t,z}^{-3} + O(u^4)$ with $|O(u^4)| \leq \frac{1}{24}u^4 a_{t,z}^{-2}$. The proof of this estimate, which involves a decomposition into Poisson processes and the use of Fourier transform, can be found in the proof of Lemma 4.2 of [BQR]. Part (i) corresponds to Lemma 4.2 of [BQR].

Let us next prove part (ii). First note that $P_x(\tau > t) = P_0(-|x| < Y(t) \leq |x|)$. So let $-|x| \leq y \leq |x|$. Then, from (9.4) with $z = 0$, we have

$$P_0(Y(t) = y) = \frac{1}{\sqrt{t}} \int_{-\pi\sqrt{t}}^{\pi\sqrt{t}} \exp\left\{-iuy/\sqrt{t} - \frac{1}{2}u^2 + O(u^4)\right\} \frac{du}{2\pi}, \tag{9.5}$$

where $|O(u^4)| \leq \frac{1}{24}u^4 t^{-1}$. Let I_1 be the above integral restricted to $[-\log t, \log t]$ and I_2 the integral over the rest. Expressing $1/\sqrt{2\pi}$ as the sum of $\int_{-\log t}^{\log t} \exp(-\frac{1}{2}u^2)\,\frac{du}{2\pi}$ and $2\int_{\log t}^{\infty} \exp(-\frac{1}{2}u^2)\,\frac{du}{2\pi}$, and using the bound $O(u^4) \leq \frac{1}{24}u^4 t^{-1}$, we can easily get

$$\left| I_1 - \frac{1}{\sqrt{2\pi}} \right| < \log t \left(\frac{|y|\log t}{\sqrt{t}} + \frac{(\log t)^4}{t} \right) + e^{-(\log t)^2/4}.$$

Now, since $|y| \leq |x| \leq t^{1/2-\epsilon}$, this last expression is bounded by $(\log t) \cdot 3/t^{\epsilon/2}$. On the other hand, using again the bound on the error $O(u^4)$, we can bound $|I_2|$ by $4/t^{(\log t)/40}$. Plugging these estimates into (9.5) we see that $P_0(Y(t) = y) = \frac{1}{\sqrt{2\pi t}}[1 + R_1(t)]$, where $|R_1(t)| \leq 8/t^{\epsilon/4}$. Summing $P_0(Y(t) = y)$ over $-|x| \leq y \leq |x|$, we get (9.1).

Let us now prove parts (iii) and (iv). We first consider the case $d \geq 3$ of (iv). Let us state the following estimate for the Green function $G(x) = \int_0^\infty P_0(Y(t) = x)\,dt$ of a d-dimensional random walk of total jump rate 1 when $d \geq 3$ and $x \in \mathbb{Z}^d$:

$$G(x) = \frac{a_d}{|x|^{d-2}}\left[1 + R_4(x, d)\right],$$

where $|R_4(x,d)| \le c_{100}(d)/|x|$ for some dimension dependent constant $c_{100}(d)$. This can be deduced from (i) and (ii) (see also Theorem 1.5.4 of [La], observing that the continuous and discrete time Green functions coincide). Now let $\eta = \inf\{t \ge 0 : Y(t) \in \{0\} \cup B(0, (t/d)^{1/2-\epsilon/2})^c\}$. As in Proposition 1.5.9 of [La], it can be shown using the Green function estimate that

$$P_x(Y(\eta)=0) = a_d\left(\frac{1}{|x|^{d-2}} - \frac{1}{(t/d)^{(1-\epsilon)(d-2)/2}}\right) + R_5(x,d), \tag{9.6}$$

where $|R_5(x,d)| \le c_{13}(d)/|x|^{d-1}$ and $c_{13}(d)$ is some constant. Next, for $v \ge 0$, let $T_v = \inf\{t \ge 0 : |Y(t)| \ge v\}$. Then

$$P_x(\tau < t) \ge P_x(\tau < T_{(t/d)^{(1-\epsilon)/2}}) - P_x(t < T_{(t/d)^{(1-\epsilon)/2}}). \tag{9.7}$$

But $P_x(t < T_{(t/d)^{(1-\epsilon)/2}}) \le P(\sup_{0\le s\le t}|Y_1(s)| \le (t/d)^{(1-\epsilon)/2})$, where Y_1 is the first coordinate of Y, and in turn this is $c_{14}((\lambda t/d)^{1/2}+1)e^{-\lambda t/d}$, where λ is the principal Dirichlet eigenvalue of the discrete Laplacian on $[-(t/d)^{(1-\epsilon)/2}, (t/d)^{(1-\epsilon)/2}] \cap \mathbb{Z}$ and c_{14} is a positive constant. Here we have used a standard estimate (see [BR] or the Brownian motion version in Theorem 1.2, Chapter 3, of [Sz1]). On the other hand, $\lambda \ge 3/(t/d)^{1-\epsilon}$. Therefore, from the bound $\sqrt{x}+1 \le \exp\{x/2\}$, valid for $x \ge 0$, we get $P_x(t < T_{(t/d)^{(1-\epsilon)/2}}) \le c_{14}e^{-(t/d)^{\epsilon/2}}$. Substituting this into (9.7) we see that

$$P_x(\tau < t) \ge \left(1 - c_{14}e^{-(t/d)^{\epsilon/2}}\frac{1}{P_x(\tau < T_{(t/d)^{(1-\epsilon)/2}})}\right)P_x(\tau < T_{(t/d)^{(1-\epsilon)/2}}).$$

Now, from (9.6) and the assumption $|x| \le \frac{1}{2}(t/d)^{1/2-\epsilon}$, we get the lower bound

$$P_x(\tau < T_{(t/d)^{1-\epsilon)/2}}) \ge \frac{1}{|x|^{d-1}}\left(\frac{a_d|x|}{2} - c_{13}\right).$$

Combining this estimate with (9.6) again we get

$$P_x(\tau < t) \ge \frac{a_d}{|x|^{d-2}}[1 + R_3(x,d)],$$

where $\lim_{|x|\to 0} R_3(x,d) = 0$. To finish the proof of (iv) we now remark that $P_x(\tau < t) \le G(x)/G(0) = a_d/|x|^{d-2} + R_6(x,d)$, where $\lim_{|x|\to\infty} R_6(x,d) = 0$.

The proof of part (iii) follows a similar scheme. However, instead of relying on the Green function, here we need a potential kernel estimate, $a(x) = (2/\pi)(\log|x| + R_7(x))$, where $a(x) = \int_0^\infty (P_0(Y(t)=0) - P_0(Y(t)=x))\,dt$ is the potential kernel and $|R_8(x)| \le c_{15}/|x|$ for some constant $c_{15} > 0$. Again, this is a consequence of (i) and (ii) (see also Proposition 1.6.2 of [La]). Next, as in Proposition 1.6.7 of [La], using the potential kernel estimate we deduce that

$$P_x(Y(\eta)=0) = 1 - \frac{\log|x| + R_7(x)}{\log t} + R_8(t),$$

where $|R_8(t)| \le (\log t)^{-2}$. Finally, a calculation analogous to the one of part (iv) finishes the proof. □

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References

[A] Antal, P.: Trapping problems for the simple random walk. Thesis, ETH, Zürich (1994)

[AMP1] Alves, O., Machado, F., Popov, S.: The shape theorem for the frog model. Ann. Appl. Probab. **12**, 533–546 (2002) Zbl 1013.60081 MR 2003c:60159

[AMP2] Alves, O., Machado, F., Popov, S.: Phase transition for the frog model. Electronic J. Probability no. 16, 21 pp. (2002) Zbl 1012.60081 MR 2004a:60156; The shape theorem for the frog model with random initial configuration. Markov Process. Related Fields **7**, 525–539 (2001) Zbl 0991.60097 MR 2003f:60171

[BCL] Bramson, M., Cox, J. T., Le Gall, J.-F.: Super-Brownian limits of voter model clusters. Ann. Probab. **29**, 1001–1032 (2001) Zbl pre01906008 MR 2003c:60160

[BG] Bramson, M., Griffeath, D.: On the Williams–Bjerknes tumor growth model: I, II. Ann. Probab. **9**, 173–185 (1981) Zbl 0459.92012 MR 82f:60231a; Math. Proc. Cambridge Philos. Soc. **88**, 339–357 (1980) Zbl 0459.92013 MR 2f:60231b

[BE] Bebernes, J., Eberly, D.: Mathematical Problems from Combustion Theory. Springer, (1989) Zbl 0692.35001 MR 91d:35165

[BQR] Ben Arous, G., Quastel, J., Ramírez, A. F.: Internal DLA in a random environment. Ann. Inst. H. Poincaré Probab. Statist. **39**, 301–324 (2003) Zbl pre01901043 MR 2003m:60285

[BR] Ben Arous, G., Ramírez, A. F.: Asymptotic survival probabilities in the random saturation process. Ann. Probab. **28**, 1470–1527 (2000) Zbl pre01905952 MR 2001m:60221

[CD] Cox, J. T., Durrett, R.: Some limit theorems for percolation processes with necessary and sufficient conditions. Ann. Probab. **9**, 583–603 (1981) Zbl 0462.60012 MR 82k:60208

[HS] Hara, T., Slade, G.: Self-avoiding walk in five or more dimensions. Rev. Math. Phys. **4**, 235–327 (1992) Zbl 0755.60054 MR 93j:82033

[K] Kesten, H.: Aspects of first passage percolation. In: Ecole d'été de probabilités de Saint-Flour XIV-1984, Lecture Notes Math. 1180, Springer, 125–264 (1986) Zbl 0602.60098 MR 88h:60201

[KS] Krug, J., Spohn, H.: Kinetic roughening of growing surfaces. In: Solids far from Equilibrium: Growth, Morphology and Defects, C. Godrèche (ed.), Cambridge Univ. Press, 479-582 (1991)

[La] Lawler, G.: Intersections of Random Walks. Birkhäuser, Boston (1996). Zbl 0925.60078

[LBG] Lawler, G., Bramson, M., Griffeath, D.: Internal diffusion limited aggregation. Ann. Probab. **20**, 2117–2140 (1992) Zbl 0762.60096 MR 94a:60105

[Li] Liggett, T.: Interacting Particle Systems. Springer, New York (1985) Zbl 0559.60078 MR 86e:60089

[NP] Newman, C., Piza, M.: Divergence of shape fluctuations in two dimensions. Ann. Probab. **23**, 977–1005 (1995) Zbl 0835.60087 MR 96g:82052

[PPZ] Pisztora, A., Povel, T., Zeitouni, O.: Precise large deviation estimates for a one-dimensional random walk in a random environment. Probab. Theory Related Fields **113**, 191–219 (1999) Zbl 0922.60059 MR 99m:60048

[R] Richardson, D.: Random growth in a tessellation. Proc. Cambridge Philos. Soc. **74**, 515–528 (1973) Zbl 0295.62094 MR 48 #7421

[Sz1] Sznitman, A. S.: Brownian Motion, Obstacles and Random Media. Springer, Berlin (1998) Zbl 0973.60003 MR 2001h:60147

[Sz2] Sznitman, A. S.: Slowdown estimates and central limit theorem for random walks in random environment. J. Eur. Math. Soc. **2**, 93–143 (2000) Zbl 0976.60097 MR 2001j:60192

[W] Williams, F.: Combustion Theory. Addison-Wesley (1965).