

On the Kneser–Tits problem for triality forms

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Dedicated to G. A. Margulis

Abstract. The purpose of this paper is to provide a concrete description of the “Whitehead group” $W(k, G) := G(k)/G(k)^+$ for the simply connected triality forms G of k -rank 1, and to use this description to prove that if k is a global field, then the Kneser–Tits problem for these forms has an affirmative solution.

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1. Introduction

Let G be a semi-simple simply connected algebraic group defined, simple, and isotropic over a field k . Let $G(k)^+$ be the normal subgroup of $G(k)$ generated by the k -rational elements of the unipotent radicals of parabolic k -subgroups of G . The Kneser–Tits problem is the question whether $G(k)^+ = G(k)$, or, equivalently, whether $G(k)$ is projectively simple (i.e., the only proper normal subgroups of $G(k)$ are central). In [8] it has been shown that the Kneser–Tits problem has an affirmative solution for all semi-simple simply connected groups defined, simple, and isotropic over k , if and only if, the problem has an affirmative solution for every such group of k -rank 1.

It is known that for all G as above, the Kneser–Tits problem has an affirmative solution if k is a local field (see [8]); if k is a global field, then for all G as above of k -rank ≥ 2 , the Kneser–Tits problem has an affirmative solution. In fact, the only absolutely simple simply connected isotropic groups over global fields for which a solution of the Kneser–Tits problem does not exist in the literature are the triality forms ${}^{3,6}D_{4,1}^9$, and the outer form ${}^2E_{6,1}^{35}$, we have used here the notation for indices introduced in [14], this notation will be used throughout the paper. (Though the groups of type ${}^2E_{6,1}^{29}$ are not included in [15] among the groups for which the Kneser–Tits problem was known to have an affirmative solution, Skip Garibaldi pointed out to me that in [16] Veldkamp has shown that for every absolutely simple simply connected

group of type ${}^2E_{6,1}^{29}$ over an arbitrary field of characteristic different from 2 and 3, the Kneser–Tits problem has an affirmative solution. For a modern rendition of Veldkamp’s proof, see [4]. We note here that over a global function field, groups of type ${}^2E_{6,1}^{29}$ do not exist since Harder’s Galois-cohomology vanishing theorem implies that over these fields the only absolutely simple anisotropic groups are of (inner or outer) type A_n for some n .)

The purpose of this paper is two fold. First, in §2, we provide a concrete description of the “Whitehead group” $W(k, G) := G(k)/G(k)^+$ for the simply connected triality forms G of k -rank 1. Next, in §3 we use this description to prove that if k is a global field, then for a simply connected triality form G/k of k -rank 1, $G(k)^+ = G(k)$ (Theorem 3.1). Theorem 3.1 settles the Kneser–Tits problem for the triality forms over global fields in the affirmative. It was proved jointly with M. S. Raghunathan in the late eighties but that proof has not been published.

We note here that over any field k , a triality form of k -rank > 1 is quasi-split (and is actually of k -rank 2). It is known (due to C. Chevalley and R. Steinberg, see [15]) that for a semi-simple simply connected quasi-split k -group H , $H(k)^+ = H(k)$.

2. Description of $G(k)/G(k)^+$

In this section, k is an arbitrary infinite field and G is an absolutely simple simply connected algebraic group of type ${}^{3,6}D_4$ defined over k and of k -rank 1.

Let S be a nontrivial k -split torus of G , $M = Z_G(S)$ its centralizer, and $M' = [M, M]$ the commutator subgroup of M . Let $P = M \rtimes U$ and $P^- = M \rtimes U^-$ be the two minimal parabolic k -subgroups of G containing M . It is known (see, for example, [1, 6.2(v) and 6.11(i)]) that $G(k)^+ = \langle U(k), U^-(k) \rangle$, i.e., $G(k)^+$ is generated by $U(k)$ and $U^-(k)$, and $G(k) = \langle P(k), P^-(k) \rangle = M(k)G(k)^+$. Let $M(k)^+ = M(k) \cap G(k)^+$. Then $W(k, G) = G(k)/G(k)^+ \cong M(k)/M(k)^+$.

From the description of G as the Galois-twist of the corresponding simply connected quasi-split group by a 1-cocycle taking values in a parabolic subgroup corresponding to P in the *adjoint* group of the quasi-split group, we see that there exists a quaternion division algebra D , whose center K is a separable cubic extension of k , and whose *corestriction*¹ to k is trivial, such that M is the inverse image of the unique 1-dimensional k -split torus $\mathrm{GL}_{1,k}$ contained in $R_{K/k}(\mathrm{GL}_1) = \mathrm{GL}_{1,K}$ under the reduced norm map $\mathrm{GL}_{1,D} \xrightarrow{\mathrm{Nrd}} \mathrm{GL}_{1,K}$; M is a connected reductive k -group of

¹For a description of the *corestriction* $\mathrm{Br}(K) \rightarrow \mathrm{Br}(k)$, from the Brauer group $\mathrm{Br}(K)$ of K to the Brauer group $\mathrm{Br}(k)$ of k , see [2, §8]. The following is an alternate description of this corestriction. If we identify $\mathrm{Br}(K)$ (resp., $\mathrm{Br}(k)$) with the Galois-cohomology group $H^2(K, \mathrm{GL}_1) = H^2(k, R_{K/k}(\mathrm{GL}_1))$ (resp., $H^2(k, \mathrm{GL}_1)$), then the corestriction is the homomorphism $H^2(k, R_{K/k}(\mathrm{GL}_1)) \rightarrow H^2(k, \mathrm{GL}_1)$ induced by the norm map $N_{K/k}: R_{K/k}(\mathrm{GL}_1) \rightarrow \mathrm{GL}_1$.

dimension 10 and $M' = \text{SL}_{1,D}$. From this description of M it is obvious that

$$M(k) = \{x \in D^\times \mid \text{Nrd}(x) \in k^\times\}.$$

Remark 1. If k is a local field and K is a separable cubic extension of k , then the corestriction to k of the unique quaternion division algebra with center K is nontrivial (see [7, Proposition 6.14]). Hence, over a local field k , any k -isotropic triality form is quasi-split.

In particular, if v is a place of a global field k , k_v is the completion of k at v , and $K_v := K \otimes_k k_v$ is a field, then G is quasi-split over k_v .

Remark 2. A quaternion division algebra D with center K contains an element $x \in D \setminus K$ such that $x^2 \in K$. (In fact, for any separable quadratic extension L of K contained in D , by the Skolem–Noether theorem there is $x \in D^\times$ such that conjugation by x keeps L stable, and restricted to L it coincides with the nontrivial automorphism of L/K . Then $x^2 \in K$.) This implies that there does not exist a quaternion division algebra whose center is a perfect field of characteristic 2. Now if k is a perfect field, then so is its separable cubic extension K , and we conclude that there does not exist a triality form of rank 1 over a perfect field k of characteristic 2.

Let

$$\mathfrak{M} := \langle \ell^\times \mid \ell \text{ is a separable quadratic extension of } k \text{ contained in } D \rangle.$$

Our goal in this section is to prove that \mathfrak{M} is contained in $M(k)^+$, and these two groups are equal if, for example, k is not of characteristic 2. As \mathfrak{M} is clearly a normal subgroup of D^\times , it would follow, in particular, that if $\text{char}(k) \neq 2$, then $M(k)^+ \triangleleft D^\times$. We begin with the following:

Proposition 2.1. *Given a separable quadratic extension ℓ of k contained in D , there is an absolutely simple simply connected k -subgroup H of G of type 2A_2 which contains S and which is quasi-split over k , so that $T(k) = \ell^\times$, where $T (\subset M)$ is the centralizer of S in H .*

Proof. Let T be the Zariski-closure of $\ell^\times (\subset M(k))$ in M . Then T is a 2-dimensional torus containing S , and it splits over ℓ . We now observe that since $D \otimes_k \ell$ is a matrix algebra over the field $K \otimes_k \ell$, M is quasi-split over ℓ and T is a maximal ℓ -split torus of M . Hence, there exists a Borel subgroup B of M defined over ℓ and containing the centralizer Z of T in M . Note that Z is a maximal torus of M , and hence also of G , and it is defined over k since T is. Moreover, T is clearly the maximal ℓ -split subtorus of Z .

We now work with the ordering on the root system of G , with respect to the maximal torus Z , determined by the Borel subgroup $B \times U$ of G . Let $\{\alpha_i, i = 0, 1, 2, 3\}$

be the set of simple roots enumerated so that for all nonzero i , α_i is trivial on S . Then α_1, α_2 and α_3 , and their negatives, are the roots of M with respect to Z . Let L be the splitting field of Z over k ; L contains ℓ since Z contains T . Let Γ be the Galois group of L/k and Γ' be the (normal) subgroup of Γ of index two consisting of the elements which fix ℓ pointwise. Γ' fixes the root α_0 and acts transitively on the set $\{\alpha_1, \alpha_2, \alpha_3\}$. Also, since U is defined over k , the set Θ of positive roots $\alpha = \sum_{i=0}^3 m_i \alpha_i$ such that m_0 is nonzero, which is the set of roots of U with respect to Z , is stable under Γ . The roots $\alpha_0, \sum_{i=0}^3 \alpha_i$, and the highest root $2\alpha_0 + \sum_{i=1}^3 \alpha_i$ are the only roots in the set Θ which are fixed under Γ' , and the highest root is actually fixed by all of Γ . We conclude that the set $\{\alpha_0, \sum_{i=0}^3 \alpha_i\}$ is stable under Γ . Hence the subgroup H of G generated by the 1-dimensional unipotent root subgroups corresponding to the roots $\alpha_0, \sum_{i=0}^3 \alpha_i, 2\alpha_0 + \sum_{i=1}^3 \alpha_i$, and their negatives, is a connected subgroup of G defined over k , split over ℓ , and it is clearly of type A_2 . It can be easily checked that H is simply connected. As it contains the subgroup generated by the root subgroups corresponding to the highest root $2\alpha_0 + \sum_{i=1}^3 \alpha_i$ and its negative, which is a group k -isomorphic to SL_2 , we conclude that H is of k -rank 1 and hence it is a quasi-split special unitary group in three variables defined in terms of the quadratic extension ℓ . As T is the maximal ℓ -split subtorus of Z , T is contained in H and it equals the centralizer of S in H . \square

Proposition 2.2. $\mathfrak{M} \subset M(k)^+$.

Proof. Let ℓ be a separable quadratic extension of k contained in D , and H and T be as in the above proposition. Since H is quasi-split over k and is simply connected, $H(k) = H(k)^+ \subset G(k)^+$. This implies that $\ell^\times = T(k) \subset M(k)^+$, which in turn implies that $\mathfrak{M} \subset M(k)^+$. \square

Notation 2.3. For $u \in U(k) \setminus \{1\}$, let $u = f(u)n(u)g(u)$ be the Bruhat decomposition of u (with respect to P^-), with $f(u), g(u) \in U^-(k)$ and $n(u) \in N(k)$, where N is the normalizer of S in G . Note that as G is of k -rank 1, $N(k)/M(k)$ is of order 2, and $n(u) \in N'(k)$, where $N' = N \setminus M$.

Let a be the generator of the character group of S which is a root of U with respect to S . Then $\pm a, \pm 2a$ are the roots of G with respect to S . Let U_b denote the connected unipotent root subgroup corresponding to the root $b \in \{\pm a, \pm 2a\}$. It is easy to see that $U_{\pm a}$ are 9-dimensional, $U_{\pm 2a}$ are 1-dimensional, and so U_a/U_{2a} is an 8-dimensional M -module defined over k ; $U = U_a, U^- = U_{-a}$.

From a consequence of Proposition 1.8 of [3], stated in the first sentence on p. 311 of that paper (note that the notation used in [3] is different from ours), we at once obtain that

$$M(k)^+ = \langle n(u)n(v)^{-1} \mid u \in U_a(k) \setminus U_{2a}(k), v \in U_{2a}(k) \setminus \{1\} \rangle. \quad (*)$$

Let \mathfrak{U} denote the set of elements $u \in U_a(k) \setminus U_{2a}(k)$ such that there is an absolutely simple simply connected quasi-split k -subgroup $H(u)$ of G of type 2A_2 containing S , $U_{\pm 2a}$, and the element u (and hence also $n(u)$). We note that if T ($\subset M$) is the centralizer of S in $H(u)$, and ℓ is the separable quadratic extension of k over which T splits, then the k -subalgebra of D generated by $T(k)$ is isomorphic to ℓ and will be identified with it. With this identification, $T(k) = \ell^\times$. Hence, for all $v \in U_{2a}(k) \setminus \{1\}$, $n(u)n(v)^{-1} \in T(k) = \ell^\times \subset \mathfrak{M}$.

We will now prove the following:

Proposition 2.4. $\mathfrak{M} = \langle n(u)n(v)^{-1} \mid u \in \mathfrak{U}, v \in U_{2a}(k) \setminus \{1\} \rangle$.

Proof. To prove the proposition, it would suffice to show that given a separable quadratic extension ℓ of k contained in D , the group generated by $\{n(u)n(v)^{-1} \mid u \in \mathfrak{U}, v \in U_{2a}(k) \setminus \{1\}\}$ contains ℓ^\times . According to Proposition 2.1, there is an absolutely simple simply connected k -subgroup H of G of type 2A_2 , which contains S and which is quasi-split over k , so that if T is the centralizer of S in H , then $T(k) = \ell^\times$. This subgroup is k -isomorphic to the special unitary group $SU(h)$ of the hermitian form h on ℓ^3 given by

$$h(x, y, z) = x\bar{z} + \bar{x}z + y\bar{y} \quad \text{for } x, y, z \in \ell,$$

where for $\lambda \in \ell$, $\bar{\lambda}$ denotes its conjugate over k . We identify H with $SU(h)$ in terms of a k -isomorphism which carries $T(k)$ onto the diagonal subgroup of $SU(h)(k)$.

It is obvious that $H(k) \cap (U_a(k) \setminus U_{2a}(k)) \subset \mathfrak{U}$ (for every $u \in H(k) \cap (U_a(k) \setminus U_{2a}(k))$, we can take $H(u)$ appearing in the definition of \mathfrak{U} to be H). We easily see, by a direct computation in $SU(h)(k)$, for any field k , that the elements $n(u)n(v)^{-1}$, as u varies over $H(k) \cap (U_a(k) \setminus U_{2a}(k))$ and v varies over $H(k) \cap (U_{2a}(k) \setminus \{1\})$, generate $T(k) = \ell^\times$. This proves the proposition. \square

It follows from Theorem 5.1 of Raghunathan [9] that given an element $u \in U_a(k) \setminus U_{2a}(k)$, $u^2 \neq 1$, as U_{2a} is 1-dimensional, there is an absolutely simple simply connected quasi-split k -subgroup $H = H(u)$, of type 2A_2 , containing the subgroups S , $U_{\pm 2a}$, and the element u . Thus $\mathfrak{U} \supset \{u \in U_a(k) \setminus U_{2a}(k) \mid u^2 \neq 1\}$. In particular, if $\text{char}(k) \neq 2$, then as every nontrivial element of $U_a(k)$ is of order at least 3, $\mathfrak{U} = U_a(k) \setminus U_{2a}(k)$. Now combining this with the description of \mathfrak{M} given by Proposition 2.4 and that of $M(k)^+$ provided by (*), we conclude the following.

Theorem 2.5. *If the characteristic of k is different from 2, then $M(k)^+ = \mathfrak{M}$. Hence, $W(k, G) \cong M(k)/\mathfrak{M}$.*

The following remark is due to Andrei S. Rapinchuk.

Remark 3. If the Whitehead group $W(k, G) (\cong M(k)/M(k)^+)$ is finite and $M(k)^+ = \mathfrak{M}$, then $W(k, G)$ is solvable. In fact, $M(k)$ and $M(k)^+ (= \mathfrak{M})$ are normal subgroups of D^\times , and the conjugation action of D^\times on $M(k)$ and $M(k)^+$ induces an action of D^\times on $M(k)/M(k)^+$. Let C be the kernel of this action. Then C is subgroup of D^\times of finite index, and it clearly contains $M(k)^+$. Hence, according to the main theorem of [11], the group D^\times/C is solvable. This implies, in particular, that $M(k)/C \cap M(k)$ is solvable. On the other hand, the commutator subgroup $[M(k), C]$ is contained in $M(k)^+$, and hence, $(C \cap M(k))/M(k)^+$ is abelian, and we conclude that $M(k)/M(k)^+$ is solvable.

For the proof of Theorem 3.1, the following proposition (Proposition 2.7), which is an immediate consequence of Raghunathan’s result mentioned above, suffices. For the sake of completeness, we give its proof which was obtained with help from Rapinchuk.

Let k be an arbitrary field. The central k -split torus S of M acts on the 8-dimensional M -module $V := U_a/U_{2a}$ by homotheties by the nontrivial character a . Over the algebraic closure \bar{k} of k , $M' = [M, M]$ is the direct product of three copies of SL_2 , $M' = M'_1 \times M'_2 \times M'_3$, where $M'_i \cong SL_2$ for $i \leq 3$. Over \bar{k} , as an M' -module, V is the tensor product $V_1 \otimes V_2 \otimes V_3$, where V_i is the standard 2-dimensional representation of $M'_i (\cong SL_2)$, and S acts faithfully on V by homotheties.

We fix a basis $\{e_1^{(i)}, e_2^{(i)}\}$ of V_i , and let T_i be the maximal torus of M'_i represented by diagonal matrices with respect to this basis; we will denote $\text{diag}(t, t^{-1})$ by $\delta(t)$. Set

$$e = e_1^{(1)} \otimes e_1^{(2)} \otimes e_1^{(3)} + e_2^{(1)} \otimes e_2^{(2)} \otimes e_2^{(3)}.$$

Lemma 2.6. *The identity component M_e° of the isotropy subgroup M_e of e in M coincides with*

$$T := \{(\delta(t_1), \delta(t_2), \delta(t_3)) \in T_1 \times T_2 \times T_3 \mid t_1 t_2 t_3 = 1\},$$

and the annihilator of e in the Lie algebra of M is precisely the Lie algebra of the two dimensional torus T . Consequently, the M -orbit $\Omega := M \cdot e$ of e is a Zariski-open subvariety of $V = U_a/U_{2a}$, and for any k -rational element v of Ω , the identity component M_v° of the isotropy subgroup M_v is a 2-dimensional torus defined over k and contained in M' .

Proof. It is easy to see that to prove the first assertion, it is enough to show that

$$M'_e \subset N_1 \times N_2 \times N_3, \tag{\diamond}$$

where N_i is the normalizer of T_i in M'_i . Assume that (\diamond) does not hold. Then there exists an $i \in \{1, 2, 3\}$ such that $\pi_i(M'_e) \not\subset N_i$, where $\pi_i : M' \rightarrow M'_i$ is the natural

projection. Since $\pi_i(M'_e) \supset \pi_i(T) = T_i$, we see that $\pi_i(M'_e)$ contains either an upper or lower triangular element. Then, by considering commutators of elements of M'_e with appropriate elements of T , we see that M'_e contains the subgroup that has the full upper or lower unitriangular group as the i -th and the identity as the remaining two components. But it is obvious that such a subgroup cannot fix e , and so we have proved the first assertion.

Since $\dim M \cdot e = 10 - 2 = 8 = \dim V$, the orbit $M \cdot e$ is open in V .

The annihilator of e in the Lie algebra of M can be easily computed and shown to be equal to the Lie algebra of T . The assertion that for a k -rational element v of $\Omega = M \cdot e$, M_v° is defined over k follows now from the fact that the Lie algebra of M_v coincides with the annihilator of v in the Lie algebra of M . \square

Since the M -orbit $\Omega = M \cdot e$, being an open subvariety of the k -vector space V , contains a k -rational element, we conclude that Ω is in fact defined over k . Let \mathcal{U} be the open k -subvariety of U_a consisting of the elements whose image in V lies in the open subvariety Ω . Now let $u \in \mathcal{U}(k)$ and v be its image in V . Then $T_v := M_v^\circ$ is a torus of dimension 2 which is defined and anisotropic over k . The derived subgroup $H(u)$ of the centralizer of T_v in G is an absolutely simple simply connected quasi-split k -group of type 2A_2 which contains S , $U_{\pm 2a}$, and the element u . Thus we have proved the following.

Proposition 2.7. *Let \mathcal{U} be the nontrivial open k -subvariety of U_a defined above. Then $\mathcal{U}(k) \subset \mathcal{U}$.*

3. The Kneser–Tits problem for triality forms over global fields

The purpose of this section is to prove the following result.

Theorem 3.1 (Prasad and Raghunathan). *Let k be a global field (i.e., either a number field, or the function field of a curve over finite field) and G be an absolutely simple simply connected algebraic k -group of type ${}^{3,6}D_4$ of k -rank 1. Then $G(k)^+ = G(k)$.*

We will continue to use the notation introduced in §2, however, henceforth k will denote a global field. For a global field L , and a place w of L , we will denote by L_w the completion of L at w . For a place v of k , we will denote $D \otimes_k k_v$ by D_v in the sequel.

As observed at the beginning of §2, we may identify $M(k)$ with $\{x \in D^\times \mid \text{Nrd}(x) \in k^\times\}$. In particular, $\text{SL}_1(D) \subset M(k) \subset G(k)$. To prove Theorem 3.1, we must show that $M(k)^+ = M(k)$. Our first goal is to show that $\text{SL}_1(D) \subset M(k)^+$.

3.2. Let \mathcal{T} be the finite set of places of k where G is *not* quasi-split. Let $v \in \mathcal{T}$. Then $K \otimes_k k_v$ cannot be a field, for if it is a field, then G is a triality form over k_v , and according to Remark 1, it is quasi-split at v . Hence, there are only the following two possibilities.

(1) There are three distinct places w_1, w_2 and w_3 of K lying over v , i.e., $K \otimes_k k_v = K_{w_1} \oplus K_{w_2} \oplus K_{w_3}$, and each K_{w_i} is equal to k_v . As the corestriction of D_v to k_v is trivial, $D_v = D \otimes_k k_v = D \otimes_K (K \otimes_k k_v) = M_2(k_v) \oplus \mathcal{D} \oplus \mathcal{D}$, where \mathcal{D} is the quaternion division algebra with center k_v . In this case, over k_v , G is an inner-form of rank 2 of the simply connected Chevalley group of type D_4 .

From the description of M as an algebraic k -group given in §2, it is obvious that an element $(x, y, z) \in D_v^\times = \text{GL}_2(k_v) \times \mathcal{D}^\times \times \mathcal{D}^\times$ is in $M(k_v)$ if and only if $\det(x) = \text{Nrd}(y) = \text{Nrd}(z)$.

(2) There are exactly two places w_1, w_2 of K lying over v , i.e., $K \otimes_k k_v = K_{w_1} \oplus K_{w_2}$, we can assume that $K_{w_1} = k_v$, and K_{w_2} is a separable quadratic extension of k_v . In this case, K is not a Galois extension of k , G is of type 6D_4 over k and of type 2D_4 over k_v . As the corestriction of D_v to k_v is trivial, $D_v = D \otimes_k k_v = D \otimes_K (K \otimes_k k_v) = \mathcal{D}_1 \oplus \mathcal{D}_2$, where \mathcal{D}_i is a quaternion division algebra with center K_{w_i} , $i = 1, 2$. In particular, v is necessarily nonarchimedean in this case.

From the description of M as an algebraic k -group given in §2, we see that $M(k_v)$ is the set of elements $(x, y) \in D_v^\times = \mathcal{D}_1^\times \times \mathcal{D}_2^\times$ such that $\text{Nrd}(x) = \text{Nrd}(y)$.

Lemma 3.3. *For every place $v \in \mathcal{T}$, we have $[D_v^\times, M(k_v)] = \text{SL}_1(D_v)$.*

Proof. The lemma can be proved easily using the description of D_v^\times and $M(k_v)$ given above and the following well-known results: (i) $[\text{SL}_2(k_v), \text{SL}_2(k_v)] = \text{SL}_2(k_v)$. For any central division algebra \mathcal{D} over a local field ℓ , (ii) $[\mathcal{D}^\times, \mathcal{D}^\times] = \text{SL}_1(\mathcal{D})$ (this is a theorem of Matsushima and Nakayama), (iii) $\text{Nrd}(\mathcal{D}^\times) = \ell^\times$ if ℓ is nonarchimedean, (iv) $[\text{SL}_1(\mathcal{D}), \text{SL}_1(\mathcal{D})]$ is the first congruence subgroup of $\text{SL}_1(\mathcal{D})$ (see the corollary in §5 of [13]). (v) If case (2) of 3.2 occurs and K_{w_2} is an unramified extension of k_v , then in the quaternion division algebra \mathcal{D}_2 we can find a uniformizer which normalizes an unramified quadratic extension of K_{w_2} contained in \mathcal{D}_2 and whose square is an element of k_v . □

Let $G_{\mathcal{T}} = \prod_{v \in \mathcal{T}} G(k_v)$. We will think of $G(k)$ as a subgroup of $G_{\mathcal{T}}$ in terms of its diagonal embedding in the latter. The \mathcal{T} -adic topology on a subset of $G(k)$ is the topology induced on it by the product (locally compact Hausdorff) topology on $G_{\mathcal{T}}$.

Now we observe that the map $U(k) \setminus \{1\} \rightarrow N'(k)$, which maps u onto $n(u)$ (see Notation 2.3), is continuous in the \mathcal{T} -adic topology. To see this, we note that $\Omega' := U^- N' U^-$ is an open k -subvariety of G , $\Omega'(k)$ contains $U(k) \setminus \{1\}$, and the product morphism $U^- \times N' \times U^- \rightarrow G$ is an isomorphism onto Ω' . The inverse of the product morphism induces a homeomorphism from $\Omega'_{\mathcal{T}} := \prod_{v \in \mathcal{T}} U^-(k_v) N'(k_v) U^-(k_v)$

onto $\prod_{v \in \mathcal{T}} U^-(k_v) \times \prod_{v \in \mathcal{T}} N'(k_v) \times \prod_{v \in \mathcal{T}} U^-(k_v)$. Let ν be the composite of this homeomorphism and the projection onto the middle factor. Then the map $u \mapsto n(u)$ is simply the restriction of ν to $U(k) \setminus \{1\}$, and hence it is continuous in the \mathcal{T} -adic topology.

Lemma 3.4. (1) $G(k)^+$ is dense in $G_{\mathcal{T}} = \prod_{v \in \mathcal{T}} G(k_v)$.
 (2) $M(k)^+$ and \mathfrak{M} are dense in $M_{\mathcal{T}} := \prod_{v \in \mathcal{T}} M(k_v)$.

Proof. (1): It is known that G has the weak approximation property (Theorem 7.8 of [7]) and hence $G(k)$ embedded diagonally in $G_{\mathcal{T}}$ is dense in the latter. As G is k -isotropic, $G(k_v)$ does not contain any proper noncentral normal subgroups (Theorems 7.1 and 7.6 of [7]). This implies that the noncentral normal subgroup $G(k)^+$ of $G(k)$ is also dense in $G_{\mathcal{T}}$.

(2): Since $G(k)^+$ is dense in $\prod_{v \in \mathcal{T}} G(k_v)$, it follows that $G(k)^+ \cap \Omega_{\mathcal{T}} = U(k)(M(k) \cap G(k)^+)U^-(k)$ is dense in the open Bruhat-cell

$$\Omega_{\mathcal{T}} := \prod_{v \in \mathcal{T}} U(k_v)M(k_v)U^-(k_v).$$

Projecting on the middle component we conclude that $M(k)^+ = M(k) \cap G(k)^+$ is dense in $M_{\mathcal{T}} = \prod_{v \in \mathcal{T}} M(k_v)$.

Now to prove that \mathfrak{M} is dense in $M_{\mathcal{T}}$, it suffices to show that it is dense in $M(k)^+$ in the \mathcal{T} -adic topology. Let \mathcal{U} be the nontrivial open k -subvariety of U_a as in Proposition 2.7. Then according to that proposition $\mathcal{U}(k) \subset \mathfrak{U}$, which implies that the subset \mathfrak{U} of $U_a(k) \setminus U_{2a}(k)$ is dense in the latter in the \mathcal{T} -adic topology. Now the density of \mathfrak{M} in $M(k)^+$ is a consequence of the description of \mathfrak{M} provided by Proposition 2.4, the description of $M(k)^+$ given by (*), and the continuity of the map $u \mapsto n(u)$ established above. \square

Proposition 3.5. $SL_1(D) \subset \mathfrak{M}$.

Proof. Consider the subgroup $\mathcal{H} = [D^\times, \mathfrak{M}]$ of \mathfrak{M} . Since D^\times is dense in $\prod_{v \in \mathcal{T}} D_v^\times$ and \mathfrak{M} is dense in $\prod_{v \in \mathcal{T}} M(k_v)$ (Lemma 3.4 (2)), \mathcal{H} is dense in $\prod_{v \in \mathcal{T}} [D_v^\times, M(k_v)] = \prod_{v \in \mathcal{T}} SL_1(D_v)$ (where the last equality is given by Lemma 3.3). Therefore, \mathcal{H} is dense in $SL_1(D)$ in the \mathcal{T} -topology. However, \mathcal{H} is a noncentral normal subgroup of $SL_1(D)$, so it is \mathcal{T} -adically open (and hence closed), see [5], and for k of characteristic 2, see [10]; note that any place w of K such that $D \otimes_K K_w$ is a division algebra lies over a place v of k contained in \mathcal{T} . This implies that $\mathcal{H} = SL_1(D)$. \square

Let now

$$\mathcal{A} = \text{Nrd}_{D/K}(M(k)) \subset k^\times, \quad \mathcal{B} = \text{Nrd}_{D/K}(\mathfrak{M}).$$

Since $SL_1(D) \subset \mathfrak{M} \subset M(k)^+$, to prove Theorem 3.1, it is enough to show that $\mathcal{B} = \mathcal{A}$ (this will also imply that $\mathfrak{M} = M(k)$).

Lemma 3.6. *An element $a \in k^\times$ lies in \mathcal{A} if and only if a is positive in $k_v = \mathbb{R}$ for all real places $v \in \mathcal{T}$.*

Proof. This follows from the norm theorem of Hasse, Schilling and Maass (see Theorem 33.15 of [12]) which asserts that for a finite dimensional division algebra A with center a global field K , an element $x \in K^\times$ is the reduced norm of an element of A^\times if and only if x is positive at all real places w of K such that $A \otimes_K K_w$ is a division algebra. □

Lemma 3.7. *Suppose ℓ is a separable quadratic extension of k such that for all $v \in \mathcal{T}$ for which case (1) of 3.2 occurs, $\ell \otimes_k k_v$ is a field, and for $v \in \mathcal{T}$ for which case (2) of 3.2 occurs, $\ell \otimes_k k_v$ is a field which is not isomorphic to K_{w_2} over k_v . Then the composite $K \cdot \ell$, and so also ℓ , is embeddable in D .*

Proof. Recall that a quadratic extension L of K is embeddable in D if and only if for all places w of K such that $D \otimes_K K_w$ is a division algebra, $L \otimes_K K_w$ is a field (see [12]).

Let $L := K \cdot \ell$ and let w be a place of K such that $D \otimes_K K_w$ is a division algebra. Then w lies over a place of k belonging to \mathcal{T} . Now we observe that $L \otimes_K K_w = \ell \otimes_k K \otimes_K K_w = \ell \otimes_k K_w$. Thus it would suffice to show that $\ell \otimes_k K_w$ is a field for any place w of K lying over a place $v \in \mathcal{T}$.

Given a place $v \in \mathcal{T}$ for which case (1) of 3.2 occurs, for the places $w_i, i = 1, 2, 3$, of K lying over v , K_{w_i} is isomorphic to k_v , so $\ell \otimes_k K_{w_i}$ is a field by hypothesis. On the other hand, if v is a place of k for which case (2) of 3.2 occurs, then w_1, w_2 mentioned there are the places of K lying over v , K_{w_1} is isomorphic to k_v , and hence again, by hypothesis, $\ell \otimes_k K_{w_1}$ is a field. Finally, as $\ell \otimes_k k_v$ is a field which is not isomorphic to K_{w_2} , $\ell \otimes_k K_{w_2}$ is a field as well. □

We will also need the following well known result of local class field theory.

Lemma 3.8. *Let \mathcal{K} be a local field. Then*

- (1) *The correspondence $\mathcal{L} \mapsto N_{\mathcal{L}/\mathcal{K}}(\mathcal{L}^\times)$ is a bijection between the set of separable quadratic extensions \mathcal{L} of \mathcal{K} and the set of open subgroups \mathcal{N} of \mathcal{K}^\times of index 2.*
- (2) *$\mathcal{K}^\times / (\mathcal{K}^\times)^2$ is a direct sum of at least two copies of $\mathbb{Z}/2\mathbb{Z}$, furthermore, if the characteristic of the residue field of \mathcal{K} is 2, then it is a direct sum of at least three copies of $\mathbb{Z}/2\mathbb{Z}$. ($\mathcal{K}^\times / (\mathcal{K}^\times)^2$ is infinite if and only if \mathcal{K} is of characteristic 2.)*

Let \mathcal{T}_∞ be the set of archimedean places in \mathcal{T} , and let \mathcal{T}_0 be the set of nonarchimedean places $v \in \mathcal{T}$ for which 3.2 (2) holds and the characteristic of the residue field of k_v is odd. To show that an element of \mathcal{A} belongs to \mathcal{B} we will use:

Lemma 3.9. *Let $a \in \mathcal{A}$ and assume that for each $v \in \mathcal{T}_0$, there exists a separable quadratic extension $\mathcal{L}(v)$ of k_v such that $\mathcal{L}(v)$ is not isomorphic (over k_v) to the field K_{w_2} of 3.2 (2), and such that a is the norm of an element of $\mathcal{L}(v)$. Then there exists $b \in k$ such that the polynomial $x^2 - bx + a$ is irreducible, separable and its splitting field ℓ is embeddable in D .*

Proof. For any real place $v \in \mathcal{T}$, let $b_v = 0$, so that by Lemma 3.6, $x^2 - b_v x + a$ is irreducible over $k_v = \mathbb{R}$, and its splitting field, which we denote by $\mathcal{L}(v)$, is equal to \mathbb{C} .

For a nonarchimedean place $v \in \mathcal{T} \setminus \mathcal{T}_0$, let $\mathcal{L}(v)$ be a separable quadratic extension of k_v such that a is the norm of an element of $\mathcal{L}(v)$, and moreover if 3.2 (2) holds, then $\mathcal{L}(v)$ is not isomorphic to the quadratic extension K_{w_2} over k_v . Note that if v is a nonarchimedean place lying in $\mathcal{T} \setminus \mathcal{T}_0$ such that 3.2 (2) holds, then the characteristic of the residue field of k_v is 2, and as according to Lemma 3.8 (2), $k_v^\times / (k_v^\times)^2$ is a direct sum of at least three copies of $\mathbb{Z}/2\mathbb{Z}$, there exists a separable quadratic extension $\mathcal{L}(v)$ of k_v which is not isomorphic to K_{w_2} over k_v and such that a is a norm of an element of $\mathcal{L}(v)$.

For $v \in \mathcal{T}_0$, let $\mathcal{L}(v)$ be as in the statement of the lemma. For $v \in \mathcal{T}$, nonarchimedean, let b_v be the trace of an element of $\mathcal{L}(v) \setminus k_v$ whose norm is a (not all elements of norm 1 in $\mathcal{L}(v)^\times$ are contained in k_v , so there is an element in $\mathcal{L}(v) \setminus k_v$ whose norm is a). The splitting field of $x^2 - b_v x + a$ over k_v is $\mathcal{L}(v)$.

Choose $b \in k$ which approximates b_v so closely, for all $v \in \mathcal{T}$, that by Krasner’s Lemma the splitting field of $x^2 - bx + a$ over k_v is isomorphic to $\mathcal{L}(v)$.

Let ℓ be the splitting field of $x^2 - bx + a$ over k . Then the above choice of b guarantees that ℓ satisfies the hypothesis of Lemma 3.7 and hence it is embeddable in D . □

Now to prove that $\mathcal{B} = \mathcal{A}$, we will use Lemma 3.9. This lemma implies that under certain circumstances, an element $a \in \mathcal{A}$ is actually the norm of an element of a separable quadratic extension ℓ of k contained in D , and hence it belongs to \mathcal{B} . We would like to show that such elements generate \mathcal{A} . For this purpose, we need the following lemma.

For $v \in \mathcal{T}_0$, let K_{w_2} be the field as in 3.2 (2), and let $\mathcal{N}_v \subset k_v^\times$ be the subgroup of norms of elements of $K_{w_2}^\times$.

Lemma 3.10. *Consider the following condition on an element $x \in k^\times$.*

$$\text{For } v \in \mathcal{T}_0, \text{ either } x \notin \mathcal{N}_v \text{ or } x \in (k_v^\times)^2. \tag{\diamond}$$

Then for each $a \in \mathcal{A}$, there are elements $a_1, a_2 \in \mathcal{A}$, satisfying condition (\diamond) , such that $a = a_1 a_2$.

Proof. Let $a \in \mathcal{A}$. Then for $v \in \mathcal{T}_\infty$, a is positive in $k_v = \mathbb{R}$. Observe now that for $v \in \mathcal{T}_0$, \mathcal{N}_v is an open (and hence closed) subgroup of index two in k_v^\times . So it contains the subgroup $(k_v^\times)^2$ and $k_v^\times \setminus \mathcal{N}_v$ is open; moreover, $(k_v^\times)^2$ is an open subgroup of k_v^\times since (for all $v \in \mathcal{T}_0$) the characteristic of the residue field of k_v is odd.

Consider the open subset

$$\mathcal{O} := \prod_{v \in \mathcal{T}_\infty} k_v^{>0} \prod_{\{v \in \mathcal{T}_0 | a \in \mathcal{N}_v\}} (k_v^\times \setminus \mathcal{N}_v) \prod_{\{v \in \mathcal{T}_0 | a \notin \mathcal{N}_v\}} (k_v^\times)^2,$$

of $\prod_{v \in \mathcal{T}_0 \cup \mathcal{T}_\infty} k_v^\times$. Since k^\times is dense in the latter group, it follows that \mathcal{O} contains an element $a_1 \in k^\times$. Let $a_2 = a/a_1$. Then it is easy to check that both a_1 and a_2 satisfy condition (\diamond) . Also, for all $v \in \mathcal{T}_\infty$, a_i is positive in $k_v = \mathbb{R}$, for $i = 1, 2$. According to Lemma 3.6, both a_1 and a_2 lie in \mathcal{A} and we are done. \square

We can finally prove

Proposition 3.11. $\mathcal{B} = \mathcal{A}$.

Proof. Let $a \in \mathcal{A}$. In view of Lemma 3.10, we may assume that a satisfies (\diamond) . We claim that a is the norm of an element in a separable quadratic extension ℓ of k embeddable in D . To prove this claim, it suffices to show that a satisfies the hypothesis of Lemma 3.9. By condition (\diamond) , for $v \in \mathcal{T}_0$, either $a \notin \mathcal{N}_v$ or $a \in (k_v^\times)^2$, and using Lemma 3.8, we can find an open subgroup \mathcal{M}_v of k_v^\times of index 2 which contains a but is different from \mathcal{N}_v . Then the quadratic Galois extension $\mathcal{L}(v)$ of k_v determined by \mathcal{M}_v is not isomorphic to K_{w_2} and a is the norm of an element in $\mathcal{L}(v)$. \square

Proposition 3.11 completes the proof of Theorem 3.1.

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Note added in proof. Philippe Gille has recently proved that for an absolutely simple simply connected isotropic triality form G over an arbitrary field k , the Whitehead group $W(k, G)$ is trivial, i.e., $G(k)^+ = G(k)$. He has also settled the Kneser–Tits problem for the last remaining case over global fields of absolutely simple simply connected groups with Tits-index ${}^2E_{6,1}^{35}$ in the affirmative. See the write-up of his Séminaire Bourbaki talk “Le problème de Kneser–Tits” delivered in November 2007.

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