

Index theory and partitioning by enlargeable hypersurfaces

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Abstract. In this paper we state and prove a higher index theorem for an odd-dimensional connected spin Riemannian manifold (M, g) which is partitioned by an oriented closed hypersurface N . This index theorem generalizes a theorem due to N. Higson in the context of Hilbert modules. Then we apply this theorem to prove that if N is area-enlargeable and if there is a smooth map from M into N such that its restriction to N has non-zero degree, then the scalar curvature of g cannot be uniformly positive.

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1. Introduction

Given a compact manifold M , it is certainly an interesting problem to decide whether it carries a Riemannian metric with *everywhere positive scalar curvature* or not. This problem is revealed to be also very difficult. For constructing a metric with positive scalar curvature the most powerful technique is the Gromov–Lawson and Schoen–Yau surgery theorem asserting that if M has a metric with positive scalar curvature and if M' is obtained from M by performing surgeries in co-dimension greater than or equal to 3, then M' carries a metric with positive scalar curvature, too (see [3] and [18]). In the other direction, i.e., to find obstruction for the existence of such metrics, the Atiyah–Singer index theorem and all its variants come into play through the Lichnerowicz formula (see e.g. [10]). Even, it has been believed that all obstruction for the existence of metrics with positive scalar curvature on a spin manifold M can be encapsulated in a sophisticated index $\alpha_{\max}^{\mathbb{R}}(M)$ which takes its value in $KO_n(C_{\max, \mathbb{R}}^*(G))$, where G is the fundamental group of M (see e.g. [15]). This assertion is known as the Gromov–Lawson–Rosenberg conjecture and is shown, by T. Schick in [16], not to be true in this general form. Nevertheless, this index might subsume all index theoretic obstructions for the existence of metrics with positive scalar curvature on spin manifolds. One obstruction for the existence of metrics with positive scalar curvature is the enlargeability, which was introduced by Gromov and Lawson (see [4], [5]). Enlargeability is a homotopy invariance of smooth manifolds, and the category of enlargeable manifolds forms a rich and interesting

family containing, for example, all hyperbolic manifolds and all sufficiently large 3-dimensional manifolds.

Definition. Let N be a closed oriented manifold of dimension n with a fixed Riemannian metric g . The manifold N is enlargeable if for each real number $\varepsilon > 0$ there is a Riemannian spin cover (\tilde{N}, \tilde{g}) , with lifted metric, and a smooth map $f: \tilde{N} \rightarrow S^n$ such that the function f is constant outside a compact subset K of \tilde{N} , the degree of f is non-zero, and the map $f: (\tilde{N}, \tilde{g}) \rightarrow (S^n, g_0)$ is ε -contracting, where g_0 is the standard metric on S^n . Being ε -contracting means that $\|T_x f\| \leq \varepsilon$ for each $x \in \tilde{N}$, where $T_x f: T_x \tilde{N} \rightarrow T_{f(x)} S^n$. The manifold N is said to be area-enlargeable if there exists a function f as above which is ε -area contracting. This means that $\|\bigwedge^2 T_x f\| \leq \varepsilon$ for each $x \in \tilde{N}$, where $\bigwedge^2 T_x f: \bigwedge^2 T_x \tilde{N} \rightarrow \bigwedge^2 T_{f(x)} S^n$.

It turns out that a closed area-enlargeable manifold cannot carry a positive scalar curvature, and the basic tool to prove this theorem is a relative version of the Atiyah–Singer index theorem, cf. [5], Theorem 4.18. So one may expect that the enlargeability obstruction be recovered by the index theoretic obstruction $\alpha_{\max}^{\mathbb{R}}$. In fact T. Schick and B. Hanke in [6, 7] proved that $\alpha_{\max}^{\mathbb{R}}(N) \neq 0$ if N is enlargeable.

Given a complete Riemannian manifold (M, g) it is interesting to decide whether the scalar curvature of g is uniformly positive. Besides its interest in itself, this question has clearly applications to the compact case too. The following theorem is the main result of this paper (see Theorem 3.1 in Section 3):

Theorem 3.1. *Let (M, g) be a complete Riemannian spin manifold, and let N be a closed area-enlargeable submanifold of M with co-dimension 1. If there is a smooth map $\phi: M \rightarrow N$ such that its restriction to N is of non-zero degree then the scalar curvature of g cannot be uniformly positive.*

To prove this assertion, we have put together some basic results and methods introduced by N. Higson, J. Roe, B. Hanke and T. Schick concerning index theory in the context of operator algebras. This result seems not to be obtained easily by means of the geometric methods of [5]. So it shows also the efficiency of operator algebraic index theory to prove results on the non-existence of metrics with positive scalar curvature.

With the above notation, let E be a Clifford bundle over M and put $H = L^2(M, E)$. This a Hilbert space which is assumed to be acted upon by a Dirac-type operator D . The operator $U = (D + i)(D - i)^{-1}$ is bounded on H . Let N be a closed oriented hypersurface which partitions M into two submanifolds M_- and M_+ with common boundary N . The restriction of D to N defines a Dirac-type operator D_N with Fredholm index $\text{ind } D_N$. Let ϕ_+ denote a smooth function on M which coincides with the characteristic function of M_+ outside a compact set and put $\phi_- = 1 - \phi_+$. It turns out (see [8]) that the bounded operator $U_+ = \phi_- + \phi_+ U$ is Fredholm and its index is denoted by $\text{ind}(D, N)$. With an appropriate choice of

orientations the following relation holds between this index and the Fredholm index of D_N ; see [8], Theorem 1.5, and [13], Theorem 3.3:

$$\text{ind}(D, N) = \text{ind } D_N. \tag{1}$$

This formula has two immediate applications. The first one is to provide a proof for the cobordism invariance of the analytical index of a Dirac-type operator, while the second one is the following: if $\hat{A}(M) \neq 0$, then the scalar curvature of g cannot be uniformly positive. We generalize the above theorem in the context of Hilbert Modules over C^* -algebras, i.e., instead of E we consider a Clifford Hilbert A -module bundle W , for example, $W = S(M) \otimes V$, where $S(M)$ is the spin bundle of M and V is a Hilbert A -module bundle over M . In this case the twisted Dirac operator is denoted by D^V and its restriction to N by D_N^V . The indices $\text{ind}(D^V, N)$ and D_N^V are elements of the K-group $K_0(A)$, and we show in Theorem 2.6 of Section 2 that

$$\text{ind}(D^V, N) = \text{ind } D_N^V.$$

As above this relation can be used to prove the cobordism invariance of $\text{ind } D_N^V$, cf. Corollary 2.7. We have already mentioned Theorem 3.1 in Section 3. This theorem should be considered as a counterpart of the second application of the relation (1).

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2. Index theory on odd dimensional partitioned complete manifolds

Let A be a unital C^* -algebra and let (M, g) be an oriented complete non-compact manifold and let W be a (locally trivial, smooth) Clifford bundle whose fiber is endowed with a compatible structure of a Hilbert A -module, that is as an A -module finitely generated and projective. We assume that this bundle is equipped with a connection which is compatible with the Clifford action of TM and denote the corresponding A -linear Dirac type operator by D . Because of the Clifford structure $W_0 = S_0 \otimes V_0$, where S_0 is the irreducible spin representation space in dimension $\dim M$ and V_0 is a finitely generated projective A -module, endowed with a compatible Hilbert A -module structure. For σ and η two compactly supported smooth sections of W put

$$\langle \sigma, \eta \rangle = \int_M \langle \sigma(x), \eta(x) \rangle d\mu_g(x) \in A.$$

It is easy to show that $|\sigma| = \|\langle \sigma, \sigma \rangle\|^{1/2}$ is a norm on the space $C_c^\infty(M, W)$ of compactly supported smooth sections of the bundle W . The completion of $C_c^\infty(M, W)$ with respect to this norm is the Hilbert A -module $H = L^2(M, W)$. The operator D is a formally self adjoint densely defined operator so it is closable. From now on we replace D by its closure and denote it by the same letter D .

Lemma 2.1. *The operator D is regular and self adjoint.*

Proof. Recall that a closed formally self adjoint operator T on a Hilbert module is regular and self adjoint if and only if $T \pm i$ is surjective. Note also that $T \pm i$ is injective and has closed range: for all $\xi \in \text{Dom } T$ we have $\langle (T \pm i)\xi, (T \pm i)\xi \rangle = \langle \xi, \xi \rangle + \langle D\xi, D\xi \rangle$. Let then $\{\xi_n\}$ be a sequence in $\text{Dom } T$ such that $(T - i)\xi_n$ converges to η . Just because $\|(T - i)\xi\| \geq \|\xi\|$ the sequence ξ_n is Cauchy, thus converges also, say to $\xi \in H$. Since $T - i$ is closed (ξ, η) is in the graph of D , whence η is in the range of $T - i$, i.e this operator has closed range. The same is true for $T + i$.

We therefore have just to prove that $D \pm i$ has dense range. We prove that $D - i$ has dense range; the same proof holds for $D + i$. Let $\xi \in H$ be compactly supported and $n \in \mathbb{N}$. Let M_n be a compact smooth manifold which coincides with M up to distance n from the support of ξ , endowed a A -module bundle W_n and a Dirac operator D_n that coincide with corresponding structures on M . This data can be constructed, e.g. by doubling a sufficiently big compact part of M .

We claim that the operator D_n has the finite propagation speed property: for a section η , the section $e^{isD_n}\eta$ is supported within distance s of the support of η . We first notice that this condition does not depend on the connection used to construct D_n . Indeed, assume D'_n has finite propagation speed and write $D_n = D'_n + T$ where T is a section of the bundle $\text{End}_A(W)$ (a 0-th order differential operator). In the Trotter formula

$$e^{isD_n} = \lim_{n \rightarrow \infty} \left(e^{i\frac{s}{n}D'_n} e^{i\frac{s}{n}T} \right)^n$$

the operator $e^{i\frac{s}{n}T}$ does not change the supports because it is a pointwise element of $\text{End}(V_0)$. The wave operator $e^{i\frac{s}{n}D'_n}$ moves away the supports at most by the distance $\frac{s}{n}$ by the finite propagation speed property. This proves the finite propagation speed property for e^{isD_n} .

Next if W_n is of the form $S \otimes V_0$ where S is an ordinary spin bundle and V_0 a fixed Hilbert- A module, one can take $D'_n = D_0 \otimes 1$ where D_0 is the ordinary Dirac operator on S . We then apply finite propagation for D_0 . The general case follows since W_n is a direct summand of such a bundle.

Define the function f_n on \mathbb{R} by putting

$$f_n(x) = -i \int_0^n e^{-t} e^{-itx} dt = (x - i)^{-1} (1 - e^{-n(1+ix)}).$$

The section ξ considered as a section of $W_n \rightarrow M$ is denoted by ξ_n . Consider the section $f_n(D_n)(\xi_n)$ of W_n . Using the following formula

$$f_n(D_n)(\xi_n) = \int_0^n e^{-s} e^{-isD_n} \xi_n ds$$

and the finite propagation property of the wave operator e^{isD_n} , this section is supported in the region on M_n that coincides with a region in M . Therefore $f_n(D_n)(\xi_n)$

can be considered as a compactly supported section η_n of $W \rightarrow M$ and the following relation holds

$$\|\xi - (D - i)\eta_n\| = \|\xi_n - [(D_n - i)f_n(D)]\xi_n\|.$$

Now $(D_n - i)f_n(D_n) - Id = g_n(D)$, where $g_n(x) = (x - i)f_n(x) - 1 = e^{-n(1+ix)}$. It follows that $\|(D_n - i)f_n(D_n) - Id\| \leq \|g_n\|_\infty = e^{-n}$. So the right hand side of the above relations converges to 0 when $n \rightarrow \infty$. Therefore the range of $D - i$ is dense. \square

For bounded operators P and Q in $\text{End}_A(H)$ by $P \sim Q$ we mean that the difference $P - Q$ is compact in the sense of [11]. The following simple lemma is a key tool in what follows.

Lemma 2.2. (1) *The operator $(D + i)^{-1}$ is a bounded operator on H . Moreover, $\theta(D + i)^{-1} \sim 0$, where θ is a compactly supported function on M .*

(2) *If ϕ is a smooth function of M which is locally constant outside a compact subset, then*

$$[(D \pm i)^{-1}, \phi] \sim 0.$$

Proof. For σ a smooth element of H we have

$$\langle (D + i)\sigma, (D + i)\sigma \rangle = \langle (D^2 + 1)\sigma, \sigma \rangle \geq \langle \sigma, \sigma \rangle,$$

therefore $\|(D + i)^{-1}\| \leq 1$, i.e., $(D + i)^{-1}$ is bounded.

The operator $(D + i)$ is an elliptic differential A -operator in the sense of [11], so there is pseudodifferential A -operators Q and S of negative order such that

$$Q(D + i) - \text{id} = S,$$

which gives rise to the relation

$$\theta Q - \theta(D + i)^{-1} = \theta S(D + i)^{-1}.$$

This shows that it is enough to prove the compactness of the operators θQ and θS . Moreover for each positive number δ we may assume Q and S be of δ -propagation speed. This can be done by multiplying the kernels of this operator with smooth functions which are supported around the diagonal of $M \times M$. Given a bounded sequence $\{\gamma_j\}$ in H we show that the sequence $(Q\gamma_j)|_{\text{supp } \theta}$ contains a convergent subsequence and similarly for the sequence $(S\gamma_j)|_{\text{supp } \theta}$. For this purpose let M' be an open submanifold of M with compact closure such that it contains 2δ -neighborhood of $\text{supp }(\theta)$. Deform all geometric structures in the δ neighborhood of $\partial M'$ to product structures and consider the double compact manifold $(M' \sqcup_{\partial M'} M'^{-})$ with Clifford Hilbert A -module bundle induced from W . Corresponding to the sequence $\{\gamma_j\}$ in

H consider the sequence $\{\gamma'_j\}$ in $(M' \sqcup_{\partial M'} M'^-)$ such that $\gamma'_{j|M'} = \gamma_{j|M'}$. Since Q has propagation speed δ , we have

$$(Q\gamma_j)|_{\text{supp } \theta} = (Q'\gamma'_j)|_{\text{supp } \theta}. \tag{2}$$

Here Q' is any pseudodifferential A -operator of negative order on double space whose kernel is equal to the kernel of Q on M' . Being of negative order, Q' is a compact operator on H (cf. [11], p. 109), so $\{Q'\gamma'_j\}$ has a L^2 -convergent subsequence. The restriction of this subsequence to $\text{supp } \theta$ is convergent too, which proves what we wanted in view of the relation (2). The compactness of θS can be shown similarly.

To prove the second part notice that

$$[(D \pm i)^{-1}, \phi] = (D \pm i)^{-1}[\phi, D](D \pm i)^{-1}.$$

On the other hand, $[D, \phi]\sigma = \text{grad}(\phi) \cdot \sigma$, where “ \cdot ” denotes the Clifford action. Since the gradient vector field $\text{grad}(\phi)$ vanishes outside a compact set, for an appropriate compactly supported function θ on M one has $[D, \phi] = [D, \phi]\theta$, so the first part of the lemma can be used to deduce the compactness of $[\phi, D](D \pm i)^{-1}$ and hence the compactness of $[(D \pm i)^{-1}, \phi]$. \square

Suppose now that M is partitioned by an oriented compact hypersurface N into two parts M_+ and M_- . We assume that the positive direction of the unit normal vector \vec{n} to $N \subset M$ points from M_- toward M_+ . Let ϕ_+ be a smooth function on M which is equal to the characteristic function of M_+ outside a compact subset of M , and put $\phi_- = 1 - \phi_+$. To state the next theorem we recall from [11], p. 96, that a bounded operator $P \in \text{End}_A(H)$ is a Fredholm A -operator if there is a decomposition $H = H_0 \oplus H_1$ of the source space and a decomposition $H = H'_0 \oplus H'_1$ of the target space such that H_0 and H'_0 are finitely generated A -modules and if the matrix form of P , with respect to these decompositions, is given by

$$\begin{pmatrix} P_0 & 0 \\ 0 & P_1 \end{pmatrix},$$

where $P_1: H_1 \rightarrow H'_1$ is an isomorphism with bounded inverse. The Fomenko–Mishchenko index of the Fredholm A -operator P is given by

$$\text{ind } P = [H_0] - [H'_0] \in K_0(A)$$

and turns out to be independent of the choices made in its definition. This index is a homotopy invariant and is invariant with respect to perturbations by compact A -operators, cf. Lemmas 1.5 and 2.3 in [11]. Moreover, if Q is another Fredholm A -operator then

$$\text{ind}(PQ) = \text{ind } P + \text{ind } Q \in K_0(A);$$

see [11], Lemma 2.3.

Theorem 2.3. *Let $U = (D - i)(D + i)^{-1}$ be the Cayley transform of D . The following holds:*

- (1) *The operators $U_+ = \phi_- + \phi_+U$ and $U_- = \phi_+ + \phi_-U$ are Fredholm operators in $\text{End}_A(H)$.*
- (2) $\text{ind}(U_+) = -\text{ind}(U_-)$.
- (3) *The value of $\text{ind}(U_+)$ does not depend on ϕ_+ but on the cobordism class of the partitioning manifold N .*

Proof. Clearly both U_{\pm} are A -linear and bounded. Since $U = 1 - 2i(D + i)^{-1}$, the second part of the above lemma implies that

$$\phi_+U = \phi_+ - 2i\phi_+(D + i)^{-1} \sim \phi_+ - 2i(D + i)^{-1}\phi_+ = U\phi_+$$

and similarly $\phi_-U \sim U\phi_-$. Since the support of $\phi_+\phi_-$ is compact, it follows from the first part of the previous lemma that

$$\phi_+\phi_-U = \phi_+\phi_- - 2i\phi_+\phi_-(D + i)^{-1} \sim \phi_+\phi_-.$$

Using these relation one has $U_{\pm}U_{\pm}^* \sim \text{id}$ and $U_{\pm}^*U_{\pm} \sim \text{id}$, so both U_+ and U_- are Fredholm A -operator according to [11], Theorem 2.4. Since $U_+U_- = \text{id}$, we get the relation

$$\text{ind } U_+ + \text{ind } U_- = \text{ind } U = 0 \in K_0(A).$$

A different choice for ϕ_+ is of the form $\phi_+ + \phi$, where ϕ is a compactly supported smooth function. The corresponding operator differs from U_+ by

$$-\phi + \phi U = -\phi + \phi(1 - 2i(D + i)^{-1}) \sim 0,$$

where the equivalence is coming from first part of the previous lemma. Consequently the index of U_+ does not depend on the choice of ϕ_+ . The third part of the theorem is a direct consequence of the second part. □

The Fomenko–Mishchenko index of the operator U_+ is denoted by $\text{ind}(D, N) \in K_0(A)$. The following property of this index is crucial for our purposes.

Theorem 2.4. *If D is an isomorphism, that is, has the bounded inverse, then $\text{ind}(D, N) = 0$. In particular, if D^V is the spin Dirac operator on M twisted by the Hilbert A -module bundle V and if the scalar curvature κ of g is uniformly positive and the curvature of V is sufficiently small, then $\text{ind}(D^V, N) = 0 \in K_0(A)$.*

Proof. If D has bounded inverse, then for $0 \leq s \leq 1$ the families $(D - si)$ and $(D + si)^{-1}$ of bounded operator are continuous. So $U(s) := (D - si)(D + si)^{-1}$ is a homotopy of bounded operators and $U_+(s) = \phi_- + \phi_+U(s)$ is a homotopy of Fredholm A -operators between U_+ and id . By the homotopy invariance of the Fomenko–Mishchenko index we get $\text{ind}(D, N) = 0$.

To prove the second part, assume that there exists $\kappa_0 > 0$ such that everywhere $\kappa \geq \kappa_0$. By the Lichnerowicz formula

$$(D^V)^2 = \nabla^* \nabla + \frac{1}{4} \kappa + R,$$

where $R \in \text{End}_A(V)$. Assume R is sufficiently small, say $\|R\| \leq \frac{1}{8} \kappa_0$. Thus we obtain the following inequality in A , where σ is a smooth L^2 -section of $S(M) \otimes V$,

$$\langle D^V(\sigma), D^V(\sigma) \rangle \geq \frac{1}{8} \kappa_0 \langle \sigma, \sigma \rangle,$$

which implies the boundedness of $(D^V)^{-1}$. The assertion follows now from the first part. □

Remark. The proof of the above theorem may be slightly modified to show that $\text{ind}(D, N) = 0$ if there is a gap in the L^2 -spectrum of D .

The following lemma shows that the index $\text{ind}(D, N)$ is invariant with respect to modifications of data at each partition M_+ or M_- .

Lemma 2.5. *For $j = 1, 2$ let M_j be a complete manifold partitioned by compact hypersurface $N_j \subset M_j$ and let W_j be a Clifford Hilbert A -module bundle on M_j . If there is an isometry $\gamma: M_{2+} \rightarrow M_{1+}$ which is lifted to an isomorphism of Clifford and Hilbert module structures, then*

$$\text{ind}(D_1, N_1) = \text{ind}(D_2, N_2).$$

A similar assertion is true for M_{j-} .

Proof. Let ϕ_1 be a smooth function on M_1 that vanishes in a neighborhood of M_{1-} and is equal to 1 outside a compact subset of M_{1+} . Notice that $\phi_2 = \phi_1 \circ \gamma$ is defined only on M_{2+} but can be extended by zero to whole M_2 . As in Theorem 2.3 we have

$$\begin{aligned} U_{1+} &= 1 + 2i\phi_1(D_1 + i)^{-1}, \\ U_{2+} &= 1 + 2i\phi_2(D_2 + i)^{-1} \sim 1 + 2i(D_2 + i)^{-1}\phi_2. \end{aligned}$$

The map γ provides a unitary isomorphism $\Gamma: L^2(M_{1+}, W) \rightarrow L^2(M_{2+}, W)$. By taking an arbitrary isomorphism $L^2(M_{1-}, W) \rightarrow L^2(M_{2-}, W)$ and using the direct decomposition $L^2(M_j, W) = L^2(M_{j-}, W) \oplus L^2(M_{j+}, W)$ we get an isomorphism

$$T: L^2(M_1, W) \rightarrow L^2(M_2, W).$$

One has $(D_2 + i)\Gamma\phi_1 = \Gamma(D_1 + i)\phi_1$ and $\phi_2\Gamma(D_1 + i) = \Gamma\phi_1(D_1 + i)$, so

$$\begin{aligned} T U_{1+} - U_{2+} T &\sim T(1 + 2i\phi_1(D_1 + i)^{-1}) - (1 + 2i(D_2 + i)^{-1}\phi_2)T \\ &= 2i(\Gamma\phi_1(D_1 + i)^{-1} - (D_2 + i)^{-1}\phi_2\Gamma) \\ &= 2i(D_2 + i)^{-1}\Gamma[D_1, \phi_1](D_1 + i)^{-1}. \end{aligned}$$

Now one can proceed as in the proof of the second part of the Lemma 2.2 to deduce that the last expression is a compact operator. Consequently $TU_{1+}T^{-1} \sim U_{2+}$, which implies

$$\text{ind } U_{1+} = \text{ind } U_{2+} \in K_0(A). \quad \square$$

The Clifford action of $i\vec{n}$ provides a \mathbb{Z}_2 -grading for $W|_N$. Let D_N denote the Dirac-type operator acting on smooth sections of $W|_N \rightarrow N$. This is a A -linear elliptic operator and has an index $\text{ind } D_N \in K_0(A)$.

The following theorem, as well as its proof, is a generalization of Theorem 1.5 of [8] to the context of the Hilbert module bundles (see also [13]).

Theorem 2.6. *In the K -group $K_0(A)$ the equality*

$$\text{ind } D_N = \text{ind}(D, N).$$

holds.

Proof. As a first step we show that it is enough to prove the theorem for the cylindrical manifold $\mathbb{R} \times N$ with product metric $(dx)^2 + g_N$ and pull-back bundle $p^*(V|_N)$, where p is the projection of $\mathbb{R} \times N$ onto the second factor. Consider a collar neighborhood $(-1, 1) \times N$ in M . Using Lemma 2.5 we may change M_- to the product form $(-\infty, 1/2) \times N$ without changing the index $\text{ind}(M, N)$. By applying the third part of Theorem 2.3 we may assume that the partitioning manifold is $\{1/2\} \times N$, then Lemma 2.5 can be used again to replace M with the cylinder $\mathbb{R} \times N$ without changing neither $\text{ind}(M, N)$ nor $\text{ind } D_N$. Consequently, to prove the theorem it suffices to prove it in the special case of a cylinder. At first we prove the theorem for the very special case of the Euclidean Dirac operator $-id/dx$ twisted by the finitely projective A -module V_0 . We denote this twisted Dirac operator by $D_{\mathbb{R}}^{V_0}$ and prove the relation

$$\text{ind}(\pm D_{\mathbb{R}}^{V_0}) = \pm [V_0] \in K_0(A). \quad (3)$$

For this purpose, let ϕ_+ be a smooth function on \mathbb{R} satisfying the conditions of Theorem 2.3 and put $\psi = 2\phi_+ - 1$. One has

$$U_+ = (D_{\mathbb{R}}^{V_0} - i\psi)(D + i)^{-1}.$$

Thus the L^2 -kernels of U_+ and U_+^* , as A -modules, are isomorphic to $\ker(D - i\psi)$ and to $\ker(D + i\psi)$, respectively. The space of L^2 -solutions of $U_+^* = -id/dx + i\psi$ is null, while the space of L^2 -solutions of $U_+ = -id/dx - i\psi$ consists of the smooth functions

$$f(x) = \exp\left(-\int_0^x \psi(t) dt\right)v, \quad v \in V_0.$$

Consequently the L^2 -kernels of U_+ and U_+^* are isomorphic to the finitely generated projective A -modules V_0 and 0 , so $\text{ind}(U_+) = [V_0] - 0$, which is the desired relation. The case of $-D_{\mathbb{R}}^{V_0}$ is similar.

Now we are going to prove the theorem for the cylinder $(\mathbb{R} \times N, (dx)^2 + g_N)$, which completes the proof as is explained above. The operator D^V has the form

$$D^V = \begin{pmatrix} i\partial_x & (D_N^V)^- \\ (D_N^V)^+ & -i\partial_x \end{pmatrix}.$$

Consider the Dirac-type operator D_N^V as an unbounded operator on $L^2(N, W|_N)$. As it has been pointed out just before Theorem 2.3, there is a decomposition of $L^2(N, W|_N) = \mathcal{W}_0 \oplus \mathcal{W}_1$ into a direct sum of invariant A -modules such that \mathcal{W}_0 is finitely generated and projective. Moreover, the restriction of D_N^V to \mathcal{W}_1 has a bounded inverse. The operator $\mathcal{R} := (D_N^V)|_{\mathcal{W}_0} \oplus 0$ is a compact A -operator on $L^2(N, W|_N)$, so the operator

$$\tilde{D}_N := \begin{pmatrix} 0 & 0 \\ 0 & (D_N^V)|_{\mathcal{W}_1} \end{pmatrix}$$

is a compact perturbation of D_N^V , consequently

$$\text{ind } D_N^V = \text{ind } \tilde{D}_N = [\mathcal{W}_0^+] - [\mathcal{W}_0^-] \in K_0(A). \tag{4}$$

The Cayley transform of the family of operators

$$D_s = \epsilon i \frac{d}{dx} + \begin{pmatrix} s\mathcal{R} & 0 \\ 0 & (D_N^V)|_{\mathcal{W}_1} \end{pmatrix}$$

is a continuous family of bounded operators to which the Lemma 2.2 is applicable. Here ϵ stands for the grading operator. Therefore we have the homotopy $U_+(D_s)$ of Fredholm operators with the same index in $K_0(A)$. Therefore

$$\text{ind}(D^V, N) = \text{ind}(\bar{D}, N), \tag{5}$$

where

$$\bar{D} = \epsilon i \frac{d}{dx} + \begin{pmatrix} 0 & 0 \\ 0 & (D_N^V)|_{\mathcal{W}_1} \end{pmatrix}.$$

Remark. Notice that \bar{D} is not a twisted Dirac operator acting on sections of a Hilbert module bundle. Nevertheless since the Lemma 2.2 is applicable to this operator $U_+(\bar{D})$ is a Fredholm A -operator on $L^2(M, \mathcal{W}_1)$ and we can define the index $\text{ind}(\bar{D}, N) = \text{ind } U_+(\bar{D})$ as an element in $K_0(A)$.

With respect to the direct sum $L^2(\mathbb{R} \times N, W) = L^2(\mathbb{R}, \mathcal{W}_0) \oplus L^2(\mathbb{R}, \mathcal{W}_1)$ we put $\bar{D} = \bar{D}_0 \oplus \bar{D}_1$. In the view of the above remark it is clear that

$$\text{ind}(\bar{D}, N) = \text{ind}(\bar{D}_0, N) + \text{ind}(\bar{D}_1, N) \in K_0(A).$$

Notice that \bar{D}_0 is the Euclidean Dirac operator (up to sign) twisted by a finitely generated module, so $\text{ind}(\bar{D}_0, N)$ is well defined. If σ is a smooth element of $L^2(\mathbb{R}, \mathcal{W}_1)$, then

$$\begin{aligned} \|(\bar{D}_1 \pm i\psi)\sigma\|^2 &= \langle (\bar{D}_1 \mp i\psi)(\bar{D}_1 \pm i\psi)\sigma, \sigma \rangle \\ &= \langle (D_N^V)^* D_N^V \sigma + ((id/dx \pm i\psi)^*(id/dx \pm i\psi))\sigma \rangle \\ &\geq \|D_N^V \sigma\|^2 \\ &\geq \delta \|\sigma\|^2 \quad \text{for } \delta > 0, \end{aligned}$$

where the last inequality results from the fact that $(D_N^V)|_{\mathcal{W}_1}$ has a continuous inverse. Now the argument in the proof of the theorem can be applied to deduce that $\text{ind}(\bar{D}_1, N) = 0 \in K_0(A)$. On the other hand, the operator \bar{D}_0 has the form

$$D_{\mathbb{R}}^{\mathcal{W}_0} \oplus -D_{\mathbb{R}}^{\mathcal{W}_1},$$

acting on $L^2(\mathbb{R}, \mathcal{W}_0^+) \oplus L^2(\mathbb{R}, \mathcal{W}_0^-)$. Therefore the equality

$$\text{ind}(\bar{D}_0, N) = [\mathcal{W}_0^+] - [\mathcal{W}_0^-] \in K_0(A)$$

follows by applying the relations (3). This relation together with (4) and (5) prove the theorem for the cylindrical case and so completes the proof of the theorem. \square

The following theorem is an immediate application of the previous theorem.

Corollary 2.7. *Let N be a closed even-dimensional manifold and W be a Clifford Hilbert A -module bundle on N . Let D be a Dirac-type operator acting on sections of W . If there is a compact manifold M with $N = \partial M$ and if all geometric structures extend to M , then $\text{ind } D = 0 \in K_0(A)$.*

3. Partitioning by enlargeable manifolds

In this section we apply Theorem 2.6 to prove the following theorem concerning the existence of complete metrics on non-compact manifolds with uniformly positive scalar curvature.

Theorem 3.1. *Let (M, g) be a non-compact orientable complete Riemannian spin n -dimensional manifold where $n \geq 2$. Let N be an $(n - 1)$ -dimensional area-enlargeable closed submanifold of M which partitions M . If there is a map $\phi: M \rightarrow N$ such that its restriction to N has non-zero degree, then the scalar curvature of g cannot be uniformly positive.*

Proof. At first notice that M , hence N , may be assumed to be spin. If not, we consider the finite spin Riemannian covering (\tilde{M}, \tilde{g}) where \tilde{g} is the lifting of g . Let

$\tilde{N} \subset \tilde{M}$ be the induced covering for N which is area-enlargeable. Its normal bundle is trivial, so \tilde{N} is spin, too. The function ϕ has a lifting to a function $\tilde{\phi}: \tilde{M} \rightarrow \tilde{N}$ such that its restriction to \tilde{N} is of non-zero degree. Moreover, the scalar curvature of g is uniformly positive if and only if the scalar curvature of \tilde{g} is uniformly positive. This shows that if we prove the assertion of the theorem for \tilde{M} , then it follows for M as well.

We can also assume n to be an odd integer. If not, consider the complete manifold $(M \times S^1, g \oplus g_0)$ where g_0 is any Riemannian metric on S^1 . If M has a finite spin covering \tilde{M} , then $\tilde{M} \times S^1$ is a finite spin covering for $M \times S^1$. The restriction of the map $\phi \times \text{id}: M \times S^1 \rightarrow N \times S^1$ has non-zero degree and $N \times S^1$ is area-enlargeable. Moreover, if the scalar curvature of g is uniformly positive, then the scalar curvature of $g \oplus g_0$ is uniformly positive, so it suffices to prove the theorem for the odd-dimensional complete manifold $(M \times S^1, g \oplus g_0)$.

To prove this theorem, we use some methods and constructions introduced by B. Hanke and T. Schick in [6] and [7]. Following [7], Proposition 1.5, since N is area-enlargeable, for each positive natural number k there exists a C^* -algebra A_k and a Hilbert A_k -module bundle $V_k \rightarrow N$ with connection ∇_k with following properties: The curvature Ω_k of V_k satisfies

$$\|\Omega_k\| \leq \frac{1}{k},$$

and there exists a split extension

$$0 \rightarrow \mathbb{K} \rightarrow A_k \rightarrow \Gamma_k \rightarrow 0,$$

where \mathbb{K} denotes the algebra of compact operators on an infinite-dimensional and separable Hilbert space, and Γ_k is a certain C^* -algebra. In particular, each $K_0(A_k)$ canonically splits off a $\mathbb{Z} = K_0(\mathbb{K})$ -summand. Denote by W_k the spin bundle twisted by V_k , and by $D_N^{V_k}$ the associated twisted Dirac operator. If $a_k \in K_0(A_k)$ denotes the index of $D_N^{V_k}$, then the $\mathbb{Z} = K_0(\mathbb{K})$ -component z_k of a_k is non-zero. Moreover, there is a dense subalgebra \mathcal{A}_k of A_k which is closed under holomorphic calculus and there is a continuous trace $\alpha_k: \mathcal{A}_k \rightarrow \mathbb{C}$ such that $z_k = \text{ind}_{\alpha_k} D_N^{V_k}$. Here we use the fact that $\text{ind} D_N^{V_k} \in K_0(A_k) = K_0(\mathcal{A}_k)$, so the expression $\text{ind}_{\alpha_k} D_N^{V_k}$ makes sense. This index can be calculated in terms of the geometry of (M, g) and of the bundle (V_k, ∇^{V_k}) . Theorem 9.2 of [17] gives the explicit formula

$$z_k = \text{ind}_{\alpha_k} D_N^{V_k} = \int_N A(TM) \wedge [\text{Ch}_{\alpha_k}(V_k, \nabla^{V_k})]_+ \tag{6}$$

for that index. Here $[\omega]$ denotes the positive degree part of the differential form $\omega \in \Omega^*(N)$, and $\text{Ch}_{\alpha_k}(V_k, \nabla^{V_k})$ is defined in terms of the curvature Ω_k by the relation

$$\text{Ch}_{\alpha_k}(V_k, \nabla^{V_k}) := \alpha_k \left(\text{str} \sum_{j=0}^{\infty} \frac{\Omega_k \wedge \dots \wedge \Omega_k}{j!} \right), \tag{7}$$

which provides a closed differential form on N . The class of $\text{Ch}_{\alpha_k}(V_k, \nabla^{V_k})$ in the de Rham cohomology of N is independent of the connection ∇^{V_k} , so it is determined by the class of V_k in $K_0(A_k)$. Since $K_0(\mathcal{A}_k) \simeq K_0(A_k)$, one can assume that the value of the expression between the parentheses on the right-hand side of (7) belongs to the space \mathcal{A}^{ab} (quotient of \mathcal{A} by additive commutators), so is in the domain of α_k . This justifies the definition of the Chern character.

It is clear from the definition that for a smooth function $\psi : N \rightarrow N$ one has

$$\text{Ch}_{\alpha_k}(\psi^* V_k, \nabla^{\psi^* V_k}) = \psi^* \text{Ch}_{\alpha_k}(V_k, \nabla^{V_k}).$$

The main feature of the virtual bundle V_k , coming from area-enlargeability of N , is that the Chern character $\text{Ch}_\gamma(V_k, \nabla^{V_k})$ is, in fact, a closed differential $(n - 1)$ -form. So given a smooth map $\psi : N \rightarrow N$, the explicit formula (6) implies the relation

$$\text{ind}_{\alpha_k} D_N^{\psi^* V_k} = \text{deg}(\psi) \cdot \text{ind}_{\alpha_k} D_N^{V_k}. \tag{8}$$

To use Theorem 2.6, we need to work with flat bundles on N . For this purpose we use another fundamental construction introduced in [6]. This construction consists of *assembling* the algebras A_k , the almost flat sequence of bundles V_k and the connections ∇^{V_k} to construct a C^* -algebra A , a Hilbert A -module bundle V and a flat connection ∇^V such that the index of the twisted Dirac operator D_N^V , acting on smooth sections of $W = S \otimes V$ and taking its value in $K_0(A)$, keeps track of the index theoretic information of $D_N^{V_k}$ when k goes toward infinity. Denote by $\prod^b A_k$ the C^* -algebra consisting of all uniformly bounded sequences (a_1, a_2, \dots) with $a_k \in A_k$ and by $\prod^0 A_k$ the C^* -algebra consisting of all sequences (a_1, a_2, \dots) such that the sequence $\{\|a_k\|\}_k$ converges to 0. The C^* -algebra A is defined by the quotient

$$A := \frac{\prod^b A_k}{\prod^0 A_k}.$$

The C^* -algebras Γ and \mathcal{K} are constructed from $\{\Gamma_k\}_k$ and from $\{\mathbb{K}\}_k$ by similar quotients. Clearly one has the split exact sequence

$$0 \rightarrow \mathcal{K} \rightarrow A \rightarrow \Gamma \rightarrow 0, \tag{9}$$

which gives rise to the following split exact sequence

$$0 \rightarrow K_0(\mathcal{K}) \rightarrow K_0(A) \rightarrow K_0(\Gamma) \rightarrow 0. \tag{10}$$

It turns out that

$$K_0(\mathcal{K}) \simeq \frac{\prod K_0(\mathbb{K})}{\bigoplus K_0(\mathbb{K})} \simeq \frac{\prod \mathbb{Z}}{\bigoplus \mathbb{Z}}.$$

Proposition 1.5 of [6] and the subsequent discussion show that the component of $\text{ind } D_N^V$ in $K_0(\mathcal{K})$ can be represented, with respect to the above isomorphism, by

$z = [(z_1, z_2, \dots)]$, where $z_k = \alpha_k(\text{ind } D_N^{V_k}) \neq 0$ for all $k \in \mathbb{N}$. This implies the non-vanishing result $\text{ind } D_N^V \neq 0 \in K_0(\mathcal{K})$.

For ψ a smooth function on N it turns out from the construction of the bundle V that the pull-back bundle $\psi^*(V)$ may be constructed by assembling the bundles $\psi^*(V_k)$. Using (8) and the above description of the $K_0(\mathcal{K})$ -component of the higher index $\text{ind}(D_N^{\psi^*V}) \in K_0(A)$, we conclude that the $K_0(\mathcal{K})$ -component of $\text{ind } D_N^{\psi^*V}$ is equal to $\text{deg}(\psi)$ -times of the $K_0(\mathcal{K})$ -component of $\text{ind } D_N^V$. Since this last component is non-vanishing, we conclude that

$$\text{ind } D_N^{\psi^*V} \neq 0 \in K_0(\mathcal{A}), \quad (11)$$

provided that $\text{deg}(\psi) \neq 0$. Now we are able to apply Theorem 2.6. Using the map $\phi: M \rightarrow N$ we construct the pull-back bundle ϕ^*V and the pull-back connection $\phi^*\nabla$ which is flat. Let D^{ϕ^*V} be the spin Dirac operator of M twisted by the flat Hilbert A -module bundle $(\phi^*V, \phi^*\nabla)$. The restriction of this bundle to N is ψ^*V , where $\psi := \phi|_N$ is of non-zero degree. By Theorem 2.6 we have

$$\text{ind}(D^{\phi^*V}, N) = \text{ind } D_N^{\psi^*V},$$

which, by relation (11), gives the non-vanishing result

$$\text{ind}(D^{\phi^*V}, N) \neq 0 \in K_0(A). \quad (12)$$

On the other hand, by Lemma 2.4, if the scalar curvature of g is uniformly positive, then odd- $\text{ind } D^{\phi^*V}$ vanishes. This is in contradiction with the above non-vanishing result and completes the proof of the theorem. \square

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