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Functional analysis. — *A spectral Schwarz lemma*, by EDOARDO VESENTINI.

ABSTRACT. — The classical Schwarz lemma for any scalar-valued holomorphic function h mapping the open unit disc $\Delta \subset \mathbb{C}$ into itself is generalized by replacing h by a holomorphic map f of Δ into a unital associative Banach algebra $\mathcal{A}$, and $|h(z)|$ by the spectral radius of $f(z)$ $(z \in \Delta)$. If $\mathcal{A} = \mathcal{L}(\mathcal{H})$ and $\mathcal{H}$ is a complex Hilbert space, the behaviour of the numerical radius of $f(z)$ is also investigated.

KEY WORDS: Banach algebra; spectrum; spectral radius; numerical radius.

MATHEMATICS SUBJECT CLASSIFICATION (2000): Primary 30E25.

Let $\mathcal{A}$ be a unital, associative Banach algebra and let f be a holomorphic map of the open unit disc Δ of $\mathbb{C}$ into $\mathcal{A}$. In the "classical" case, where $\mathcal{A} = \mathbb{C}$, the Schwarz lemma is the main tool in the construction of a geometric framework—offered by the Poincaré metric—for a comprehensive analysis of the behaviour of f. In the general case, when $\mathbb{C}$ is replaced by the Banach algebra $\mathcal{A}$, the spectral radius, numerical radius, spectrum, numerical range, etc. are gauges of the behaviour of the $\mathcal{A}$-valued holomorphic function f. In both settings, the theory of subharmonic functions—and in particular the maximum principle for these functions—play a crucial role; the connections between spectrum and spectral radius, numerical range and numerical radius, disclose new insights into the behaviour of f.

The main part of this article is devoted to elaborating on some of these new insights, investigating in particular spectrum-valued functions associated to holomorphic maps of Δ into $\mathcal{A}$, with special attention to the case in which $\mathcal{A}$ is commutative. In the final sections of the paper, Δ will be replaced by a domain $E \subset \mathbb{C}$ and the euclidean distance on Δ by the Carathéodory distance on E; the spectral invariants will be expressed in terms of the Hausdorff distance between spectra.

The results concerning the holomorphic map $\Delta \to \mathcal{A}$ yield—*via* Dunford integral —an approach which might lead to a "Schwarz lemma" for holomorphic maps of the open unit ball of $\mathcal{A}$ into itself.

1. SPECTRAL VERSIONS OF THE SCHWARZ LEMMA

Let $\mathcal{A}$ be a unital, associative Banach algebra,[1] and let $\mu : \mathcal{A} \to \mathbb{R}_+$ be a plurisubharmonic function on $\mathcal{A}$ such that $\mu(zx) = |z|\mu(x)$ for all $z \in \mathbb{C}$ and $x \in \mathcal{A}$.

[1] All algebras in this paper will be tacitly assumed to be associative.

Let $f : \Delta \to \mathcal{A}$ be a holomorphic map of the open unit disc $\Delta = \{z \in \mathbb{C} : |z| < 1\}$ into $\mathcal{A}$ such that

$$\mu(f(z)) \leq 1 \quad \forall z \in \Delta. \tag{1}$$

If

$$f(z) = a_0 + za_1 + z^2 a_2 + \cdots, \qquad \text{with } a_0, a_1, a_2, \ldots \in \mathcal{A}, \tag{2}$$

is the power-series expansion of f in Δ, then

$$f'(z) = a_1 + 2za_2 + \cdots \quad \forall z \in \Delta.$$

If

$$f(0) = a_0 = 0, \tag{3}$$

the function

$$g : z \mapsto \frac{1}{z} f(z) = a_1 + za_2 + \cdots$$

is holomorphic on Δ.

Suppose that

$$\sup\{\mu(g(z)) : z \in \Delta\} > 1,$$

i.e., there are $z_0 \in \Delta$ and $\vartheta > 0$ such that

$$\mu(g(z_0)) \geq 1 + \vartheta.$$

By the maximum principle for subharmonic functions, for any $r \in (|z_0|, 1)$ there is some $z \in \Delta$ with $|z| = r$ such that

$$\mu(g(z)) \geq 1 + \vartheta,$$

and therefore

$$\mu(f(z)) = |z|\mu(g(z)) \geq r(1 + \vartheta) > 1$$

whenever r is sufficiently close to 1, contradicting (1). Thus,

$$\mu(f(z)) \leq |z| \quad \forall z \in \Delta, \tag{4}$$

i.e.

$$\mu(za_1 + z^2 a_2 + \cdots) \leq |z|, \tag{5}$$

whence

$$\mu(f'(0)) = \mu(a_1) \leq 1. \tag{6}$$

Again by the maximum principle, if $\mu(a_1) = 1$, then

$$\mu\left(\frac{1}{z} f(z)\right) = 1,$$

that is to say,

$$\mu(f'(0)) = 1 \;\Rightarrow\; \mu(f(z)) = |z| \quad \forall z \in \Delta.$$

That proves

LEMMA 1. *If $f(0) = 0$ and if* (1) *holds, then* (4) *and* (6) *are satisfied. If either $\mu(f'(0)) = 1$ or there is some $z \in \Delta \setminus \{0\}$ such that*

(7) $$\mu(f(z)) = |z|,$$

then this latter equality holds for all $z \in \Delta$.

This lemma yields a "spectral version" of the classical Schwarz lemma for holomorphic scalar-valued functions of one complex variable.

Let $\sigma(x)$ and $\varrho(x)$ denote respectively the spectrum and the spectral radius of any $x \in \mathcal{A}$. Since the function $\log \circ \varrho$ is plurisubharmonic on $\mathcal{A}$ ([7], [8]), and therefore ϱ is plurisubharmonic on $\mathcal{A}$, Lemma 1 (with $\mu = \varrho$) yields the first part of the following

THEOREM 1. *Let f be a holomorphic map of Δ into $\mathcal{A}$. If $f(0) = 0$ (for example, if $\mathcal{A}$ contains no non-zero topologically nilpotent element and $\sigma(f(0)) = \{0\}$) and if*

(8) $$\sigma(f(z)) \subset \overline{\Delta} \quad \forall z \in \Delta,$$

or equivalently, $\varrho(f(z)) \leq 1$ for all $z \in \Delta$, then

(9) $$\varrho(f(z)) \leq |z| \quad \forall z \in \Delta$$

and

(10) $$\varrho(f'(0)) \leq 1.$$

Moreover, if either $\varrho(f'(0)) = 1$ or there is some $z \in \Delta \setminus \{0\}$ such that

(11) $$\varrho(f(z)) = |z|,$$

then this latter equality holds for all $z \in \Delta$, and the intersection

(12) $$L = \sigma\left(\frac{1}{z} f(z)\right) \cap \partial\Delta$$

(is not empty and) does not depend on $z \in \Delta$, i.e. the peripheral spectrum of $f(z)$ is zL for all $z \in \Delta$.

The final statement is a consequence of the maximum principle for the spectral radius ([7, Proposition 2], or, e.g., [8, Proposition 2.7]), according to which, if the map $h : \Delta \to \mathcal{A}$ is holomorphic and $\varrho(h)$ is equal to a constant c on Δ, then the *peripheral spectrum* of $h(z)$ (i.e. the intersection of $\sigma(h(z))$ with the circle with center 0 and radius $\varrho(h(z))$) does not depend on z.

Suppose now that there is $z_0 \in \Delta \setminus \{0\}$ such that the inner spectral radius[2] of $f(z_0)$ is

$$\kappa(f(z_0)) = |z_0|, \tag{13}$$

i.e., $f(z_0) \in \mathcal{A}^{-1}$ and

$$\frac{1}{\varrho(f(z_0)^{-1})} = |z_0|.$$

Since $f(z_0)$ and $f(z_0)^{-1}$ commute, we have

$$\varrho(f(z_0)) \geq \frac{1}{\varrho(f(z_0)^{-1})} = |z_0|,$$

and therefore, by Theorem 1,

$$\varrho(f(z_0)) = |z_0|,$$

proving thereby

LEMMA 2. *Under the hypotheses of Theorem* 1, *if* (13) *holds at some* $z_0 \in \Delta \setminus \{0\}$, *then there is a closed subset* L *of* $\partial\Delta$ *such that the peripheral spectrum of* $f(z)$ *is* zL *for all* $z \in \Delta$.

EXAMPLE. Let $\mathcal{A} = \mathcal{L}(\mathbb{C}^2)$ and let

$$f(z) = \begin{pmatrix} z & 0 \\ cz & z\varphi(z) \end{pmatrix} \tag{14}$$

with $c \in \mathbb{C}$, $\varphi : \Delta \to \mathbb{C}$ holomorphic and $|\varphi(z)| < 1$ for all $z \in \Delta$. Then

$$\sigma(f(z)) = \left\{\zeta \in \mathbb{C} : \det\begin{pmatrix} z-\zeta & 0 \\ cz & z\varphi(z)-\zeta \end{pmatrix} = 0\right\} = \{z, z\varphi(z)\},$$

and therefore

$$\varrho(f(z)) = \max\{|z|, |z|\,|\varphi(z)|\};$$

moreover,

$$f'(z) = \begin{pmatrix} 1 & 0 \\ c & \varphi(z) + z\varphi'(z) \end{pmatrix},$$

$$\sigma(f'(z)) = \left\{\zeta \in \mathbb{C} : \det\begin{pmatrix} 1-\zeta & 0 \\ c & \varphi(z) + z\varphi'(z) - \zeta \end{pmatrix} = 0\right\},$$

[2] The *inner spectral radius* $\kappa(x)$ of any element x of a unital Banach algebra $\mathcal{A}$ is, by definition, $\kappa(x) = \inf\{|\zeta| : \zeta \in \sigma(x)\}$, or equivalently, $\kappa(x) = 1/\varrho(x^{-1})$ if x is invertible in $\mathcal{A}$, and $\kappa(x) = 0$ otherwise.

whence

$$\sigma(f'(z)) = \{1, \varphi(z) + z\varphi'(z)\}, \quad \sigma(f'(0)) = \{1, \varphi(0)\}, \quad \varrho(f'(0)) = 1.$$

If $z \in \Delta \setminus \{0\}$ then

$$\frac{1}{z} f(z) = \begin{pmatrix} 1 & 0 \\ c & \varphi(z) \end{pmatrix}, \quad \sigma\left(\frac{1}{z} f(z)\right) = \{1, \varphi(z)\}, \quad \sigma\left(\frac{1}{z} f(z)\right) \cap \partial\Delta = \{1\}.$$

The spectral radius and the inner spectral radius of $(1/z) f(z)$ are

$$\varrho\left(\frac{1}{z} f(z)\right) = 1 \quad \text{and} \quad \kappa\left(\frac{1}{z} f(z)\right) = |\varphi(z)|.$$

REMARK. The condition $\sigma(f(0)) = \{0\}$ is not sufficient to grant the conclusion of Theorem 1, as the following example shows.

Let f be given by (14), with c and φ as above, and let K be a two by two complex constant matrix, $\neq 0$, given by

$$K = \begin{pmatrix} \alpha & \beta \\ \gamma & -\alpha \end{pmatrix},$$

with

$$\det K = -(\alpha^2 + \beta\gamma) = 0. \tag{15}$$

The function $g : \Delta \ni z \mapsto K + f(z)$, i.e.

$$g(z) = \begin{pmatrix} z + \alpha & \beta \\ cz + \gamma & z\varphi(z) - \alpha \end{pmatrix}, \tag{16}$$

is such that $\sigma(g(0)) = \{0\}$ but does not necessarily satisfy the conclusion of Theorem 1, as will be shown now.

The spectrum of $g(z)$ consists of the roots, ζ_1, ζ_2, of the characteristic equation of the matrix on the right-hand side of (16), i.e.

$$\zeta^2 - z(\varphi(z) + 1)\zeta + z((z + \alpha)\varphi(z) - \alpha - \beta c) - (\alpha^2 + \beta\gamma) = 0,$$

which, by (15), reads

$$\zeta^2 - z(\varphi(z) + 1)\zeta + z((z + \alpha)\varphi(z) - \alpha - \beta c) = 0.$$

Since

$$\zeta_1\zeta_2 = z((z + \alpha)\varphi(z) - \alpha - \beta c),$$

if $\varrho(g(z)) \leq |z|$ for all $z \in \Delta$, then

$$|z((z + \alpha)\varphi(z) - \alpha - \beta c)| \leq |z|^2,$$

and therefore

$$|(z+\alpha)\varphi(z)-\alpha-\beta c| \leq |z|,$$

$$\left|\varphi(z)+\frac{1}{z}(\alpha(\varphi(z)-1)-\beta c)\right| \leq 1 \tag{17}$$

for all $z \in \Delta \setminus \{0\}$.

Choosing φ constant:

$$\varphi(z)=\varphi_0 \in \Delta,$$

and such that

$$\alpha(\varphi_0-1)-\beta c \neq 0,$$

and letting $z \to 0$, (17) yields a contradiction.

Some of the conclusions of Theorem 1 can be rephrased in terms of Oka's set-valued analytic functions ([5], [4], [10]). According to Theorem IV of [6], if F is an analytic set-valued function on Δ such that $F(z)$ is uniformly bounded on Δ, then there is a separable Hilbert space $\mathcal{H}$ and a holomorphic map $f : \Delta \to \mathcal{L}(\mathcal{H})$ such that

$$\sigma(f(z)) = F(z) \quad \forall z \in \Delta.$$

Thus, Theorem 1 yields

COROLLARY 1. *If $F(z) \subset \overline{\Delta}$ for all $z \in \Delta$, and $F(0)=\{0\}$, then*

$$F(z) \subset \overline{\Delta}_{|z|} = \overline{\{\zeta \in \Delta : |\zeta| < |z|\}} \quad \forall z \in \Delta.$$

If $F(z) \subset \partial\Delta_{|z|}$ for some $z \in \Delta \setminus \{0\}$, the same inclusion holds for all $z \in \Delta$.

2. A SCHWARZ LEMMA FOR THE NUMERICAL RANGE AND NUMERICAL RADIUS

Let now $\mathcal{A} = \mathcal{L}(\mathcal{H})$, where $\mathcal{H}$ is a complex Hilbert space, $\| \; \|$ and $(\, | \,)$ being the norm and inner product in $\mathcal{H}$. Let $W(x)$ and $w(x)$ be the numerical range and numerical radius of any $x \in \mathcal{L}(\mathcal{H})$:

$$W(x) = \{\zeta = (x\xi|\xi) : \xi \in \mathcal{H}, \ \|\xi\| = 1\},$$
$$w(x) = \sup\{\|\xi\| : \xi \in W(x)\} = \sup\{|(x\xi|\xi)| : \xi \in \mathcal{H}, \ \|\xi\| = 1\}.$$

If the map $f : \Delta \to \mathcal{L}(\mathcal{H})$ is holomorphic, the function

$$\Delta \ni z \mapsto \log w(f(z))$$

is subharmonic ([1], [8]), and therefore satisfies the maximum principle. If $w \circ f$ reaches a maximum, c, at some point of Δ, and therefore

$$(w \circ f)(z) = c \quad \forall z \in \Delta,$$

then the intersection of $W(f(z))$ with the circle with centre 0 and radius c is independent of z.[3]

A similar argument to the proof of Theorem 1 yields

THEOREM 2. *If* $f(0) = 0$ *and* $w(f(z)) \leq 1$ *for all* $z \in \Delta$*, then*

$$w(f(z)) \leq |z| \quad \forall z \in \Delta, \quad \textit{and} \quad w(f'(0)) \leq 1.$$

If $w(f'(0)) = 1$ *or there is some* $z \in \Delta \setminus \{0\}$ *for which*

$$w(f(z)) = |z|,$$

then this latter equality holds for all $z \in \Delta$*, and the (non-empty) intersection*

$$W\left(\frac{1}{z}f(z)\right) \cap \partial\Delta$$

does not depend on z.

3. THE COMMUTATIVE CASE

Going back to the case of the spectral radius, it turns out—as will be shown now—that, if the unital Banach algebra $\mathcal{A}$ is commutative, some of the conclusions of Theorem 1 can be refined.

If $\Sigma(\mathcal{A})$ is the Gelfand spectrum of the unital Banach algebra $\mathcal{A}$, endowed with the Gelfand topology, then (2) yields

$$\langle f(z), \chi\rangle = \langle a_0, \chi\rangle + \langle a_1, \chi\rangle\, z + \langle a_2, \chi\rangle\, z^2 + \cdots \quad \forall z \in \Delta, \chi \in \Sigma(\mathcal{A}),$$
$$\langle f'(z), \chi\rangle = \langle a_1, \chi\rangle + 2\langle a_2, \chi\rangle\, z + \cdots \quad \forall z \in \Delta, \chi \in \Sigma(\mathcal{A}).$$

If (8) holds, i.e.

$$|\langle f(z), \chi\rangle| \leq 1 \quad \forall z \in \Delta, \chi \in \Sigma(\mathcal{A}),$$

then, by the Cauchy integral formula,

$$|\langle f'(0), \chi\rangle| \leq 1 \quad \forall \chi \in \Sigma(\mathcal{A}),$$

and therefore $\varrho(f'(0)) \leq 1$.

If $\varrho(f'(0)) = 1$, the set Q of all $\chi \in \Sigma(\mathcal{A})$ for which $|\langle f'(0), \chi\rangle| = 1$ is not empty.

Suppose now that the holomorphic map $f : \Delta \to \mathcal{A}$ satisfies (8), and furthermore that $\sigma(f(0)) = \{0\}$, i.e.

$$\langle a_0, \chi\rangle = 0 \quad \forall \chi \in \Sigma(\mathcal{A}).$$

[3] For further information on the behaviour of the set-valued function $z \mapsto W(f(z))$, see [1], [8].

The Schwarz lemma applied to $\langle f(\cdot), \chi\rangle$ for all $\chi \in \Sigma(\mathcal{A})$ then yields (9), and since now

$$\langle f(0), \chi\rangle = \langle f'(0), \chi\rangle,$$

it follows that

$$L = \{\langle f'(0), \chi\rangle : \chi \in Q\},$$

where L is defined by (12).

Hence the weaker condition $\sigma(f(0)) = \{0\}$ suffices to obtain the conclusion of Theorem 1.

Summing up, the following theorem improves Theorem 1 in the commutative case.

THEOREM 3. *If the holomorphic map f of Δ into the unital, commutative Banach algebra $\mathcal{A}$ is such that $\sigma(f(z)) \subset \overline{\Delta}$ for all $z \in \Delta$, then $\varrho(f'(0)) \leq 1$. If moreover $\sigma(f(0)) = \{0\}$, then $\varrho(f(z)) \leq |z|$ for all $z \in \Delta$.*

It will now be shown how a similar approach, based on Gelfand's theory of commutative Banach algebras, yields a "Schwarz lemma" for the inner spectral radius.

If $\kappa(f(0)) = 0$, there is $\chi_0 \in \Sigma(\mathcal{A})$ for which

$$\langle f(0), \chi_0\rangle = 0.$$

By the Schwarz lemma,

$$|\langle f(z), \chi_0\rangle| \leq |z| \quad \forall z \in \Delta,$$

which implies

LEMMA 3. *If the holomorphic map $f : \Delta \to \mathcal{A}$ is such that $\sigma(f(z)) \subset \overline{\Delta}$ for all $z \in \Delta$ and $\kappa(f(0)) = 0$ (i.e. $0 \in \sigma(f(0))$), then*

$$\kappa(f(z)) \leq |z| \quad \forall z \in \Delta.$$

If moreover $\kappa(f(z_0)) = |z_0|$ for some $z_0 \in \Delta \setminus \{0\}$, i.e. if

$$\inf\{|\langle f(z_0), \chi\rangle| : \chi \in \Sigma(\mathcal{A})\} = |z_0|,$$

then the set $\Sigma(z_0) \subset \Sigma(\mathcal{A})$ consisting of all characters χ of $\mathcal{A}$ for which

$$|\langle f(z_0), \chi\rangle| = |z_0|$$

is non-empty. By the Schwarz lemma, for every $\chi \in \Sigma(z_0)$ there is $\theta_\chi \in \mathbb{R}$ such that

$$\langle f(z), \chi\rangle = e^{i\theta_\chi} z \quad \forall z \in \Delta.$$

This conclusion can be made more precise in the following example, in which $\mathcal{A}$ is a uniform algebra on a compact Hausdorff space X.

Let $f : \Delta \times X \to \mathbb{C}$ be such that for every $z \in \Delta$ the function $f_z : X \ni x \mapsto f(z, x)$ is contained in $\mathcal{A}$, with

$$\sup\{|f(z, x)| : x \in X\} \le 1,$$

and for every $x \in X$ the function $\Delta \ni z \mapsto f(z, x)$ is holomorphic on Δ. By Dunford's theorem, the map $z \mapsto f_z$ of Δ into $\mathcal{A}$ is holomorphic on Δ.

If we identify each $x \in X$ with the evaluation δ_x at x, then X becomes a closed subset of the Gelfand spectrum $\Sigma(\mathcal{A})$ of $\mathcal{A}$, and the Shilov boundary $\partial\mathcal{A}$ of $\mathcal{A}$ is a closed subset of X.

If $f(0, x) = 0$ for some $x \in X$ (i.e. $\langle f_0, \chi\rangle = 0$ for some $\chi \in \Sigma(\mathcal{A})$), then by Lemma 3,

$$\kappa(f_z) \le |z| \quad \forall z \in \Delta,$$

i.e., for every $z \in \Delta$,

$$|\langle f_z, \chi\rangle| \le |z| \quad \text{for some } \chi \in \Sigma(\mathcal{A}). \tag{18}$$

The fact that $\partial\mathcal{A}$ can be identified with a closed subset of X implies that if $f(0, X) = \{0\}$, then $f_0 = 0$ on $\partial\mathcal{A}$, and therefore also on $\Sigma(\mathcal{A})$, whence $\sigma(f_0) = \{0\}$. By Theorem 1, $\varrho(f_z) \le |z|$ for all $z \in \Delta$, hence $|\langle f_z, \chi\rangle| \le |z|$ for all $\chi \in \Sigma(\mathcal{A})$, and therefore

$$\sup\{|f(z, x)| : x \in X\} \le |z| \quad \forall z \in \Delta.$$

Suppose now that there exist $z_0 \in \Delta \setminus \{0\}$ and $x_0 \in X$ for which

$$|f(z_0, x_0)| = |z_0|, \tag{19}$$

i.e.

$$\zeta_0 := \frac{1}{z_0}\langle f_{x_0}, \delta_{z_0}\rangle \in \partial\Delta.$$

Since

$$\left|\frac{1}{z} f(z, x_0)\right| \le 1 \quad \forall z \in \Delta,$$

the maximum principle yields:

LEMMA 4. *If $f(0, x) = 0$ for all $x \in X$ and if there exist $z_0 \in \Delta \setminus \{0\}$ and $x_0 \in X$ satisfying* (19)*, then there is $\zeta \in \partial\Delta$ such that $f(z, x_0) = z\zeta$ for all $z \in \Delta$.*

4. A SPECTRAL SCHWARZ LEMMA FOR THE UNIT BALL

Let $B = \{x \in \mathcal{A} : \|x\| < 1\}$ be the open unit ball of a unital Banach algebra $\mathcal{A}$ with no non-zero topologically nilpotent element, and let $F : B \to B$ be a holomorphic map such that $\varrho(F(0)) = 0$.

If $u \in \partial B$, then $1 > \varrho(u) > 0$, and, for $z \in \mathbb{C}$,

$$\varrho(zu) = |z|\varrho(u) \le |z|\,\|u\| = |z|.$$

The holomorphic map $f : \Delta \ni z \mapsto F(zu)$ is such that $f(0) = 0$, and

$$\varrho(f(z)) = \varrho(F(zu)) \leq \|F(zu)\| \leq 1.$$

By Theorem 1,

$$\varrho(f(z)) \leq |z| \quad \forall z \in \Delta,$$

i.e.,

$$\varrho(F(zu)) \leq \|zu\| \quad \forall z \in \Delta, u \in \partial B,$$

and in conclusion

$$\varrho(F(x)) \leq \|x\| \quad \forall x \in B. \tag{20}$$

If $\varrho(F(x_0)) = \|x_0\|$ for some $x_0 \in B \setminus \{0\}$, i.e., upon setting $x_0 = \|x_0\|u_0$ with $u_0 \in \partial B$,

$$\varrho(f(\|x_0\|)) = \varrho(F(\|x_0\|u_0)) = \varrho(F(x_0)) = \|x_0\|,$$

then $\varrho(f(z)) = |z|$ for all $z \in \Delta$, that is to say,

$$\varrho\left(F\left(\frac{z}{z_0} z_0 u_0\right)\right) = |z| = \left\|\frac{z}{z_0} z_0 u_0\right\| = \left\|\frac{z}{z_0} x_0\right\|,$$

i.e.,

$$\varrho(F(zx_0)) = |z|\,\|x_0\| \quad \forall z \in \Delta_{1/\|x_0\|}. \tag{21}$$

Since

$$f'(0) = \frac{d}{dz} F(zu_0)\bigg|_0 = F'(0)u_0,$$

(20) also holds if $\varrho(F'(0)u_0) = 1$, i.e.

$$\varrho(F'(0)x_0) = |z_0| = \|x_0\|.$$

Summing up:

THEOREM 4. *If $\sigma(F(0)) = \{0\}$, then* (20) *holds. If either*

$$\varrho(F(x_0)) = \|x_0\| \quad or \quad \varrho(F'(0)x_0) = \|x_0\|$$

for some $x_0 \in B \setminus \{0\}$, then (21) *holds.*

Now, let the unital Banach algebra $\mathcal{A}$ be commutative and semisimple; as before, let the holomorphic map $F : B \to B$ be such that $\sigma(F(x)) \subset \overline{\Delta}$ for all $x \in B$. By Lemma 3,

$$\kappa(F(0)) = 0 \;\Rightarrow\; \kappa(F(x)) \leq \|x\| \quad \forall x \in B.$$

Going back to the example at the end of the previous section, let $\mathcal{A}$ be a uniform algebra on a compact Hausdorff space X, and let $F : B \times X \to \mathbb{C}$ be such that:

- for every $\xi \in B$ the function $X \ni x \mapsto F(\xi, x)$ is an element of $\mathcal{A}$, with

$$\sup\{|F(\xi, x)| : x \in X\} \leq 1; \tag{22}$$

- for every $x \in X$ the function $B \ni \xi \mapsto F(\xi, x)$ is holomorphic on B.

In view of (22), Dunford's theorem implies that the map $\xi \mapsto F(\xi, \cdot)$ is holomorphic on B.

Set $f_z = F(zu, x)$ for $u \in \partial B$. Then the function $f_z : \Delta \to \mathcal{A}$ is holomorphic on Δ, and Lemma 4 then yields

PROPOSITION 1. *If $F(0, x) = 0$ for all $x \in X$ and if there exist $\xi_0 \in B \setminus \{0\}$ and $x_0 \in X$ with*

$$|F(\xi_0, x_0) = \|\xi_0\|,$$

then there is $\zeta \in \partial\Delta$ such that

$$F\left(\frac{z}{\|\xi_0\|}\xi_0, x_0\right) = \zeta z \quad \forall\, z \in \Delta.$$

5. THE HAUSDORFF DISTANCE

If X is a metric space with a distance d, let $\delta(K_1, K_2)$ be the Hausdorff distance of two compact subsets K_1 and K_2 of X:

$$\delta(K_1, K_2) = \max\big\{\sup\{d(x_1, K_2) : x_1 \in K_1\}, \sup\{d(K_1, x_2) : x_2 \in K_2\}\big\}.$$

If $X = \mathbb{C}$ and d is the euclidean distance in $\mathbb{C}$, and if $\mathcal{A}$ is a unital Banach algebra, then, for $x \in \mathcal{A}$,

$$\varrho(x) = \delta(\{0\}, \sigma(x)),$$

and, more generally, for any $\zeta \in \mathbb{C}$,

$$\begin{aligned}\varrho(\zeta 1_{\mathcal{A}} - x) &= \sup\{|\tau| : \tau \in \sigma(\zeta 1_{\mathcal{A}} - x)\} = \sup\{|\tau| : \tau \in \zeta - \sigma(x)\}\\ &= \sup\{|\zeta - \tau| : \tau \in \Sigma(x)\} = \delta(\{\zeta\}, \sigma(x)),\end{aligned}$$

so that, for $x_1, x_2 \in \mathcal{A}$,

$$\begin{aligned}\delta(\sigma(x_1), \sigma(x_2)) &= \max\big\{\sup\{d(\zeta_1, \sigma(x_2)) : \zeta_1 \in \sigma(x_1)\},\\ &\qquad \sup\{d(\sigma(x_1), \zeta_2) : \zeta_2 \in \sigma(x_2)\}\big\}\\ &= \max\big\{\sup\{\varrho(\zeta_1 1_{\mathcal{A}} - x_2) : \zeta_1 \in \sigma(x_1)\},\\ &\qquad \sup\{\varrho(\zeta_2 1_{\mathcal{A}} - x_1) : \zeta_2 \in \sigma(x_2)\}\big\}\\ &= \max\big\{\sup\{\delta(\{\zeta_1\}, \sigma(x_2)) : \zeta_1 \in \sigma(x_1)\},\\ &\qquad \sup\{\delta(\sigma(x_1), \{\zeta_2\}) : \zeta_2 \in \sigma(x_2)\}\big\}.\end{aligned}$$

If $\mathcal{A}$ is commutative, then

$$\begin{aligned}\delta(\sigma(x_1), \sigma(x_2)) &= \max\big\{\sup\{d(\langle x_1, \chi\rangle, \sigma(x_2)) : \chi \in \Sigma(\mathcal{A})\},\\ &\qquad \sup\{d(\sigma(x_1, \langle x_2, \chi\rangle) : \chi \in \Sigma(\mathcal{A})\}\big\}\\ &= \max\big\{\sup\{\inf\{|\langle x_1, \chi\rangle - \langle x_2, \lambda\rangle| : \lambda \in \Sigma(\mathcal{A})\} : \chi \in \Sigma(\mathcal{A})\},\\ &\qquad \sup\{\inf\{|\langle x_1, \lambda\rangle - \langle x_2, \chi\rangle| : \lambda \in \Sigma(\mathcal{A})\} : \chi \in \Sigma(\mathcal{A})\}\big\}.\end{aligned}$$

Let ω be the Poincaré distance in Δ, and let $\delta = \delta_\omega$ now be the Hausdorff distance defined by ω. Let $\mathcal{A}$ be a unital Banach algebra and let $f : \Delta \to \mathcal{A}$ be, as before, a holomorphic map such that $\sigma(f(z)) \subset \Delta$ for all $z \in \Delta$.

For any $z_0 \in \Delta$ and any $x \in \mathcal{A}$ with $\sigma(x) \subset \Delta$,

$$\begin{aligned}\delta_\omega(z_0, \sigma(x)) &= \max\big\{\omega(z_0, \sigma(x)), \sup\{\omega(z_0, z) : z \in \sigma(x)\}\big\}\\ &= \sup\{\omega(z_0, z) : z \in \sigma(x)\}.\end{aligned}$$

For $z_0 \in \Delta$, let ϕ be the Möbius transformation

$$\phi : z \mapsto \frac{z - z_0}{1 - \bar{z}_0 z}.$$

By the invariance of the Poincaré distance, if $\sigma(f(z)) \subset \Delta$, then

$$\begin{aligned}\delta_\omega(z_0, \sigma(f(z))) &= \delta_\omega(\phi(z_0), \phi(\sigma(f(z)))) = \delta_\omega(0, \phi(\sigma(f(z))))\\ &= \delta_\omega(0, \sigma(\phi(f(z)))) = \varrho(\hat{\phi}(f(z))),\end{aligned}$$

where

$$\hat{\phi}(f(z)) = (1_{\mathcal{A}} - \bar{z}_0 f(z))^{-1}(f(z) - z_0 1_{\mathcal{A}}),$$

and if

$$\sigma(f(z_0)) = \{z_0\}, \tag{23}$$

then

$$\varrho(\hat{\phi}(f(z_0))) = \varrho(\phi(z_0)) = 0.$$

Let $g : \Delta \to \mathcal{A}$ be the holomorphic map defined by

$$g(z) = \hat{\phi}(f(\phi^{-1}(z))).$$

Since $\phi(z_0) = 0$, (23) implies

$$\begin{aligned}g(0) &= \hat{\phi}(f(\phi^{-1}(0))) = \hat{\phi}(f(z_0)),\\ \sigma(g(0)) &= \phi(\sigma(f(z_0))) = \{\phi(z_0)\} = \{0\}.\end{aligned}$$

If $\mathcal{A}$ contains no non-zero topologically nilpotent element, then $g(0) = 0$, and, by Theorem 1,

$$\varrho(g(z)) \le |z| \quad \forall z \in \Delta,$$

i.e.

$$\varrho(\hat{\phi}(f(\phi^{-1}(z))) \leq |z| \quad \forall z \in \Delta.$$

Setting $z = \phi(w)$ with $w \in \Delta$ yields

$$\begin{aligned}\varrho(\hat{\phi}(f(w))) \leq |\phi(w)| &= \delta_\omega(\{0\}, \{\phi(w)\}) = \delta_\omega(\{\phi^{-1}(0)\}, \{w\}) \\ &= \delta_\omega(\{z_0\}, \{w\}) \quad \forall w \in \Delta\end{aligned}$$

i.e.

$$\begin{aligned}\delta_\omega(\{\phi^{-1}(0)\}, \sigma(\phi(w))) &= \delta_\omega(\{0\}, \sigma(\hat{\phi}(f(w)))) \\ &\leq \delta_\omega(\{z_0\}, \{w\}) = \omega(z_0, w),\end{aligned}$$

proving thereby

THEOREM 5. *Let $\mathcal{A}$ be a unital, Banach algebra containing no non-zero topologically nilpotent element, and let $f : \Delta \to \mathcal{A}$ be a holomorphic map such that $\sigma(f(z)) \subset \Delta$ for all $z \in \Delta$. If* (23) *holds at some point $z_0 \in \Delta$, then*

$$\delta_\omega(\{z_0\}, \sigma(f(w))) \leq \omega(z_0, w) \quad \forall w \in \Delta.$$

If either

$$\varrho\left(\frac{f'(z_0)}{(1 - \bar{z}_0 f(z_0))^2}\right) = \frac{1}{(1 - |z_0|^2)^2},$$

or there is some $w \in \Delta \setminus \{z_0\}$ such that

$$\delta_\omega(\{z_0\}, \sigma(f(w))) = \omega(z_0, w),$$

then this latter equality holds for all $w \in \Delta$.

The last part of the theorem follows from Theorem 1 and from the fact that

$$g'(0) = \frac{(1 - |z_0|^2)^2 f'(z_0)}{(1 - \bar{z}_0 f(z_0))^2}.$$

6. A SCHWARZ LEMMA FOR THE CARATHÉODORY SPECTRAL RADIUS

The above results can be restated in terms of the Carathéodory distance on a bounded domain in $\mathbb{C}$. Let $\mathcal{A}$ be a unital Banach algebra containing no non-zero topologically nilpotent element. For any $x \in \mathcal{A}$, let E be a domain in $\mathbb{C}$ containing $\sigma(x)$. Let c_E be the Carathéodory distance in E, and let δ_{c_E} be the Hausdorff distance defined by c_E. If $\zeta_0 \in E$, let

$$\tau_E(\zeta_0, x) = \max\{c_E(\zeta_0, \zeta) : \zeta \in \sigma(x)\} = \delta_{c_E}(\{\zeta_0\}, \sigma(x)).$$

Let f be a holomorphic map of a domain $U \subset \mathbb{C}$ into $\mathcal{A}$ such that $\sigma(f(z)) \subset E$ for all $z \in U$.

According to Theorem II (p. 60) of [9], the function

$$z \mapsto \log \tau_E(\zeta_0, f(z)) = \log \delta_{c_E}(\{\zeta_0\}, \sigma(f(z)))$$

is subharmonic on U.

If $E = \Delta$ and $\zeta_0 = 0$, and if ω is the Poincaré distance in Δ, then

$$\tau_\Delta(0, x) = \omega(0, \varrho(x)) = \frac{1}{2} \frac{1 + \varrho(x)}{1 - \varrho(x)},$$

and denoting by $\mathrm{Hol}(E, \Delta)$ the set of all holomorphic maps of E into Δ, we have

$$\begin{aligned}\tau_E(\zeta_0, x) &= \sup\{\omega(\varphi(\zeta_0), \varphi(\zeta)) : \zeta \in \sigma(x),\ \varphi \in \mathrm{Hol}(E, \Delta)\} \\ &\leq \{c_E(\zeta_0, \zeta) : \zeta \in \sigma(x)\} = \delta_{c_E}(\{\zeta_0\}, \sigma(x)).\end{aligned}$$

THEOREM 6. *Let E be a domain in $\mathbb{C}$, bi-holomorphically homeomorphic to Δ. Let $f : \Delta \to \mathcal{A}$ be a holomorphic map such that $\sigma(f(z)) \subset E$ for all $z \in \Delta$, $\sigma(f(0)) = \{\zeta_0\}$ for some $\zeta_0 \in E$, and $f(\zeta_0) = 0$. Then*

$$\tau_E(\zeta_0, f(z)) \leq |z| \quad \forall z \in \Delta. \tag{24}$$

PROOF. If ψ is a bi-holomorphic homeomorphism of E onto Δ such that $\psi(\zeta_0) = 0$, then

$$\begin{aligned}\tau_E(\zeta_0, f(z)) &= \sup\{c_E(\zeta_0, \zeta) : \zeta \in \sigma(f(z))\} \\ &= \sup\{c_E(\psi^{-1}(0), (\psi^{-1} \circ \psi)(\zeta)) : \zeta \in \sigma(f(z))\} \\ &= \sup\{\omega(0, \psi(\zeta)) : \zeta \in \sigma(f(z))\} \\ &= \sup\{|\lambda| : \lambda \in \psi(\sigma(f(z)))\} \\ &= \sup\{|\lambda| : \lambda \in \sigma(\hat{\psi}(f(z)))\},\end{aligned}$$

where $\hat{\psi}(f(z)) \in \mathcal{A}$ is the image of ψ defined by the Dunford integral.

Since

$$\sigma(\hat{\psi}(f(z))) = \psi(\sigma(f(z))) \subset \psi(E) \subset \Delta,$$

the conclusion follows from Theorem 1. □

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