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Transcendental submanifolds of projective space

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Abstract. Given integers m and c satisfying $m-2 \geq c \geq 2$, we explicitly construct a nonsingular m-dimensional algebraic subset of $\mathbb{P}^{m+c}(\mathbb{R})$ that is not isotopic to the set of real points of any nonsingular complex algebraic subset of $\mathbb{P}^{m+c}(\mathbb{C})$ defined over $\mathbb{R}$. The first examples of this type were obtained by Akbulut and King in a more complicated and nonconstructive way, and only for certain large integers m and c.

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1. Introduction

Denote by $\mathbb{P}^n(\mathbb{R})$ and $\mathbb{P}^n(\mathbb{C})$ real and complex projective n-spaces. We regard $\mathbb{P}^n(\mathbb{R})$ as a subset of $\mathbb{P}^n(\mathbb{C})$. A smooth (of class $\mathcal{C}^\infty$) submanifold M of $\mathbb{P}^n(\mathbb{R})$ is said to be of *algebraic type* if it is isotopic in $\mathbb{P}^n(\mathbb{R})$ to the set of real points of a nonsingular complex algebraic subset of $\mathbb{P}^n(\mathbb{C})$ defined over $\mathbb{R}$; otherwise M is said to be *transcendental*. It is not at all obvious that transcendental submanifolds exist. However, Akbulut and King [2] proved the existence of transcendental submanifolds M of $\mathbb{P}^n(\mathbb{R})$ which can even be realized as nonsingular algebraic subsets of $\mathbb{P}^n(\mathbb{R})$. Their examples are obtained in a nonconstructive way, by a method which requires both $m = \dim M$ and $n - m$ to be large integers satisfying $2m - n \geq 2$. In the present paper we explicitly construct such examples, assuming only $n - m \geq 2$ and $2m - n \geq 2$. Moreover, we verify that M is a transcendental submanifold of $\mathbb{P}^n(\mathbb{R})$ using only the Barth–Larsen theorem [6, Corollary 6.5] and completely avoiding all results of [1], [2]. More precisely, denote by S^k the unit k-sphere,

$$S^k = \{(y_1, \dots, y_{k+1}) \in \mathbb{R}^{k+1} \mid y_1^2 + \dots + y_{k+1}^2 = 1\}.$$

In Section 3 we prove the following:

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Theorem 1.1. *Let m and n be positive integers satisfying $n-m \geq 2$ and $2m-n \geq 2$. Let*

$$\varphi\colon \mathbb{P}^2(\mathbb{R}) \times S^{m-2} \longrightarrow \mathbb{P}^n(\mathbb{R})$$

be defined by

$$\begin{aligned}&\varphi((x_1 : x_2 : x_3), (y_1, \dots, y_{m-1}))\\ &\quad = (x_1^2 + x_2^2 + x_3^2 : x_1x_2 : x_1x_3 : x_2x_3 : \sigma y_1 : \dots : \sigma y_{m-1} : 0 : \dots : 0),\end{aligned}$$

where 0 is repeated $n-m-2$ times and $\sigma = x_1^2 + 2x_2^2 + 3x_3^2$. Then:

(i) *The image $M = \varphi(\mathbb{P}^2(\mathbb{R}) \times S^{m-2})$ is an m-dimensional nonsingular algebraic subset of $\mathbb{P}^n(\mathbb{R})$.*

(ii) *$\varphi\colon \mathbb{P}^2(\mathbb{R}) \times S^{m-2} \to M$ is a biregular isomorphism.*

(iii) *M is a transcendental submanifold of $\mathbb{P}^n(\mathbb{R})$.*

It follows directly from Theorem 1.1 that for any integers m and c satisfying $m-2 \geq c \geq 2$, there is a nonsingular algebraic set M in $\mathbb{P}^{m+c}(\mathbb{R})$ such that $\dim M = m$ and M is a transcendental submanifold. In particular, there are transcendental submanifolds of arbitrary dimension $m \geq 4$. The existence of transcendental submanifolds of dimension 2 or 3 remains unsettled at this time. There are no transcendental submanifolds of dimension 1 or of codimension 1. The last assertion is a special case of the following well known fact.

Remark 1.2. Let M be a smooth m-dimensional submanifold of $\mathbb{P}^n(\mathbb{R})$. If either $n-m=1$ or $2m+1 \leq n$, then there exists a smooth embedding $e\colon M \to \mathbb{P}^n(\mathbb{R})$, arbitrarily close in the $\mathcal{C}^\infty$ topology to the inclusion map $M \hookrightarrow \mathbb{P}^n(\mathbb{R})$, such that $e(M)$ is the set of real points of a nonsingular complex algebraic subset of $\mathbb{P}^n(\mathbb{C})$ defined over $\mathbb{R}$.

If $n-m=1$, the claim is explicitly established for example in [3, Theorem 7.1].

For the second case, consider $\mathbb{P}^n(\mathbb{R})$ as a subset of $\mathbb{P}^k(\mathbb{R})$, where k is a large integer. By [8], there exists a smooth embedding $j : M \to \mathbb{P}^k(\mathbb{R})$, arbitrarily close in the $\mathcal{C}^\infty$ topology to the inclusion map $M \hookrightarrow \mathbb{P}^k(\mathbb{R})$, such that $j(M)$ is a nonsingular algebraic subset of $\mathbb{P}^k(\mathbb{R})$. Increasing k if necessary and making use of Hironaka's resolution of singularities theorem [7], we may assume that the Zariski complex closure of $j(M)$ in $\mathbb{P}^k(\mathbb{C})$ is nonsingular. If $2m+1 \leq n$, we obtain an embedding $e\colon M \to \mathbb{P}^n(\mathbb{R})$ with the required properties by composing j with an appropriate generic projection onto $\mathbb{P}^n(\mathbb{R})$.

2. A criterion for transcendence

First we need some results related to the Picard group. Following the current custom, we state them in the language of schemes.

Let V be a smooth projective scheme over $\mathbb{R}$. Assume that the set $V(\mathbb{R})$ of $\mathbb{R}$-rational points of V is nonempty. We regard $V(\mathbb{R})$ as a compact smooth manifold. Every invertible sheaf $\mathcal{L}$ on V determines a real line bundle on $V(\mathbb{R})$, denoted $\mathcal{L}(\mathbb{R})$. The correspondence which assigns to each invertible sheaf $\mathcal{L}$ on V the first Stiefel–Whitney class $w_1(\mathcal{L}(\mathbb{R}))$ of $\mathcal{L}(\mathbb{R})$ gives rise to a canonical homomorphism

$$w_1\colon \operatorname{Pic}(V) \longrightarrow H^1(V(\mathbb{R}), \mathbb{Z}/2),$$

defined on the Picard group $\operatorname{Pic}(V)$ of isomorphism classes of invertible sheaves on V. We set

$$H^1_{\mathrm{alg}}(V(\mathbb{R}), \mathbb{Z}/2) = w_1(\operatorname{Pic}(V)).$$

It will be convenient to recall another description of $\operatorname{Pic}(V)$. Consider the scheme $V_{\mathbb{C}} = V \times_{\mathbb{R}} \mathbb{C}$ over $\mathbb{C}$ and its Picard group $\operatorname{Pic}(V_{\mathbb{C}})$. The Galois group $G = \operatorname{Gal}(\mathbb{C}/\mathbb{R})$ of $\mathbb{C}$ over $\mathbb{R}$ acts on $\operatorname{Pic}(V_{\mathbb{C}})$. We denote by $\operatorname{Pic}(V_{\mathbb{C}})^G$ the subgroup of $\operatorname{Pic}(V_{\mathbb{C}})$ consisting of the elements fixed by G. Given an invertible sheaf $\mathcal{L}$ on V, we write $\mathcal{L}_{\mathbb{C}}$ for the corresponding sheaf on $V_{\mathbb{C}}$. The correspondence $\mathcal{L} \to \mathcal{L}_{\mathbb{C}}$ defines a canonical group homomorphism

$$\alpha\colon \operatorname{Pic}(V) \longrightarrow \operatorname{Pic}(V_{\mathbb{C}})^G.$$

It follows from the general theory of descent [4] that α is an isomorphism (a simple treatment of the case under consideration can also be found in [5]).

As usual, we set $\mathbb{P}^n_{\mathbb{R}} = \operatorname{Proj}(\mathbb{R}[T_0, \dots, T_n])$ and identify $\mathbb{P}^n_{\mathbb{R}}(\mathbb{R})$ with $\mathbb{P}^n(\mathbb{R})$. Thus if V is a subscheme of $\mathbb{P}^n_{\mathbb{R}}$, then $V(\mathbb{R})$ is a subset of $\mathbb{P}^n(\mathbb{R})$.

Proposition 2.1. *Let V be a closed smooth m-dimensional subscheme of $\mathbb{P}^n_{\mathbb{R}}$. If $2m - n \geq 2$, then*

$$H^1_{\mathrm{alg}}(V(\mathbb{R}), \mathbb{Z}/2) = i^*(H^1(\mathbb{P}^n(\mathbb{R}), \mathbb{Z}/2)),$$

where $i\colon V(\mathbb{R}) \hookrightarrow \mathbb{P}^n(\mathbb{R})$ is the inclusion map.

Proof. Let $j\colon V \hookrightarrow \mathbb{P}^n_{\mathbb{R}}$ and $j_{\mathbb{C}}\colon V_{\mathbb{C}} \hookrightarrow \mathbb{P}^n_{\mathbb{C}} = \mathbb{P}^n_{\mathbb{R}} \times_{\mathbb{R}} \mathbb{C}$ be the inclusion morphisms. By the Barth–Larsen theorem [6, Corollary 6.5], the induced homomorphism

$$j^*_{\mathbb{C}}\colon \operatorname{Pic}(\mathbb{P}^n_{\mathbb{C}}) \longrightarrow \operatorname{Pic}(V_{\mathbb{C}})$$

is an isomorphism. Since $j^*_{\mathbb{C}}$ is G-equivariant, the restriction

$$j^*_{\mathbb{C}}\colon \operatorname{Pic}(\mathbb{P}^n_{\mathbb{C}})^G \longrightarrow \operatorname{Pic}(V_{\mathbb{C}})^G$$

is an isomorphism. We have the following commutative diagram:

$$\begin{array}{ccc}
\mathrm{Pic}(\mathbb{P}^n_{\mathbb{C}})^G & \xrightarrow{j^*_{\mathbb{C}}} & \mathrm{Pic}(V_{\mathbb{C}})^G \\
\uparrow{\scriptstyle \alpha} & & \uparrow{\scriptstyle \alpha} \\
\mathrm{Pic}(\mathbb{P}^n_{\mathbb{R}}) & \xrightarrow{j^*} & \mathrm{Pic}(V) \\
\downarrow{\scriptstyle w_1} & & \downarrow{\scriptstyle w_1} \\
H^1(\mathbb{P}^n(\mathbb{R}), \mathbb{Z}/2) & \xrightarrow{i^*} & H^1(V(\mathbb{R}), \mathbb{Z}/2).
\end{array}$$

Since the homomorphisms α are isomorphisms and

$$H^1(\mathbb{P}^n(\mathbb{R}), \mathbb{Z}/2) = H^1_{\mathrm{alg}}(\mathbb{P}^n(\mathbb{R}), \mathbb{Z}/2),$$

it follows that

$$H^1_{\mathrm{alg}}(V(\mathbb{R}), \mathbb{Z}/2) = i^*(H^1(\mathbb{P}^n(\mathbb{R}), \mathbb{Z}/2)),$$

as required. $\square$

Note that a smooth submanifold of $\mathbb{P}^n(\mathbb{R})$ is of algebraic type if and only if it is isotopic in $\mathbb{P}^n(\mathbb{R})$ to $V(\mathbb{R})$ for some closed smooth subscheme V of $\mathbb{P}^n_{\mathbb{R}}$. Hence Proposition 2.1 yields the following criterion for transcendence.

Proposition 2.2. *Let M be a compact smooth m-dimensional submanifold of $\mathbb{P}^n(\mathbb{R})$. Assume that the inclusion map $e\colon M \hookrightarrow \mathbb{P}^n(\mathbb{R})$ induces a trivial homomorphism*

$$e^*\colon H^1(\mathbb{P}^n(\mathbb{R}), \mathbb{Z}/2) \longrightarrow H^1(M, \mathbb{Z}/2),$$

that is, $e^* = 0$. If M is nonorientable and $2m - n \geq 2$, then M is a transcendental submanifold of $\mathbb{P}^n(\mathbb{R})$.

Proof. Suppose to the contrary that M is of algebraic type. Let V be a closed smooth subscheme of $\mathbb{P}^n_{\mathbb{R}}$ with $V(\mathbb{R})$ isotopic to M in $\mathbb{P}^n(\mathbb{R})$. Then the homomorphism

$$i^*\colon H^1(\mathbb{P}^n(\mathbb{R}), \mathbb{Z}/2) \longrightarrow H^1(V(\mathbb{R}), \mathbb{Z}/2),$$

induced by the inclusion map $i\colon V(\mathbb{R}) \hookrightarrow \mathbb{P}^n(\mathbb{R})$, is trivial. Since $\dim V = m$ and $2m - n \geq 2$, Proposition 2.1 implies

$$H^1_{\mathrm{alg}}(V(\mathbb{R}), \mathbb{Z}/2) = 0.$$

On the other hand, the first Stiefel–Whitney class $w_1(V(\mathbb{R}))$ of $V(\mathbb{R})$ is nonzero, $V(\mathbb{R})$ being a nonorientable manifold. Moreover, $w_1(V(\mathbb{R})) = w_1(\mathcal{K}(\mathbb{R}))$, where $\mathcal{K}$ is the canonical invertible sheaf of V, and hence, $w_1(V(\mathbb{R}))$ is in $H^1_{\mathrm{alg}}(V(\mathbb{R}), \mathbb{Z}/2)$. In view of this contradiction, the proof is complete. $\square$

3. Transcendental submanifolds

We begin with some preliminary observations. Identify $\mathbb{R}^n$ with its image under the map

$$\mathbb{R}^n \longrightarrow \mathbb{P}^n(\mathbb{R}), \quad (x_1, \dots, x_n) \longmapsto (1 : x_1 : \dots : x_n);$$

thus $\mathbb{R}^n \subset \mathbb{P}^n(\mathbb{R})$. An algebraic subset X of $\mathbb{R}^n$ is said to be *projectively closed* if X is also an algebraic subset of $\mathbb{P}^n(\mathbb{R})$. One readily checks that X is projectively closed if and only if it can be defined by a real polynomial equation

$$f(x_1, \dots, x_n) = 0,$$

where the homogeneous form of top degree in f vanishes only at 0 in $\mathbb{R}^n$.

Lemma 3.1. *Let X be an algebraic subset of $\mathbb{R}^k$ contained in the open half-space*

$$H = \{(x_1, \dots, x_k) \in \mathbb{R}^k \mid x_k > 0\}.$$

Then the map $\psi \colon X \times S^\ell \to \mathbb{R}^{k+\ell}$ defined by

$$\psi((x_1, \dots, x_k), (y_1, \dots, y_{\ell+1})) = (x_1, \dots, x_{k-1}, x_k y_1, \dots, x_k y_{\ell+1})$$

is an algebraic embedding, that is, the image $Y = \psi(X \times S^\ell)$ is an algebraic subset of $\mathbb{R}^{k+\ell}$ and $\psi \colon X \times S^\ell \to Y$ is a biregular isomorphism. Moreover, if X is projectively closed in $\mathbb{R}^k$, then Y is projectively closed in $\mathbb{R}^{k+\ell}$.

Proof. Let

$$f(u, v) = 0$$

be a real polynomial equation defining X, where $u = (x_1, \dots, x_{k-1})$ and $v = x_k$. Since

$$X \subset H, \tag{1}$$

the subset Y of $\mathbb{R}^{k+\ell}$ is defined by the equation

$$f(u, \rho) = 0, \tag{2}$$

where

$$\rho = (x_k^2 + x_{k+1}^2 + \dots + x_{k+\ell}^2)^{\frac{1}{2}}.$$

We will now show that (2) can be replaced by a polynomial equation in $x_1, \dots, x_{k-1}$, $x_k, \dots, x_{k+\ell}$. To this end we write

$$f(u, v) = g(u, v^2) + v h(u, v^2), \tag{3}$$

where g and h are real polynomials in (u, v). Then (2) is equivalent to

$$g(u,\rho^2) + \rho h(u,\rho^2) = 0, \tag{4}$$

and in view of (1) also to

$$(g(u,\rho^2))^2 - \rho^2 (h(u,\rho^2))^2 = 0, \tag{5}$$

which is a polynomial equation, as required. Consequently, Y is an algebraic subset of $\mathbb{R}^{k+\ell}$.

It is clear that ψ is injective and $\theta\colon Y \to X$,

$$\theta(x_1,\dots,x_{k-1},x_k,\dots,x_{k+\ell}) = \Big(x_1,\dots,x_{k-1},\frac{x_k}{\rho},\dots,\frac{x_{k+\ell}}{\rho}\Big),$$

is the inverse of $\psi\colon X \to Y$. By (4),

$$\rho = -\frac{g(x_1,\dots,x_{k-1},x_k^2+\dots+x_{k+\ell}^2)}{h(x_1,\dots,x_{k-1},x_k^2+\dots+x_{k+\ell}^2)}$$

for $(x_1,\dots,x_{k-1},x_k,\dots,x_{k+\ell})$ in Y, and hence θ is a regular map. Thus $\psi\colon X \to Y$ is a biregular isomorphism.

Assume now that X is projectively closed in $\mathbb{R}^k$. We may also assume that the homogeneous form of top degree in f, denoted F, vanishes only at 0 in $\mathbb{R}^k$. Note that $F(u,\rho^2)F(u,-\rho^2)$ is the homogeneous form of top degree in equation (5). This form vanishes only at 0 in $\mathbb{R}^{k+\ell}$, and hence Y is projectively closed in $\mathbb{R}^{k+\ell}$. □

Lemma 3.2. *The map* $g\colon \mathbb{P}^2(\mathbb{C}) \to \mathbb{P}^4(\mathbb{C})$,

$$g((x_1 : x_2 : x_3)) = (x_1^2 + x_2^2 + x_3^2 : x_1x_2 : x_1x_3 : x_2x_3 : x_1^2 + 2x_2^2 + 3x_3^2),$$

is an algebraic embedding. In particular, the restriction $f\colon \mathbb{P}^2(\mathbb{R}) \to \mathbb{P}^4(\mathbb{R})$ *of* g *is an algebraic embedding.*

Proof. One readily checks that g is injective. Moreover, the (complex) differential of g at each point of $\mathbb{P}^2(\mathbb{C})$ is of rank 2. It follows that g is an algebraic embedding, and hence f is an algebraic embedding. □

Proof of Theorem 1.1. Let $f\colon \mathbb{P}^2(\mathbb{R}) \to \mathbb{P}^4(\mathbb{R})$ be the algebraic embedding of Lemma 3.2. Note that the image $X = f(\mathbb{P}^2(\mathbb{R}))$ is a projectively closed algebraic subset of $\mathbb{R}^4 \subset \mathbb{P}^4(\mathbb{R})$, contained in the open half-space

$$\{(u_1,u_2,u_3,u_4) \in \mathbb{R}^4 \mid u_4 > 0\}.$$

Let

$$\psi\colon X \times S^{m-2} \longrightarrow \mathbb{R}^{4+(m-2)} = \mathbb{R}^{m+2} \subset \mathbb{P}^{m+2}(\mathbb{R})$$

be the algebraic embedding of Lemma 3.1 (with $k = 4$ and $\ell = m - 2$). Note that $\psi(X \times S^{m-2})$ is projectively closed in $\mathbb{R}^{m+2}$, and hence is an algebraic subset of $\mathbb{P}^{m+2}(\mathbb{R})$.

Clearly, if $i : S^{m-2} \to S^{m-2}$ is the identity map, then

$$f \times i : \mathbb{P}^2(\mathbb{R}) \times S^{m-2} \longrightarrow X \times S^{m-2}$$

is a biregular isomorphism. Denoting by $j : \mathbb{P}^{m+2}(\mathbb{R}) \to \mathbb{P}^n(\mathbb{R})$ the standard embedding,

$$j((v_0 : \ldots : v_{m+2})) = (v_0 : \ldots : v_{m+2} : 0 : \ldots : 0),$$

we obtain

$$\varphi = j \circ \psi \circ (f \times i),$$

which implies that φ is an algebraic embedding. In other words, conditions (i) and (ii) are satisfied. Moreover, $M \subset \mathbb{R}^n \subseteq \mathbb{P}^n(\mathbb{R})$. Since M is nonorientable and $2m - n \geq 2$, condition (iii) follows from Proposition 2.2. □

References

[1] S. Akbulut and H. King, Transcendental submanifolds of $\mathbb{R}^n$. *Comment. Math. Helv.* **68** (1993), 308–318. Zbl 0806.57017 MR 1214234

[2] S. Akbulut and H. King, Transcendental submanifolds of $\mathbb{RP}^n$. *Comment. Math. Helv.* **80** (2005), 427–432. Zbl 1071.57026 MR 2142249

[3] J. Bochnak, M. Buchner and W. Kucharz, Vector bundles over real algebraic varieties. *K-Theory* **3** (1989), 271–298; Erratum, *K-Theory* **4** (1990), 103. Zbl 0761.14020 MR 1076527

[4] A. Grothendieck, *Technique de descente et théorèmes d'existence en géométrie algebrique*, I–VI. Seminaire Bourbaki, Exposés 190, 195, 212, 221, 232, 236, 1959–62. Zbl 0234.14007 MR 1603475

[5] J. van Hamel, Algebraic cycles and topology of real algebraic varieties. Dissertation, Vrije Universiteit Amsterdam. CWI Tract. 129, Stichting Mathematisch Centrum, Centrum voor Wiscunde en informatica, Amsterdam 2000. Zbl 0986.14042 MR

[6] R. Hartshorne, Equivalence relations on algebraic cycles and subvarieties of small codimension. *Proc. Sympos. Pure Math* **29** (1975), 129–164. Zbl 0314.14001 MR 0369359

[7] H. Hironaka, Resolution of singularities of an algebraic variety over a field of characteristic zero. *Ann. of Math.* **79** (1964), 109–203. Zbl 0122.38603 MR 0199184

[8] H. King, Approximating submanifolds of real projective space by varieties. *Topology* **15** (1976), 81–85. Zbl 0316.57015 MR 0396572

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