

Visual decompositions of Coxeter groups

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Abstract. A Coxeter system is an ordered pair (W, S) where S is the generating set in a particular type of presentation for the Coxeter group W . A subgroup of W is called special if it is generated by a subset of S . Amalgamated product decompositions of a Coxeter group having special factors and special amalgamated subgroup are easily recognized from the presentation of the Coxeter group. If a Coxeter group is a subgroup of the fundamental group of a given graph of groups, then the Coxeter group is also the fundamental group of a graph of special subgroups, where each vertex and edge group is a subgroup of a conjugate of a vertex or edge group of the given graph of groups. A vertex group of an arbitrary graph of groups decomposition of a Coxeter group is shown to split into parts conjugate to special groups and parts that are subgroups of edge groups of the given decomposition. Several applications of the main theorem are produced, including the classification of maximal FA subgroups of a finitely generated Coxeter group as all conjugates of certain special subgroups.

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1. Introduction

We take a *Coxeter presentation* to be given as

$$P = \langle S : (st)^{m(s,t)} \ (s, t \in S, m(s,t) < \infty) \rangle,$$

where $m: S^2 \rightarrow \{1, 2, \dots, \infty\}$ is such that $m(s, t) = 1$ iff $s = t$ and $m(s, t) = m(t, s)$. In the group with this presentation, the elements of S represent distinct elements of order 2 and a product st of generators has order $m(s, t)$. A *Coxeter group* W is a group having a Coxeter presentation and a *Coxeter system* (W, S) is a Coxeter group W with generating subset S corresponding to the generators in a Coxeter presentation of W . When the order of the product of a pair of generators is infinite there will be no defining relator for that pair of generators and we will say that the generators are *unrelated*. A Coxeter group W belongs to some Coxeter system

(W, S) , and though S need not be uniquely determined up to an automorphism of W , we often take such an S as given. Our basic reference for Coxeter groups is Bourbaki [3]. A *special subgroup* of a Coxeter system (W, S) , is a subgroup of W generated by a subset of S (see [5]). If W' is the special subgroup generated by $S' \subseteq S$ in a Coxeter system (W, S) , then (W', S') is also a Coxeter system.

The information given by a Coxeter presentation may be conveniently expressed in the form of a labeled graph. We define the *presentation diagram* of the system (W, S) to be the labeled graph $\Gamma(W, S)$ with vertex set S , and an (undirected) edge labeled $m(s, t)$ between distinct vertices s and t when $m(s, t) < \infty$. The connected components of the presentation diagram $\Gamma(W, S)$ correspond to special subgroups which are the factors in a free product decomposition of W . (This is in contrast to a *Coxeter graph* with vertex set S and labeled edges when $m(s, t) \neq 2$, and having components corresponding to direct product factors of W . The Coxeter graph is not used in this paper.) The presentation diagram of the special subgroup of W generated by a subset $S' \subseteq S$ is the induced subgraph of $\Gamma(W, S)$ with vertex set S' (and in this sense, special subgroups could as well be termed *visual* subgroups since we can see the presentation diagram of such a subgroup in $\Gamma(W, S)$).

Suppose that $\Gamma(W, S) = \Gamma_1 \cup \Gamma_2$ is a union of induced subgraphs and let $\Gamma_0 = \Gamma_1 \cap \Gamma_2$ (so vertices and edges of $\Gamma(W, S)$ are in Γ_1 or Γ_2 or both, and Γ_0 is the induced subgraph consisting of the vertices and edges in both). Equivalently, suppose Γ_0 is an induced subgraph with $\Gamma(W, S) - \Gamma_0$ having at least two components, Γ_1 is Γ_0 together with some of these components and Γ_2 is Γ_0 together with the other components. We say in this case that Γ_0 *separates* $\Gamma(W, S)$ (separates it into at least two components). Then it is evident from the Coxeter presentation that W is an amalgamated product of special subgroups corresponding to Γ_1 and Γ_2 over the special subgroup corresponding to Γ_0 . Amalgamated product decompositions with special factors and special amalgamated subgroup are easily seen in the presentation diagram and we call such an amalgamated product a *visual splitting* of W . Other amalgamated product decompositions may also be possible, and we want to understand such splittings in terms of visual splittings.

More generally, we are interested in when a Coxeter group W can be realized as the fundamental group of a graph of groups (as explained in the next section). We show that the graph must actually be a tree and so this generalizes amalgamated products. We say that Ψ is a *visual graph of groups decomposition* of W (for a given S), if each vertex and edge group of Ψ is a special subgroup of W , the injections of each edge group into its endpoint vertex groups are given simply by inclusion, and the fundamental group of Ψ is isomorphic to W by the homomorphism induced by the inclusion map of vertex groups into W . A sequence of compatible visual splittings of W will result in such a decomposition. Our main result shows that an arbitrary graph of group decomposition of a Coxeter group can be refined (in a certain sense) to a visual graph of groups decomposition.

Theorem 1 (Main theorem). *Suppose that (W, S) is a Coxeter system and W is a subgroup of the fundamental group of a graph of groups Λ . Then W has a visual graph of groups decomposition Ψ where each vertex group of Ψ is a subgroup of a conjugate of a vertex group of Λ , and each edge group of Ψ is a subgroup of a conjugate of an edge group of Λ . Moreover, Ψ can be taken so that each special subgroup of W that is a subgroup of a conjugate of a vertex group of Λ is a subgroup of a vertex group of Ψ .*

For (W, S) and Λ as in Theorem 1 and Ψ satisfying the full conclusion of Theorem 1 (including the moreover clause), we say that Ψ is a (W, S) -visual decomposition from Λ . Suppose that G is a group decomposed as $A *_C B$. If H is a subgroup of B , then $\langle A \cup H \rangle$ decomposes as $A *_C \langle C \cup H \rangle$ (consider the action of $\langle A \cup H \rangle$ on the Bass–Serre tree for $A *_C B$). Furthermore, the decomposition $(A *_C \langle C \cup H \rangle)_{\langle C \cup H \rangle} B$ reduces to $A *_C B$, so G decomposes as $\langle A \cup H \rangle *_C B$. In this way some visual decompositions of Coxeter groups can be “artificially” altered to decompositions that have no visual vertex or edge groups. In Section 3, we exhibit a Coxeter system (W, S) and a reduced graph of groups decomposition Λ_N , for (W, S) with N vertices for any positive integer N . Even so, the following theorem defines limits on how far an arbitrary graph of groups decomposition for a finitely generated Coxeter system can stray from a visual decomposition for that system.

Theorem 2. *Suppose that (W, S) is a finitely generated Coxeter system, Λ is a graph of groups decomposition of W and Ψ is a reduced graph of groups decomposition of W such that each vertex group of Ψ is a subgroup of a conjugate of a vertex group of Λ . Then for each vertex V of Λ , the vertex group $\Lambda(V)$ has a graph of groups decomposition Φ_V such that each vertex group of Φ_V is either*

- (1) *conjugate to a vertex group of Ψ , or*
- (2) *a subgroup of $v\Lambda(E)v^{-1}$ for some $v \in \Lambda(V)$ and E some edge of Λ adjacent to V .*

When Ψ is visual, vertex groups of the first type in Theorem 2 are visual. Those of the second type seem somewhat artificial.

It is easy to recognize whether a finitely generated Coxeter group is 2-ended or infinite ended, by refining Stallings’ theorem (from [18]) to a visual splitting theorem. A Dunwoody decomposition of a finitely generated Coxeter group is a graph of groups with finite or 1-ended vertex groups and finite edge groups. These refine to a visual Dunwoody decomposition, and we get a simple argument for why finitely generated Coxeter groups are accessible with respect to splittings over finite groups (as in [10]). In separate papers [14] and [15], we developed the fundamental ideas of this paper to prove a strong accessibility result for Coxeter groups with respect to splittings over “minimal” splitting subgroups, and a JSJ result for splittings of Coxeter groups over

virtually abelian groups. At the bottom level, the strong accessibility result is a visual version of Dunwoody's result in the Coxeter setting.

J.-P. Serre gives an account of FA groups in [17]. In particular, an FA group has no nontrivial splittings. We apply the main theorem to show that the maximal FA subgroups of a finitely generated Coxeter group W are those conjugate to a special subgroup with presentation diagram a maximal complete subgraph of $\Gamma(W, S)$.

Our results identify certain properties of a Coxeter group recognizable directly from a particular presentation of the group, properties apparent in the presentation diagram. There can be different Coxeter systems (W, S) and (W, S') where S' is not simply a conjugate of S (and may not even correspond under any automorphism of W), and so conjugates of special subgroups with respect to S need not correspond to conjugates of special subgroups with respect to S' . However, these results do imply that certain special subgroups are, up to conjugation, special for any Coxeter system. This investigation thus has significant application to rigidity questions and the determination of when Coxeter groups are isomorphic.

As a final applications of these ideas we give a visual classification of finitely generated virtually free Coxeter groups in Section 8.

2. Graphs of groups

Our main tool in this investigation is the connection between group actions on trees and fundamental groups of graphs of groups. As reference the reader is referred to [9] and [17]. We review some of the pertinent definitions and results.

A graph of groups Λ consists of a set $V(\Lambda)$ of vertices, a set $E(\Lambda)$ of edges, and maps $\iota, \tau: E(\Lambda) \rightarrow V(\Lambda)$ giving the initial and terminal vertices of each edge in a connected graph, together with vertex groups $\Lambda(V)$ for $V \in V(\Lambda)$, edge groups $\Lambda(E)$ for $E \in E(\Lambda)$, with $\Lambda(E) \subset \Lambda(\iota(E))$ and an injective group homomorphism $t_E: \Lambda(E) \rightarrow \Lambda(\tau(E))$, called the edge map of E and denoted by $t_E: g \mapsto g^{t_E}$. The fundamental group $\pi_1(\Lambda)$ of a graph of groups Λ is the group with presentation having generators the disjoint union of $\Lambda(V)$ for $V \in V(\Lambda)$, together with a symbol t_E for each edge $E \in E(\Lambda)$, and having as defining relations the relations for each $\Lambda(V)$, the relations $gt_E = t_E g^{t_E}$ for $E \in E(\Lambda)$ and $g \in \Lambda(\iota(E))$, and relations $t_E = 1$ for E in a given spanning tree of Λ (the result, up to isomorphism, is independent of the spanning tree taken).

An amalgamated product $A *_C B$ is realized as the fundamental group of the graph of groups with 2 vertices having vertex groups A and B , a single edge between, with edge group (the image in A of) C , and edge map t_E determined by the injection of the edge group into B . Similarly, an HNN-extension of A by $t_E: C \rightarrow A$ is realized as the fundamental group of a graph of groups with a single vertex and single edge. In general, the fundamental group of a graph of groups can be understood as taking amalgamated products of the vertex groups along the edge groups in the spanning tree,

followed by HNN-extensions over the remaining edge groups. The t_E for edges not in the spanning tree correspond to a stable letter of an HNN extension. For edges in the spanning tree the relations amount to identifying $\Lambda(E)$ in $\Lambda(\iota(E))$ with its image $\Lambda(E)^{t_E}$ in $\Lambda(\tau(E))$ as in an amalgamated product. Each vertex and edge group of a graph of groups Λ injects into the fundamental group of Λ (Britton's lemma) and we usually identify these with the corresponding subgroups of the fundamental group of Λ .

A graph of groups on a graph which is not simply connected will have \mathbb{Z} as a homomorphic image (take a t_E not in the spanning tree to 1 and all other generators to 0). Since the generators of a Coxeter group are of order 2, a homomorphism into \mathbb{Z} must have trivial image. Thus if a Coxeter group is the fundamental group of a graph of groups the graph must be a tree and the group arises as successive amalgamated products. In working with such trees of groups we will often simply assume suitable identifications have been made and the edge maps are simply inclusion maps.

Given a graph of groups Λ , the Bass–Serre tree for Λ is defined with vertices the disjoint union over $V \in V(\Lambda)$ of the different cosets $g\Lambda(V)$ of $\Lambda(V)$ in $\pi_1(\Lambda)$, and edges the disjoint union over $E \in E(\Lambda)$ of the different cosets $g\Lambda(E)$ of $\Lambda(E)$ in $\pi_1(\Lambda)$, taken with $\iota(g\Lambda(E)) = g\Lambda(\iota(E))$ and $\tau(g\Lambda(E)) = g\Lambda(\tau(E))$. The Bass–Serre tree is in fact a tree and the fundamental group of Λ acts on this tree by taking for $h \in \pi_1(\Lambda)$, $h(g\Lambda(V)) = (hg)\Lambda(V)$ and $h(g\Lambda(E)) = (hg)\Lambda(E)$.

If a group G acts on a tree T (as a directed graph, the action preserving the orientation of edges), then a transversal for this action consists of a vertex and edge from each orbit of the action of G on vertices and edges. There must always exist a transversal having a spanning subtree such that each other edge of the transversal originates in the subtree. (For instance, if Λ has a single vertex and single edge, giving $G = \pi_1(\Lambda)$ as an HNN extension, then G acts on the Bass–Serre tree T for Λ with transversal consisting of a single vertex and an edge originating at that vertex, that does not include the terminal vertex and has spanning subtree just the single vertex.) From such a transversal, a graph of groups is defined by taking as graph the quotient of T under the action of G , and taking vertex and edge groups to be the stabilizers of the corresponding vertices and edges in the transversal. For edges in the spanning subtree of the transversal the edge maps are given by inclusion. An edge of the transversal not in the spanning subtree connects a vertex in the transversal to a translate by an element $g \in G$ of a vertex in the transversal and the edge map is given by conjugation by g . Then the fundamental group of this graph of groups is naturally isomorphic to G by the homomorphism extending the inclusion map of vertex groups into G . Choosing a different transversal in T gives rise to a different graph of groups decomposition of G with vertex groups each conjugate to vertex groups of the first graph of groups, i.e., having isomorphic vertex groups but corresponding with different subgroups of G .

Suppose that Λ is a graph of groups and X is a subset of the edges in Λ . Contracting the graph of Λ along the edges in X gives rise to a graph whose vertices are the equivalence classes of vertices modulo the equivalence relation defined by identifying

endpoints of edges in X , taking edges $E \in E(\Lambda) - X$ with endpoints the equivalence classes of endpoints of E . Define a graph of groups Λ' on this graph, having the same fundamental group as Λ , by taking as vertex groups $\Lambda'(V)$ the fundamental group of the graph of groups with vertices of Λ identified to V and edges in X between these vertices, with edge groups $\Lambda'(E)$ corresponding to the remaining edge groups in Λ with corresponding edge maps. We call Λ' the graph of groups resulting from collapsing the edges X in Λ . If Λ is a tree of groups, then collapsing edges in Λ results in another tree of groups.

A graph of groups will be called *reduced* if no edge group is equal to its originating vertex group nor has image equal to its terminating vertex group, *except* for an edge which is a loop at a vertex. If a graph of groups is not reduced, then we may collapse a vertex across an edge, where the edge group is (or has image) the same as the endpoint vertex group, giving a smaller graph of groups with vertex and edge groups among the original vertex and edge groups and having the same fundamental group. Repeated reductions of this sort in a finite graph of groups must eventually end with a reduced graph of groups all of whose vertex and edge groups were present in the original graph of groups and having the same fundamental group.

The following is well known. We include a proof as this result is frequently referenced throughout the article.

Lemma 3. *Suppose that Λ is a reduced graph of groups decomposition of a group G , the underlying graph for Λ is a tree, V and U are vertices of Λ , and $g\Lambda(V)g^{-1} \subset \Lambda(U)$ for some $g \in G$, then $V = U$ and $g \in \Lambda(V)$.*

Proof. If $U \neq V$ or if $U = V$ and $g \notin \Lambda(V)$, then in the Bass–Serre tree for Λ , $\Lambda(V)$ stabilizes the distinct vertices $\Lambda(V)$ and $g^{-1}\Lambda(U)$. But then $\Lambda(V)$ also stabilizes a geodesic path between these vertices and hence stabilizes the first edge $h\Lambda(E)$ for an edge E of Λ at V . This would mean that $\Lambda(V)$ is equal to $\Lambda(E)$ (or its image in $\Lambda(V)$), and we could collapse V across E , contradicting that Λ was reduced. \square

In the other direction, given a graph of groups Λ and a graph of groups decomposition of a vertex group $\Lambda(V)$ as $\pi_1(\Phi)$, we would like to see when Λ results from collapsing out Φ in a larger graph of groups Λ' . Say that Φ is a *compatible decomposition* of $\Lambda(V)$ if each edge group of Λ incident at V is a subgroup of a $\Lambda(V)$ conjugate of a vertex group of Φ . In general then, given Λ and Φ a compatible decomposition of $\Lambda(V)$, construct Λ' by replacing V in Λ by Φ and attaching edges of Λ incident at V instead to vertices in Φ with edge groups and edge maps appropriately adjusted. So each such edge group has image in a vertex group of Φ (for edges ending in Φ in Λ' or V in Λ) or is an isomorphic subgroup of a vertex group of Φ (for edges originating in Φ in Λ' or V in Λ). This adjustment of edge maps means that the image in $\pi_1(\Lambda')$ of a vertex group of Λ' not in Φ is conjugate to the image of the corresponding vertex group of Λ in $\pi_1(\Lambda) = \pi_1(\Lambda')$.

Alternatively, the Bass–Serre tree T' for Λ' can be constructed from the Bass–Serre tree for Λ with each coset of $\Lambda(V)$ replaced by the Bass–Serre tree for Φ with a $\pi_1(\Lambda)$ group action. Contracting the edges in the orbit of Φ in T' gives T , and a transversal of T' with a spanning subtree contracts to a transversal of T with a spanning subtree. Starting with a transversal with spanning subtree of T and a transversal with spanning subtree of the Bass–Serre tree for Φ we may need to translate the parts of the T transversal in different components of $T - \{V\}$ so that they attach to the transversal of the Bass–Serre tree for Φ in T' . The result is still a transversal of T with the same quotient graph as Λ but with vertex groups identified with conjugates of the vertex groups of $\pi_1(\Lambda)$.

Suppose that (W, S) is a Coxeter system. A visual graph of groups decomposition Ψ of W has special vertex and edge groups, edge maps given by inclusion, and is such that the inclusion of vertex groups in W extends to an isomorphism of $\pi_1(\Psi)$ with W . As noted above, Ψ is a tree. In terms of presentation diagrams, $\Gamma(W, S)$ must be the union of the subdiagrams corresponding to the vertex groups of Ψ , with the edge groups corresponding to the intersections of adjacent vertex subdiagrams. To understand in visual terms exactly when a graph of special subgroups has fundamental group isomorphic to W we have the following essential lemma.

Lemma 4. *Suppose that (W, S) is a Coxeter system. A graph of groups Ψ with graph a tree, where each vertex group and edge group is a special subgroup and each edge map is given by inclusion, is a visual graph of groups decomposition of W iff each edge in the presentation diagram of W is an edge in the presentation diagram of a vertex group and, for each generator $s \in S$, the set of vertices and edges with groups containing s is a nonempty subtree in Ψ .*

Proof. Suppose that Ψ is a visual graph of groups decomposition of W so that the homomorphism ψ extending the inclusion map on vertex groups is an isomorphism of $\pi_1(\Psi)$ and W . Since each vertex group is generated by the elements of S it contains, the image of ψ is generated by the set of all $s \in S$ that belong to some vertex group. Since a proper subset of S generates a proper subgroup of W , each element of S must be an element of a vertex group of Ψ . Let T be the Bass–Serre tree for Ψ . Then since the edge maps are inclusions, the subgraph of T consisting of the identity cosets of vertex and edge groups of Ψ is a transversal of T , a subtree of T , and the stabilizers of the vertices and edges of the transversal are simply the corresponding vertex and edge groups of Ψ . If $s \in S$ belongs to two vertex groups, then $\psi^{-1}(s)$ stabilizes the corresponding vertices of the transversal, hence stabilizes all the edges and vertices in a geodesic path in T between these vertices, and so s belongs to all the vertex and edge groups in the path in Ψ between these vertices. Hence the vertices and edges with groups containing s form a subtree in Ψ . If there is an edge in $\Gamma(W, S)$ between $s, t \in S$, then the subgroup U of W generated by s and t is finite. But then $\psi^{-1}(U)$ is a finite subgroup of $\pi_1(\Psi)$ acting on T and so must stabilize a vertex V of T . Let V'

be the vertex of the transversal closest to V in T . Then V' lies between V and a vertex of the transversal stabilized by s and so is also stabilized by s , and similarly for t . Thus V' corresponds to a vertex of Ψ having vertex group containing both s and t and this special subgroup has an edge between s and t in its presentation diagram as well.

Conversely, suppose Ψ is a graph of groups with each edge of the presentation diagram of W in the presentation diagram of some vertex group, and such that, for each generator $s \in S$, the vertices and edges of Ψ with groups containing s form a subtree in Ψ . All of the occurrences of an $s \in S$ in different vertex groups of Ψ are identified by the relators of $\pi_1(\Psi)$ along edges in the subtree of vertex and edge groups containing s . Take $\psi: \pi_1(\Psi) \rightarrow W$ extending the inclusion map of the generators S in $\pi_1(\Psi)$ into W . We get that ψ is an isomorphism by checking that each of the defining relators of W is already a relator of $\pi_1(\Psi)$. But each generator of W belongs to some vertex group of Ψ where it has order 2, and the other relators of W correspond to edges of the presentation diagram and these relators also already hold in some vertex group of Ψ . \square

If Ψ is a visual graph of groups decomposition for the Coxeter system (W, S) , it is convenient to label a vertex of Ψ by the subset of S that generates the corresponding vertex group. So if $Q \subset S$ is a vertex label of Ψ , then $\Psi(Q) = \langle Q \rangle$. If Ψ is reduced, its vertex labels are distinct and we identify vertices with their labels. Even if Ψ is reduced, two distinct edges may have the same edge group, so we do not extend this labeling to edges.

The following two technical results are useful tools.

Corollary 5. *Suppose that (W, S) is a finitely generated Coxeter system, Ψ is a visual graph of groups decomposition of W , and $E \subset S$ is such that $\langle E \rangle = \Psi(E')$ for E' an edge of Ψ . If $\{x, y\} \subset S - E$, and $x \in X$ and $y \in Y$ for X and Y labels of vertices of Ψ on opposite sides of E' , then as a subset of $\Gamma(W, S)$, E separates x and y in Γ .*

Proof. Otherwise, among all such x, y that fail the conclusion, let α be a shortest path in Γ from x to y avoiding E . Note that by Lemma 4, $x \neq y$. If $x \equiv x_0, x_1, \dots, x_n \equiv y$ are the consecutive vertices of α , then, by Lemma 4, $\{x_0, x_1\} \subset V$ for some vertex label V of Ψ . Since $x_0 \notin E$, Lemma 4 implies that V and X label vertices on the same side of E' in Ψ . But then x_1 and y satisfy the hypothesis of the corollary, contradicting the minimality of α . \square

Corollary 6. *Suppose that (W, S) is a finitely generated Coxeter system, and Ψ is a visual graph of groups decomposition of W . If C is a complete subset of the presentation diagram $\Gamma(W, S)$, then there is a vertex labeled $V(\subset S)$ of Ψ such that $C \subset V$.*

Proof. We may assume that Ψ is reduced. If C is not a subset of a vertex of Ψ , then there is a vertex $U \subset S$ of Ψ containing a maximal number of elements of C and $V \subset S$ a vertex of Ψ closest to U such that V contains an element c of $C - U$. If E' is the last edge of the Ψ -geodesic from U to V and $\Psi(E') = \langle E \rangle$ for $E \subset S$, then $(U \cap C) \not\subset E$ and $c \notin E$. By Corollary 5, E separates $(U \cap C) - E$ from c in Γ . This is impossible since C is complete. \square

3. Proof of main results

We begin this section with a proof of Theorem 1 and conclude with a proof of Theorem 2. Several examples are presented to introduce the reader to visual decompositions.

Proof of Theorem 1. Suppose that (W, S) is a Coxeter system and suppose Λ is a graph of groups with W a subgroup of $\pi_1(\Lambda)$. We may identify the vertex and edge groups of Λ with subgroups of $\pi_1(\Lambda)$. Then $\pi_1(\Lambda)$ acts on the Bass–Serre tree T such that the stabilizers of vertices of the tree are conjugates of the vertex groups of Λ and stabilizers of edges of the tree are conjugates of the edge groups of Λ .

We build a visual graph of groups decomposition Ψ of W with this tree as its graph. For each vertex V (resp. edge E) of Ψ take vertex group $\Psi(V)$ (resp. edge group $\Psi(E)$) to be the subgroup of W generated by the $s \in S$ stabilizing V (resp. E). The edge groups inject into the vertex groups by inclusion maps. Clearly, each vertex group (resp. edge group) of Ψ is a special subgroup of W and a subgroup of a conjugate of a vertex group (resp. edge group) of Λ . Each generator $s \in S$ is of order 2 and so stabilizes some vertex of T . If two vertices of T are stabilized by s , then s stabilizes the geodesic path between the vertices, thus the subgraph of T of vertices and edges stabilized by s is a subtree. Suppose that there is an edge in the presentation diagram for (W, S) between s and t in S . The subgroup generated by s and t has finite order, so stabilizes some vertex of T , with s and t belonging to that vertex group. By Lemma 4 then, $\pi_1(\Psi)$ is isomorphic to W and Ψ is a visual graph of groups decomposition of W .

Finally, if U is a special subgroup and a subgroup of a conjugate of one of the vertex groups of Λ , then U stabilizes a vertex in T and so U is a subgroup of the vertex group of that vertex. \square

Generally, Ψ will be an infinite tree. If W is finitely generated, by taking the subtree spanning a finite set of vertices whose vertex groups between them include all of the elements of S , we get a finite visual graph of groups decomposition of W . Reducing this graph of groups results in a reduced visual graph of groups decomposition for W .

Remark 1. If W , Λ and Ψ are as in Theorem 1, then each edge E of Λ defines a splitting of W as $A *_{\Lambda(E)} B$ where A is generated by the vertex groups of the vertices on one side of E in Λ and B is generated by those on the other side of E in Λ . There is a quotient map from the Bass–Serre tree for Λ to that of $A *_{\Lambda(E)} B$ which respects the action of W . If $A \neq \Lambda(E) \neq B$ then W is not a subgroup of a conjugate of A or B and the proof above shows that there is an edge of Ψ such that the edge divides Ψ into two components, neither alone having all of the generators of W in its vertex groups, and such that the corresponding edge group lies in a conjugate of $\Lambda(E)$ (and so also after applying possible reductions to Ψ).

Example 1. As an example of the main theorem consider the Coxeter group W with presentation

$$P = \langle s_1, s_2, s_3, s_4, s_5 : s_i^2, (s_1s_2)^2 = (s_2s_3)^2 = (s_3s_4)^2 = (s_4s_5)^2 = 1 \rangle.$$

Note that $(s_3s_5s_3)^2 = (s_4s_3s_5s_3)^2 = 1$ so the map $\phi: W \rightarrow W$ defined on generators by $\phi(s_i) = s_i$ for $i < 5$, and $\phi(s_5) = s_3s_5s_3$, extends to a homomorphism, is surjective, is its own inverse, and so is an automorphism of W . Consider subgroups $A = \langle s_1, s_2, s_4, s_3s_5s_3 \rangle$, $B = \langle s_2, s_3, s_4 \rangle$, and $C = \langle s_2, s_4 \rangle$. Then $W = A *_C B$ is the image under ϕ of a visual amalgamated product decomposition of W , but A is not a visual subgroup of W . We refine this decomposition to a visual graph of groups decomposition following the proof of Theorem 1.

The Bass–Serre tree on which W acts has vertices corresponding to the different cosets of A and B and edges corresponding to the different cosets of C with hC connecting hA and hB for each $h \in W$. The vertex A (respectively B) of this tree has one edge for each coset aC (resp. bC) of C in A (resp. B). An element $g \in W$ acts on this tree by mapping hA to ghA and similarly for cosets of B and C . The stabilizer of gA is the conjugate gAg^{-1} . The graph of groups Ψ is defined on this tree by taking the vertex and edge groups generated by the s_i which stabilize that vertex or edge. Thus s_1, s_2 , and s_4 stabilize the A vertex in the tree, s_2 and s_4 stabilize both the C edge and the s_3C edge, s_2, s_3 , and s_4 stabilize the $B = s_3B$ vertex, s_2, s_4 , and s_5 stabilize the s_3A vertex (etc.). Since the vertex groups of these three vertices include all of the generators of W , the vertex and edge groups for the rest of the tree collapse back to this three vertex subgraph with the same fundamental group as the graph of groups. The visual decomposition reduces to a three factor amalgamated product

$$W = \langle s_1, s_2, s_4 \rangle *_C B *_C \langle s_2, s_4, s_5 \rangle,$$

where the first and third factors are generated by those generators of W stabilizing A and s_3A , respectively, are subgroups of conjugates of A , and are special subgroups of W .

This simple example illustrates that an amalgamated product decomposition of a Coxeter group need not refine to a visual amalgamated product decomposition of only

two factors. Instead the visual graph of groups decomposition we have produced is sometimes necessary.

Proof of Theorem 2. Let Φ_V be the decomposition of $\Lambda(V)$ given by selecting a fundamental transversal for its action on T_Ψ , the Bass–Serre tree for Ψ . That is, take the graph of Φ_V to be a subtree of T_Ψ containing a single vertex and edge from each orbit of the action of $\Lambda(V)$ on T_Ψ and take the group of a vertex or edge to be the subgroup of $\Lambda(V)$ stabilizing that vertex or edge. If U is a vertex of Φ_V , then $U \in T_\Psi$ corresponds to a coset $g\Psi(X)$ of a vertex group $\Psi(X)$ of Ψ . The stabilizer in W of U is simply the conjugate $g\Psi(X)g^{-1}$, and so $\Phi_V(U) = \Lambda(V) \cap (g\Psi(X)g^{-1})$. The group $\Psi(X)$ is a subgroup of a conjugate $h\Lambda(Y)h^{-1}$ of a vertex group $\Lambda(Y)$ of Λ . Hence $\Phi_V(U)$ is a subgroup of $\Lambda(V)$ and of a conjugate $gh\Lambda(Y)h^{-1}g^{-1}$ of $\Lambda(Y)$, and so stabilizes the vertices of T_Λ corresponding to $\Lambda(V)$ and $gh\Lambda(Y)$. If $V = Y$ and $gh \in \Lambda(V)$, then these are the same vertex of T_Λ , and $g\Psi(X)g^{-1} \subset gh\Lambda(Y)h^{-1}g^{-1} = \Lambda(V)$. Thus $\Phi_V(U) = \Lambda(V) \cap (g\Psi(X)g^{-1}) = g\Psi(X)g^{-1}$ is equal to a conjugate of a vertex group of Ψ . Otherwise, $\Lambda(V)$ and $gh\Lambda(Y)$ are different vertices of T_Λ . The group $\Phi_V(U)$ stabilizes both vertices and so a geodesic path between the two. In particular, $\Phi_V(U)$ stabilizes the first edge $k\Lambda(E)$ in this path, where E is an edge of Λ incident with V and $k \in \Lambda(V)$. Thus $\Phi_V(U) \subset k\Lambda(E)k^{-1}$. Reducing Φ_V gives the desired graph of groups decomposition. \square

The next corollary and example examine technical aspects of the decompositions of Theorem 2.

Corollary 7. *Suppose that (W, S) is a finitely generated Coxeter system, Λ is a graph of groups decomposition of W , Ψ is a reduced (W, S) -visual decomposition from Λ and Φ_V is a reduced graph of groups decomposition for the vertex group $\Lambda(V)$ as given in the proof of Theorem 2. If X is a vertex of Ψ , V is a vertex of Λ , and $g\Psi(X)g^{-1}$ is a subgroup of $\Lambda(V)$ for some $g \in W$, then $vg\Psi(X)g^{-1}v^{-1}$ is a vertex group of Φ_V for some $v \in \Lambda(V)$.*

Proof. Let Φ'_V be the decomposition of $\Lambda(V)$ given by selecting a fundamental transversal for its action on T_Ψ , the Bass–Serre tree for Ψ , and let Φ_V be reduced from Φ'_V . By the definition of transversal, there is $v \in \Lambda(V)$ such that the coset $vg\Psi(X)$ corresponds to a vertex B of the transversal. Then $\Phi'_V(B) = \Lambda(V) \cap vg\Psi(X)g^{-1}v^{-1} = vg\Psi(X)g^{-1}v^{-1}$. It remains to show that this vertex group survives reduction. Otherwise, there is a vertex Q of Φ'_V such that $vg\Psi(X)g^{-1}v^{-1}$ is a proper subgroup of $\Phi'_V(Q) \equiv \Lambda(V) \cap h\Psi(Z)h^{-1}$ (where $Q \in T_\Psi$ corresponds to the coset $h\Psi(Z)$). But then $\Psi(X)$ is a proper subgroup of $g^{-1}v^{-1}h\Psi(Z)h^{-1}vg$ which is impossible by Lemma 3. \square

The next example shows that the previous corollary cannot be extended to show that a vertex group of Ψ is conjugate to a vertex group of Φ_V for a unique vertex V

of Λ . In fact, all vertex groups of a particular Φ_V may be equal to vertex groups of other Φ_Q for Q a vertex of Λ .

Example 2. Consider the group W with Coxeter presentation $\langle w, x, y, z : w^2 = x^2 = y^2 = z^2 = 1 \rangle$ (and free product decomposition $\langle w \rangle * \langle x \rangle * \langle y \rangle * \langle z \rangle$). There is an automorphism of W that fixes w and y , and maps x to $wyxyw$ and z to $y(wyxyw)z(wyxyw)y$. W has the decomposition $\langle w, x \rangle *_{\langle x \rangle} \langle x, z \rangle *_{\langle z \rangle} \langle y, z \rangle$. Hence the automorphism induces a decomposition Λ , with vertices V_1 and V_2 and V_3 , where V_1 and V_2 are connected by an edge E_1 , and V_2 and V_3 are connected by an edge E_2 . The decomposition $\Lambda \equiv \Lambda(V_1) *_{\Lambda(E_1)} \Lambda(V_2) *_{\Lambda(E_2)} \Lambda(V_3)$ of W is such that

$$\begin{aligned} \Lambda(V_1) &= \langle w, wyxyw \rangle, \\ \Lambda(E_1) &= \langle wyxyw \rangle, \\ \Lambda(V_2) &= \langle wyxyw, y(wyxyw)z(wyxyw)y \rangle, \\ \Lambda(E_2) &= \langle y(wyxyw)z(wyxyw)y \rangle, \\ \Lambda(V_3) &= \langle y, y(wyxyw)z(wyxyw)y \rangle. \end{aligned}$$

A visual decomposition Ψ , from Λ , is $\langle w \rangle * \langle x \rangle * \langle y \rangle * \langle z \rangle$. The decomposition Φ_{V_1} of Theorem 2 is $\langle w \rangle * \langle wyxyw \rangle$, Φ_{V_2} is $\langle wyxyw \rangle * \langle y(wyxyw)z(wyxyw)y \rangle$, and Φ_{V_3} is $\langle y \rangle * \langle y(wyxyw)z(wyxyw)y \rangle$. Hence the vertex groups $\langle x \rangle$ and $\langle z \rangle$ of Ψ are conjugate to vertex groups of Φ_V for more than one vertex V of Λ . Furthermore, both vertex groups of Φ_{V_2} are equal to vertex groups of other Φ_{V_i} .

Example 3. Consider the Coxeter presentation $\langle a, b, c : a^2 = b^2 = c^2 = 1 \rangle$. This group splits as $\Lambda \equiv \langle a, bc \rangle *_{\langle bc \rangle} \langle b, c \rangle$. The visual decomposition from this splitting is $\Psi \equiv \langle a \rangle * \langle b, c \rangle$. If $\Lambda(V)$ is the vertex group $\langle a, bc \rangle$, then its graph of groups decomposition Φ_V , induced by its action on T_Ψ , the Bass–Serre tree for Ψ , is $\mathbb{Z}_2 * \mathbb{Z}$. This group also has a decomposition with underlying graph not a tree, but it is not induced by its action on T_Ψ . We wonder if every decomposition of a finitely generated Coxeter group and corresponding visual decomposition can only induce Φ_V having as underlying graph a tree.

In their accessibility paper [2], Bestvina and Feighn limit the number of vertex groups that can occur in a reduced graph of groups decomposition with “small” edge groups, for a given finitely presented group. Strong accessibility results for Coxeter groups over “minimal” splittings are the focus of [14]. While Theorem 2 limits how far a vertex group of an arbitrary decomposition of a Coxeter group can stray from visual, the following example shows that one cannot expect an accessibility result for Coxeter groups over arbitrary splittings.

Example 4. For a virtually free (non-virtually cyclic) group, the following technique is standard for creating non-trivial reduced graph of groups decompositions with n

vertex groups for any $n \in \{1, 2, \dots\}$. Let W be the Coxeter group with presentation $\langle s_1, s_2, s_3, s_4, s_5 : s_i^2 = 1 \rangle$. Then W has free product decomposition $A * B$ where $A = \langle s_1, s_2, s_3 \rangle$ and $B = \langle s_4, s_5 \rangle$. The element $b \equiv s_4 s_5$ generates an infinite cyclic subgroup of finite index in B and A contains a non-abelian free subgroup of finite index. Let $\{a_1, a_2, \dots\}$ be a free generating set for an infinite rank free subgroup of A . Then W has the decomposition

$$\Lambda_1 = \langle A \cup \{b\} \rangle *_{\langle a_1, b \rangle} \langle B \cup \{a_1\} \rangle.$$

The group $\langle A \cup \{b\} \rangle$ has the following non-trivial reduced graph of groups decomposition, which is compatible with Λ_1 :

$$\langle A \cup \{b^2\} \rangle *_{\langle a_1, a_2, b^2 \rangle} \langle a_1, a_2, b \rangle.$$

Hence W has the reduced decomposition

$$\Lambda_2 = \langle A \cup \{b^2\} \rangle *_{\langle a_1, a_2, b^2 \rangle} \langle a_1, a_2, b \rangle *_{\langle a_1, b \rangle} \langle B \cup \{a_1\} \rangle.$$

Let $E_n = \langle a_1, \dots, a_n, b^{(2^{n-2})} \rangle$, let $C_n = \langle a_1, \dots, a_n, b^{(2^{n-1})} \rangle$, and let $A_n = \langle A \cup \{b^{(2^{n-1})}\} \rangle$. Then we have a reduced decomposition for W with $n + 1$ vertex groups:

$$\Lambda_n = A_n *_{C_n} E_n *_{C_{n-1}} E_{n-1} *_{C_{n-2}} \dots *_{C_1} E_2 *_{C_1} \langle B \cup \{a_1\} \rangle.$$

For each n , $A * B$ is a visual decomposition from Λ_n .

4. Technical results

As mentioned in the introduction, our basic reference for Coxeter groups is [3]. The technical results of this section are well known and can be derived from two fundamental facts: the first concerning special subgroups of Coxeter groups and the second called the “deletion condition”.

Proposition 8. *Suppose that (W, S) is a Coxeter system and $P = \langle S : (st)^{m(s,t)}$ for $m(s, t) < \infty \rangle$ (where $m : S^2 \rightarrow \{1, 2, \dots, \infty\}$) is a Coxeter presentation for W . If $A \subset S$, then $(\langle A \rangle, A)$ is a Coxeter system with Coxeter presentation $\langle A : (st)^{m'(s,t)}$ for $m'(s, t) < \infty \rangle$ (where $m' = m|_{A^2}$).*

Given a group G and a generating set S , an S -geodesic for $g \in G$ is a shortest word in $S \cup S^{-1}$ such that the product of the letters of this word is g . The number of letters in an S -geodesic for g is the S -length of g . We include an elementary proof of the deletion condition pointed out to us by A. Yu. Ol’shanskii and based on the theory of van Kampen diagrams (see Chapter 5 of Lyndon and Schupp’s book [13] for a basic introduction to van Kampen diagrams).

Proposition 9 (The deletion condition). *Suppose that (W, S) is a Coxeter system and $w = a_1 \dots a_n$ for $a_i \in S$. If $a_1 \dots a_n$ is not geodesic then there are indices $i < j$ in $\{1, 2, \dots, n\}$ such that $w = a_1 \dots a_{i-1} a_{i+1} \dots a_{j-1} a_{j+1} \dots a_n$, i.e., a_i and a_j can be deleted.*

Proof. Let $b_1 \dots b_m$ be an S -geodesic for w , so that $m < n$. Let \mathcal{D} be a van Kampen diagram for the word $a_1 \dots a_n b_m \dots b_1$. The relation 2-cell r_1 containing a_1 is of even length and so there is an edge e_1 of r_1 “opposite” a_1 . The two subpaths of the boundary of r_1 separated by a_1 and e_1 have the exact same labeling. Let r_2 be the relation 2-cell of \mathcal{D} that shares e_1 with r_1 , and let e_2 be the edge of r_2 opposite e_1 . Again the labeling of the two subpaths of the boundary of r_2 separated by e_1 and e_2 are the same. Continue until e_k is on the boundary of \mathcal{D} . The “strip” determined by $\bigcup_{i=1}^k r_i$ is such that the two subpaths of the boundary of this strip separated by a_1 and e_k have the same labeling. This creates a unique pairing of edges on the boundary of \mathcal{D} . Since $m < n$ there are indices $i < j$ in $\{1, 2, \dots, n\}$ such that a_i is paired with a_j . If the two subpaths of the boundary of the strip for a_i separated by a_i and a_j are labeled β , then (as the product of edge path labels of loops in \mathcal{D} is trivial in W) $a_{i+1} \dots a_{j-1} = \beta = a_i \dots a_j$, i.e., a_i and a_j are deleted. \square

Lemma 10. *Suppose that (W, S) is a Coxeter system, $A \subset S$, and $a \in \langle A \rangle$. If $a = a_1 \dots a_n = b_1 \dots b_n$ are S -geodesics then $\{a_1, \dots, a_n\} = \{b_1, \dots, b_n\}$.*

Proof. Assume n is minimal among all counterexamples to the lemma. By Proposition 8, we have $n \neq 1$. Note that $a_1 \dots a_n b_n = b_1 \dots b_{n-1}$. By the deletion condition, b_n is deleted together with some a_i in the first expression. Inductively it follows that $\{b_1, \dots, b_{n-1}\} \subset \{a_1, \dots, a_n\}$. Similarly, $b_1 a_1 \dots a_n = b_2 \dots b_n$ so that $\{b_2, \dots, b_n\} \subset \{a_1, \dots, a_n\}$. \square

If G is a group with generating set S , then the *Cayley graph* $\mathcal{K}(G, S)$ is a labeled graph with vertex set G and a directed edge (labeled s) from the vertex g to the vertex gs for each $s \in S$. Given a vertex x of $\mathcal{K}(G, S)$, there is a bijective correspondence between edge paths at x and words in $S \cup S^{-1}$, where traversing an edge labeled s opposite its orientation is read as s^{-1} . Hence S -geodesics and $\mathcal{K}(G, S)$ -geodesics at a given vertex are the same.

The next result is a straightforward application of the deletion condition.

Lemma 11. *Suppose that (W, S) is a Coxeter system, $\{x, y\} \subset W$ and $A \subset S$. Then in the Cayley graph $\mathcal{K}(W, S)$ there is a unique closest point z of the coset $y\langle A \rangle$ to x . Furthermore, if α is the S -geodesic from x to z , and β is any A -geodesic at z , then $\alpha\beta$ is an S -geodesic.*

The next lemma follows by a result of Kilmoyer (see Theorem 2.7.4 of [6]), but we include a direct proof for completeness.

Lemma 12. *Suppose that (W, S) is a Coxeter system, $I, J \subseteq S$, and d is a minimal length double coset representative in $\langle I \rangle d \langle J \rangle$. Then $\langle I \rangle \cap d \langle J \rangle d^{-1} = \langle K \rangle$ where $K = I \cap (dJd^{-1})$. Hence $g \langle I \rangle g^{-1} \cap h \langle J \rangle h^{-1} = f \langle K \rangle f^{-1}$ for $K \subset I$.*

Proof. With $K = I \cap (dJd^{-1})$, clearly, $\langle K \rangle \subset \langle I \rangle \cap d \langle J \rangle d^{-1}$. Suppose that a is a shortest element in $\langle I \rangle \cap d \langle J \rangle d^{-1}$ but not in $\langle K \rangle$. Write $a = a_1 a_2 \dots a_m$ geodesically with $a_i \in I$ and $d^{-1} a d = b_1 b_2 \dots b_n$ geodesically for $b_i \in J$. Write $d = d_1 d_2 \dots d_k$ geodesically for $d_i \in S$. Then since d is a minimal length double coset representative, $ad = a_1 a_2 \dots a_m d_1 d_2 \dots d_k = d_1 d_2 \dots d_k b_1 b_2 \dots b_n$ are each geodesic. Hence $m = n$. Clearly $a \neq 1$, and $a \notin S$ else $a = a_1 = db_1 d^{-1} \in K$ by definition. Instead, $m > 1$. Now $b_1 b_2 \dots b_m d^{-1}$ and $d^{-1} a_m$ are geodesic, but $d^{-1} a_1 \dots a_{m-1} = b_1 b_2 \dots b_m d^{-1} a_m$, so this last is not geodesic and, by the deletion condition, a_m is deleted together with some b_i to give $b_1 b_2 \dots b_m d^{-1} a_m = b_1 \dots b_{i-1} b_{i+1} \dots b_m d^{-1}$ (if $1 < i < m$ and similarly if $i = 1$ or $i = m$). But then $a_1 a_2 \dots a_{m-1} = db_1 \dots b_{i-1} b_{i+1} \dots b_m d^{-1} \in \langle I \rangle \cap d \langle J \rangle d^{-1}$ and, by the minimality of a and Lemma 10, we have $\{a_1, a_2, \dots, a_{m-1}\} \subset K$. Likewise $\{a_m, \dots, a_2\} \subset \langle I \rangle \cap d \langle J \rangle d^{-1}$, so $a_m \in K$. But then $a \in \langle K \rangle$, contradicting the choice of a . Instead, every $a \in \langle I \rangle \cap d \langle J \rangle d^{-1}$ is in $\langle K \rangle$.

Given conjugates $g \langle I \rangle g^{-1}$ and $h \langle J \rangle h^{-1}$ of special subgroups, take d of minimal length in $\langle I \rangle g^{-1} h \langle J \rangle$, so $g^{-1} h = adb^{-1}$ for $a \in \langle I \rangle$ and $b \in \langle J \rangle$, $\langle I \rangle g^{-1} h \langle J \rangle = \langle I \rangle d \langle J \rangle$, and $\langle I \rangle \cap d \langle J \rangle d^{-1} = \langle K \rangle$ as above. Then

$$\begin{aligned} g \langle I \rangle g^{-1} \cap h \langle J \rangle h^{-1} &= ga \langle I \rangle a^{-1} g^{-1} \cap hb \langle J \rangle b^{-1} h^{-1} \\ &= ga (\langle I \rangle \cap d \langle J \rangle d^{-1}) a^{-1} g^{-1} \\ &= ga \langle K \rangle a^{-1} g^{-1}. \end{aligned} \quad \square$$

We will use the following corollary in Section 6 in our analysis of FA subgroups.

Corollary 13. *Suppose that (W, S) is a finitely generated Coxeter system, $I, J \subseteq S$, the induced subgraph on I is a maximal complete subgraph of $\Gamma(W, S)$, and $I \subseteq w \langle J \rangle w^{-1}$ for some $w \in W$. Then $I \subseteq J$ and $w \in \langle J \rangle$.*

Proof. Take d a minimal length double coset representative in $\langle I \rangle w \langle J \rangle$ so $w = adb^{-1}$ for $a \in \langle I \rangle$ and $b \in \langle J \rangle$. Then since

$$\begin{aligned} \langle I \rangle \cap d \langle J \rangle d^{-1} &= a^{-1} (a \langle I \rangle a^{-1} \cap adb^{-1} \langle J \rangle b d^{-1} a^{-1}) a \\ &= a^{-1} (\langle I \rangle \cap w \langle J \rangle w^{-1}) a \\ &= \langle I \rangle, \end{aligned}$$

we have $I \subseteq dJd^{-1}$ and so $d^{-1} I d \subseteq J$. If $d \neq 1$ and $d = d_1 d_2 \dots d_k$ geodesically, then by the minimality of d , $d_1 \notin I$. There is some $s \in I$ unrelated to d_1 , since I induces a maximal complete subgraph of the Coxeter diagram. But if $a_1 \dots a_k$ is

any geodesic and s is unrelated to a_k then $a_1 \dots a_k s$ is geodesic (apply Lemma 11 with $x = 1$, $y = a_1 \dots a_{k-1}$, $A = \{a_k, s\}$ and $\beta = \beta_1 a_k s$, where β_1 is the $\{a_k, s\}$ geodesic from z to y). Hence $d^{-1} s d$ cannot have length 1 and cannot be in J . Instead then, $d = 1$, $I \subseteq J$ and $w = a d b^{-1} \in \langle J \rangle$. \square

The following corollaries find application in the analysis of rigidity of Coxeter groups.

Corollary 14. *If (W, S) is a finitely generated Coxeter system, $I \subseteq S$, and $w^{-1} \langle I \rangle w \subset \langle I \rangle$ for some $w \in W$, then $w^{-1} \langle I \rangle w = \langle I \rangle$.*

Proof. Let d be a minimal length double coset representative in $\langle I \rangle w \langle I \rangle$, so $w = a d b^{-1}$ for $a, b \in \langle I \rangle$. Then

$$\langle I \rangle = \langle I \rangle \cap w \langle I \rangle w^{-1} = a(\langle I \rangle \cap d \langle I \rangle d^{-1}) a^{-1}.$$

Hence $\langle I \rangle = \langle I \rangle \cap d \langle I \rangle d^{-1} = \langle I \cap d I d^{-1} \rangle$, by Lemma 12. By Propositions 8 and 9, a proper subset of I cannot generate the same Coxeter group as I , so $I = d I d^{-1}$. Hence $d^{-1} I d = I$ and $w^{-1} \langle I \rangle w = b d^{-1} \langle I \rangle d b^{-1} = b \langle I \rangle b^{-1} = \langle I \rangle$. \square

Corollary 15. *Suppose that (W, S) and (W, S') are finitely generated Coxeter systems for the same Coxeter group W . Suppose that $I \subset S$ is such that the induced subgraph of I separates $\Gamma(W, S)$ (its complement has at least two components). Then there are sets $J \subset S$ and $J' \subset S'$ such that the induced subgraphs of J and J' separate $\Gamma(W, S)$ and $\Gamma(W, S')$, respectively, $\langle J \rangle$ and $\langle J' \rangle$ are conjugate, and $\langle J \rangle$ is conjugate to a subgroup of $\langle I \rangle$.*

Proof. As noted in the introduction, since I separates $\Gamma(W, S)$, $W = A *_I B$, for A the (S) special subgroup generated by I and the generators in some components of the complement of I , and B the (S) special subgroup generated by I and the generators in the other components of the complement of I in $\Gamma(W, S)$, $A \neq \langle I \rangle \neq B$. Taking this amalgamated product decomposition of W to be Λ , we consider a corresponding visual decomposition Ψ with respect to the alternate generating set S' . As noted in Remark 1, there must be an edge of Ψ , with not all of the generators S' appearing on one side, having edge group $\langle I'_1 \rangle$, for an $I'_1 \subset S'$, which is a subgroup of a conjugate of $\langle I \rangle$. That is, there is an $I'_1 \subset S'$ which separates $\Gamma(W, S')$ and for which $\langle I'_1 \rangle$ is a subset of a conjugate of $\langle I \rangle$.

But then I'_1 gives a proper splitting of W and by the same reasoning there is an $I_2 \subset S$ which separates $\Gamma(W, S)$ and which is a subset of a conjugate of $\langle I'_1 \rangle$. Continuing in this fashion, we can find $I'_{2k+1} \subset S'$ separating $\Gamma(W, S')$, generating a subgroup of a conjugate of $\langle I_{2k} \rangle$, and an $I_{2k+2} \subset S$ separating $\Gamma(W, S)$ and generating a subgroup of a conjugate of $\langle I'_{2k+1} \rangle$.

Since there are only finitely many subsets of S or S' , for some $k_1 < k_2 < k_3$, $I'_{k_1} = I'_{k_3}$. Take $J' = I'_{k_1} = I'_{k_3}$ and $J = I_{k_2}$. Since $\langle I_j \rangle$ and $\langle I'_{j+1} \rangle$ are subgroups of conjugates of $\langle I_i \rangle$ and $\langle I'_{i+1} \rangle$ for $i < j$, take $a_1, a_2 \in W$ such that

$$\langle I'_{k_3} \rangle \subset a_2 \langle I_{k_2} \rangle a_2^{-1} \subset a_2 a_1 \langle I'_{k_1} \rangle a_1^{-1} a_2^{-1}.$$

Then $a_2^{-1} a_1^{-1} \langle J' \rangle a_1 a_2 \subset \langle J' \rangle$, so by Corollary 14 these are equal, and we get

$$\langle J' \rangle \subset a_2 \langle J \rangle a_2^{-1} \subset a_2 a_1 \langle J' \rangle a_1^{-1} a_2^{-1} = \langle J' \rangle.$$

Thus $\langle J' \rangle = a_2 \langle J \rangle a_2^{-1}$, and J and J' generate subgroups of conjugates of $\langle I \rangle$ that separate $\Gamma(W, S)$ and $\Gamma(W, S')$ respectively. \square

5. Ends

Stalling's theorem [18] states that if a finitely generated group has more than one end then it splits nontrivially as an amalgamated product or HNN-extension over a finite group. The following result is then an easy consequence of our main theorem. It can also be obtained from work of M. Davis [8].

Corollary 16. *For any finitely generated Coxeter group W with presentation diagram Γ , the following assertions are equivalent:*

- (1) W has more than one end.
- (2) W decomposes as a nontrivial amalgamated product $A *_C B$ where C is finite and A, B , and C are special subgroups.
- (3) Γ contains a complete separating subgraph, the vertices of which generate a finite subgroup of W .

Proof. If W is not 1-ended or finite, then by Stallings' splitting theorem, $W = A *_C B$ with C finite (and not as an HNN-extension since an HNN-extension maps onto \mathbb{Z} but a homomorphism of a Coxeter group into \mathbb{Z} must take generators to the identity). Then by Theorem 1, W has a reduced visual graph of groups decomposition in which each edge group is a subgroup of a conjugate of C and so is finite. This decomposition cannot be trivial since each vertex group is a subgroup of a conjugate of A or B neither of which is W in the given nontrivial splitting, and so no vertex group can be W . Hence there is at least one edge after reducing Ψ and collapsing the other edges gives W as a visual amalgamated product of special subgroups over a finite special subgroup. The remaining implications are easy. \square

By Stallings' theorem and Theorem 1, a 2-ended Coxeter group splits as a visual amalgamated product over a finite group which is of index two in each factor. The following result then characterizes 2-ended Coxeter groups.

Corollary 17. *A Coxeter group with system (W, S) and diagram Γ is 2-ended iff Γ contains a separating subdiagram Γ_0 which is the presentation diagram of a finite group, and $\Gamma - \Gamma_0$ consists of two vertices each of which is connected to each vertex of Γ_0 by edges labeled 2 (but not connected to each other). Equivalently, $W = \langle x, y \rangle \times \langle H \rangle$ where $\{x, y\} \cup H = S$, x and y are unrelated and $\langle H \rangle$ is finite.*

Thus the number of ends of a Coxeter group can be easily determined from an analysis of separating subdiagrams of a presentation diagram, and checking which subdiagrams correspond to finite subgroups. A Coxeter group whose presentation diagram is complete is either finite or 1-ended. The finite Coxeter groups have been enumerated [3].

A Dunwoody decomposition of a finitely presented group is a graph of groups decomposition of the group with finite edge groups and 1-ended and finite vertex groups. In [10], Dunwoody shows that any finitely presented group has such a decomposition.

Corollary 18. *Suppose that (W, S) is a finitely generated Coxeter system and W is the fundamental group of a graph of groups Λ where each edge group is finite. Then W has a visual decomposition Ψ where each vertex group is 1-ended or finite and a subgroup of a conjugate of a vertex group of Λ , and where each edge group is finite.*

Proof. Take Ψ the reduced visual graph of groups from Λ as given by Theorem 1. Then each edge group of Ψ is a subgroup of a conjugate of an edge group of Λ and so is finite. Suppose that some vertex $V(\subset S)$ of Ψ is such that $\langle V \rangle$ is not 1-ended or finite. Then $\langle V \rangle$ visually splits nontrivially over a finite subgroup. If $E(\subset S)$ is an edge of Ψ with endpoint V , then $E \subset V$ and $\langle E \rangle$ is finite. In particular, E induces a complete subgraph in $\Gamma(W, S)$. By Corollary 6, E must be contained in a vertex group of the visual decomposition of $\langle V \rangle$, i.e., the splitting of $\langle V \rangle$ is visually compatible with Ψ . Replacing $\langle V \rangle$ in Ψ by this splitting gives a visual graph of groups decomposition with finite edge groups. Since a special vertex group is replaced by special vertex groups with fewer generators, repeating this process eventually must end with a visual graph of groups decomposition having finite edge groups and finite or 1-ended vertex groups. \square

One might wonder how “visual” a Dunwoody decomposition must be.

Theorem 19. *Suppose that (W, S) is a finitely generated Coxeter system. If Λ is a reduced Dunwoody graph of groups decomposition of W and Ψ is a reduced visual decomposition for (W, S) such that each edge group of Ψ is finite and each vertex group of Ψ is a subgroup of a conjugate of a vertex group of Λ (in particular if Ψ is a reduced visual graph of groups decomposition from Λ), then*

- (1) Ψ is a Dunwoody decomposition;

- (2) *there is a (unique) bijection α of the vertices of Λ to the vertices of Ψ such that for each vertex V of Λ , $\Lambda(V)$ is conjugate to $\Psi(\alpha(V))$;*
- (3) *each edge group of Λ is conjugate to a special subgroup for (W, S) .*

Proof. If Φ is a graph of groups decomposition of W with finite edge groups, then any finite or 1-ended subgroup of W is a subgroup of a conjugate of a vertex group of Φ (otherwise, the action of this group on the Bass–Serre tree for Φ would induce a non-trivial splitting over a finite group). Hence each vertex group of Λ is a subgroup of a conjugate of a vertex group of Ψ . If a vertex group $A = \Lambda(V)$ of Λ is a subgroup of a conjugate of $\Psi(V')$ for V' a vertex of Ψ , then since $\Psi(V')$ is a subgroup of a conjugate of a vertex group of Λ , A is a subgroup of a conjugate of a vertex group of a vertex V'' of Λ . As noted in Lemma 3, in a reduced graph of groups, the vertex group A at V is a subgroup of a conjugate of a vertex group at V'' only if $V = V''$ and the conjugate is by an element of A . But then A is conjugate to $\Psi(V')$. Again, since no vertex group of Ψ is contained in a conjugate of another, V' is uniquely determined, and we set $\alpha(V) = V'$. Since each vertex group $\Psi(V')$ is contained in a conjugate of some $\Lambda(V)$ which is in turn conjugate to $\Psi(\alpha(V))$, we must have $V' = \alpha(V)$ and each V' is in the image of α . In particular, each vertex group of Ψ is 1-ended or finite and so Ψ is a Dunwoody decomposition of W .

Since Λ is a tree, we can take each edge group of Λ as contained in its endpoint vertex groups taken as subgroups of W . Hence each edge group is simply the intersection of its adjacent vertex groups (up to conjugation). Since vertex groups of Λ correspond to conjugates of vertex groups in Ψ , their intersection is conjugate to a special subgroup by Lemma 12. □

In Example 1 we have a visual Dunwoody decomposition

$$\langle s_1, s_2 \rangle *_{\langle s_2 \rangle} \langle s_2, s_3 \rangle *_{\langle s_3 \rangle} \langle s_3, s_4 \rangle *_{\langle s_4 \rangle} \langle s_4, s_5 \rangle,$$

which is carried by the automorphism ϕ to the Dunwoody decomposition where the last factor is replaced by $\langle s_4, s_3s_5s_3 \rangle$. Thus in this theorem we cannot expect a single element to conjugate all factors of a Dunwoody decomposition to the corresponding factors of the corresponding visual decomposition. The connection between an arbitrary Dunwoody decomposition and a visual Dunwoody decomposition is however clearly quite close.

It is worthwhile to see how this analysis of Coxeter groups leads to an understanding of why finitely generated Coxeter groups are accessible. While this argument only re-proves a special case of Dunwoody’s accessibility theorem, it is the base case of the main theorems of our papers [14] and [15], where we prove a strong accessibility result for Coxeter groups and splittings over “minimal” splitting subgroups, and a JSJ result for Coxeter groups and splittings over virtually abelian subgroups.

Lemma 20. *Suppose that (W, S) is a finitely generated Coxeter system and Λ is a graph of groups decomposition of W with finite edge groups. Suppose that a vertex group of Λ splits nontrivially as $A *_C B$ over a finite C . Then there is a special subgroup or a subgroup of a finite special subgroup of W contained in a conjugate of B which is not also contained in a conjugate of A (and then also with A and B reversed).*

Proof. Let Λ' be the graph of groups resulting from replacing the vertex whose group splits by the graph corresponding to $A *_C B$. Let Ψ be the corresponding visual graph of groups decomposition of W and let T be the Bass–Serre tree for Ψ . The intersection of any conjugates of A and B , or the intersection of distinct conjugates of B , is contained in a conjugate of an edge group and so is finite. If an infinite vertex group of Ψ lies in a conjugate of B , it cannot also lie in a conjugate of A , so suppose that no infinite vertex group of Ψ lies in a conjugate of B . From the action of B on T we get a reduced graph of groups decomposition of B with vertex and edge groups contained in conjugates of vertex and edge groups of Ψ , in particular all of the edge groups are finite. If any vertex group B_1 of this decomposition is infinite, it is contained in a conjugate of an infinite vertex group of Ψ which is in turn contained in a conjugate of a vertex group of Λ' other than B . But B_1 would then be an infinite subgroup of B and contained in a conjugate of another vertex group of Λ' , which is impossible. Instead, the vertex groups of the decomposition of B are finite and conjugate to subgroups of finite special subgroups. Replace B in Λ' by this graph of groups decomposition and collapse the edge C if it equals one of the vertex groups of the decomposition of B to get a new graph of groups decomposition Λ'' where A is adjacent to a vertex group B_1 of the decomposition of B by an edge labeled C_1 , a proper subgroup of B_1 (either with $C_1 = C$, if no collapse happens, or with C_1 an edge group of the decomposition of B). Now B_1 is finite, contained in B , but B_1 cannot also be contained in a conjugate of A , since otherwise B_1 would stabilize a path from a coset of A to B_1 in the Bass–Serre tree for Λ'' and hence stabilize a coset of C_1 , i.e., would be contained in a conjugate of C_1 which has fewer elements than B_1 . \square

As mentioned earlier, our next theorem follows from Dunwoody’s accessibility theorem, and the proof of this theorem leads to a general approach to more complex accessibility and JSJ results.

Theorem 21. *Finitely generated Coxeter groups are accessible.*

Proof. Suppose that (W, S) is a finitely generated Coxeter system. There are only finitely many special subgroups of W and finitely many subgroups of finite special subgroups. For G a subgroup of W let $n(G)$ be the number of special subgroups or subgroups of finite special subgroups which are contained in any conjugate of G

(which includes the trivial group), so $1 \leq n(G) \leq n(W)$. For Λ a finite graph of groups decomposition of W let $c(\Lambda) = (c_{n(W)}, \dots, c_2, c_1)$ where c_i is the count of vertex groups G of Λ with $n(G) = i$. Let $<$ order these $n(W)$ -tuples lexicographically, a well ordering of $n(W)$ -tuples of nonnegative integers. If Λ reduces to Λ' then clearly no c_i increases and some c_i must decrease. If a vertex group G of Λ splits as $A *_C B$ to produce a new Λ' , then every subgroup of a conjugate of A or B is a subgroup of a conjugate of G , but, by the last lemma, some special subgroup or subgroup of a finite special subgroup is contained in a conjugate of B , and so of G , but not in a conjugate of A . Hence $n(A) < n(G)$, and similarly $n(B) < n(G)$, and so $c(\Lambda') < c(\Lambda)$ since $c_{n(G)}$ decreases by 1 in going from Λ to Λ' and the only other components that change are $c_{n(A)}$ and $c_{n(B)}$ which are later in the tuples. Since $<$ is a well ordering, there can be no infinite sequence of graph of group decompositions of W resulting from successive reductions or splittings over finite subgroups, i.e., W is accessible. \square

6. Maximal FA subgroups of Coxeter groups

Let (W, S) be a finitely generated Coxeter system. A set of vertices of a complete subgraph of $\Gamma(W, S)$ is called a *simplex* of (W, S) and the subgroup of W generated by a simplex of (W, S) is called a *simplex group* of (W, S) . If V is a vertex of a graph Γ , define $lk(V)$, *the link of V* , to be the set of all vertices V' of Γ such that V' is connected to V by an edge. Define $st(V)$, *the star of V* , to be $lk(V) \cup \{V\}$.

A group G is called FA if for every tree on which G acts, the set of fixed points of G in the tree is non-empty. In §6 of [17], we find basic results on FA groups. In particular we find:

Proposition 22. *If G is denumerable then G is FA iff G is finitely generated, no quotient of G is isomorphic to the infinite cyclic group and G does not split non-trivially as an amalgamated product.*

Proposition 23. *If an FA group G is a subgroup of $A *_C B$ then G is a subgroup of a conjugate of A or B .*

Proposition 24. *Every simplex subgroup of a Coxeter system is FA.*

We have then the following result.

Lemma 25. *Suppose that (W, S) is a Coxeter system and G is an FA subgroup of W , then G is a subgroup of a conjugate of a simplex subgroup of (W, S) .*

Proof. If the presentation diagram $\Gamma(W, S)$ is not complete, choose vertices s and t that are not related. Then $lk(s)$ separates s and t in $\Gamma(W, S)$. Hence

$$W \cong \langle st(s) \rangle *_{\langle lk(s) \rangle} \langle S - \{s\} \rangle$$

By Proposition 23, G is a subgroup of a conjugate of $\langle st(s) \rangle$ or $\langle S - \{s\} \rangle$. Continue splitting until the conclusion is realized. \square

The following theorem is the main result of this section. Our proof is based in group actions on trees. A more combinatorial approach works equally well.

Theorem 26. *The maximal FA subgroups of a Coxeter group W are precisely the conjugates of the special subgroups whose diagrams are the maximal complete sub-diagrams of a (and equivalently any) presentation diagram for W .*

Proof. We have that simplex subgroups are FA and any FA subgroup is contained in a conjugate of a simplex subgroup. If A is a maximal simplex subgroup contained in wBw^{-1} for B another simplex subgroup, then by Corollary 13, $A = B$ and $w \in B$. \square

By a result of Tits [3], up to conjugacy there are only finitely many elements of order 2 in a finitely generated Coxeter group. Hence we have:

Proposition 27. *Suppose that (W, S) is a finitely generated Coxeter system. Then the subgroup of automorphisms a of W such that $a(s)$ is a conjugate of s for all $s \in S$ is of finite index in $\text{Aut}(W)$. In particular, there exists an integer $n(W, S)$ such that for any $a \in \text{Aut}(W)$, $a^n(s)$ is conjugate to s for all $s \in S$.*

Next we prove:

Theorem 28. *Suppose that (W, S) is a finitely generated Coxeter system with maximal simplices $\sigma_1, \dots, \sigma_m$. If C is the subgroup of all $c \in \text{Aut}(W)$ such that for $i \in \{1, \dots, m\}$ there exists a $w_{i,c} \in W$ with $c(x) = w_{i,c}xw_{i,c}^{-1}$ when $x \in \langle \sigma_i \rangle$ (i.e., c restricted to $\langle \sigma_i \rangle$ is conjugation by a $w_{i,c} \in W$ depending only on c and σ_i), then C has finite index in $\text{Aut}(W)$.*

Proof. If $a \in \text{Aut}(W)$ and $\sigma_1, \dots, \sigma_m$ are the maximal simplicies of (W, S) , then $a(\langle \sigma_i \rangle) = w_{i,a} \langle \sigma_{\alpha(i)} \rangle w_{i,a}^{-1}$ for some $\alpha(i) \in \{1, \dots, m\}$, by Theorem 26. Observe that $\sigma_{\alpha(j)} \neq \sigma_{\alpha(k)}$ for $k \neq j$, since otherwise

$$\begin{aligned} a^{-1}(w_{k,a}^{-1} \langle \sigma_k \rangle a^{-1}(w_{k,a})) &\equiv a^{-1}(\langle \sigma_{\alpha(k)} \rangle) \\ &= a^{-1}(\langle \sigma_{\alpha(j)} \rangle) \equiv a^{-1}(w_{j,a}^{-1} \langle \sigma_j \rangle a^{-1}(w_{j,a})) \end{aligned}$$

implies that $\sigma_k = \sigma_j$ by Corollary 13. Hence α is a permutation of $(1, \dots, m)$. If $b \in \text{Aut}(W)$ and $b(\langle \sigma_j \rangle) = w_{j,b} \langle \sigma_{\beta(j)} \rangle w_{j,b}^{-1}$ then

$$ba(\langle \sigma_i \rangle) = b(w_{i,a} \langle \sigma_{\alpha(i)} \rangle w_{i,a}^{-1}) = b(w_{i,a}) w_{\alpha(i),b} \langle \sigma_{\beta(\alpha(i))} \rangle w_{\alpha(i),b}^{-1} b(w_{i,a})^{-1}.$$

Hence the map of $\text{Aut}(W)$ into the group of permutations of $(1, \dots, m)$ defined by $a \mapsto \alpha$ is a homomorphism. If K is the kernel of this homomorphism, then K has finite index in $\text{Aut}(W)$ and for all $a \in K$, $a(\langle \sigma_i \rangle) = w_{i,a} \langle \sigma_i \rangle w_{i,a}^{-1}$ for all i .

For each $a \in K$ and for each i , we see that $a(x) = w_{i,a} \tau_i(x) w_{i,a}^{-1}$ for all $x \in \langle \sigma_i \rangle$, where $\tau_i \in \text{Aut}(\langle \sigma_i \rangle)$. The map $q_i : K \rightarrow \text{Aut}(\langle \sigma_i \rangle)$ defined by $a \mapsto \tau_i$ is a homomorphism. The main result of [12] shows that $I(\langle \sigma_i \rangle)$, the inner automorphism group of $\langle \sigma_i \rangle$, has finite index in $\text{Aut}(\langle \sigma_i \rangle)$. Hence $C \equiv \bigcap_{i=1}^m q_i^{-1}(I(\langle \sigma_i \rangle))$ is a subgroup of finite index in $\text{Aut}(W)$. □

Example 5. The group

$$\langle a, b, c, d : a^2 = \dots = d^2 = 1, (ab)^2, (ac)^2, (cd)^2, (bc)^3, (ad)^3 \rangle.$$

has a subgroup generated by an edge in the Coxeter diagram that is conjugate to that of a disjoint edge.

The element bc conjugates $\langle a, b \rangle$ to $\langle a, c \rangle$ and the element ad conjugates $\langle a, c \rangle$ to $\langle c, d \rangle$. Hence the subgroup $\langle a, b \rangle$ is conjugate to $\langle c, d \rangle$. Neither group is a maximal FA subgroup.

7. Visually stable subgroups

If (W, S) is a finitely generated Coxeter system and A is a special subgroup for this system, then A is *W-visually stable* (or *W-VS*) if for any other Coxeter system (W, S') for W , A is conjugate to a special subgroup for (W, S') .

Knowledge of the visually stable subgroups of a Coxeter group is of interest in several important questions related to the isomorphism problem for Coxeter groups. In particular, this knowledge is useful for “rigidity” questions (see for example [7] and [4] and the references therein) and questions about when reflections are preserved when passing between different Coxeter systems for W (see [1]).

Maximal finite special subgroups are visually stable (see [11]). Clearly W and the trivial group are *W-VS*. The following elementary observation is useful.

Lemma 29. *If H is W-VS and K is H-VS, then K is W-VS.*

The next result is a direct corollary to Theorem 26.

Corollary 30. *The maximal FA subgroups of a finitely generated Coxeter group W are W-VS.*

Suppose that (W, S) is finitely generated. By Dunwoody's accessibility theorem, W decomposes as a graph of groups Λ , with finite edge groups such that each vertex group is finite or 1-ended. Suppose that Ψ is a reduced visual decomposition for (W, S) derived from Λ as given in Theorem 1. Any finite or 1-ended subgroup of W is a subgroup of a conjugate of a vertex group of Ψ and so by Theorem 19 there is a bijection γ from the vertices of Λ to the vertices of Ψ such that $\Lambda(V)$ is conjugate to $\Psi(\gamma(V))$. Note that the vertex groups of Ψ are finite or maximal 1-ended special subgroups of (W, S) . In particular we have

Proposition 31. *The maximal 1-ended special subgroups of (W, S) are W - VS .*

The results of [14] and [15] imply that vertex groups of the JSJ decompositions and strong accessibility splittings considered there are VS . One example of these implications is the following

Proposition 32. *If Ψ is a visual and irreducible with respect to 2-ended splittings decomposition for a 1-ended Coxeter system (W, S) and V is a vertex of Ψ , then $\Psi(V)$ is W - VS .*

An interesting situation occurs during repeated applications of Lemma 29, Proposition 31 and Proposition 32 to an arbitrary finitely generated Coxeter system. Suppose that H_1 is a maximal 1-ended special subgroup for (W, S) , and H_2 is a maximal 1-ended special subgroup of a vertex group of Ψ , a visual irreducible with respect to 2-ended splittings decomposition of H_1 . Then H_2 may split over a 2-ended group (just not in a way compatible with Ψ). The vertex groups of an irreducible with respect to 2-ended splittings decomposition of H_2 are W - VS . Continuing on this line we have

Theorem 33. *Suppose that (W, S) is a finitely generated Coxeter system, and $A \subset S$ is maximal in the set of all $A' \subset S$ such that the induced diagram for A' is not separated by a subdiagram for a finite or 2-ended special subgroup for (W, S) . Then $\langle A \rangle$ is W - VS .*

8. A final application

Theorem 1 and Corollary 16 can be applied to “visually” characterize virtually free Coxeter groups.

Theorem 34. *The following conditions are equivalent for any Coxeter system (W, S) with presentation diagram $\Gamma(W, S)$:*

- (1) W is virtually free;

- (2) W has a visual graph of groups decomposition in which each vertex group is finite;
- (3) a) every complete subgraph of $\Gamma(W, S)$ is that of a finite Coxeter group, and
 b) no induced subgraph of $\Gamma(W, S)$ is a circuit of more than three vertices.

Proof. Virtually free groups cannot contain 1-ended subgroups. So by Dunwoody's accessibility theorem, every finitely generated virtually free group is the fundamental group of a graph of groups with finite vertex groups. By Theorem 1, (1) implies (2). Conversely, any graph of groups with finite vertex groups has fundamental group which is virtually free, and (2) implies (1).

Again, as W contains no 1-ended subgroup, Corollary 16 can be applied to show (2) implies (3a). A circuit of more than three vertices determines a 1-ended group.

Now suppose condition (3) holds in $\Gamma(W, S)$. If $\Gamma(W, S)$ were a complete graph then by a) it would be finite. Instead take x and y in $\Gamma(W, S)$ not connected by an edge. Consider a component K of the complement of $st(x)$ in $\Gamma(W, S)$. Let S_1 be the vertices in $st(x)$ adjacent to a vertex in K . We claim that any two vertices a and b in S_1 are connected by an edge in $\Gamma(W, S)$, otherwise a minimal length path from a to b in $S_1 \cup K$ meeting S_1 only at its endpoints, together with the edges from b to x and x to a would be a circuit in $\Gamma(W, S)$ of more than three vertices contradicting b). Thus S_1 is a complete graph, and so generates a finite subgroup C . Since S_1 separates $\Gamma(W, S)$, W splits as a nontrivial visual amalgamated product $A *_C B$. Inductively, we obtain that (3) implies (2). \square

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