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On the dimension of secant varieties

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Abstract. In this paper we generalize Zak’s theorems on tangencies and on linear normality as well as Zak’s definition and classification of Severi varieties. In particular we find sharp lower bounds for the dimension of higher secant varieties of a given variety X under suitable regularity assumptions on X , and we classify varieties for which the bound is attained.

Keywords. Higher secant varieties, tangential projections, special varieties

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Introduction

Let $X \subset \mathbb{P}^r$ be an irreducible, projective, non-degenerate variety of dimension n . For any non-negative integer k one can consider the k -secant variety of X , which is the Zariski closure in \mathbb{P}^r of the union of all k -dimensional subspaces of \mathbb{P}^r that are spanned by $k + 1$ independent points of X . Secant varieties are basic projective invariants related to a given variety X and their understanding is of primary importance in the study of the geometry and topology of X . As such, they have been, for more than a century, the object

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of important research in algebraic geometry. For instance, the classification of *defective varieties*, i.e. the ones for which some secant variety has dimension smaller than expected, goes back to several classical authors, like Terracini [25], Palatini [18], [19], and Scorza [22], [23] to mention but a few. For recent developments on this classical theme see [5], [6], [7] and the basic reference [27].

In more recent times the interest in the geometry of projective varieties has been revived by Zak's epochal work (see [27]). Specifically, Zak first proved his so-called theorem on tangencies, a basic tool which, although very classical in spirit, completely escaped the consideration of the classics. This theorem was used by Zak to prove a sharp lower bound for the dimension of the first secant variety to a smooth variety X , as well as the classification of those varieties achieving the bound, i.e. the so-called *Severi varieties*.

In this paper we present an extension of these results of Zak's. Namely we first extend the theorem on tangencies, then we provide, under suitable regularity assumptions on a variety X , a lower bound for the dimension of its higher secant varieties, and finally we classify the varieties for which the bound is attained.

To be specific, we introduce in §5 the notion of J_k -tangency extending the concept of J -tangency which is one of the cornerstones of Zak's theorem on tangencies. The notion of J_k -tangency is crucial for us, so we devote to it, and to related concepts, all §5. In §6 we prove Theorem 6.1, which is the announced generalization of the theorem on tangencies. In §7 we prove our extension of Zak's theorem on linear normality, i.e. Theorem 7.5 providing a sharp lower bound for the dimension of the k -secant variety to varieties having a suitable tangential behaviour which we call R_k -property, where R stands for *regularity* (see Definition 5.5). Basic tools in the proof are the generalized theorem on tangencies, as well as a few basic facts about secant varieties, defects and contact loci, which we present in §§2 and 3.

Notice that, without suitable regularity assumptions, it seems quite unlikely to get good bounds for the dimension of higher secant varieties. Examples, together with a nice account of the general theory, can be found in [17], where several partial results are given.

In §8 we define k -Severi varieties as the irreducible R_k -varieties for which the bound in the extended theorem on linear normality is attained. Smoothness is not required in the definition. However we prove that k -Severi varieties are smooth (see Theorem 8.7). The classification of k -Severi varieties is given in Theorem 8.3. The main point here is to observe that k -Severi varieties are *Scorza varieties* in the sense of Zak (see [27, Chapter VI]). Then our classification theorem follows from Zak's classification of Scorza varieties in [27]. However, a crucial point here is the smoothness of certain contact loci (ensured by Lemma 8.5 and Claims 8.9 and 8.12 in the proof of Theorem 8.7), which is essential in Zak's analysis of Scorza varieties. It is well known that strong motivations for Zak's work have been Hartshorne's conjectures. One of them, Hartshorne's Conjecture 7.2 on linear normality, has been proved by Zak. The other (see Conjecture 9.1) is still unsolved. In §9 we speculate on a possible extension of this conjecture which may be suggested by the results of the present paper.

We want to finish by observing that, besides the intrinsic interest of the subject, defective varieties, or more generally properties of secant varieties, are relevant also in other fields of mathematics, such as expressions of polynomials as sums of powers, Waring

type problems, polynomial interpolation, rank tensor computations and canonical forms, Bayesian networks, algebraic statistics etc. (see [9] as a general reference, [4], [13], [15], [20]). This classical subject is therefore still very lively and widely open to future research.

1. Preliminaries and notation

We work over the field \mathbb{C} of complex numbers and we consider the projective space $\mathbb{P}^r = \mathbb{P}_{\mathbb{C}}^r$, equipped with the tautological line bundle $\mathcal{O}_{\mathbb{P}^r}(1)$.

If $Y \subset \mathbb{P}^r$ is a subset, we denote by $\langle Y \rangle$ the span of Y . We say that Y is *non-degenerate* if $\langle Y \rangle = \mathbb{P}^r$. A linear subspace of dimension n of \mathbb{P}^r will be called an *n-subspace* of \mathbb{P}^r .

Given a subscheme $X \subset \mathbb{P}^r$, I_X will denote its homogeneous ideal and \mathcal{I}_X the ideal sheaf of X .

Let $X \subseteq \mathbb{P}^r$ be a scheme. By a *general point* of X we mean a point varying in some dense open Zariski subset of some irreducible component of X . We will denote by $\dim(X)$ the maximum of the dimensions of the irreducible components of X . We will often assume that X is *pure*, i.e. all the irreducible components of X have the same dimension. If X is projective, reduced and pure, we will say it is a *variety*.

Let $X \subset \mathbb{P}^r$ be a variety. We will denote by $\text{Sing}(X)$ the closed Zariski subset of singular points of X . Let $x \in X - \text{Sing}(X)$ be a smooth point. We will denote by $T_{X,x}$ the embedded tangent space to X at x , which is an n -subspace of \mathbb{P}^r ($n = \dim(X)$). More generally, if x_1, \dots, x_k are smooth points of X , we will set

$$T_{X,x_1,\dots,x_k} = \left\langle \bigcup_{i=1}^k T_{X,x_i} \right\rangle.$$

We will denote by $V_{n,d}$ the *d-Veronese variety* of \mathbb{P}^n , i.e. the image of \mathbb{P}^n via the *d-Veronese embedding*

$$v_{n,d} : \mathbb{P}^n \rightarrow \mathbb{P}^r, \quad r = \binom{n+d}{d} - 1.$$

Given positive integers $0 < m_1 \leq \dots \leq m_h$ we will denote by $\text{Seg}(m_1, \dots, m_h)$ the *Segre variety* of type (m_1, \dots, m_h) , i.e. the image of $\mathbb{P}^{m_1} \times \dots \times \mathbb{P}^{m_h}$ in \mathbb{P}^r , $r = (m_1 + 1) \dots (m_h + 1) - 1$, via the *Segre embedding*

$$s_{m_1,\dots,m_h} : \mathbb{P}^{m_1} \times \dots \times \mathbb{P}^{m_h} \rightarrow \mathbb{P}^r.$$

Let $0 \leq a_1 \leq \dots \leq a_n$ be integers and set $\mathbb{P}(a_1, \dots, a_n) := \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(a_1) \oplus \dots \oplus \mathcal{O}_{\mathbb{P}^1}(a_n))$. Set $r = a_1 + \dots + a_n + n - 1$ and consider the morphism

$$\phi_{a_1,\dots,a_n} : \mathbb{P}(a_1, \dots, a_n) \rightarrow \mathbb{P}^r$$

defined by the sections of the line bundle $\mathcal{O}_{\mathbb{P}(a_1,\dots,a_n)}(1)$. We denote the image of ϕ_{a_1,\dots,a_n} by $S(a_1, \dots, a_n)$. As soon as $a_n > 0$, the morphism ϕ_{a_1,\dots,a_n} is birational to its image.

Then the dimension of $S(a_1, \dots, a_n)$ is n , its degree is $a_1 + \dots + a_n = r - n + 1$ and $S(a_1, \dots, a_n)$ is a *rational normal scroll*, which is smooth if and only if $a_1 > 0$.

We will denote by $\mathbb{G}(m, n)$ the *Grassmann variety* of m -subspaces in \mathbb{P}^n , embedded in \mathbb{P}^r , $r = \binom{n+1}{m+1} - 1$, via the *Plücker embedding*.

For all integers $k \geq 1$, we will denote by S_k the *k-spinor variety*, which parametrizes the family of $(k - 1)$ -subspaces contained in a smooth quadric of dimension $2k - 1$. The variety S_k is smooth, of dimension $\binom{k+1}{2}$, and its Picard group is generated by a very ample divisor which embeds S_k in \mathbb{P}^{2^k-1} .

2. Joins, secant varieties and defects

Let X_0, \dots, X_k be varieties in \mathbb{P}^r . The *join* $J(X_0, \dots, X_k)$ of X_0, \dots, X_k is the closure in \mathbb{P}^r of the set

$$\{x \in \mathbb{P}^r : x \text{ lies in the span of } k + 1 \text{ independent points } p_i \in X_i, 0 \leq i \leq k\}.$$

We will use the exponential notation $J(X_1^{m_1}, \dots, X_h^{m_h})$ if X_i is repeated m_i times, $1 \leq i \leq h$. If X_0, \dots, X_k are irreducible, their join is also irreducible. The definition is independent of the order of X_0, \dots, X_k and one has

$$\dim(J(X_0, \dots, X_k)) \leq \min\left\{r, k + \sum_{i=0}^k \dim(X_i)\right\}.$$

The right hand side is called the *expected dimension* of the join.

If X is irreducible of dimension n , we will set $S^k(X) = J(X^{k+1})$, and we will call $S^k(X)$ the *k-secant variety* of X . This is an irreducible variety of dimension

$$s^{(k)}(X) := \dim(S^k(X)) \leq \min\{r, n(k + 1) + k\} =: e^{(k)}(X). \tag{2.1}$$

Again, the right hand side is called the *expected dimension* of $S^k(X)$.

If X is reducible of dimension n , then $J(X^{k+1})$ is in general reducible and not pure. In this case, we consider the union of all joins $J(X_0, \dots, X_k)$, where X_0, \dots, X_k are distinct irreducible components of X . It is convenient for us to denote this by $S^k(X)$ and call it the *k-secant variety* of X . With this convention, formula (2.1) still holds. The varieties X we will be considering next, even if reducible, will have the property that $S^k(X)$ is pure. We will therefore often make this assumption.

One says that X is *k-defective* when strict inequality holds in (2.1). One calls

$$\delta_k(X) := e^{(k)}(X) - s^{(k)}(X)$$

the *k-secant defect* of X . There is however a slightly different concept of *k-defect*, which will be useful for us, i.e. the concept of *k-fibre defect* $f_k(X)$, defined as

$$f_k(X) = (k + 1)n + k - s^{(k)}(X). \tag{2.2}$$

Notice that $f_k(X) = \delta_k(X)$ if $r \geq (k + 1)n + k$, while otherwise $f_k(X) = \delta_k(X) + (k + 1)n + k - r$, thus $f_k(X)$ can be positive even if $\delta_k(X) = 0$.

Remark 2.1. The reason for the name *fibre defect* is the following. Assume $S^k(X)$ is pure. Then $f_k(X)$ equals the dimension of the family of $(k + 1)$ -secant k -spaces to X passing through a general point of $S^k(X)$.

Indeed, consider the *abstract secant variety* $\mathcal{S}^k(X)$ which is the union of the closures of the sets

$$\{(p_0, \dots, p_k, x) \in X_0 \times \dots \times X_k \times \mathbb{P}^r : x \in \langle p_0, \dots, p_k \rangle \simeq \mathbb{P}^k\}$$

with X_0, \dots, X_k distinct irreducible components of X . Then the image of the projection $p : \mathcal{S}^k(X) \rightarrow \mathbb{P}^r$ is $S^k(X)$ and $f_k(X)$ is the dimension of a general fibre of p . Hence one may have $f_k(X) > 0$ even if X is not k -defective: this happens when $S^k(X) = \mathbb{P}^r$ and $r < (k + 1)n + k$.

We will use abbreviated notation like $s^{(k)}, e^{(k)}, \delta_k, f_k$ instead of $s^{(k)}(X), e^{(k)}(X), \delta_k(X), f_k(X)$ if there is no danger of confusion. Also, we may drop the index k when $k = 1$.

3. Secant varieties and contact loci

If X_0, \dots, X_k are projective varieties in \mathbb{P}^r , then Terracini’s Lemma describes the tangent space to their join at a general point of it (see [25] or, for modern versions, [1], [5], [8], [10], [27]).

Theorem 3.1. *Let X_0, \dots, X_k be varieties in \mathbb{P}^r . If $p_i \in X_i, 0 \leq i \leq k$, are general points and $x \in \langle p_0, \dots, p_k \rangle$ is a general point, then*

$$T_{J(X_0, \dots, X_k), x} = \langle T_{X_0, p_0}, \dots, T_{X_k, p_k} \rangle.$$

In particular, if $X \subset \mathbb{P}^r$ is an irreducible, projective variety, and if $p_0, \dots, p_k \in X$ are general points and $x \in \langle p_0, \dots, p_k \rangle$ is a general point, then

$$T_{S^k(X), x} = T_{X, p_0, \dots, p_k}.$$

We recall a useful consequence of Terracini’s Lemma, which is well known in the irreducible case (see [27, p. 106]). The easy proof can be left to the reader.

Proposition 3.2. *Let $X \subset \mathbb{P}^r$ be a non-degenerate variety. If we have $\dim(J(X^k)) = \dim(J(X^{k+1}))$ then $J(X^k) = \mathbb{P}^r$. Similarly, if $\dim(S^k(X)) = \dim(S^{k-1}(X))$, then $S^k(X) = \mathbb{P}^r$.*

Given a variety $X \subset \mathbb{P}^r$ of dimension n , the *Gauss map* of X is the rational map

$$g_X : X \dashrightarrow \mathbb{G}(n, r)$$

defined at the smooth points of X by mapping $x \in X - \text{Sing}(X)$ to $T_{X, x}$. It is well known that if $x \in X$ is a general point, then the closure of the fibre of g_X through x is a linear subspace $\Gamma_{X, x}$ of \mathbb{P}^r .

Definition 3.3. In the above setting, $\Gamma_{X,x}$ is called the general *Gauss fibre* of X and its dimension the *tangential defect* of X , denoted by $t(X)$. We will set $t_k(X) = t(S^k(X))$.

Note that if X is smooth, then $t(X) = 0$ (see [27]).

Let $X \subset \mathbb{P}^r$ be a non-degenerate variety such that $s^{(k)}(X) < r$. Terracini’s Lemma implies that $t_k(X) \geq k$. More precise information about $t_k(X)$ will be provided in a while. First we are going to introduce a few remarkable families of subvarieties of X related to $S^k(X)$.

Given a general point $x \in S^k(X)$, i.e. $x \in \langle p_0, \dots, p_k \rangle$, with $p_0, \dots, p_k \in X$ general points, consider the Zariski closure of the set

$$\{p \in X - \text{Sing}(X) : T_{X,p} \subseteq T_{S^k(X),x}\}.$$

We denote by Γ_{p_0, \dots, p_k} the union of all irreducible components of this locus containing p_0, \dots, p_k , and by $\gamma_k(X)$ its dimension, which clearly does not depend on p_0, \dots, p_k . Note that $\Gamma_{p_0, \dots, p_i} \subseteq \Gamma_{p_0, \dots, p_k}$ for all $i = 1, \dots, k$. We set

$$\Pi_{p_0, \dots, p_k} = \langle \Gamma_{p_0, \dots, p_k} \rangle.$$

We will use the abbreviated notation $\Gamma_k, \Pi_k, \gamma_k$ if no confusion arises. Note that Π_k contains $\langle p_0, \dots, p_k \rangle$, hence it contains x .

Definition 3.4. In the above setting, we will call Γ_{p_0, \dots, p_k} the *tangential k -contact locus* of X at p_0, \dots, p_k . We will call $\gamma_k(X)$ the *k -tangential defect* of X .

Let $X \subset \mathbb{P}^r$ be an irreducible, projective variety as above and let again $p_0, \dots, p_k \in X$ be general points. Consider the projection of X with centre T_{X,p_1, \dots, p_k} . We call this a *general k -tangential projection* of X , and we denote it by τ_{X,p_1, \dots, p_k} , or τ_{p_1, \dots, p_k} , or τ_k . We denote its image by X_{p_1, \dots, p_k} , or simply X_k . By Terracini’s Lemma, the map τ_k is generically finite to its image if and only if $s^{(k)}(X) = s^{(k-1)}(X) + n + 1$.

Definition 3.5. Let $p_0 \in X$ be a general point. Let Ψ_{p_0, \dots, p_k} be the component of the fibre of τ_{X,p_1, \dots, p_k} containing p_0 . We will denote it by Ψ_k if no confusion arises. It is called the *projection k -contact locus* of X at p_0, \dots, p_k and we will denote by $\psi_k(X)$, or ψ_k , its dimension, which is independent of p_0, \dots, p_k . This will be called the *projection k -defect*.

Remark 3.6. Notice that Γ_{p_0, \dots, p_k} contains Ψ_{p_0, \dots, p_k} . Indeed T_{X,p_0, \dots, p_k} projects, via τ_k , to the tangent space of X_k at the point $\tau_k(p_0)$, thus it is tangent along the component of the fibre containing p_0 . In particular we get $\gamma_k \geq \psi_k$. Equality holds if and only if the Gauss map of X_k is generically finite to its image, which is equivalent to being birational to its image.

One has $\Psi_{p_0, \dots, p_i} \subseteq \Psi_{p_0, \dots, p_k}$ for all $i = 1, \dots, k$.

One of the main consequences of Terracini’s Lemma is that if X is k -defective, then $\psi_k > 0$, so that γ_k is also positive.

Other relevant items related to the secant variety $S^k(X)$ are the so called *entry loci*.

Definition 3.7. Let $x \in S^k(X)$ be a point. We define the *entry locus* $E_{k,x}$ of x with respect to X as the closure of the set

$$\{z \in X : \text{there is } x' \in S^{k-1}(X) \text{ with } x' \neq z \text{ and } x \in \langle z, x' \rangle\}.$$

Alternatively, consider the fibre F_x of $p : S^k(X) \rightarrow S^k(X)$ over x . The entry locus $E_{k,x}$ is the image of F_x under the projection $p_1 : S^k(X) \rightarrow X$ to the first factor.

If $x \in S^k(X)$ is a general point, we may denote $E_{k,x}$ simply by E_k .

Next we can provide interesting information about the k -contact loci.

Lemma 3.8. *Let $X \subset \mathbb{P}^r$ be an irreducible, non-degenerate variety such that $s^{(k)}(X) < r$. If $p_0, \dots, p_k \in X$ are general points and q_0, \dots, q_k are general points on Γ_{p_0, \dots, p_k} such that q_i specializes to p_i for all $i = 0, \dots, k$, then $\Gamma_{p_0, \dots, p_k} = \Gamma_{q_0, \dots, q_k}$.*

Proof. One has $T_{X,q_i} \subset T_{S^k(X),x} = T_{X,p_0, \dots, p_k}$ for all $i = 0, \dots, k$, thus $T_{X,q_0, \dots, q_k} = T_{X,p_0, \dots, p_k}$. This immediately implies the assertion. \square

Proposition 3.9. *Same hypotheses as in Lemma 3.8. Then:*

- (i) Γ_{p_0, \dots, p_k} is smooth at p_0, \dots, p_k ; moreover it is either irreducible or consists of $k + 1$ irreducible components of the same dimension γ_k , each containing one of the points p_0, \dots, p_k as its general point;
- (ii) $f_i(\Gamma_k) = f_i(X)$ for all $i = 1, \dots, k$;
- (iii) $\Pi_k = S^k(\Gamma_k)$ equals the general Gauss fibre $\Gamma_{S^k(X),x}$ of $S^k(X)$, whereas $S^i(\Gamma_k) \neq \Pi_k$ for all $i = 1, \dots, k - 1$;
- (iv) $t_k(X) = \dim(\Pi_k) = k\gamma_k + k + \gamma_k - f_k(X)$.

Proof. Part (i) follows from Lemma 3.8 and from monodromy on the general points p_0, \dots, p_k (see [7]).

Let us prove (ii). We assume $\Gamma_k = \Gamma_{p_0, \dots, p_k}$ is irreducible, otherwise the same argument works. Let $x \in \langle p_0, \dots, p_k \rangle$ be a general point of $S^k(X)$. Let also q_0, \dots, q_i be general points on Γ_k and let $y \in \langle q_0, \dots, q_i \rangle$ be a general point of $S^i(\Gamma_k)$. By the generality assumption on p_0, \dots, p_k , also q_0, \dots, q_i are general points on X , hence y is a general point of $S^i(X)$.

Since q_0, \dots, q_i are in Γ_k , we have $T_{S^i(X),y} = T_{X,q_0, \dots, q_i} \subset T_{S^k(X),x}$. Moreover, we have an $f_i(X)$ -dimensional family of $(i + 1)$ -secant i -spaces to X passing through y . Let $\langle r_0, \dots, r_i \rangle$ be a general element of such a family, with $r_0, \dots, r_i \in X$. Then $T_{X,r_0, \dots, r_i} = T_{S^i(X),y} \subset T_{S^k(X),x}$, which shows that $r_0, \dots, r_i \in \Gamma_k$. This implies (ii).

Let us prove (iii). We have $S^k(\Gamma_{p_0, \dots, p_k}) \subseteq \Pi_{p_0, \dots, p_k} \subseteq \Gamma_{S^k(X),x}$. Let y be a general point of $\Gamma_{S^k(X),x}$, hence $y \in \langle q_0, \dots, q_k \rangle$ with $q_0, \dots, q_k \in X$. Since $T_{S^k(X),y} = T_{S^k(X),x}$, we have $q_0, \dots, q_k \in \Gamma_{p_0, \dots, p_k}$, thus $y \in S^k(\Gamma_{p_0, \dots, p_k})$. This proves the first assertion. If $S^i(\Gamma_k) = \Pi_k$ for some $i < k$, then we would have $S^i(X) = S^k(X)$, contradicting $S^k(X) \neq \mathbb{P}^r$ (see Proposition 3.2).

Part (iv) easily follows. \square

Remark 3.10. It is useful to notice that the k -contact loci are *responsible* for the k -fibre defect of X . In fact Proposition 3.9 tells us that they have the same k -fibre defect of X and through $k + 1$ general points of X there is one of them passing.

Moreover, by applying Proposition 3.9 for all positive integers $i < k$, one sees that $S^i(\Gamma_i)$ is the linear subspace Π_i , which is also the general Gauss fibre of $S^i(X)$, and $t_i(X) = i\gamma_i + i + \gamma_i - f_i(X)$.

The next proposition shows an important relation between fibre and projection defects.

Proposition 3.11. *Let $X \subset \mathbb{P}^r$ be an irreducible variety of dimension n such that $s^{(k)}(X) < r$. Then*

$$f_i = \psi_1 + \dots + \psi_i, \quad \forall i \leq k.$$

The same holds if X is reducible but $S^{(k)}(X)$ is pure and $s^{(k)}(X) < r$.

Proof. The proof is by induction on i . The case $i = 1$ is an immediate consequence of Terracini’s Lemma.

Suppose $i > 1$. Consider the general i -tangential projection τ_i of X from T_{X,p_1,\dots,p_i} . One has $\dim(X_i) = n - \psi_i$ and the general tangent space to X_i is the projection of T_{X,p_0,p_1,\dots,p_i} , p_0 being a general point of X . We have $\dim(T_{X,p_0,p_1,\dots,p_i}) = in + i + n - f_i$ and $\dim(T_{X,p_1,\dots,p_i}) = in + i - 1 - f_{i-1}$. Hence $n - \psi_i = n - f_i + f_{i-1}$, and the assertion follows by induction. \square

Corollary 3.12. *In the above setting, for all $i = 1, \dots, k$, fix general points p_1, \dots, p_i in Γ_k . Then the general projection i -contact locus of Γ_k coincides with Ψ_i . In particular $\psi_i(\Gamma_k) = \psi_i(X)$.*

Proof. Note that $T_{\Gamma_k,p_1,\dots,p_i} \subseteq T_{X,p_1,\dots,p_i} \cap \Pi_k$. Moreover T_{X,p_1,\dots,p_i} does not contain Γ_k , otherwise it would contain the whole of X , since $p_0 \in \Gamma_k$ is a general point of X . Therefore it makes sense to consider the restriction to Γ_k of the i -tangential projection $\tau_{X,i}$, which factors through the i -tangential projection $\tau_{\Gamma_k,i}$. This implies that $\psi_i(\Gamma_k) \leq \psi_i(X)$. By Propositions 3.11 and 3.9(ii), equality has to hold for all $i = 1, \dots, k$ and the assertion follows. \square

Useful information about the entry loci is provided by the following:

Proposition 3.13. *Let $X \subset \mathbb{P}^r$ be an irreducible variety with $s^{(k)}(X) < r$. Then E_k is pure of dimension ψ_k . The same holds if X is reducible and $S^{k-1}(X)$ and $S^k(X)$ are pure.*

Proof. Let $x \in S^k(X)$ be a general point. The fibre F_x of $p : S^k(X) \rightarrow S^k(X)$ over x is pure of dimension f_k . The projection to the first factor yields a dominant map $q : F_x \rightarrow E_{k,x}$. Let z be a general point in a component of $E_{k,x}$, and let $F_{x,z}$ be the fibre of q over z . Let $\xi = (z, p_1, \dots, p_k, x)$ be a general point in a component Z of $F_{x,z}$. Note that $\langle p_1, \dots, p_k \rangle$ is a $(k - 1)$ -space intersecting the line $\langle x, z \rangle$ at a point p . By Proposition 3.2, the point p does not depend on ξ . Moreover one sees that, for x and z general, p is a general point in $S^{k-1}(X)$. Hence we have a map

$$Z \dashrightarrow S^{k-1}(X), \quad \xi \mapsto (p_1, \dots, p_k, p),$$

which is birational to its image, and this, in turn, is a component of the fibre of $S^{k-1}(X) \rightarrow S^{k-1}(X)$ over p . This shows that any component Z of $F_{x,z}$ has dimension f_{k-1} . Hence any component of $E_{k,x}$ has dimension $f_k - f_{k-1} = \psi_k$ (see Proposition 3.11). \square

A different proof of Proposition 3.13 follows from Proposition 2.2 of [16], which asserts that Ψ_k can be seen as a degeneration of E_k .

Remark 3.14. Terracini’s Lemma implies that for $x \in S^k(X)$ general and for a general point $z \in E_{k,x}$, one has $T_{X,z} \subseteq T_{S^k(X),x}$. Hence $E_{k,x}$ is contained in the tangential k -contact locus Γ_{p_0, \dots, p_k} for all $p_0, \dots, p_k \in E_{k,x}$ which are smooth points of X and such that $x \in \langle p_0, \dots, p_k \rangle$. Again we deduce $\psi_k \leq \gamma_k$.

We finish by recalling the following well known *subadditivity theorem* by Palatini–Zak, whose proof is an application of the previous results (see [27, Chapter V, Proposition 1.7 and Theorem 1.8]). One more piece of notation before that. Let $X \subset \mathbb{P}^r$ be an irreducible, non-degenerate variety. We set

$$k_0 := k_0(X) = \min\{k \in \mathbb{N} : S^k(X) = \mathbb{P}^r\}.$$

Theorem 3.15. *Let $X \subset \mathbb{P}^r$ be a smooth, irreducible, non-degenerate variety of dimension n . Then*

- (i) $\psi_1 \leq \dots \leq \psi_{k_0} \leq n$;
- (ii) $\psi_k \geq \psi_{k-1} + \psi_1$ for all $k \leq k_0$.

From Theorem 3.15 one deduces

$$\psi_k \geq k\psi_1, \quad \forall k \leq k_0. \tag{3.1}$$

Definition 3.16 (see [27, Chapter VI, Proposition 1.2]). A smooth, irreducible, non-degenerate variety $X \subset \mathbb{P}^r$ of dimension n with $\psi_1 > 0$ is called a *Scorza variety* if equality holds in (3.1) and in addition $k_0 = \lceil n/\psi_1 \rceil$.

The classification of Scorza varieties is contained in [27, Chapter VI].

Remark 3.17. If X is smooth, and the general entry locus E_1 is a quadric, then E_1 is smooth (see [11, pp. 964–965]). To the best of our knowledge, there is no argument for the smoothness of the general entry locus E_k , though in [27, p. 123], this is claimed to be a consequence of “usual general position arguments” which we are unable to understand. The classification of Scorza varieties in [27] seems to depend on this assertion, which is however false, in its full generality, as the following example shows.

Example 3.18. Consider the scroll $X = S(1, h) \subset \mathbb{P}^{h+2}$ with $h \geq 4$. Let L be the line directrix of X . For all $k \geq 1$, $S^k(X)$ is the cone with vertex L over $S^k(Y)$, with Y a rational normal curve in an h -space Π' which is skew with L . Therefore

$$s^{(k)}(X) = 2k + 3 < h + 2$$

as soon as $k < (h - 1)/2$, and X is k -defective if $k \geq 2$. In this case it is not difficult to see that E_k is formed by $k + 1$ general rulings of X plus the line L .

4. Zak's theorem on tangencies

Zak's theorem on tangencies is a basic tool for the study of projective varieties and their secant varieties (see [26], [27]). It says that if a linear space L in \mathbb{P}^r is tangent to a *smooth*, pure variety $X \subset \mathbb{P}^r$ along a subvariety Y , then

$$\dim(L) \geq \dim(X) + \dim(Y). \quad (4.1)$$

Zak's original formulation in [26] is more general, inasmuch as it also applies to singular varieties X : in this case the dimension of the singular locus of X enters into play. On the other hand, as proved in Zak's book [27], formula (4.1) also works for singular varieties X , provided one takes the *right* definition of tangency at singular points. Let us recall this definition.

Definition 4.1. Let X be a variety, L a linear subspace of \mathbb{P}^r , and Y a subvariety of X contained in L . One says that L is *J-tangent* to X along Y if the following holds. Let $\{(q_0(t), q_1(t))\}_{t \in \Delta}$ be any analytic curve in $X \times Y$ parametrized by the unitary disc Δ and such that:

- (i) $q_0(t) \notin L$ for any $t \in \Delta - \{0\}$;
- (ii) $q_0(0) \in L$.

Then the flat limit of the line $\langle q_0(t), q_1(t) \rangle$ lies in L .

Remark 4.2. In point (ii) of Definition 4.1 one may equivalently ask that $q_0(0) \in Y$. Indeed, if $q_0(0) \in L$ but not in Y , then it is clear that the flat limit of the line $\langle q_0(t), q_1(t) \rangle$ lies in L , since $q_1(0) \in Y$ and $q_0(0) \neq q_1(0)$.

If L is *J-tangent* to X along Y , then it is also *J-tangent* to X along any subvariety Z contained in Y .

If L is *J-tangent* to X along Y , then L is *J-tangent* to any irreducible component of X along any irreducible component of Y .

If L is *J-tangent* to X along Y , and Z is a subvariety of X containing Y but not contained in L , then L is also *J-tangent* to Z along Y .

If X is smooth along Y , being *J-tangent* is equivalent to the condition that L contains the tangent space $T_{X,y}$ to X at a general point $y \in Y$. If X is singular at some point $y \in Y$, *J-tangency* imposes further restrictions on L .

Example 4.3. If $X \subset \mathbb{P}^r$ is a non-degenerate cone with vertex v , then no proper subspace of \mathbb{P}^r can be *J-tangent* to X along a subvariety containing v .

The notion of *J-tangency* provides a suitable setting for a general formulation of Zak's theorem on tangencies valid for singular varieties.

Theorem 4.4 (Zak's theorem on tangencies). *Let $X \subset \mathbb{P}^r$ be a variety. If a linear space L of \mathbb{P}^r is *J-tangent* to X along a subvariety Y , then (4.1) holds.*

It is well known that this theorem is sharp.

Example 4.5. There are smooth surfaces X in \mathbb{P}^4 with a hyperplane H tangent to X along a curve. An example is the projection X to \mathbb{P}^4 of the Veronese surface $V_{2,2}$ in \mathbb{P}^5 from a general point $p \in \mathbb{P}^5$.

In fact, the Veronese surface has a 2-dimensional system of conics and there is a hyperplane tangent to $V_{2,2}$ along each conic. Hence, there is a 1-dimensional system of these tangent hyperplanes passing through the centre of projection p . Such hyperplanes project down to \mathbb{P}^4 to hyperplanes of \mathbb{P}^4 which are tangent to X along the corresponding conics.

To be more specific, let Y be any conic on X . Consider the pencil of hyperplanes containing Y . This pencil cuts out on X , off Y , a pencil of conics having a base point y . There is a hyperplane tangent to X along Y if and only if $y \in Y$.

Since the map associating y to Y is a projective transformation $\omega : (\mathbb{P}^2)^* \rightarrow \mathbb{P}^2$, the locus of points $y \in X$ belonging to the corresponding conic Y describes a conic in \mathbb{P}^2 and therefore a rational normal quartic on X .

Although sharp, Zak’s theorem can be improved, as we shall see. To do this, we first have to extend the notion of J -tangency. We will do this in the next section.

5. The notion of J_k -tangency

In order to improve Zak’s theorem on tangencies, we need to extend the concept of J -tangency.

Definition 5.1. Let X, L and Y be as in Definition 4.1. Let k be a positive integer such that $\dim(\langle Y \rangle) \geq k - 1$. One says that L is J_k -tangent to X along Y if the following holds. Let $\{(q_0(t), q_1(t), \dots, q_s(t))\}_{t \in \Delta}$, with $1 \leq s \leq k$, be any analytic curve in $X \times Y^s$ parametrized by the unitary disc Δ and such that:

- (i) $(q_0(t), \dots, q_s(t))$ are linearly independent for any $t \in \Delta - \{0\}$;
- (ii) $q_0(t) \notin L$ for any $t \in \Delta - \{0\}$;
- (iii) $q_0(0) \in L$.

Then the flat limit of the s -space $\langle q_0(t), q_1(t), \dots, q_s(t) \rangle$ as $t \rightarrow 0$ lies in L .

Remark 5.2. Of course J_1 -tangency is J -tangency. Remark 4.2 can be repeated verbatim replacing J -tangency with J_k -tangency.

In particular, in point (iii) of Definition 5.1 one may equivalently ask that $q_0(0) \in J(Y^s) \cap X$. Indeed, if $q_0(0)$ lies in L but not on $J(Y^s) \cap X$, then the flat limit of the s -space $\langle q_0(t), q_1(t), \dots, q_s(t) \rangle$ as $t \rightarrow 0$ lies in L , since it is spanned by the flat limit Π of the $(s - 1)$ -space $\langle q_1(t), \dots, q_s(t) \rangle$, which lies in L , and by $q_0(0)$ which lies in L and not on Π .

Finally J_k -tangency implies J_h -tangency for all $h < k$, but the converse does not hold.

Example 5.3. Let us go back to Example 4.5, from which we keep the notation. Consider a hyperplane H tangent, and therefore J -tangent, to X along a conic Y . Let us show that H is not J_2 -tangent to X along Y . Let $\Pi = \langle Y \rangle$ and let $y \in Y$ be as in Example 4.5.

Note that all hyperplanes through Π cut on X a curve of the form $Y + Z$, with Z a conic through y . This shows that $\Pi = T_{X,y}$. Take a general conic Z on X through y . Set $\Pi' = \langle Z \rangle$. Then $\Pi \cap \Pi'$ is a line ℓ which is the tangent line to Z at y and is also a general line in Π through y . Let x be the intersection of ℓ with Y other than y . Consider an analytic parametrization $p(t)$ of Z around y , so that $p(0) = y \in Y$, and consider the analytic curve $(p(t), x, y)$ in $X \times Y^2$. Then the 2-space $\langle p(t), x, y \rangle = \Pi'$ does not depend on t , and therefore its limit is Π' , which does not lie in H .

The notion of J_k -tangency will play a crucial role below. Let us add a couple of related definitions.

Definition 5.4. Let $X \subset \mathbb{P}^r$ be variety, Y a subvariety of X and let k be a positive integer. We say that X is k -smooth along Y if X is smooth along Y and any subscheme Z of X of finite length $s \leq k + 1$ supported at Y spans a linear space of dimension $s - 1$. We say that X is k -smooth if it is k -smooth along X .

Definition 5.5. Let $X \subset \mathbb{P}^r$ be a variety such that $s^{(k)}(X) < r$. We will say that X enjoys the R_k -property (or briefly that X is an R_k -variety) if the following holds. For any $i = 1, \dots, k$ and general points p_0, \dots, p_i , taken in different components of X if X is reducible, the general hyperplane tangent to X at p_0, \dots, p_i is J_i -tangent to X along Γ_{p_0, \dots, p_i} .

Remark 5.6. (a) The notion of k -smoothness is *hereditary*, i.e., if X is k -smooth along Y , then X is k -smooth along any subvariety Z of Y .

(b) The notion of k -smoothness coincides with $\mathcal{O}_X(1)$ being k -very ample (see [3]). It can also be rephrased as follows: X is k -smooth along Y if there is no linear space L of dimension $s < k$ containing a subscheme Z of X of finite length $\ell \geq s + 2$ supported at Y .

Note that k -smoothness is a rather rigid notion. For example a smooth variety containing a line is not k -smooth for any $k \geq 2$. On the other hand, if X is smooth, its d -tuple Veronese embedding, with $d \geq k + 1$, is k -smooth.

Next we will show a relationship between the notions of k -smoothness and J_k -tangency.

Lemma 5.7. Let $X \subset \mathbb{P}^r$ be a variety, and Y a subvariety of X . Assume that X is smooth along Y and let L be a linear space tangent to X along Y . Let $\{(q_0(t), q_1(t), \dots, q_s(t))\}_{t \in \Delta}$ be an analytic curve in $X \times Y^s$ parametrized by the unitary disc Δ such that $q_0(t), q_1(t), \dots, q_s(t)$ are distinct for $t \in \Delta - \{0\}$ and $q_0(0) \in L$. Let Z_0 be the flat limit of the reduced 0-dimensional scheme corresponding to the 0-cycle $Z_t = q_0(t) + q_1(t) + \dots + q_s(t)$. Then Z_0 is contained in L .

Proof. Let Z be the limit of the 0-dimensional scheme corresponding to the 0-cycle $q_1(t) + \dots + q_s(t)$. Note that Z sits in Y and therefore in L . The degrees of Z and Z_0 differ by 1. If Z and Z_0 do not share the same support, then the assertion is clear. If Z and Z_0 have the same support, then $q_0(0) \in Y$. Moreover the ideal sheaves of Z and

Z_0 behave as follows: for any point p of the common support one has $\mathcal{I}_{Z_0,p} \subseteq \mathcal{I}_{Z,p}$ and the inclusion is strict only at one point q , where $\mathcal{I}_{Z,q}/\mathcal{I}_{Z_0,q} = \mathbb{C}$. This corresponds to a single condition imposed on functions in $\mathcal{I}_{Z,q}$ in order to have functions in $\mathcal{I}_{Z_0,q}$. This is clearly a *tangential condition*, i.e. the functions in $\mathcal{I}_{Z,q}$, in order to be in $\mathcal{I}_{Z_0,q}$, are required to be annihilated by the differential operator corresponding to the tangent vector to the branch of the curve $\{q_0(t)\}_{t \in \Delta}$ at $t = 0$. Since L is tangent to X along Y , this condition is satisfied by the equations of L as well, proving the assertion. \square

Proposition 5.8. *Let $X \subset \mathbb{P}^r$ be a variety, and Y a subvariety of X . Assume that X is k -smooth along Y . Then a linear space L is J_k -tangent to X along Y if and only if it is tangent to X along Y .*

Proof. Assume L is tangent to X along Y . Let $\{(q_0(t), q_1(t), \dots, q_s(t))\}_{t \in \Delta}$, with $1 \leq s \leq k$, be any analytic curve in $X \times Y^s$ as in Definition 5.1.

Suppose the limit Π_0 of the s -space $\Pi_t = \langle q_0(t), q_1(t), \dots, q_s(t) \rangle$ does not lie in L . Then $\Pi_0 \cap L$ would be a linear space of dimension $t < s$ containing the scheme Z_0 (see Lemma 5.7). This contradicts the k -smoothness assumption. \square

Remark 5.9. The converse of Proposition 5.8 does not hold, i.e. there are varieties which are k -regular but not k -smooth. For example, the Segre variety $\text{Seg}(n, n)$, with $n \geq 3$, is not 2-smooth because it contains lines, hence also triples of collinear points, but it is an R_{n-1} -variety (see Example 5.14 below).

Next we point out a couple of easy lemmata.

Lemma 5.10. *Let X, Y, k be as in Definition 5.4. Let L be a linear space which is J_k -tangent to X along Y . Fix a point $p \notin S^k(X)$ and let π be the projection from p . Then $L' = \pi(L)$ is J_k -tangent to $X' = \pi(X)$ along $Y' = \pi(Y)$.*

Proof. Let $\{(q_0(t), q_1(t), \dots, q_s(t))\}_{t \in \Delta}$, with $1 \leq s \leq k$, be an analytic curve as in Definition 5.1. Then the limit Π of the s -space $\langle q_0(t), q_1(t), \dots, q_s(t) \rangle$ as $t \rightarrow 0$ lies in L and does not contain p .

Consider the curve $\{(\pi(q_0(t)), \pi(q_1(t)), \dots, \pi(q_s(t)))\}_{t \in \Delta}$ in $X' \times Y'^s$. It enjoys properties (i)–(iii) of Definition 5.1 with Y, L replaced by Y', L' . The limit of the s -space $\langle \pi(q_0(t)), \pi(q_1(t)), \dots, \pi(q_s(t)) \rangle$ is the projection of Π from p , hence it is an s -space contained in L' . On the other hand, any curve in $X' \times Y'^s$ enjoying properties (i)–(iii) of Definition 5.1 with Y, L replaced by Y', L' can be obtained in this way. \square

Lemma 5.11. *Let X, Y, k be as in Definition 5.4 with $k \geq 2$. Let L be a linear space which is J_k -tangent to X along Y . Let p be a point on Y and let π be the projection from p . Then $L' = \overline{\pi(L)}$ is J_{k-1} -tangent to $X' = \overline{\pi(X)}$ along $Y' = \overline{\pi(Y)}$.*

Proof. Suppose the assertion is not true. Then we can find an analytic curve $\{(q_0(t), \dots, q_s(t))\}_{t \in \Delta}$, with $1 \leq s \leq k - 1$, in $X' \times Y'^s$ satisfying (i)–(iii) of Definition 5.1 with Y, L replaced by Y', L' , and such that the limit Π' of the s -space $\langle q_0(t), q_1(t), \dots, q_s(t) \rangle$ as $t \rightarrow 0$ does not lie in L' . By slightly perturbing this curve if necessary, we may assume that it can be lifted to an analytic curve $\{(p_0(t), \dots, p_s(t))\}_{t \in \Delta}$ in $X \times Y^s$. Consider then

the analytic curve $\{(p_0(t), \dots, p_s(t), p_{s+1}(t))\}_{t \in \Delta}$ in $X \times Y^{s+1}$, with $p_{s+1}(t)$ independent of t and equal to p . This curve satisfies (i)–(iii) of Definition 5.1 and therefore the limit Π of the $(s+1)$ -space $\langle p_0(t), \dots, p_s(t), p \rangle$ as $t \rightarrow 0$ lies in L . But then its projection from p , which is Π' , should lie in L' , a contradiction. \square

Remark 5.12. As in Lemma 5.10, one has the following. Let X be an R_k -variety [resp. a k -smooth variety] and fix a point $p \notin S^k(X)$. Then the image of the projection π of X from p is again an R_k -variety [resp. a k -smooth variety].

Similar considerations hold for Lemma 5.11.

The following provides a simple criterion for the R_k -property.

Proposition 5.13. *Let $X \subset \mathbb{P}^r$ be an irreducible, non-degenerate variety such that $s^{(k)}(X) < r$. Assume that $\gamma_i = \psi_i$ for all $i = 1, \dots, k$. Assume moreover that the intersection of the indeterminacy loci of all tangential projections τ_{p_0, \dots, p_i} is empty for p_0, \dots, p_i being general points in X . Then X is an R_k -variety.*

Proof. Fix any $i = 1, \dots, k$ and general points p_0, \dots, p_i . Consider the i -tangential projection τ_i of X from T_{X, p_1, \dots, p_i} and set $p = \tau_i(p_0)$. Note that $\gamma_i = \psi_i$ implies that Ψ_{p_0, \dots, p_i} is the irreducible component of Γ_{p_0, \dots, p_i} containing p_0 . Assume $\Gamma_i = \Gamma_{p_0, \dots, p_i}$ is irreducible. The argument in the reducible case is the same, and can be left to the reader. Thus p is the image via τ_i of $\Gamma_i = \Gamma_{p_0, \dots, p_i}$. Consider a general hyperplane H tangent to X along Γ_i . Let H' be the image of H via τ_i , tangent to X_i at p , which is a smooth point of X_i .

Take a curve $\{(q_0(t), q_1(t), \dots, q_s(t))\}_{t \in \Delta}$, with $1 \leq s \leq i$, in $X \times \Gamma_i^s$ satisfying (i)–(iii) of Definition 5.1 with $Y = \Gamma_i$. Choose p_0, \dots, p_i on Γ_i in such a way that none of the points $q_0(t), q_1(t), \dots, q_s(t)$, for t general in Δ , sits in the indeterminacy locus of τ_i . Then the projection via τ_i of the limit Π of the s -space $\langle q_0(t), q_1(t), \dots, q_s(t) \rangle$ as $t \rightarrow 0$ sits in $T_{X_i, p}$, which in turn sits in H' . This implies that Π sits in H , proving the assertion. \square

Example 5.14. The previous proposition implies that the following varieties are R_k -varieties:

- (i) the $(k+1)$ -dimensional Veronese variety $V_{2, k+1}$ in $\mathbb{P}^{k(k+3)/2}$;
- (ii) the $2(k+1)$ -dimensional Segre variety $\text{Seg}(k+1, k+1)$ in \mathbb{P}^{k^2+4k+3} ;
- (iii) the $4(k+1)$ -dimensional Grassmann variety $\mathbb{G}(1, 2k+3)$ in $\mathbb{P}^{\binom{2k+4}{2}-1}$.

Indeed, the i -contact locus of $V_{2, k+1}$ is the Veronese image of a general linear subspace of dimension i . This is also the fibre of the general i -tangential projection of $V_{2, k+1}$.

Similarly the i -contact locus of $\text{Seg}(k+1, k+1)$ is a subvariety of type $\text{Seg}(i, i)$, which is also the fibre of the general i -tangential projection of $\text{Seg}(k+1, k+1)$.

Finally the i -contact locus of $\mathbb{G}(1, 2k+3)$ is a subvariety of type $\mathbb{G}(1, 2i+1)$, which is also the fibre of the general i -tangential projection of $\text{Seg}(k+1, k+1)$.

The condition about the indeterminacy loci is easily seen to be satisfied in all these cases.

6. An extension of Zak’s theorem on tangencies

The notion of J_k -tangency plays a basic role in the following extension of Zak’s theorem on tangencies.

Theorem 6.1. *Let $X \subset \mathbb{P}^r$ be a non-degenerate variety and let $L \neq \mathbb{P}^r$ be a proper linear subspace of \mathbb{P}^r which is J_k -tangent to X along a pure subvariety Y . Then*

$$\dim(L) \geq \dim(X) + \dim(J^k(Y)) \geq \dim(X) + \dim(S^{k-1}(Y)). \tag{6.1}$$

In particular, (6.1) holds when L is tangent to X along Y and X is k -smooth along Y .

Proof. The proof follows Zak’s original argument.

By definition we have $\dim(\langle Y \rangle) \geq k - 1$. Fix k independent points $p_1, \dots, p_k \in Y$ such that $N := \langle p_1, \dots, p_k \rangle$ lies in a component $Z \subseteq J^k(Y)$ of maximal dimension. Fix $p_0 \in X$ general, so that $p_0 \notin L$. Set $M := \langle p_0, p_1, \dots, p_k \rangle$, so that $\dim(M) = k$.

We may assume that M is not contained in X , otherwise X would be a cone with vertex N , contradicting the fact that L is J_k -tangent to X along Y (see Example 4.3).

Pick a general point $x \in M - X$ and let $f : X \times J^k(Y) \rightarrow L \times L$ be the morphism which is the inclusion on the second coordinate and the projection π from a general linear space Π of dimension $r - \dim(L) - 1$ containing x on the first coordinate. By the generality assumption one has $\Pi \cap L = \Pi \cap X = \emptyset$.

We claim that f is finite. Suppose in fact C is a curve mapping to a point via f . Its projection on X would be a curve C' , since f is injective on the second coordinate. Moreover $\pi(C') = y$ would be a point, and therefore $C' \subset \langle y, \Pi \rangle$. But then $X \cap \Pi \supseteq C' \cap \Pi \neq \emptyset$, a contradiction.

Now consider the component X_0 of X passing through p_0 and restrict the map f to $X_0 \times Z$. If $\dim(L) < \dim(X) + \dim(J^k(Y)) = \dim(X_0) + \dim(Z)$, Fulton–Hansen’s connectedness theorem (see [12]) implies that the inverse image of the diagonal D of $L \times L$ is connected.

Notice that $\pi(p_0) = y \in N$. Thus $f(p_0, y) = (y, y)$. Since also (x, x) , with $x \in X \cap J^k(Y)$, belongs to the inverse image of D , Fulton–Hansen’s theorem implies that there is a curve $\{(q_0(t), q_1(t), \dots, q_k(t))\}_{t \in \Delta}$ in $X_0 \times Y^k$ such that:

- (i) $q_1(t), \dots, q_k(t)$ are linearly independent for any $t \in \Delta - \{0\}$;
- (ii) $q_0(t)$ is a general point in X_0 for $t \in \Delta - \{0\}$;
- (iii) $\pi(q_0(t)) \in Q_t := \langle q_1(t), \dots, q_k(t) \rangle \subset Z$ for all $t \in \Delta - \{0\}$;
- (iv) $q_0(0) \in J^k(Y) \cap X$.

The J_k -tangency hypothesis implies that the limit P of $P_t := \langle q_0(t), q_1(t), \dots, q_k(t) \rangle$ as $t \rightarrow 0$ lies in L . On the other hand, since $\pi(q_0(t)) \in Q_t$ for all $t \in \Delta - \{0\}$, we have $\Pi \cap P_t \neq \emptyset$ for all $t \in \Delta - \{0\}$, and therefore $\Pi \cap P \neq \emptyset$. Thus $L \cap \Pi \neq \emptyset$, a contradiction. □

Remark 6.2. Theorem 6.1 is sharp. The Veronese surface $V_{2,2}$ has hyperplanes L which are J_2 -tangent along a conic Y , because $V_{2,2}$ is 2-smooth. In that case $S^1(Y)$ is a plane and in (6.1) equality holds. This extends to higher Veronese varieties $V_{2,r}$, $r \geq 3$.

The J_k -tangency hypothesis is essential in Theorem 6.1. Indeed, for a general projection X of $V_{2,2}$ in \mathbb{P}^4 , the inequality (6.1) does not hold, but X is not 2-smooth (since it has trisecant lines, see Example 4.5).

7. An extension of Zak’s theorem on linear normality

A striking consequence of the theorem on tangencies is the famous:

Theorem 7.1 (Zak’s theorem on linear normality). *Let $X \subset \mathbb{P}^r$ be a smooth, irreducible, non-degenerate variety of dimension n . Then*

$$s(X) \geq \min\{r, 3n/2 + 1\}.$$

The reason for the name of the theorem is that it gives a positive answer to the following conjecture by Hartshorne (see [14]):

Conjecture 7.2 (Hartshorne’s conjecture on linear normality). *Let $X \subset \mathbb{P}^r$ be a smooth, irreducible, non-degenerate variety of dimension n . If $3n > 2(r - 1)$ then X is linearly normal.*

In this section we want to extend Zak’s Theorem 7.1 by giving a lower bound on $s^{(k)}(X)$ under suitable assumptions on the variety $X \subset \mathbb{P}^r$.

Let us prove the following key lemma:

Lemma 7.3. *Let $X \subset \mathbb{P}^r$ be a non-degenerate variety such that $s^{(k)}(X) < r$. Assume X is an R_k -variety (or X is k -smooth). Then*

$$2f_k(X) \leq kn. \tag{7.1}$$

Moreover, if equality holds, then:

$$\begin{aligned} \text{(i)} \quad & \gamma_i = \psi_i = if \quad \text{for all } i = 1, \dots, k, \\ \text{(ii)} \quad & f_i = \frac{i(i+1)}{2}f \quad \text{for all } i = 1, \dots, k, \\ \text{(iii)} \quad & n = (k + 1)f, \end{aligned} \tag{7.2}$$

where, as usual, we set $f = f_1$.

Proof. After projecting generically and applying Lemma 5.10, we may reduce ourselves to the case $r = nk + n + k - f_k + 1$.

Consider the general tangential i -contact locus $\Gamma_i = \Gamma_{p_0, \dots, p_i}$ for all $i = 1, \dots, k$. By the hypothesis, the general hyperplane H_i tangent to X at p_0, \dots, p_{i-1} is J_{i-1} -tangent to X along Γ_{i-1} . Moreover it does not contain Γ_i . Hence it is also J_{i-1} -tangent to Γ_i along Γ_{i-1} . Since $S^{i-1}(\Gamma_i)$ does not fill up Π_i (see Proposition 3.9), after projecting from a general point of $H_i \cap \Pi_i$, by Lemma 5.10 we find a linear space of dimension $\dim(H_i \cap \Pi_i) - 1 = \dim(\Pi_i) - 2$, which is J_{i-1} -tangent to the projection of Γ_i along Γ_{i-1} . Then, by Proposition 3.2 and Theorem 6.1, we deduce that

$$\dim(\Pi_i) - 2 \geq \dim(\Gamma_i) + \dim(S^{i-2}(\Gamma_{i-1})).$$

Using Proposition 3.9, this inequality translates into

$$i\gamma_i \geq (i - 1)\gamma_{i-1} + f_i - f_{i-2} \tag{7.3}$$

for all $i = 2, \dots, k$, where we put $f_0 = 0$. Adding these relations up, and taking into account that $\gamma_1 \geq f_1 = \psi_1$, we deduce

$$k\gamma_k \geq f_k + f_{k-1}. \tag{7.4}$$

Now notice that there exists a hyperplane H in \mathbb{P}^r which is J_k -tangent to X along Γ_k . With the same argument as above, we find

$$nk \geq k\gamma_k + f_k - f_{k-1}.$$

By taking into account (7.4), (7.1) follows.

Suppose $2f_k = kn$. Then equality holds in (7.3) for all $i = 1, \dots, k$. In particular we get $\gamma_1 = \psi_1 = f$. Thus (i) of 7.2 holds for $i = 1$. Assume $i \geq 2$ and proceed by induction.

By Proposition 3.11, we know that $\gamma_i \geq \psi_i = f_i - f_{i-1}$. This, together with (7.3) (where we must have an equality), yields

$$(i - 1)\gamma_i \leq (i - 1)\gamma_{i-1} + f_{i-1} - f_{i-2}$$

so that, by induction,

$$(i - 1)\gamma_i \leq (i - 1)\gamma_{i-1} + \psi_{i-1} = i\gamma_{i-1} = i(i - 1)f,$$

thus $if \geq \gamma_i$. On the other hand, by induction and subadditivity (see Theorem 3.15), we have

$$\gamma_i \geq \psi_i \geq \psi_{i-1} + \psi_1 = \gamma_{i-1} + f = if$$

so that $if = \gamma_i$. This proves (i), and (ii) immediately follows; moreover $kn = 2f_k = k(k - 1)f$ and also (iii) is proved. \square

Example 7.4. There are examples of R_k -varieties X for which formulas (7.2) (i), (ii) hold, but $2f_k(X) < kn$ and $n \neq (k + 1)f$.

For instance, take X to be the Segre variety $\text{Seg}(3, 4)$ in \mathbb{P}^{19} . The variety $S^2(X)$ has dimension 17. The first tangential projection sends X to $\text{Seg}(2, 3) \subset \mathbb{P}^{11}$. The second tangential projection sends X to $\text{Seg}(1, 2) \subset \mathbb{P}^5$. One computes $f = f_1 = \gamma_1 = \psi_1 = 2$, $\gamma_2 = \psi_2 = 4 = 2f$, $f_2 = 6 = 3f$. Moreover, using Proposition 5.13 one sees that X is an R_2 -variety.

On the other hand, $n = 7 \neq 6 = (k + 1)f$, and $2f_k = 12 < 14 = nk$.

We can now prove our extension of Zak’s linear normality theorem.

Theorem 7.5. *Let $X \subset \mathbb{P}^r$ be a non-degenerate variety of dimension n . Assume X is an R_k -variety (or X is k -smooth). Then*

$$\text{either } S^k(X) = \mathbb{P}^r \quad \text{or} \quad s^{(k)}(X) \geq \frac{k + 2}{2}n + k.$$

Proof. Let $s^{(k)}(X) < r$. By (7.1), one has

$$s^{(k)}(X) = (k+1)n + k - f_k \geq (k+1)n + k - \frac{kn}{2} = \frac{k+2}{2}n + k. \quad \square$$

Corollary 7.6. *Let $X \subset \mathbb{P}^r$ be a k -smooth variety of dimension n . If $(k+2)n > 2(r-k)$ then X is linearly normal.*

Proof. If X is not linearly normal, then it comes as an isomorphic projection of a variety $X' \subset \mathbb{P}^{r+1}$ from a point $p \notin X'$. Hence X' is also k -smooth and therefore an R_k -variety. Then Theorem 7.5 implies that $S^k(X') = \mathbb{P}^{r+1}$. Therefore there is some $(k+1)$ -secant k -space to X' passing through the centre of projection, yielding a $(k+1)$ -secant $(k-1)$ -space to X , contradicting k -smoothness. \square

We finish this section by stressing that the R_k -property in Theorem 7.5 is really essential, as the following example due to C. Fontanari shows.

Example 7.7. Consider the rational normal scroll 3-fold $X := S(1, 1, h) \subset \mathbb{P}^{h+4}$, $h \geq 2$. Note that we have two line sections on X spanning a 3-space Π . For all $k \geq 1$, $S^k(X)$ is the cone with vertex Π over $S^k(Y)$, with Y a rational normal curve in an h -space Π' which is skew with Π . Therefore

$$s^{(k)}(X) = 2k + 5 < h + 4$$

as soon as $k < (h-1)/2$. On the other hand $2k+5$ is smaller than the bound $\frac{k+2}{2}n + k = \frac{5}{2}k + 3$ as soon as $k > 4$.

Indeed, X is not an R_k -variety. For instance, consider the case $h = 2k + 2$. Then there is a unique hyperplane H which is tangent to X at $k+1$ general points p_0, \dots, p_k : it contains Π and projects to the unique hyperplane H' in Π' which is tangent to Y at the projection points p'_0, \dots, p'_k of p_0, \dots, p_k from Π . The k -contact locus Σ contains the union of the rulings of the scroll X passing through p_0, \dots, p_k . Suppose H is J_k -tangent to X along Σ . By projecting down from three general points of Π we would have J_{k-3} -tangency of the image hyperplane to a cone over Y , a contradiction (see Lemma 5.11 and Example 4.3).

8. The classification theorem

Let us start with the following definition.

Definition 8.1. A k -Severi variety is an irreducible, non-degenerate R_k -variety $X \subset \mathbb{P}^r$ such that

$$r > s^{(k)}(X) = \frac{k+2}{2}n + k. \quad (8.1)$$

A 1-Severi variety is simply called a *Severi variety*. Note we do not require smoothness of X in Definition 8.1. We will see in a moment that k -Severi varieties are smooth, thus, in case $k = 1$, our definition of Severi varieties turns out to coincide with the one of Zak (see [27]).

Remark 8.2. A trivial, but useful, remark is that k -Severi varieties are not cones, because of the R_k -property.

Another striking result of Zak’s is the famous classification theorem.

Theorem 8.3 (Zak’s Classification Theorem). *Let $X \subseteq \mathbb{P}^r$ be a smooth Severi variety. Then X is one of the following varieties:*

- (i) the Veronese surface $V_{2,2}$ in \mathbb{P}^5 ;
- (ii) the 4-dimensional Segre variety $\text{Seg}(2, 2)$ in \mathbb{P}^8 ;
- (iii) the 8-dimensional Grassmann variety $\mathbb{G}(1, 5)$ in \mathbb{P}^{14} ;
- (iv) the 16-dimensional E_6 -variety in \mathbb{P}^{26} .

Remark 8.4. Case (i) of Theorem 8.3 is due to Severi (see [24]), whence the name of Severi varieties.

Recall that Severi varieties are related to the unitary composition algebras $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$. If \mathcal{A} is one of these algebras, then take all 3×3 hermitian matrices A and impose that $\text{rk}(A) = 1$. This gives equations defining the Severi varieties. The secant variety to a Severi variety is defined by the vanishing of $\det(A)$. Note that \mathbb{O} being non-associative, the existence of this *determinant* is somewhat exceptional. Indeed there is no analogue for higher order matrices.

We devote this section to the analogous classification of k -Severi varieties for $k \geq 2$.

Lemma 8.5. *Let $X \subset \mathbb{P}^r$ be a k -Severi variety. Let $p_0, \dots, p_k \in X$ be general points and $x \in \langle p_0, \dots, p_k \rangle$ be a general point of $S^k(X)$. Then Ψ_{p_0, \dots, p_k} and $E_{k,x}$ are irreducible components of Γ_{p_0, \dots, p_k} .*

Proof. This follows from (7.2)(i) of Lemma 7.1. □

Lemma 8.6. *Let $X \subset \mathbb{P}^r$ be a k -Severi variety. Let $p_0, \dots, p_k \in X$ be general points and set, as usual, $\Gamma_i = \Gamma_{p_0, \dots, p_i}$ and $\Pi_i = \langle \Gamma_i \rangle$. Then for all $i = 1, \dots, k$ one has*

$$T_{\Gamma_i, p_1, \dots, p_i} = T_{X, p_1, \dots, p_i} \cap \Pi_i. \tag{8.2}$$

Moreover the intersection of Π_i with X coincides with Γ_i .

Proof. One has

$$T_{\Gamma_i, p_1, \dots, p_i} \subseteq T_{X, p_1, \dots, p_i} \cap \Pi_i.$$

By formulas (7.2) and Proposition 3.9, $T_{\Gamma_i, p_1, \dots, p_i}$ has codimension 1 in Π_i . Hence, if (8.2) did not hold, then T_{X, p_1, \dots, p_i} would contain Π_i , and therefore Γ_i . Thus it would contain p_0 , i.e. a general point of X , a contradiction. The final assertion follows from the fact that Γ_i is the general fibre of the general tangential projection τ_{i-1} . □

Theorem 8.7. *Let $X \subset \mathbb{P}^r$ be a k -Severi variety of dimension n . Then X is smooth.*

Proof. After a general projection, we may assume that $r = \frac{k+2}{2}n + k + 1$.

Let $p_1, \dots, p_k \in X$ be general points. Let τ_k be the k -tangential projection from T_{X,p_1,\dots,p_k} . Its image X_k has dimension $n - \psi_k$ and spans a projective space of dimension $r - 1 - s^{(k-1)} = r - 1 - (kn + k - 1 - f_{k-1})$. Using (7.2) one sees that X_k is a non-linear hypersurface in \mathbb{P}^{f+1} .

Let $q_0, q_1 \in X$ be general points. By (7.2), and since the general secant line to X is not a trisecant (see [27]), the tangential 1-contact locus $\Gamma_1 = \Gamma_{q_0,q_1}$ is an f -dimensional quadric. Indeed, by Proposition 3.9 we have $\dim(\Pi_1) = f + 1$.

Claim 8.8. *The tangential projection τ_k isomorphically maps Γ_1 to X_k . Then X_k and Γ_1 are smooth quadrics. Moreover the general tangential 1-contact locus intersects the general tangential k -contact locus $\Gamma_k = \Gamma_{p_0,\dots,p_k}$ transversally in one point.*

Proof of the Claim. In order to prove the first assertion, it suffices to show that $\Pi_1 = \langle \Gamma_1 \rangle$ does not intersect T_{X,p_1,\dots,p_k} . Aiming for a contradiction, assume that $T_{X,p_1,\dots,p_k} \cap \Pi_1 \neq \emptyset$. If this happens, then either

- (i) $\tau_k(\Gamma_1)$ is a subspace of dimension at most f , or
- (ii) Γ_1 is singular, $T_{X,p_1,\dots,p_k} \cap \Pi_1$ is a subspace of the vertex of Γ_1 and $\tau_k(\Gamma_1)$ is a quadric of dimension smaller than f .

Case (i) is impossible. Indeed, given two general points of X_k there would be a subspace $\tau_k(\Gamma_1)$ containing them and sitting inside X_k , contradicting the non-degeneracy of X_k in \mathbb{P}^{f+1} . In case (ii), the general tangential k -contact locus Γ_k , i.e. the general fibre of τ_k , intersects the singular quadric Γ_1 in a positive-dimensional subspace strictly containing $T_{X,p_1,\dots,p_k} \cap \Pi_1$ and not contained in the vertex of Γ_1 . Consider then the projection $\pi : X \dashrightarrow \mathbb{P}^n$ from $\Pi_k = \langle \Gamma_k \rangle$ and let X' be its image, which is non-degenerate in \mathbb{P}^n .

By the above considerations, the image of Γ_1 via π would be a linear subspace of dimension $f' < f$. Therefore $\dim(X') < n$ and moreover two general points of X' would be contained in a linear subspace of dimension f' contained in X' . This contradicts the non-degeneracy of X' .

As for the second assertion, note that X_k is a quadric, which is smooth, otherwise we have a contradiction to Lemma 8.5. The final assertion also follows from Lemma 8.5 since the above argument implies that Γ_1 intersects the general fibre of τ_k , i.e. Γ_k , transversally in one point. □

As a consequence we have:

Claim 8.9. *For all $i = 2, \dots, k$, the general tangential i -contact locus Γ_i is an $(i - 1)$ -Severi variety.*

Proof of the Claim. The irreducibility of the general 1-contact loci implies the irreducibility of the higher contact loci Γ_i with $i \geq 2$. Moreover, by Lemma 8.6 and by the R_k -property, Γ_i is an R_{i-1} -variety, and Lemma 7.3 and Proposition 3.9 imply that Γ_i is a $(i - 1)$ -Severi variety. □

Next we have:

Claim 8.10. *For all $i = 2, \dots, k$, and for the general tangential i -contact locus Γ_i , the secant variety $S^{i-1}(\Gamma_i)$ is not a cone. Similarly $S^k(X)$ is not a cone.*

Proof of the Claim. We prove the assertion for Γ_i , the proof for X is similar. Suppose $S^{i-1}(\Gamma_i)$ is a cone with vertex p . Since Γ_i is not a cone (see Remark 8.2), there is a maximum positive integer $j < i$ such that p is not a vertex of $S^{j-1}(\Gamma_i)$. Then p does not sit in the indeterminacy locus of the general projection $\tau_j = \tau_{X,p_1,\dots,p_j}$. The image Z of Γ_i via τ_j has dimension $(i - j)f$ and is a cone with vertex at the image of p . Hence the general tangent hyperplane to Z is tangent along a positive-dimensional variety. This implies that the general tangential j -contact locus of Γ_i , hence of X , has dimension at least $jf + 1$, which is a contradiction. \square

Remark 8.11. The previous argument provides a sort of converse to the criterion of Proposition 5.13.

Indeed, it proves that if an R_k -variety satisfies $\gamma_i = \psi_i = if$ for all $i = 1, \dots, k$, then the intersection of the indeterminacy loci of all tangential projections τ_{p_0,\dots,p_i} is empty for $i = 1, \dots, k$.

Now we improve Lemma 8.6.

Claim 8.12. *For all $i = 2, \dots, k$, the general tangential i -contact locus Γ_i is smooth. Furthermore X is smooth along Γ_k , and Γ_k is the schematic intersection of Π with X .*

Proof of the Claim. Suppose Γ_i is singular at a point p . Consider the general tangential projection $\tau_i = \tau_{X,p_1,\dots,p_i}$. By generic smoothness of τ_i and by Lemma 8.5, p has to be in the indeterminacy locus of τ_i , i.e. in the intersection of T_{X,p_1,\dots,p_i} with Γ_i . By Lemma 8.6, this coincides with the intersection of $T_{\Gamma_i,p_1,\dots,p_i}$ with Γ_i . By the genericity of p_1, \dots, p_i and Terracini’s Lemma, we deduce that $S^{i-1}(\Gamma_i)$ is a cone with vertex p , contradicting Claim 8.10.

Let now $q \in \Gamma_k$ be any point and let $p_1, \dots, p_k \in \Gamma_k$ be general points. Thus p_1, \dots, p_k are general points on X . Moreover Γ_k is the fibre of q in the tangential projection τ_k . Since the image X_k of X via τ_k is smooth of dimension f and the fibre of q via τ_k is smooth of dimension $\psi_k = kf$ at q we see that X is smooth of dimension $n = (k + 1)f$ at q . The final assertion follows by the same argument. \square

Now we go back to the projection $\pi : X \dashrightarrow \mathbb{P}^n$ from Π_k .

Claim 8.13. *The map $\pi : X \dashrightarrow \mathbb{P}^n$ is birational.*

Proof of the Claim. Let X' be the image of X . By Claim 8.8, the restriction of π to a general tangential 1-contact locus Γ_1 is birational, and its image is an f -subspace of \mathbb{P}^n contained in X' . This shows that X' is a subspace of \mathbb{P}^n , and therefore $X' = \mathbb{P}^n$.

Assume towards a contradiction that π is not birational. Then, if $x \in X$ is a general point, there is a point $y \in X$, with $x \neq y$, such that $\pi(x) = \pi(y)$. Note that there is some f -dimensional quadric Γ containing x and y and contained in X , i.e. a flat limit of $\Gamma_{x,z}$ with z a general point of X tending to y . The quadric Γ , as well as $\Gamma_{x,z}$, has a non-empty intersection with Π_k (see Claim 8.8). Since the fibre of x in π is 0-dimensional, also the

fibre of $\pi|_{\Gamma}$ is finite. This implies that $\langle \Gamma \rangle \cap \Pi_k$ is only one point $z \in \Gamma$ and that the line $\langle x, z \rangle$ is not contained in Γ . So the line $\langle x, z \rangle$ meets Γ only at z and x , contradicting $\pi(x) = \pi(y)$. \square

Let H be the hyperplane in $\mathbb{P}^{\frac{k+2}{2}n+k+1}$ which is tangent to X along Γ_k and let H' be its image via π .

Claim 8.14. *The inverse of the map $\pi : X \dashrightarrow \mathbb{P}^n$ is well defined off H' .*

Proof of the Claim. We resolve the indeterminacies of π by blowing-up Γ_k . If $f : \tilde{X} \rightarrow X$ is this blow-up, then $p = \pi \circ f : \tilde{X} \rightarrow \mathbb{P}^n$ is a morphism. Note that \tilde{X} is smooth along the exceptional divisor E by Claim 8.12. Points on the exceptional divisor E are mapped via p to points of H' .

Let $z \in \mathbb{P}^n$ be a point where the inverse of π is not defined. If $x \in X$ is a point such that $\pi(x) = z$, then the subspace $\Pi_x := \langle \Pi_k, x \rangle$ intersects X along an irreducible positive-dimensional subvariety Z containing x , which is contracted to a point via π . Since $\Pi_k \cap X = \Gamma_k$ (see Corollary 8.6), we have $Z \cap \Gamma_k \neq \emptyset$. Let Z' be the strict transform of Z on \tilde{X} . Then Z' intersects E , and is contracted to a point by p . Hence $z \in H'$. \square

Now we are able to finish the proof of Theorem 8.7. By Claim 8.14, $X - (H \cap X)$ is isomorphic to the affine space $\mathbb{P}^n - H'$. It remains to prove that there is no point $x \in X$ contained in all hyperplanes tangent to the tangential k -contact loci Γ_k . Suppose that such a point exists. Let $p_1, \dots, p_k \in X$ be general points and consider again the tangential projection τ_k from T_{X, p_1, \dots, p_k} . Since $S^{k-1}(X)$ is not a cone (see Claim 8.10), τ_k is well defined at p . Let z be its image via τ_k . Then all tangent hyperplanes to X_k would contain z , hence X_k would be a cone, a contradiction. \square

We can now prove the classification theorem. By taking into account Theorem 8.7 and Zak's Classification Theorem 8.3, we may consider only k -Severi varieties with $k \geq 2$.

Theorem 8.15. *Let $X \subseteq \mathbb{P}^r$ be a k -Severi variety with $k \geq 2$. Then X is one of the following varieties:*

- (i) *the $(k + 1)$ -dimensional Veronese variety $V_{2, k+1}$ in $\mathbb{P}^{k(k+3)/2}$;*
- (ii) *the $2(k + 1)$ -dimensional Segre variety $\text{Seg}(k + 1, k + 1)$ in \mathbb{P}^{k^2+4k+3} ;*
- (iii) *the $4(k + 1)$ -dimensional Grassmann variety $\mathbb{G}(1, 2k + 3)$ in $\mathbb{P}^{\binom{2k+4}{2}-1}$.*

Proof. The varieties in (i)–(iii) are R_k -varieties (see Example 5.14). Moreover they are k -Severi varieties, i.e. (8.1) holds for them (see [27]).

Set now $s = s^{(k)} + 1 = \frac{k+2}{2}n + k + 1$ and take a general projection X' to \mathbb{P}^s . Then X' is still a k -Severi variety. Moreover we have $k_0(X') = k + 1$, $\psi_i(X') = \psi_i(X) = if$, $f_i(X') = f_i(X) = \frac{i(i+1)}{2}f$ for all $i = 1, \dots, k$ and $n = (k + 1)f$ by Lemma 7.3. Thus X' is a Scorza variety. The classification follows from Zak's classification of Scorza varieties, which implies that Scorza varieties are linearly normal, in particular $X = X'$. Notice the smoothness of the entry loci, which follows by Lemma 8.5 and Claims 8.9 and 8.12. This is essential in Zak's argument (see Remark 3.17). \square

Remark 8.16. The k -Severi varieties, with $k \geq 2$, are related to the unitary composition algebras $\mathbb{R}, \mathbb{C}, \mathbb{H}$. In this case we take all $(k + 1) \times (k + 1)$ hermitian matrices A and impose that $\text{rk}(A) = 1$. This gives the equations defining the k -Severi varieties. Again the k -secant variety to a k -Severi variety is defined by the vanishing of $\det(A)$. The absence of the analogue of the E_6 -variety, related to the composition algebra \mathbb{O} , reflects the absence of higher order determinants on \mathbb{O} (see Remark 8.4).

A quick proof of the classification of Severi varieties can be obtained by using the beautiful ideas contained in [21]. Along the same lines one can give a proof of the classification of k -Severi varieties, alternative to the one described above based on Zak’s classification of Scorza varieties. We do not dwell on this here.

9. Speculations

Perhaps the main motivation for Zak’s beautiful piece of work was the following famous conjecture by Hartshorne (see [14]):

Conjecture 9.1 (Hartshorne’s conjecture). *Let $X \subset \mathbb{P}^r$ be a smooth, irreducible, non-degenerate variety of dimension n . If $3n > 2r$ then X is a complete intersection of $r - n$ hypersurfaces in \mathbb{P}^r .*

This in turn was motivated by Barth–Larsen’s fundamental result (see [2]) to the effect that smooth varieties $X \subset \mathbb{P}^r$ of low codimension are topologically similar to \mathbb{P}^r . Barth–Larsen’s theorem, in our context, can be stated as follows:

Theorem 9.2. *Let $X \subset \mathbb{P}^r$ be a smooth, irreducible variety. Then for any non-negative integer $i < f_1(X)$ the natural map*

$$\rho_{X,i} : H^i(\mathbb{P}^r, \mathbb{Z}) \rightarrow H^i(X, \mathbb{Z})$$

is an isomorphism. In particular, if $f_1(X) \geq 2$ then X is simply connected and if $f_1(X) \geq 3$ then $\text{Pic}(X)$ is generated by the hyperplane class.

One of the basic steps in the proof of Conjecture 9.1 would be to show that, under the hypotheses, X is projectively Cohen–Macaulay. Since linear normality is the first naïve requirement for being projectively Cohen–Macaulay, this is the motivation for Conjecture 7.2, which in turn motivates Zak’s theorems.

Now, in presence of our refined form of Zak’s linear normality theorem, one may speculate on the possibility of having an even more general view on Hartshorne’s conjecture. This is what we want to present next. To be precise, we want to propose the following:

Conjecture 9.3 (Extended Hartshorne’s conjecture). *There is a suitable function $f(r, n, k)$ such that the following happens. Let $X \subset \mathbb{P}^r$ be a k -smooth, irreducible, non-degenerate variety of dimension n . If $(k + 2)n > 2r$ then \mathcal{I}_X is generated by at most $f(r, n, k)$ elements.*

This conjecture does not make too much sense unless one specifies the form of the function $f(r, n, k)$. What we intend is that $f(r, n, k)$ should be reasonably small. If one wants to be really bold, one may even conjecture that $f(r, n, k) = k(r - n)$. A further strength-

ening of the conjecture would be to replace the ideal sheaf \mathcal{I}_X with the homogeneous ideal I_X .

Example 9.4. There are varieties *at the boundary* of Hartshorne's conjecture. One of them is $\mathbb{G}(1, 4)$, which has dimension 6 in \mathbb{P}^9 . Its homogeneous ideal is generated by five quadrics. This would fit Conjecture 9.3 for $k = 2$ and $f(r, n, 2) = 2(r - n)$, but unfortunately $\mathbb{G}(1, 4)$ is not 2-smooth, since it contains lines.

Another variety at the boundary of Hartshorne's conjecture is the 10-dimensional *spinor variety* $S_4 \subset \mathbb{P}^{15}$, its homogeneous ideal being generated by 10 quadrics. Again this would fit Conjecture 9.3 for $k = 2$ and $f(r, n, 2) = 2(r - n)$, but this variety is not 2-smooth either.

More varieties at the boundary of Hartshorne's conjecture, actually all the known ones, are deduced from these by pulling them back via a general morphism $\mathbb{P}^r \rightarrow \mathbb{P}^r$. Now these can in general be 2-smooth and Conjecture 9.3 holds for them with $f(r, n, 2) = 2(r - n)$.

The examples of $\mathbb{G}(1, 4)$ and S_4 suggest that the k -smoothness assumption in Conjecture 9.3 might even be too strong. Maybe something like the R_k -property could suffice.

Remark 9.5. At this point a related natural question arises: is there, in this same spirit, any extension of Barth–Larsen's Theorem 9.2? By taking into account (3.1) we see that

$$2f_k \geq k(k+1)f_1$$

and therefore one might ask: is the map $\rho_{X,i}$ an isomorphism for all positive integers i such that $k(k+1)(i+1) \leq 2f_k(X)$, under the assumption that $X \subset \mathbb{P}^r$ be a k -smooth, irreducible variety? Or, is $\rho_{X,i}$ an isomorphism under the condition

$$\binom{k+1}{2}i \leq (k+1)n - r - \binom{k}{2}$$

if $X \subset \mathbb{P}^r$ is a k -smooth, irreducible variety?

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