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The discrete Douglas problem: theory and numerics

PAOLA POZZI†

Centre for Mathematics and its Applications, MSI, Australian National University, Canberra, ACT 0200, Australia

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We solve the problem of finding and justifying an optimal fully discrete finite element procedure for approximating annulus-like, possibly unstable, minimal surfaces.

In this paper we introduce the general framework, some preliminary estimates, develop the ideas used for the algorithm, and give the numerical results. Similarities and differences with respect to the fully discrete finite element procedure given by G. Dziuk and J. Hutchinson in the case of the classical Plateau problem are also addressed.

In a subsequent paper we prove convergence estimates.

Keywords: Minimal surfaces; Finite elements; Order of convergence; Douglas problem.

1. Introduction

The problem of showing the existence of a *minimal surface* of a given topological type spanning a collection of disjoint closed oriented rectifiable Jordan curves is known as the *Douglas problem*. Although in this work we study the specific problem of approximating *annulus-like* minimal surfaces, we will often refer to it (with some abuse of notation) as the Douglas case, or Douglas problem. Comprehensive references for the classical theory of minimal surfaces are the books by Dierkes, Hildebrandt, Küster and Wohlrab [1], [2], and by J. C. C. Nitsche [9]; more specifically, the Douglas problem is considered in the works by J. Jost [8], [7], and the references given there.

In this paper and a subsequent one [10] we find and justify an optimal fully discrete finite element procedure for approximating annulus-like, possibly unstable, minimal surfaces. This work is a natural extension of the research done by G. Dziuk and J. Hutchinson, and the author, in the case of the classical Plateau problem: see [4], [5], and [11].

Unlike the Plateau case, where every disc-like surface is conformally equivalent to the unit disc, in the Douglas problem every annulus-like surface is conformally equivalent to a unique cylinder of radius one and length λ, for some $\lambda \in (0, \infty)$. This means that when we look for a parametrisation of our surface we have a one-parameter family of possible domains (i.e. the set of all cylinders of radius one and length λ for $\lambda \in (0, \infty)$) as opposed to the fixed disc for the classical Plateau problem.

Needless to say, the introduction of this a priori unknown parameter represents a major problem in the study of the Douglas case.

The main results can be informally stated as follows. Let $\Gamma_1, \Gamma_2 \subset \mathbb{R}^n$ be two disjoint closed Jordan curves, rectifiable and with given orientation, and set $\Gamma = (\Gamma_1, \Gamma_2)$. Let C_λ be a cylinder

†Email: paola.pozzi@maths.anu.edu.au

of unit radius and length $\lambda \in (0, \infty)$. An equivalent formulation of the notion of an annulus-like minimal surface which we will use is the following. Let $\mathcal{F}$ be the class of maps $u : C_\lambda \to \mathbb{R}^n$, for *all* possible choices of $\lambda > 0$, such that $u|_{\partial C_\lambda} : \partial C_\lambda \to \Gamma$ is monotone and u is *harmonic*. The function $u \in \mathcal{F}$ defined on C_λ is said to be a *minimal surface* if u is stationary in $\mathcal{F}$ for the Dirichlet energy $\mathcal{D}(u) = \frac{1}{2}\int_{C_\lambda} |\nabla u|^2$. Such a map u provides a harmonic and *conformal* parametrisation of the corresponding minimal surface.

Following this characterisation, a first approximation to our numerical method is as follows. For any $\lambda > 0$, let $\mathcal{G}_{\lambda h}$ be a quasi-uniform triangulation of C_λ controlled by h (in practice this is done by considering C_λ as a rectangle on the plane with the two sides of length λ identified and by triangulating the planar figure in the natural way). We can consider $\mathcal{G}_{\lambda h}$ as a one-parameter family of triangulations corresponding to the one-parameter family of domains C_λ. Let $\mathcal{F}_h$ be the class of continuous piecewise linear maps $u_h : C_\lambda \to \mathbb{R}^n$, for all possible choices of $\lambda > 0$, which are *discrete harmonic* and for which $u_h(\phi_j) \in \Gamma$ whenever ϕ_j is a boundary node of C_λ. Note that we do not require the monotonicity of $u_h|_{\partial C_\lambda}$. A function $u_h \in \mathcal{F}_h$ defined on C_{λ_h} is said to be a *discrete minimal surface* if u_h is stationary within $\mathcal{F}_h$ for the Dirichlet energy $\mathcal{D}(u_h) = \frac{1}{2}\int_{C_{\lambda_h}} |\nabla u_h|^2$. A member of $\mathcal{F}_h$ is determined by its values at the boundary nodes and by the knowledge of the length λ_h of its domain.

One of the main convergence results proved in [10] is that if $u : C_\lambda \to \mathbb{R}^n$ is a "nondegenerate", harmonic and conformally parametrised minimal surface spanning Γ, then there exist $\lambda_h \in (0, \infty)$ and a discrete minimal surface $u_h : C_{\lambda_h} \to \mathbb{R}^n$ such that if we denote by σ_μ the cylinder transformation of the form $\sigma_\mu : C_1 \to C_\mu$, $\sigma_\mu(x, \theta) = (\mu x, \theta)$, then

$$\|u \circ \sigma_\lambda - u_h \circ \sigma_{\lambda_h}\|_{H^1(C_1)} \leqslant ch, \qquad |\lambda - \lambda_h| \leqslant ch, \tag{1}$$

where c depends on a fixed parametrisation γ of Γ, λ, and the nondegeneracy constant for u but is independent of h.

Under basically the same hypotheses it is proved furthermore that

$$\|u \circ \sigma_\lambda - u_h \circ \sigma_{\lambda_h}\|_{L^2(C_1)} \leqslant ch^2|\ln h|^{3/2}, \qquad |\lambda - \lambda_h| \leqslant ch^2|\ln h|^{3/2}, \tag{2}$$

where, as above, c does not depend on h.

Once a suitable framework is established (and this is a crucial point), (1) is obtained by similar arguments used to prove the analogous estimate for the case of the Plateau problem (see [4] and [5]). We obtain (2) by extending to the present situation the results given in [11]. In both cases, techniques are developed to deal with the parameter λ.

In this paper we introduce the general framework and illustrate some of the techniques used to treat the parameter λ. Furthermore we give a constructive way to find stationary points for the Dirichlet energy: see Section 3, Proposition 3.7. This proof is not needed theoretically, because the existence of a solution to the Douglas problem has already been proved (see for example [8, Theorem 1.2.1]) and extensive literature is available on this topic. However, Proposition 3.7 motivates and justifies the construction of the so called "discrete sequence" (discussed in Section 5), on which idea the algorithm used to solve the discretised Douglas problem is based. Last but not least, this approach makes it easier to recognize differences and similarities between the Douglas case and the classical Plateau problem.

Finally, we demonstrate numerically that the orders of convergence obtained in (1) and (2) cannot generally be improved. We would like to point out that our numerical investigation does not aim to be exhaustive but rather verify the results obtained theoretically.

2. The smooth Douglas problem

2.1 *Theoretical background*

In this work we are interested in the study of annulus-like surfaces, i.e. surfaces of genus zero with two boundary curves. We can word the problem as follows.

> Given two disjoint oriented and rectifiable Jordan curves Γ_1 and Γ_2 in $\mathbb{R}^n$ ($n \geqslant 2$), find the area minimizer (or more generally, find a critical point for the area functional) among all functions which have a cylinder C_λ (of finite length) for domain and map ∂C_λ onto $\Gamma_1 \cup \Gamma_2$ in a weakly monotone way and respecting the orientation of the boundary.

It is of course not true that without further assumptions such a minimizer exists. A typical example to keep in mind is that of a catenoid: given are two equal rings placed on parallel planes at a distance d apart in such a way that one ring is the projection of the other in the direction perpendicular to the planes. If d is small enough, it can be shown that there exist two annulus-like minimal surfaces, one of which is an absolute area minimizer. If d exceeds a critical value $\tilde{d}$ and we consider an area minimizing sequence of annulus-like surfaces, it can be observed that an increasingly narrower neck is developed and the surfaces degenerate in topological type by tending to two disjoint discs. In this situation no annulus-like surface can absolutely minimize the area. The area minimizer is given instead by the union of the two flat discs bounded by the two rings. But the topological type has now changed: we have a minimizer of lower topological type.

This example shows that we need extra conditions to guarantee the existence of both minimal surfaces and area minimizers of a given topological type: in this particular case, a bound on the separation of the boundary curves would do. In general the so called *Douglas condition* is usually assumed to be true.

It is not our intention to go into more details about the Douglas condition and we refer the reader to the classical books mentioned in the Introduction for more information about it. For the sake of this paper the reader needs just to be aware that such an assumption is sufficient (but not necessary!) to prove the existence of a minimal surface of given topological type. We state briefly the existence theorem whose proof can be found in [8].

THEOREM 2.1 (Douglas Theorem) Let $\Gamma = (\Gamma_1, \Gamma_2)$ be two disjoint closed oriented rectifiable Jordan curves in $\mathbb{R}^n$. If the Douglas condition is satisfied, namely if

$$d(\Gamma, 0) < d^*(\Gamma, 0), \tag{3}$$

then Γ bounds a connected minimal surface (an area minimizer) of genus 0.

Intuitively, condition (3) guarantees the existence of an annulus-like surface whose area is strictly less than the sum of the areas of the disc-like minimal surfaces for the two given Jordan curves Γ_1 and Γ_2.

2.2 *Formulation of the problem*

Set $\Gamma = (\Gamma_1, \Gamma_2)$ and define C_λ to be the cylinder

$$C_\lambda := \{(x, \theta) \mid 0 \leqslant x \leqslant \lambda,\ \theta \in S^1\}.$$

We look at the maps

$$\mathcal{C}'(\Gamma) := \{u : C_\lambda \to \mathbb{R}^n \mid 0 < \lambda < \infty,\ u \text{ maps } \partial C_\lambda \text{ onto } \Gamma \text{ in a weakly monotone way and preserving orientation}\} \cap H^1(C_\lambda) \cap C^0(C_\lambda),$$

and we are interested in finding $u \in \mathcal{C}'(\Gamma)$ such that u is *stationary* for the area functional. It is well known that there is a one-to-one correspondence between *conformal* maps that are stationary for the area functional and maps that are stationary for the Dirichlet energy (see remark below). Since a surface of genus zero with two boundary curves is conformally equivalent to a cylinder C_λ for a specific $\lambda > 0$, it is natural to give and use the following definition.

DEFINITION 2.1 A *minimal surface* is a map which is stationary for the Dirichlet functional.

Note that such a surface does not have to be an area minimizer. For later purposes we make the following remark.

REMARK. The map u (sometimes we will write (u, λ) to remind us that $u : C_\lambda \to \mathbb{R}^n$) is stationary for the Dirichlet functional $\mathcal{D}$ if and only if

(D1) $\frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0}\mathcal{D}(u + tv) = 0$ for all $v \in H_0^1(C_\lambda)$ *(stationarity with respect to variations of the surface),*

(D2) $\frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0}\mathcal{D}(u \circ \sigma_t) = 0$ for every smooth family of diffeomorphisms $\sigma_t : C_{\lambda_t} \to C_\lambda$ with $\sigma_0 = \mathrm{id}$ and λ_t depending differentiably on t *(stationarity with respect to variations of and in the domain).*

In [8] we find the following important characterisation. A function that is stationary for the Dirichlet energy must have a natural parametrisation, namely a conformal one. The fact that by working with the Dirichlet functional we can control the parametrisation is one of the main reasons for discarding the area functional and using the Dirichlet energy instead.

PROPOSITION 2.2 (u, λ) satisfies (D1) and (D2), i.e. $u : C_\lambda \to \mathbb{R}^n$ is a minimal surface, if and only if u is harmonic and conformal in the interior of C_λ, which means

(H1) $\Delta u = 0$ in $\mathring{C}_\lambda$ *(harmonicity),*

(H2) $|u_x| = |u_\theta|$ and $\langle u_x, u_\theta\rangle = 0$ in $\mathring{C}_\lambda$ *(conformality).*

Basically the following equivalences hold:

$$(\mathrm{D1}) \Leftrightarrow (\mathrm{H1}), \qquad (\mathrm{D2}) \Leftrightarrow (\mathrm{H2}).$$

In the proof of Jost, however, it becomes clear that we have something more, namely

$$(\mathrm{D2}') + (\mathrm{D3}') \Leftrightarrow (\mathrm{H2}),$$

where

(D2′) $\frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0}\mathcal{D}(u \circ \sigma_t) = 0$ for every smooth family of diffeomorphisms $\sigma_t : C_\lambda \to C_\lambda$ such that $\sigma_0 = \mathrm{id}$ *(stationarity with respect to variations on the fixed domain),*

(D3′) $\frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0}\mathcal{D}(u \circ \sigma_t) = 0$ for $\sigma_t^{-1} : C_\lambda \to C_{\lambda_t}$ a diffeomorphism of the form $C_\lambda \ni (x, \theta) \mapsto ((1 + t)x, \theta)$.

Just to give an idea of the implications of each of these statements, let us recall the following important lemma proved in [8].

LEMMA 2.3 Let Σ be a compact Riemann surface with smooth boundary $\partial\Sigma$, $h \in H^1(\Sigma, \mathbb{R}^n)$, and suppose

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \mathcal{D}(h \circ \sigma_t) = 0$$

for all smooth families of diffeomorphisms $\sigma_t : \Sigma \to \Sigma$ with $\sigma_0 = \mathrm{id}$. Then, with $z = x + iy$ a local conformal parameter on Σ,

$$\varphi(z)\,\mathrm{d}z^2 := h_z^2\,\mathrm{d}z^2 = \tfrac{1}{4}(h_x^2 - h_y^2 - 2ih_x \cdot h_y)(\mathrm{d}x^2 - \mathrm{d}y^2 + 2i\,\mathrm{d}x\,\mathrm{d}y)$$

is a holomorphic quadratic differential on Σ which is real on $\partial\Sigma$.

That $\varphi(z)\,\mathrm{d}z^2$ is real on $\partial\Sigma$ means the following: if we choose our local conformal parameter $z = x+iy$ near $\partial\Sigma$ in such a way that $\partial\Sigma$ is locally given by $y = 0$, then along $\partial\Sigma$, $\mathrm{d}y = 0$; hence if $\varphi(z)\,\mathrm{d}z^2$ is real on $\partial\Sigma$, then

$$0 = \mathrm{Im}(\varphi\,\mathrm{d}z^2) = -\tfrac{1}{2}h_x \cdot h_y\,\mathrm{d}x^2,$$

i.e. h_x and h_y are orthogonal along $\partial\Sigma$.

On the unit disc, every holomorphic quadratic differential which is real on the boundary vanishes identically, so conformality is immediately obtained.

On the other hand, on a cylinder the holomorphic quadratic differentials real on the boundary are of the form

$$(\text{real constant}) \cdot \mathrm{d}z^2,$$

so conformality is not quite achieved yet. It is at this point that condition (D3′) comes into play.

These facts will actually become relevant at a later stage. At the moment it is sufficient to note that from now on we will consider the Dirichlet energy and therefore we can restrict our class of maps to

$$\mathcal{C}(\Gamma) = \mathcal{C}'(\Gamma) \cap \{u : C_\lambda \to \mathbb{R}^n \mid u \text{ harmonic}, \lambda \in (0, \infty)\}.$$

The big advantage of working in $\mathcal{C}(\Gamma)$ is that harmonic maps are uniquely determined by their boundary values. So essentially each map $u \in \mathcal{C}(\Gamma)$ is uniquely determined by λ and $u|_{\partial C_\lambda}$. About the boundary behaviour of a solution to the Douglas problem we have the following result.

THEOREM 2.4 Let u be a minimal surface which maps an open arc $A \subset \partial C_\lambda$ into an open portion $\Gamma' \subset \Gamma$ and assume that $\Gamma' \in C^{k,\alpha}$ for some $k \in \mathbb{N}$ and some $0 < \alpha < 1$. Then $u \in C^{k,\alpha}(\mathring{C}_\lambda \cup A)$.

Proof. See [2, §7.3]. □

2.3 *Reformulation of the problem*

Our approach to the problem uses the ideas presented in [13] and in [4] and [5]. The main goal is to transfer the nonlinearity from the class $\mathcal{C}(\Gamma)$ of competing functions to the energy functional.

To do so, take a cylinder C_1 of radius and length 1 and fix $\gamma : \partial C_1 \to \Gamma$, $\gamma = (\gamma_1, \gamma_2)$, $\gamma_i : S^1 \to \Gamma_i$ for $i = 1, 2$, to be a regular C^r-parametrisation of Γ with $r \geqslant 3$.

If $\pi_\lambda : \partial C_\lambda \to \partial C_1$ is the map that identifies ∂C_λ with ∂C_1, then $\gamma \circ \pi_\lambda$ acts on ∂C_λ exactly like γ on ∂C_1. Thus, from now on we will identify these two maps and we will write γ also when we actually mean $\gamma \circ \pi_\lambda$.

Given $u \in \mathcal{C}(\Gamma)$, $u : C_\lambda \to \mathbb{R}^n$, then $u|_{\partial C_\lambda}$ can be uniquely written in the form $\gamma \circ s$, where $s : \partial D \dot\cup \partial D \to S^1 \dot\cup S^1$ (and $\dot\cup$ denotes the disjoint union). Although $\partial D \dot\cup \partial D$, $S^1 \dot\cup S^1$ and ∂C_λ are naturally isomorphic, we will usually consider $S^1 \dot\cup S^1$ as the domain of the *fixed* parametrisation γ of Γ and $\partial D \dot\cup \partial D$ as the boundary of the parameter domains C_λ for various parametrised surfaces. See Figure 1.

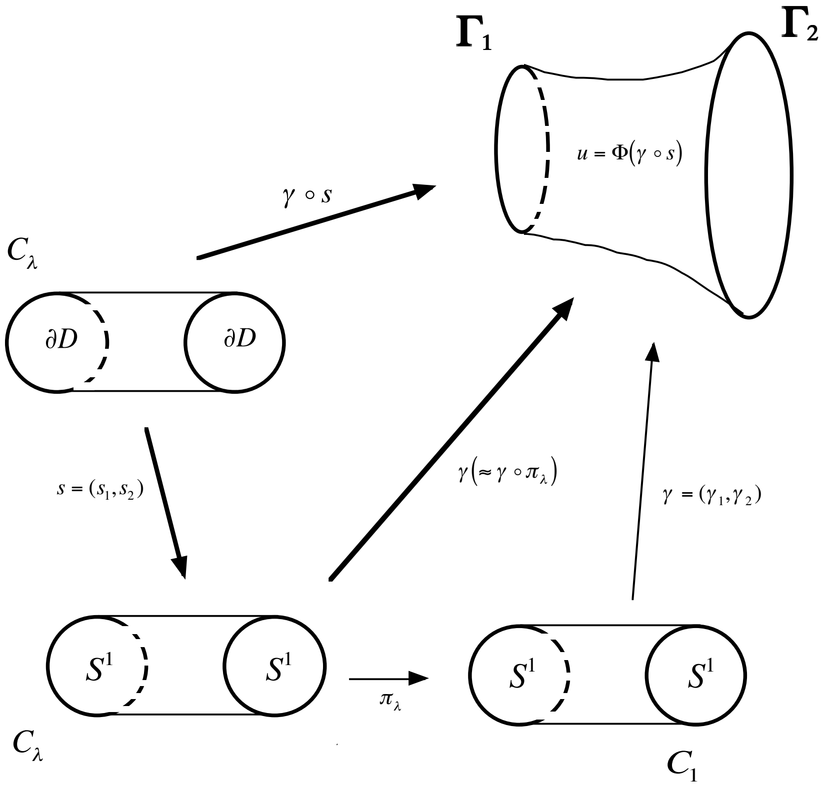

FIG. 1

A map $f \in C^0(\partial D, S^1)$ is said to be *monotone* if f is positively oriented and $f^{-1}(p)$ is connected for all $p \in S^1$. Note that a monotone f need not be injective: as it moves once around S^1 it can pause but never retraces its path. We similarly define the notion of monotone map from ∂D to Γ_i, for $i = 1, 2$. Since there is a one-one correspondence $s_i \leftrightarrow \gamma \circ s_i$ between monotone maps in $C(\partial D, S^1)$ and monotone maps in $C(\partial D, \Gamma_i)$, $i = 1, 2$, there is also a one-one correspondence $s = (s_1, s_2) \leftrightarrow \gamma \circ s = (\gamma_1 \circ s_1, \gamma_2 \circ s_2)$ between monotone maps in $C^0(\partial D, S^1) \dot\cup C^0(\partial D, S^1) \simeq C^0(\partial D \dot\cup \partial D, S^1 \dot\cup S^1)$ and monotone maps in $C^0(\partial D, \Gamma_1) \dot\cup C^0(\partial D, \Gamma_2) \simeq C^0(\partial C_\lambda, \Gamma)$.

Further note that any monotone map $s : \partial D \dot\cup \partial D \to S^1 \dot\cup S^1$ can be written in the form

$$s = (s_1, s_2) = (\mathrm{id} + \sigma_1, \mathrm{id} + \sigma_2) := \mathrm{id} + \sigma.$$

Here id : $\partial D \to S^1$ is the "identity" map $\mathrm{id}(\theta) = \theta$ (with abuse of notation we will write id also when we intend (id, id)) and $\sigma = (\sigma_1, \sigma_2) \in C^0(\partial D, \mathbb{R}) \dot\cup C^0(\partial D, \mathbb{R})$ is a 2π-periodic function defined up to a constant $c = (c_1, c_2)$ with $c_i = 2\pi k_i$, $k_i \in \mathbb{Z}$, $i = 1, 2$. Addition of such maps is performed modulo 2π.

For $w \in C^0(\partial C_\lambda, \mathbb{R}^n)$ we denote by $\Phi(w)$ the unique harmonic extension of w on C_λ.

We can now define the *energy functional E*.

DEFINITION 2.2 For $s \in C^0(\partial D \dot{\cup} \partial D, S^1 \dot{\cup} S^1)$ and $\lambda \in (0, \infty)$ let

$$E(s, \lambda) := \frac{1}{2} \int_{C_\lambda} |\nabla \Phi(\gamma \circ s)|^2 = \mathcal{D}(\Phi(\gamma \circ s)). \tag{4}$$

Thus $E(s, \lambda)$ is just the Dirichlet energy of the harmonic extension of $\gamma \circ s$ on C_λ.

Norms and function spaces. For $f : \partial D \to \mathbb{R}$ the $H^{1/2}$ seminorm can be defined by

$$|f|^2_{H^{1/2}(\partial D)} = \int_{\partial D} \int_{\partial D} \frac{|f(\phi) - f(\bar{\phi})|^2}{|\phi - \bar{\phi}|^2} \, \mathrm{d}\phi \, \mathrm{d}\bar{\phi}$$

and the corresponding norm is given by

$$\|f\|^2_{H^{1/2}(\partial D)} = \|f\|^2_{L^2(\partial D)} + |f|^2_{H^{1/2}(\partial D)}.$$

Let C denote a cylinder of radius one and fixed length. For $f = (f_1, f_2) : \partial C \simeq \partial D \dot{\cup} \partial D \to \mathbb{R} \dot{\cup} \mathbb{R}$, with $f_i : \partial D \to \mathbb{R}$, $i = 1, 2$, we define the $H^{1/2}$ seminorm to be

$$|f|_{H^{1/2}(\partial C)} = (|f_1|^2_{H^{1/2}(\partial D)} + |f_2|^2_{H^{1/2}(\partial D)})^{1/2}$$

and the norm

$$\|f\|_{H^{1/2}(\partial C)} = (\|f_1\|^2_{H^{1/2}(\partial D)} + \|f_2\|^2_{H^{1/2}(\partial D)})^{1/2}.$$

As a domain for the energy functional E one first chooses a suitable space $\mathcal{X} = \mathcal{H} \times (0, \infty)$ (see the definitions below), which basically consists of pairs (s, λ), where λ is a positive real number and s is an $H^{1/2}$ map $s : \partial D \dot{\cup} \partial D \to \partial S^1 \dot{\cup} \partial S^1$ which winds once around the boundary of the cylinder. However to obtain a differentiable functional it will be necessary to restrict E to the subspace $\mathcal{T} \times (0, \infty)$ of continuous members of $\mathcal{X}$.

DEFINITION 2.3 The Hilbert space H is defined by

$$H := H^{1/2}(\partial C, \mathbb{R} \dot{\cup} \mathbb{R}) \simeq H^{1/2}(\partial D, \mathbb{R}) \dot{\cup} H^{1/2}(\partial D, \mathbb{R}).$$

The corresponding affine Hilbert space $\mathcal{H}$ is the space of maps $s : \partial D \dot{\cup} \partial D \to \partial S^1 \dot{\cup} \partial S^1$ such that

$$s = \mathrm{id} + \sigma$$

for some $\sigma \in H$. Note that we identify σ with its equivalence class $[\sigma] = \{\bar{\sigma} \mid \sigma = \bar{\sigma} + (2\pi k_1, 2\pi k_2),\ k_1, k_2 \in \mathbb{Z}\}$.

DEFINITION 2.4 The Banach space T is defined by

$$T = H \cap C^0(\partial C, \mathbb{R} \dot{\cup} \mathbb{R})$$

with norm

$$\|\xi\|_T = \|\xi\|_{H^{1/2}(\partial C)} + \|\xi\|_{C^0(\partial C)}$$

(where we take $\|\xi\|_{C^0(\partial C)} = (\|\xi_1\|^2_{C^0(\partial D)} + \|\xi_2\|^2_{C^0(\partial D)})^{1/2}$). The corresponding affine space $\mathcal{T}$ is defined by

$$\mathcal{T} = \mathcal{H} \cap C^0(\partial C, S^1 \dot\cup S^1).$$

DEFINITION 2.5 The Hilbert space X is defined by

$$X := H \times \mathbb{R}$$

with norm

$$\|(\xi, \mu)\|_X = (\|\xi\|^2_{H^{1/2}(\partial C)} + \mu^2)^{1/2}.$$

The corresponding affine Hilbert space is given by $\mathcal{X} = \mathcal{H} \times (0, \infty)$.

DEFINITION 2.6 The Banach space TR is defined by

$$TR := T \times \mathbb{R}$$

with norm

$$\|(\xi, \mu)\|_{TR} = (\|\xi\|^2_T + \mu^2)^{1/2}.$$

The corresponding affine space is given by $\mathcal{T} \times (0, \infty)$.

The space of variations at $s \in \mathcal{H}$, $s \in \mathcal{T}$, $(s, \lambda) \in \mathcal{X}$, and $(s, \lambda) \in \mathcal{T} \times (0, \infty)$ is naturally identified with H, T, X, and TR respectively.

Notation. For $f, g : \partial C \to \mathbb{R} \dot\cup \mathbb{R}$, $s : \partial C \to \partial C$, $f = (f_1, f_2)$, $g = (g_1, g_2)$, and $s = (s_1, s_2)$ we set

$$fg := (f_1 g_1, f_2 g_2), \qquad f + g := (f_1 + g_1, f_2 + g_2), \qquad f \circ s := (f_1 \circ s_1, f_2 \circ s_2),$$

i.e. all operations are always meant componentwise. Furthermore

$$\|f\| = (\|f_1\|^2 + \|f_2\|^2)^{1/2}$$

for various norms. Finally for $s = \mathrm{id} + \sigma : \partial D \dot\cup \partial D \to \partial S^1 \dot\cup \partial S^1$ we write $\|s\| = 1 + \|\sigma\|$ for various norms on σ. (Of course, $\|s\|$ does not define a norm.)

For future references we note the following properties.

LEMMA 2.5 Suppose $f, g : \partial C \to \mathbb{R} \dot\cup \mathbb{R}$ and $s : \partial C \to \partial C$. Then

$$\|fg\|_{H^{1/2}} \leqslant c\|f\|_{C^1}\|g\|_{H^{1/2}}, \tag{5}$$

$$|fg|_{H^{1/2}} \leqslant c(\|f\|_{C^0}|g|_{H^{1/2}} + |f|_{H^{1/2}}\|g\|_{C^0}), \tag{6}$$

$$\|g \circ s\|_{H^{1/2}} \leqslant c\|g\|_{C^1}\|s\|_{H^{1/2}}. \tag{7}$$

Proof. Use the definitions of the norm and results in [13, Lemma II 2.6] and [5, Prop. 3.1–3.2]. □

It is standard that for fixed λ, $\Phi : H^{1/2}(\partial C_\lambda, \mathbb{R}^n) \to H^1(C_\lambda, \mathbb{R}^n)$ is a bounded linear map with bounded inverse. Therefore E is well defined and finite for $(s, \lambda) \in \mathcal{H} \times (0, \infty)$. In fact we have

PROPOSITION 2.6 $E(\cdot,\lambda):\mathcal{H}\to\mathbb{R}$ and

$$E(s,\lambda)\leqslant c(\lambda)\|\gamma\|_{C^1}^2\|s\|_{H^{1/2}}^2.$$

Proof. From trace theory and (7) we get

$$\begin{aligned}E(s,\lambda)&=\frac{1}{2}\int_{C_\lambda}|\nabla\Phi(\gamma\circ s)|^2\leqslant c\|\Phi(\gamma\circ s)\|_{H^1(C_\lambda)}^2\\&\leqslant c(\lambda)\|\gamma\circ s\|_{H^{1/2}}^2\leqslant c(\lambda)\|\gamma\|_{C^1}^2\|s\|_{H^{1/2}}^2.\end{aligned}$$

Note that the constant depends on the domain, i.e. on λ. □

REMARK. In the case of the classical Plateau problem, the analogous function space H of boundary maps is characterized by three additional integral conditions which correspond to the so called “three-point condition”. Such a restraint is necessary to prove compactness results. In the case of the Douglas problem we do not need such an assumption and convergence is ensured by the Douglas condition.

Differentiability properties of E. We now want to investigate the differentiability properties of $E=E(s,\lambda)$.

Conventions regarding derivatives. Derivatives with respect to the function s in the direction ξ are usually denoted $\langle E'(s,\lambda),\xi\rangle$ or $dE(s,\lambda)(\xi)$. Derivatives with respect to the parameter λ are usually written $\frac{\partial}{\partial\lambda}E(s,\lambda)$. Derivatives at (s,λ) in the direction (ξ,μ) are denoted $\langle E'(s,\lambda),(\xi,\mu)\rangle$.

First let us fix λ and compute formally the first and second derivative with respect to variations of the boundary map s. Using the notation

$$u=\Phi(\gamma\circ s),\qquad v=\Phi(\gamma'\circ s\,\xi),\qquad w=\Phi(\gamma''\circ s\,\xi^2),\tag{8}$$

for $\xi\in H$, we get from (4) and formal computation

$$E(s,\lambda)=\frac{1}{2}\int_{C_\lambda}|\nabla u|^2,\tag{9}$$

$$\langle E'(s,\lambda),\xi\rangle=\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0}E(s+t\xi,\lambda)=\int_{C_\lambda}\nabla u\nabla v,\tag{10}$$

$$E''(s,\lambda)(\xi,\xi)=\frac{d^2}{dt^2}\bigg|_{t=0}E(s+t\xi,\lambda)=\int_{C_\lambda}\nabla u\nabla w+\int_{C_\lambda}|\nabla v|^2,\tag{11}$$

with an analogous expression for $E''(s,\lambda)(\xi,\eta)$ obtained by bilinearity in the case of distinct variations.

Following the analysis of the differentiability properties of the energy functional for the classical Plateau problem (see [4] and [3]), it is not difficult to verify the following two propositions.

PROPOSITION 2.7 Let $s=\mathrm{id}+\sigma$. Then $E(\cdot,\lambda):\mathcal{T}\to\mathbb{R}$ is C^{r-1}. Moreover

$$|d^jE(s,\lambda)(\xi_1,\ldots,\xi_j)|\leqslant c(\lambda,\|\gamma\|_{C^{j+1}},\|s\|_{H^{1/2}})\|\xi_1\|_T\cdots\|\xi_j\|_T$$

for $1\leqslant j\leqslant r-1$.

These estimates cannot be improved by replacing $\|\xi\|_T$ by $\|\xi\|_{H^{1/2}}$, unless the regularity of s is increased.

PROPOSITION 2.8 If $s \in C^1$ then $dE(s, \lambda)$ extends to a bounded linear operator on H and

$$|dE(s,\lambda)(\xi)| \leqslant c(\lambda)\|\gamma\|_{C^2}^2 \|s\|_{C^1}^2 \|\xi\|_{H^{1/2}}.$$

If $s \in C^2$ then $d^2E(s, \lambda)$ extends to a bounded bilinear operator on $H \times H$ and

$$|d^2E(s,\lambda)(\xi,\eta)| \leqslant c(\lambda)\|\gamma\|_{C^2}^2 \|s\|_{C^2}^2 \|\xi\|_{H^{1/2}} \|\eta\|_{H^{1/2}}.$$

Now let us fix the boundary map s and compute the first and second derivative of $E = E(s, \lambda)$ with respect to λ. To do so, it is convenient to define the following function:

$$F : (0, \infty) \to H^1(C_1), \qquad F(\lambda) = u^\lambda = \Phi(\gamma \circ s) \circ \sigma_\lambda,$$

where $\Phi(\gamma \circ s)$ is the harmonic extension of $\gamma \circ s$ on the domain C_λ, $s \in \mathcal{H}$, and $\sigma_\lambda : C_1 \to C_\lambda$, $\sigma_\lambda(x, \theta) = (\lambda x, \theta)$, is the map that transforms the unit cylinder to a cylinder of length λ.

LEMMA 2.9 F is smooth on $(0, \infty)$. Each derivative is the unique weak solution of a partial differential equation. In particular $F'(\lambda) \in H_0^1(C_1)$ satisfies

$$\int_{C_1} \left(\frac{1}{\lambda} F'(\lambda)_x w_x + \lambda F'(\lambda)_\theta w_\theta\right) = \int_{C_1} \left(\frac{1}{\lambda^2} u_x^\lambda w_x - u_\theta^\lambda w_\theta\right)$$

for all $w \in H_0^1(C_1)$ and $\|F'(\lambda)\|_{H^1(C_1)} \leqslant c(\lambda)|u^\lambda|_{H^1(C_1)}$.

Proof. Since u^λ is a weak solution of

$$\begin{cases} -\dfrac{1}{\lambda} u_{xx} - \lambda u_{\theta\theta} = 0 & \text{in } C_1, \\ u = \gamma \circ s & \text{on } \partial C_1, \end{cases} \tag{12}$$

we immediately obtain

$$\int_{C_1} \left(\frac{1}{\lambda + h}\left(\frac{u^{\lambda+h} - u^\lambda}{h}\right)_x w_x + (\lambda + h)\left(\frac{u^{\lambda+h} - u^\lambda}{h}\right)_\theta w_\theta\right) = \int_{C_1} \left(\frac{1}{\lambda(\lambda + h)} u_x^\lambda w_x - u_\theta^\lambda w_\theta\right)$$

for all $w \in H_0^1(C_1)$. Hence, for fixed h, $\frac{u^{\lambda+h}-u^\lambda}{h}$ is the unique weak solution of

$$\begin{cases} -\dfrac{1}{\lambda + h} v_{xx} - (\lambda + h) v_{\theta\theta} = -\dfrac{1}{\lambda(\lambda + h)} u_{xx}^\lambda + u_{\theta\theta}^\lambda & \text{in } C_1, \\ v = 0 & \text{on } \partial C_1, \end{cases}$$

and $\|\frac{u^{\lambda+h}-u^\lambda}{h}\|_{H_0^1(C_1)} \leqslant c(\lambda)|u^\lambda|_{H^1(C_1)}$ for all $h < 1$. It follows that there exists a function $F'(\lambda) \in H_0^1(C_1)$ such that by passing to a subsequence of $h \to 0$,

$$\frac{u^{\lambda+h} - u^\lambda}{h} \rightharpoonup F'(\lambda) \quad \text{weakly in } H^1,$$
$$\frac{u^{\lambda+h} - u^\lambda}{h} \to F'(\lambda) \quad \text{strongly in } L^2,$$

and $F'(\lambda)$ is the weak solution of

$$\begin{cases} -\dfrac{1}{\lambda}F'(\lambda)_{xx} - \lambda F'(\lambda)_{\theta\theta} = -\dfrac{1}{\lambda^2}u^\lambda_{xx} + u^\lambda_{\theta\theta} & \text{in } C_1, \\ F'(\lambda) = 0 & \text{on } \partial C_1. \end{cases}$$

Moreover $\|F'(\lambda)\|_{H^1(C_1)} \leqslant c(\lambda)|u^\lambda|_{H^1(C_1)}$. Since the solution of such a PDE is unique, we infer that the above convergences occur for all subsequences of $h \to 0$.

Moreover, by employing the same type of argument as in Lemma 3.2, we can show that $\|\frac{u^{\lambda+h}-u^\lambda}{h} - F'(\lambda)\|_{H^1(C_1)} \to 0$ as $h \to 0$ (i.e. F is Fréchet differentiable at λ), and that F' is continuous on $(0, \infty)$. Existence and continuity of higher derivatives are shown in a similar way. □

Now suppose that $s \in \mathcal{T}$ is fixed. By performing a change of variables, we can write E as an integral over a fixed domain C_1, namely

$$E(s, \lambda) = \frac{1}{2}\int_{C_1}\left(\frac{1}{\lambda}(F(\lambda))^2_x + \lambda(F(\lambda))^2_\theta\right). \tag{13}$$

Hence

$$\begin{aligned}\frac{\partial}{\partial\lambda}E(s, \lambda) = \frac{1}{2}\int_{C_1}&\left(-\frac{1}{\lambda^2}(F(\lambda))^2_x + (F(\lambda))^2_\theta\right)\\ &+ \int_{C_1}\left(\frac{1}{\lambda}(F(\lambda))_x(F'(\lambda))_x + \lambda(F(\lambda))_\theta(F'(\lambda))_\theta\right)\end{aligned}$$

and the second term cancels out due to $F'(\lambda) \in H^1_0(C_1)$ and $F(\lambda) = u^\lambda$ satisfying (12). Therefore we can write

$$\begin{aligned}\frac{\partial}{\partial\lambda}E(s, \lambda) &= \frac{1}{2}\int_{C_1}\left(-\frac{1}{\lambda^2}\left|\frac{\partial}{\partial x}(\Phi(\gamma \circ s) \circ \sigma_\lambda)\right|^2 + \left|\frac{\partial}{\partial\theta}(\Phi(\gamma \circ s) \circ \sigma_\lambda)\right|^2\right)\\ &= \frac{1}{2\lambda}\int_{C_\lambda}\left(-\left|\frac{\partial}{\partial x}(\Phi(\gamma \circ s))\right|^2 + \left|\frac{\partial}{\partial\theta}(\Phi(\gamma \circ s))\right|^2\right).\end{aligned} \tag{14}$$

In a similar way we calculate

$$\begin{aligned}\frac{\partial^2}{\partial\lambda^2}E(s, \lambda) &= \frac{1}{\lambda^3}\int_{C_1}\left|\frac{\partial}{\partial x}(\Phi(\gamma \circ s) \circ \sigma_\lambda)\right|^2 - \int_{C_1}\left(\frac{1}{\lambda}(F'(\lambda))^2_x + \lambda(F'(\lambda))^2_\theta\right)\\ &= \frac{1}{\lambda^2}\int_{C_\lambda}\left|\frac{\partial}{\partial x}(\Phi(\gamma \circ s))\right|^2 - \int_{C_\lambda}|\nabla\Psi|^2,\end{aligned} \tag{15}$$

where $\Psi \in H^1_0(C_\lambda)$ solves

$$\int_{C_\lambda}\nabla\Psi\nabla g = \frac{1}{\lambda}\int_{C_\lambda}\left(\frac{\partial}{\partial x}(\Phi(\gamma \circ s))\,\frac{\partial g}{\partial x} - \frac{\partial}{\partial\theta}(\Phi(\gamma \circ s))\,\frac{\partial g}{\partial\theta}\right) \tag{16}$$

for all $g \in H^1_0(C_\lambda)$. Note that $\|\Psi\|_{H^1(C_\lambda)} \leqslant c(\lambda)|\Phi(\gamma \circ s)|_{H^1(C_\lambda)}$.

For the mixed variations (where one variable is kept fixed at each step) we can show with the same type of argument as above that

$$\left\langle \left(\frac{\partial E}{\partial \lambda}\right)'(s,\lambda),\xi \right\rangle = \frac{\partial}{\partial \lambda}\langle E'(s,\lambda),\xi\rangle$$
$$= \frac{1}{\lambda}\int_{C_\lambda}\left(-\frac{\partial}{\partial x}(\Phi(\gamma\circ s))\,\frac{\partial}{\partial x}(\Phi(\gamma'\circ s\,\xi)) + \frac{\partial}{\partial \theta}(\Phi(\gamma\circ s))\,\frac{\partial}{\partial \theta}(\Phi(\gamma'\circ s\,\xi))\right). \quad (17)$$

Finally, let us compute formally the first and second variation for E. Using again the notation

$$u = \Phi(\gamma\circ s), \qquad v = \Phi(\gamma'\circ s\,\xi), \qquad w = \Phi(\gamma''\circ s\,\xi^2),$$

and letting $\Psi \in H_0^1(C_\lambda)$ be a solution of (16), and $(\xi,\mu)\in X$, from (10) and (14) we get

$$\langle E'(s,\lambda),(\xi,\mu)\rangle = \int_{C_\lambda}\nabla u\nabla v + \frac{\mu}{2\lambda}\int_{C_\lambda}\left(\left|\frac{\partial u}{\partial \theta}\right|^2 - \left|\frac{\partial u}{\partial x}\right|^2\right). \quad (18)$$

Furthermore using (11), (15), and (17) we can write

$$E''(s,\lambda)(\xi,\mu)^2 = \int_{C_\lambda}(|\nabla v|^2 + \nabla u\nabla w)$$
$$+ \frac{2\mu}{\lambda}\int_{C_\lambda}\left(-\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial \theta}\frac{\partial v}{\partial \theta}\right) + \left(\frac{\mu}{\lambda}\right)^2\int_{C_\lambda}\left|\frac{\partial u}{\partial x}\right|^2 - \mu^2\int_{C_\lambda}|\nabla\Psi|^2, \quad (19)$$

with an analogous expression for $E''(s,\lambda)(\xi,\mu)(\eta,\sigma)$ obtained by bilinearity in the case of distinct variations.

PROPOSITION 2.10 The functional $E(s,\lambda):\mathcal{T}\times(0,\infty)\to\mathbb{R}$ is C^3. Moreover

$$|\langle E'(s,\lambda),(\xi,\mu)\rangle| \leqslant c(\lambda,\|\gamma\|_{C^2},\|s\|_{H^{1/2}})\|(\xi,\mu)\|_{TR}$$

and

$$|E''(s,\lambda)(\xi,\mu)(\eta,\sigma)| \leqslant c(\lambda,\|\gamma\|_{C^3},\|s\|_{H^{1/2}})\|(\xi,\mu)\|_{TR}\|(\eta,\sigma)\|_{TR}.$$

Proof. Through the previous analysis we were able to describe the variations of $E = E(s,\lambda)$ when fixing a variable. Roughly speaking we found the "gradient" and "Hessian" for E and used these to formally derive an expression for the first and second variation of E. By using Lemma 3.2 below it can be checked directly in a routine manner that E is three times Fréchet differentiable, its derivatives are continuous, and (18) and (19) are the correct expressions for the first and second variation respectively.

The last part of the proposition follows immediately by applying trace theory, (7), and (6) to the expressions (18) and (19). □

PROPOSITION 2.11 If $s\in C^1$ and $\lambda\in(0,\infty)$ then $E'(s,\lambda)$ extends to a bounded linear operator on X and

$$\langle E'(s,\lambda),(\xi,\mu)\rangle \leqslant c(\lambda)\|\gamma\|_{C^2}^2\|s\|_{C^1}^2\|(\xi,\mu)\|_X.$$

If $s\in C^2$ then $E''(s,\lambda)$ extends to a bounded bilinear operator on $X\times X$ and

$$|E''(s,\lambda)(\xi,\mu)(\eta,\sigma)| \leqslant c(\lambda)\|\gamma\|_{C^2}^2\|s\|_{C^2}^2\|(\xi,\mu)\|_X\|(\eta,\sigma)\|_X.$$

Proof. The first inequality follows by (18), trace theory, (5), and (7). The second inequality follows by (19), trace theory, (5), (7), and Proposition 2.8. □

Definition of minimal surface in terms of E. Now we are ready to give the formulation of minimal surface which we will use throughout this work.

DEFINITION 2.7 The harmonic function

$$u = \Phi(\gamma \circ s)$$

defined on C_λ is a *minimal surface spanning Γ* if s is monotone and the pair $(s, \lambda) \in \mathcal{T} \times (0, \infty)$ is stationary for E, i.e. if the following two statements are true:

(E1) s is monotone and stationary for $E(\cdot, \lambda)$ in the sense that

$$\langle E'(s, \lambda), \xi \rangle = 0 \quad \forall \xi \in T, \tag{20}$$

(E2) λ is such that (what we could call) "equipartition of energy" holds, namely

$$\int_{C_\lambda} \left| \frac{\partial u}{\partial x} \right|^2 \mathrm{d}x\, \mathrm{d}\theta = \int_{C_\lambda} \left| \frac{\partial u}{\partial \theta} \right|^2 \mathrm{d}x\, \mathrm{d}\theta. \tag{21}$$

REMARK. Note that by (18) the pair $(s, \lambda) \in \mathcal{T} \times (0, \infty)$ is stationary for E if and only if $\langle E'(s, \lambda), (\xi, \mu) \rangle = 0$ for all $(\xi, \mu) \in T \times \mathbb{R}$.

PROPOSITION 2.12 Definitions 2.1 and 2.7 are equivalent. In other words, (s, λ) is stationary for E if and only if $u = \Phi(\gamma \circ s)$ defined on C_λ is stationary for the Dirichlet functional (or equivalently u is harmonic and conformal).

Proof. First note that (E2) and (D3′) are equivalent, then apply similar arguments to those used by Struwe in the proof of [13, II Proposition 2.9]. The most difficult step consists in proving that condition (E1) implies that $u \in H^2(C_\lambda)$. This regularity result is achieved by using the same arguments applied by Struwe in [13, II §5]. □

PROPOSITION 2.13 If $\gamma \in C^{k,\alpha}$ where $k \geqslant 2$, $0 < \alpha < 1$ and (s, λ) is stationary for E, then

$$\|s\|_{C^{k,\alpha}} \leqslant c = c(\|\gamma\|_{C^{k,\alpha}}, \| |\gamma'|^{-1} \|_{L^\infty}).$$

Proof. This follows directly from the regularity result given in Theorem 2.4 and Proposition 2.12. □

Nondegeneracy for the energy functional E. We will need to consider the second order behaviour of E near a stationary point $(s, \lambda) \in \mathcal{T} \times (0, \infty)$. For $s \in C^2$, $\lambda \in (0, \infty)$ and $\gamma \in C^2$ (in particular, by regularity theory, for (s, λ) stationary for E and $\gamma \in C^3$) let us consider the bilinear form $E''(s, \lambda)$ as given in (19). By Proposition 2.11 we know that $E''(s, \lambda)$ extends to a bounded bilinear operator on $X \times X$. Hence, by the Riesz representation theorem, we introduce the bounded self-adjoint map $\nabla^2 E(s, \lambda) : X \to X$ defined by

$$\langle \nabla^2 E(s, \lambda)(\xi, \mu), (\eta, \sigma) \rangle_X = E''(s, \lambda)(\xi, \mu)(\eta, \sigma)$$

for all $(\eta, \sigma), (\xi, \mu) \in X$, where $\langle \cdot, \cdot \rangle_X$ is the inner product defined on the Hilbert space X. Write

$$X = X^- \oplus X^0 \oplus X^+ \tag{22}$$

for the orthogonal decomposition generated by the eigenfunctions of $\nabla^2 E(s,\lambda)$ having negative, zero, and positive eigenvalues, respectively.

For $(\xi,\lambda) \in X$, we will write

$$(\xi,\mu) = (\xi^-,\mu^-) + (\xi^0,\mu^0) + (\xi^+,\mu^+), \tag{23}$$

where $(\xi^-,\mu^-) \in X^-$, $(\xi^0,\mu^0) \in X^0$, and $(\xi^+,\mu^+) \in X^+$.

DEFINITION 2.8 If (s,λ) is a stationary point for E, we say that

$$(s,\lambda) \text{ is } \textit{nondegenerate} \quad \text{if} \quad X^0 = \{0\}$$

The corresponding minimal surface $u = \Phi(\gamma \circ s)$ is also said to be *nondegenerate*.

If (s,λ) is a nondegenerate stationary point for E, it follows that the eigenvalues of $\nabla^2 E(s,\lambda)$ are bounded away from zero and $\nabla^2 E(s,\lambda)$ is invertible with bounded inverse (see [3, Proposition 4.9]). In particular there exists a $\kappa > 0$ such that

$$\begin{aligned} E''(s,\lambda)(\xi,\mu)(\xi^+ - \xi^-, \mu^+ - \mu^-) \\ = E''(s,\lambda)(\xi^+,\mu^+)^2 - E''(s,\lambda)(\xi^-,\mu^-)^2 \geqslant \kappa(\|\xi\|^2_{H^{1/2}} + \mu^2). \end{aligned} \tag{24}$$

We call κ the *nondegeneracy constant* for (s,λ).

3. The "smooth sequence"

In this section we are concerned with the problem of giving a constructive method for finding stationary points of the energy functional E.

As mentioned in the Introduction this is not needed in order to establish the main error estimates (1) and (2) discussed in [10]. However it motivates the construction of the so called "discrete sequence" (see Section 5) on which the numerical algorithm given in Section 6 is based.

Let us build the following sequence of points $(s_n,\lambda_n) \in \mathcal{T} \times (0,\infty)$. Choose $\lambda_0 \in (0,\infty)$, then repeat the following two steps.

Step 1. Given λ_n, find a monotone map $s_n \in \mathcal{T}$ such that s_n is stationary for $E(\cdot,\lambda_n)$. In other words find s_n such that

$$\langle E'(s_n,\lambda_n),\xi\rangle = 0, \qquad \forall \xi \in T. \tag{25}$$

Using (10), we see that (25) can be written as

$$\int_{C_{\lambda_n}} \nabla\Phi(\gamma \circ s_n)\nabla\Phi(\gamma' \circ s_n\,\xi) = 0 \tag{26}$$

for all $\xi \in T$, or equivalently

$$\begin{aligned} \frac{1}{\lambda_n}\int_{C_1} \frac{\partial}{\partial x}(\Phi(\gamma \circ s_n) \circ \sigma_{\lambda_n})\,\frac{\partial}{\partial x}(\Phi(\gamma' \circ s_n\,\xi) \circ \sigma_{\lambda_n})\,\mathrm{d}x\,\mathrm{d}\theta \\ + \lambda_n \int_{C_1} \frac{\partial}{\partial\theta}(\Phi(\gamma \circ s_n) \circ \sigma_{\lambda_n})\,\frac{\partial}{\partial\theta}\big(\Phi(\gamma' \circ s_n\,\xi) \circ \sigma_{\lambda_n})\,\mathrm{d}x\,\mathrm{d}\theta = 0 \end{aligned} \tag{27}$$

for all $\xi \in T$, where $\sigma_{\lambda_n} : C_1 \to C_{\lambda_n}$ is the diffeomorphism of the form $\sigma_{\lambda_n}(x, \theta) := (\lambda_n x, \theta)$. For later use, let us denote by

$$h_n := \Phi(\gamma \circ s_n)$$

the harmonic extension of $\gamma \circ s_n$ on C_{λ_n}.

Step 2. Given s_n, λ_n, and h_n, find $\lambda_{n+1} \in (0, \infty)$ such that h_n, reparametrised to the domain $C_{\lambda_{n+1}}$, satisfies "equipartition of energy" (see Definition 2.7 again). Precisely this means that if we denote by $k_{n,n+1}$ the function $k_{n,n+1} : C_{\lambda_{n+1}} \to C_{\lambda_n}$ which maps $(\tilde{x}, \tilde{\theta})$ to $\left(\frac{\lambda_n}{\lambda_{n+1}}\tilde{x}, \tilde{\theta}\right) = (x, \theta)$, then λ_{n+1} must be such that

$$\int_{C_{\lambda_{n+1}}} \left|\frac{\partial}{\partial \tilde{x}}(h_n \circ k_{n,n+1})\right|^2 \mathrm{d}\tilde{x}\, \mathrm{d}\tilde{\theta} = \int_{C_{\lambda_{n+1}}} \left|\frac{\partial}{\partial \tilde{\theta}}(h_n \circ k_{n,n+1})\right|^2 \mathrm{d}\tilde{x}\, \mathrm{d}\tilde{\theta}. \tag{28}$$

Note that a change of variables in (28) gives

$$\frac{\lambda_n}{\lambda_{n+1}} \int_{C_{\lambda_n}} \left|\frac{\partial h_n}{\partial x}\right|^2 \mathrm{d}x\, \mathrm{d}\theta = \frac{\lambda_{n+1}}{\lambda_n} \int_{C_{\lambda_n}} \left|\frac{\partial h_n}{\partial \theta}\right|^2 \mathrm{d}x\, \mathrm{d}\theta, \tag{29}$$

which is easily solved since λ_n and h_n are known.

Let us first point out the following important fact.

LEMMA 3.1 Step 2 does not increase the Dirichlet energy. More precisely, we have

$$\mathcal{D}(h_n \circ k_{n,n+1}, \lambda_{n+1}) \leqslant \mathcal{D}(h_n, \lambda_n),$$

with equality holding if and only if $\lambda_n = \lambda_{n+1}$. (Here $\mathcal{D}(h_n, \lambda_n)$ denotes the Dirichlet energy of the map h_n defined on the domain C_{λ_n}.)

Proof. We can compute λ_{n+1} directly from (29). (We are assuming here that Step 2 can be realized, i.e. none of the integrals in expression (29) vanishes.) Note also that the solution is unique.

To prove the assertion, let us recall that the Dirichlet energy $\mathcal{D}(h, \lambda)$ for $h : C_\lambda \to \mathbb{R}^n$ can be written as

$$\mathcal{D}(h, \lambda) = |h(C_\lambda)| + E_C(h, \lambda),$$

where $|h(C_\lambda)|$ is the area of the image $h(C_\lambda)$ and $E_C(h, \lambda)$ is the conformal energy as defined in [6], namely

$$E_C(h, \lambda) := \frac{1}{2} \int_{C_\lambda} \left|J(h)\frac{\partial h}{\partial x} - \frac{\partial h}{\partial \theta}\right|^2 \mathrm{d}x\, \mathrm{d}\theta.$$

Here $J(h)$ is rotation through $\pi/2$ in the oriented tangent plane to the image of h.

Since $|h_n(C_{\lambda_n})| = |h_n \circ k_{n,n+1}(C_{\lambda_{n+1}})|$, all we have to prove is that $E_C(h_n \circ k_{n,n+1}, \lambda_{n+1}) \leqslant E_C(h_n, \lambda_n)$. This follows by a direct computation. □

REMARKS. 1) During the first step, we fix λ_n and we find a boundary map s_n for which condition (E1) of Definition 2.7 holds. This is very much like solving the classical Plateau problem. Bear in mind that since we find a stationary map for $E(\cdot, \lambda_n)$, the functional E need not decrease. Furthermore the surface that we get, namely $\Phi(\gamma \circ s_n)$ (where the harmonic extension is taken over C_{λ_n}), generally fails to be conformal. Note that if we look back at Lemma 2.3 and the comments there, and we consider again the proof of Proposition 2.12, we come to realize that $\Phi(\gamma \circ s_n)$ *just* fails to be conformal.

In the second step, we fix the surface just computed, i.e. $\Phi(\gamma \circ s_n)$, and parametrise it from a different cylinder $C_{\lambda_{n+1}}$ in such a way that the "equipartition of energy" (see (E2) in Definition 2.7 and (28)) holds. In other words, we are trying to make up for the lack of conformality. Note that now we have the problem that s_n is not necessarily stationary for $E(\cdot, \lambda_{n+1})$, so we need to "keep going" with our construction.

Our wish is to derive a sequence of points (s_n, λ_n) that will "approximate" conditions (E1) and (E2) of Definition 2.7 more and more accurately as n increases.

2) It is clear that if the constructed sequence stops for some $n \in \mathbb{N}$, then what we obtain is exactly a stationary point for E, since (E1) and (E2) are satisfied at the same time.

3) Lemma 3.1 is interesting because if we are able to decrease the Dirichlet energy also during Step 1 for each n (say, we find a Dirichlet energy minimizer for the fixed domain C_{λ_n}), then we end up with a sequence (s_n, λ_n) for which

$$E(s_{n+1}, \lambda_{n+1}) \leqslant E(s_n, \lambda_n)$$

is true for all $n \in \mathbb{N}$, i.e. the sequence is energy decreasing.

4) We pointed out in 1) that Step 1 is basically equivalent to solving the classical Plateau problem. On the other hand, Step 2 is performed with a very easy computation (see (29)). It becomes clear then that since the problem of implementing a program that solves the Plateau problem has already been solved by G. Dziuk and J. Hutchinson (see [4]), the investigation of the convergence of the "smooth sequence" is appealing also from a computational point of view.

Motivated by the remarks just made, we now tackle the problem of finding under which conditions we can ensure the convergence of the sequence to a stationary point for the energy functional E.

Let us first give a few useful lemmas. The first establishes that if we take the harmonic extension of the same boundary map on two different cylinders whose difference in length is small, then the difference in the H^1 norm of the rescaled maps is also small.

LEMMA 3.2 For $f \in H^{1/2}(\partial C)$, $\sigma_\mu : C_1 \to C_\mu$ a diffeomorphism of the form $\sigma_\mu(x, \theta) = (\mu x, \theta)$ for $\mu > 0$, and $\lambda_n \to \lambda \in (0, \infty)$, we have

$$\|\Phi(f) \circ \sigma_{\lambda_n} - \Phi(f) \circ \sigma_\lambda\|_{H^1(C_1)} \to 0$$

as $n \to \infty$. More precisely, we have

$$\|\Phi(f) \circ \sigma_{\lambda_n} - \Phi(f) \circ \sigma_\lambda\|_{H^1(C_1)} \leqslant c(\lambda)|\lambda - \lambda_n|\,|\Phi(f) \circ \sigma_\lambda|_{H^1(C_1)}.$$

REMARK. Note that $\Phi(f) \circ \sigma_{\lambda_n}$ and $\Phi(f) \circ \sigma_\lambda$ of Lemma 3.2 are two different functions: the first $\Phi(f)$ is the harmonic extension of f on C_{λ_n} whereas the second $\Phi(f)$ is the harmonic extension of the same boundary values f on C_λ.

Proof. Set $u^n := \Phi(f) \circ \sigma_{\lambda_n}$ and $u := \Phi(f) \circ \sigma_\lambda$. Then $u^n \in H^1(C_1)$ is the unique solution of

$$\begin{cases} L^n v = 0 & \text{in } C_1, \\ v = f & \text{on } \partial C_1, \end{cases}$$

where $L^n v = -\frac{1}{\lambda_n} v_{xx} - \lambda_n v_{\theta\theta}$. The map $u \in H^1(C_1)$ is the unique solution of

$$\begin{cases} Lv = 0 & \text{in } C_1, \\ v = f & \text{on } \partial C_1, \end{cases} \tag{30}$$

where $Lv = -\frac{1}{\lambda} v_{xx} - \lambda v_{\theta\theta}$. By subtraction we get

$$\begin{cases} L^n(u^n - u) = \left(\frac{1}{\lambda_n} - \frac{1}{\lambda}\right) u_{xx} + (\lambda_n - \lambda) u_{\theta\theta} \equiv f_n & \text{in } C_1, \\ u^n - u = 0 & \text{on } \partial C_1. \end{cases}$$

Let us write $v^n := u^n - u \in H_0^1(C_1)$. By definition, v^n is such that

$$\int_{C_1} \left(\frac{1}{\lambda_n} v_x^n w_x + \lambda_n v_\theta^n w_\theta \right) = \int_{C_1} \left(\left(\frac{1}{\lambda} - \frac{1}{\lambda_n} \right) u_x w_x + (\lambda - \lambda_n) u_\theta w_\theta \right)$$

for all $w \in H_0^1(C_1)$. Choose $w = v^n$. Then for n sufficiently large,

$$c(\lambda) \|\nabla v^n\|^2_{L^2(C_1)} \leqslant C(\lambda) |\lambda - \lambda_n| \, \|\nabla u\|_{L^2(C_1)} \, \|\nabla v^n\|_{L^2(C_1)},$$

which implies

$$\|u^n - u\|_{H^1(C_1)} \leqslant c(\lambda) |\lambda - \lambda_n| \, |u|_{H^1(C_1)} \to 0$$

as $n \to \infty$. Note that by extending canonically f to a map $\tilde{f} \in H^1(C_1)$ so that $\tilde{f}|_{\partial C_1} = f$ and $\|\tilde{f}\|_{H^1(C_1)} \leqslant c\|f\|_{H^{1/2}(\partial C_1)}$, by using $\tilde{f}$ to reduce (30) to a system with homogeneous boundary conditions and by applying arguments similar to those above, it is not difficult to show that $\|u\|_{H^1(C_1)} \leqslant c\|f\|_{H^{1/2}(\partial C_1)}$. □

Next we derive a Poincaré type inequality.

PROPOSITION 3.3 Assume that U is an open bounded subset of $\mathbb{R}^n$ with $\partial U \in C^1$. Then for every $g \in H^1(U)$,

$$\int_U |g|^2 \leqslant C \int_U |\nabla g|^2 + C \left(\int_{\partial U} g \right)^2 \tag{31}$$

with C independent of g. In particular

$$\|g\|_{H^1(U)} \leqslant C(|g|_{H^1(U)} + \|g\|_{L^2(\partial U)}).$$

Proof. Suppose that (31) is not true. Then there exist $g_n \in H^1(U)$ such that

$$1 = \int_U |g_n|^2 \geqslant n \int_U |\nabla g_n|^2 + n \left(\int_{\partial U} g_n \right)^2 \quad \forall n \in \mathbb{N}. \tag{32}$$

Since $(g_n)_{n \in \mathbb{N}}$ is a bounded sequence in $H^1(U)$, there exists $g \in H^1(U)$ to which g_n converge weakly in $H^1(U)$ and strongly in $L^2(U)$. In particular $\|g\|_{L^2(U)} = 1$. On the other hand, (32) implies that $g_n \to 0$ in the $L^1(\partial U)$ norm and $\nabla g_n \to 0$ strongly in $L^2(U)$, hence $g \equiv 0$. This yields a contradiction. □

Keeping in mind the notation used so far (see definition of λ_n, s_n, h_n, σ_{λ_n} given in Steps 1 and 2 at the beginning of Section 3), let us start with some basic observations.

LEMMA 3.4 Suppose that $\|h_n\|_{H^1(C_{\lambda_n})} \leqslant C$ for all $n \in \mathbb{N}$ and $\lambda_n \to \lambda \in (0, \infty)$. Then

$$\|h_n \circ \sigma_{\lambda_n}\|_{H^1(C_1)} \leqslant C(\lambda), \qquad |s_n|_{H^{1/2}(\partial C)} \leqslant C(\lambda, \gamma).$$

Proof. The first estimate follows from $\|h_n\|_{H^1(C_{\lambda_n})} \leqslant C$ by a change of variable and the fact that the sequence of λ_n is bounded. The second follows from the first one and the regularity of γ. □

If in addition we know that $s_n \rightrightarrows s$ uniformly, then in particular we have $\|s_n\|_{L^2(\partial C)} \leqslant C(\|s\|_{C^0})$ and $\|s_n\|_{H^{1/2}(\partial C)} \leqslant C(\lambda, \gamma, \|s\|_{C^0})$. Therefore

$$s_n \rightharpoonup s \quad \text{weakly in } H^{1/2}(\partial C).$$

One way to guarantee the uniform convergence of the maps s_n is to require that, together with the assumption $E(s_n, \lambda_n) \leqslant C$, the maps h_n satisfy the *condition of cohesion*, which means that

> *there exists a real number $\alpha > 0$ independent of n such that each closed curve lying on $h_n(C_{\lambda_n})$ whose diameter does not exceed α can be continuously shrunk to a point (inside $h_n(C_{\lambda_n})$).*

Furthermore, under these same conditions, it can proved that (a subsequence of) λ_n converges to some $\lambda \in (0, \infty)$. For more details see [9, §§559–560].

Let us define

$$h := \Phi(\gamma \circ s)$$

to be the harmonic extension of $\gamma \circ s$ on C_λ and let $\sigma_\lambda : C_1 \to C_\lambda$ be the usual cylinder transformation (replace λ_n with λ in the definition of σ_{λ_n} on page 233).

LEMMA 3.5 Suppose that $s_n \rightrightarrows s$ uniformly, $\lambda_n \to \lambda \in (0, \infty)$ and $\|h_n\|_{H^1(C_{\lambda_n})} \leqslant C$ for all $n \in \mathbb{N}$. Then

$$h_n \circ \sigma_{\lambda_n} \rightharpoonup h \circ \sigma_\lambda \quad \text{weakly in } H^1(C_1).$$

Proof. From $\|h_n \circ \sigma_{\lambda_n}\|_{H^1(C_1)} \leqslant C(\lambda)$ it follows that there exists a function $g \in H^1(C_1)$ to which, by passing to a subsequence, $h_n \circ \sigma_{\lambda_n}$ converges weakly in the H^1 norm. For $f \in H^1_0(C_1)$ we have

$$\frac{1}{\lambda_n} \int_{C_1} \frac{\partial}{\partial x}(h_n \circ \sigma_{\lambda_n}) \frac{\partial f}{\partial x} + \lambda_n \int_{C_1} \frac{\partial}{\partial \theta}(h_n \circ \sigma_{\lambda_n}) \frac{\partial f}{\partial \theta} = 0$$

for all n. Letting $n \to \infty$ and using the weak convergence we obtain

$$\frac{1}{\lambda} \int_{C_1} \frac{\partial g}{\partial x} \frac{\partial f}{\partial x} + \lambda \int_{C_1} \frac{\partial g}{\partial \theta} \frac{\partial f}{\partial \theta} = 0,$$

which implies that $g \circ \sigma_\lambda^{-1}$ is harmonic on C_λ and therefore fully determined by its value on the boundary. Set $g := \Phi(\tilde{g}) \circ \sigma_\lambda$, where $\tilde{g} = g \circ \sigma_\lambda^{-1}|_{\partial C_\lambda}$.

For $f \in H^2(C_\lambda)$ harmonic on C_λ we have

$$\frac{1}{\lambda_n}\int_{C_1}\frac{\partial}{\partial x}(h_n\circ\sigma_{\lambda_n})\,\frac{\partial}{\partial x}(f\circ\sigma_\lambda)+\lambda_n\int_{C_1}\frac{\partial}{\partial\theta}(h_n\circ\sigma_{\lambda_n})\,\frac{\partial}{\partial\theta}(f\circ\sigma_\lambda)$$
$$=\frac{1}{\lambda_n}\int_{\partial C_1}\frac{\partial}{\partial x}(f\circ\sigma_\lambda)\,\nu_1\,h_n\circ\sigma_{\lambda_n}+\lambda_n\int_{\partial C_1}\frac{\partial}{\partial\theta}(f\circ\sigma_\lambda)\,\nu_2\,h_n\circ\sigma_{\lambda_n}$$
$$+\int_{C_1}\left(-\frac{1}{\lambda_n}\frac{\partial^2}{\partial x^2}(f\circ\sigma_\lambda)-\lambda_n\frac{\partial^2}{\partial\theta^2}(f\circ\sigma_\lambda)\right)h_n\circ\sigma_{\lambda_n},$$

where $\nu=(\nu_1,\nu_2)$ is the outward unit normal vector field defined on ∂C_1. Again by letting $n\to\infty$, using the weak convergence of $h_n\circ\sigma_{\lambda_n}$ and uniform convergence of s_n we obtain

$$\frac{1}{\lambda}\int_{C_1}\frac{\partial g}{\partial x}\frac{\partial}{\partial x}(f\circ\sigma_\lambda)+\lambda\int_{C_1}\frac{\partial g}{\partial\theta}\frac{\partial}{\partial\theta}(f\circ\sigma_\lambda)$$
$$=\frac{1}{\lambda}\int_{\partial C_1}\frac{\partial}{\partial x}(f\circ\sigma_\lambda)\,\nu_1\,\gamma\circ s(\lambda x,\theta)+\lambda\int_{\partial C_1}\frac{\partial}{\partial\theta}(f\circ\sigma_\lambda)\,\nu_2\,\gamma\circ s(\lambda x,\theta)$$
$$=\frac{1}{\lambda}\int_{C_1}\frac{\partial}{\partial x}(h\circ\sigma_\lambda)\,\frac{\partial}{\partial x}(f\circ\sigma_\lambda)+\lambda\int_{C_1}\frac{\partial}{\partial\theta}(h\circ\sigma_\lambda)\,\frac{\partial}{\partial\theta}(f\circ\sigma_\lambda),$$

where the last equality is obtained by integrating by parts again and using the harmonicity of f. By a change of variables we have

$$\int_{C_\lambda}\nabla f\,\nabla(\Phi(\tilde g)-h)=\int_{\partial C_\lambda}\frac{\partial f}{\partial\nu}(\tilde g-\gamma\circ s)=0$$

for all $f\in H^2(C_\lambda)$ harmonic on C_λ. Since we are able to solve the Neumann problem

$$\begin{cases}\Delta f=0 & \text{in } C_\lambda,\\ \partial f/\partial\nu=h & \text{on } \partial C_\lambda,\end{cases}$$

for all $h\in C^\infty(\partial C_\lambda)$ such that $\int_{\partial C_\lambda} h=0$, it follows easily that $\tilde g=\gamma\circ s$.

Finally, it is not difficult to see that the whole sequence $h_n\circ\sigma_{\lambda_n}$ converges weakly to $h\circ\sigma_\lambda$ in the H^1 norm. □

Now we can use the tools developed so far to prove the following statement.

LEMMA 3.6 Suppose that $s_n\rightrightarrows s$ uniformly, $\lambda_n\to\lambda\in(0,\infty)$ and $\|h_n\|_{H^1(C_{\lambda_n})}\leqslant C$ for all $n\in\mathbb{N}$. Then

$$s_n\to s\quad\text{strongly in } H^{1/2}(\partial C).$$

Proof. First note that for each boundary component we can write

$$\gamma(s_n)-\gamma(s)=\gamma'(s_n)(s_n-s)-\int_s^{s_n}\int_u^{s_n}\gamma''(\tilde u)\,\mathrm{d}\tilde u\,\mathrm{d}u$$
$$=\gamma'(s_n)(s_n-s)+I_n,$$

and as shown in [13, II 2.11],

$$|I_n|_{H^{1/2}}\leqslant C\|s-s_n\|_{C^0}(|s_n|_{H^{1/2}}+|s|_{H^{1/2}}).$$

Recall that any weakly convergent sequence is bounded, hence this inequality is meaningful. Since by its definition I_n converges also to zero in the C^0 topology, it follows that in particular

$$\|I_n\|_{H^{1/2}} \to 0 \quad \text{as } n \to \infty.$$

Now consider

$$\begin{aligned}
&\int_{C_{\lambda_n}} |\nabla(\Phi(\gamma \circ s_n) - \Phi(\gamma \circ s))|^2 \\
&\qquad\qquad \text{(note that the harmonic extension is taken on } C_{\lambda_n} \text{ !)} \\
&\qquad = \int_{\partial C_{\lambda_n}} \frac{\partial}{\partial \nu}(\Phi(\gamma \circ s_n))(\gamma(s_n) - \gamma(s)) - \int_{C_{\lambda_n}} \nabla\Phi(\gamma \circ s)\nabla\Phi(\gamma \circ s_n - \gamma \circ s) \\
&\qquad = J_1 + J_2.
\end{aligned}$$

But

$$J_1 = \int_{\partial C_{\lambda_n}} \frac{\partial}{\partial \nu}(\Phi(\gamma \circ s_n))\gamma'(s_n)(s_n - s) + \int_{C_{\lambda_n}} \nabla\Phi(\gamma \circ s_n)\nabla\Phi(I_n).$$

The first term is zero because of the stationarity of s_n (recall Step 1 in the definition of the "smooth sequence"). Hence

$$|J_1| \leqslant \|h_n\|_{H^1(C_{\lambda_n})}\|\Phi(I_n)\|_{H^1(C_{\lambda_n})} \leqslant C(\lambda)\|I_n\|_{H^{1/2}} \to 0$$

as $n \to \infty$. With the usual change of variable we can write

$$\begin{aligned}
-J_2 = &\frac{1}{\lambda_n}\int_{C_1} \frac{\partial}{\partial x}(\Phi(\gamma \circ s) \circ \sigma_{\lambda_n}) \frac{\partial}{\partial x}(\Phi(\gamma(s_n) - \gamma(s)) \circ \sigma_{\lambda_n}) \\
&+ \lambda_n \int_{C_1} \frac{\partial}{\partial \theta}(\Phi(\gamma \circ s) \circ \sigma_{\lambda_n}) \frac{\partial}{\partial \theta}(\Phi(\gamma(s_n) - \gamma(s)) \circ \sigma_{\lambda_n}).
\end{aligned}$$

Let us look at the first term of the above expression:

$$\begin{aligned}
&\int_{C_1} \frac{\partial}{\partial x}(\Phi(\gamma \circ s) \circ \sigma_{\lambda_n}) \frac{\partial}{\partial x}(\Phi(\gamma(s_n) - \gamma(s)) \circ \sigma_{\lambda_n}) \\
&\quad = \int_{C_1} \frac{\partial}{\partial x}(\Phi(\gamma \circ s) \circ \sigma_{\lambda_n} - \Phi(\gamma \circ s) \circ \sigma_\lambda) \frac{\partial}{\partial x}(\Phi(\gamma \circ s_n) \circ \sigma_{\lambda_n} - \Phi(\gamma \circ s) \circ \sigma_\lambda) \\
&\quad\quad + \int_{C_1} \frac{\partial}{\partial x}(\Phi(\gamma \circ s) \circ \sigma_\lambda) \frac{\partial}{\partial x}(\Phi(\gamma \circ s_n) \circ \sigma_{\lambda_n} - \Phi(\gamma \circ s) \circ \sigma_\lambda) \\
&\quad\quad + \int_{C_1} \frac{\partial}{\partial x}(\Phi(\gamma \circ s) \circ \sigma_{\lambda_n}) \frac{\partial}{\partial x}(\Phi(\gamma \circ s) \circ \sigma_\lambda - \Phi(\gamma \circ s) \circ \sigma_{\lambda_n}).
\end{aligned}$$

Using Lemmas 3.4, 3.5 and 3.2 we see that each term goes to zero when n approaches infinity. Using the same arguments also for the other terms in J_2, we deduce that $J_2 \to 0$ as $n \to \infty$. Hence we have shown that

$$|\Phi(\gamma \circ s_n) - \Phi(\gamma \circ s)|_{H^1(C_{\lambda_n})} \to 0.$$

By Proposition 3.3 and $s_n \rightrightarrows s$ it follows that $\|\Phi(\gamma \circ s_n) - \Phi(\gamma \circ s)\|_{H^1(C_{\lambda_n})} \to 0$, which implies

$$\|\gamma \circ s_n - \gamma \circ s\|_{H^{1/2}(\partial C)} \to 0.$$

If we write $\gamma \circ s - \gamma \circ s_n = \gamma'(s)(s - s_n) + \tilde{I}_n$, then $\tilde{I}_n$ behaves like I_n, i.e. $\|\tilde{I}_n\|_{H^{1/2}} \to 0$ as $n \to \infty$ and

$$\|\gamma'(s)(s - s_n)\|_{H^{1/2}(\partial C)} \to 0.$$

For each boundary component we have

$$\begin{aligned}
&|\gamma'(s)(s - s_n)|^2_{H^{1/2}(\partial D)} \\
&\quad = \int_{\partial D}\int_{\partial D} \frac{|\gamma'(s)(s - s_n)(\phi) - \gamma'(s)(s - s_n)(\bar{\phi})|^2}{|\phi - \bar{\phi}|^2}\,\mathrm{d}\phi\,\mathrm{d}\bar{\phi} \\
&\quad = \int_{\partial D}\int_{\partial D} \frac{|[\gamma'(s)(\phi) - \gamma'(s)(\bar{\phi})](s - s_n)(\phi) - \gamma'(s)(\bar{\phi})[(s - s_n)(\bar{\phi}) - (s - s_n)(\phi)]|^2}{|\phi - \bar{\phi}|^2} \\
&\quad = \int_{\partial D}\int_{\partial D} \frac{[\gamma'(s)(\phi) - \gamma'(s)(\bar{\phi})]^2((s - s_n)(\phi))^2}{|\phi - \bar{\phi}|^2}\,\mathrm{d}\phi\,\mathrm{d}\bar{\phi} \\
&\qquad + \int_{\partial D}\int_{\partial D} \frac{(\gamma'(s)(\bar{\phi}))^2[(s - s_n)(\bar{\phi}) - (s - s_n)(\phi)]^2}{|\phi - \bar{\phi}|^2}\,\mathrm{d}\phi\,\mathrm{d}\bar{\phi} \\
&\qquad - 2\int_{\partial D}\int_{\partial D} \frac{[\gamma'(s)(\phi) - \gamma'(s)(\bar{\phi})](s - s_n)(\phi)}{|\phi - \bar{\phi}|^2}\gamma'(s)(\bar{\phi})[(s - s_n)(\bar{\phi}) - (s - s_n)(\phi)]\,\mathrm{d}\phi\,\mathrm{d}\bar{\phi} \\
&\quad = B_1 + B_2 + B_3.
\end{aligned}$$

Now, since $s_n \rightrightarrows s$,

$$|B_1| \leqslant \|s_n - s\|^2_{C^0}\|\gamma\|_{C^2}|s|^2_{H^{1/2}} \to 0.$$

Furthermore

$$\begin{aligned}
|B_3| &\leqslant 2\|s_n - s\|_{C^0}\|\gamma\|_{C^1} \\
&\quad \cdot \int_{\partial D}\int_{\partial D} \frac{[\gamma'(s)(\phi) - \gamma'(s)(\bar{\phi})][(s - s_n)(\bar{\phi}) - (s - s_n)(\phi)]}{|\phi - \bar{\phi}|^2}\,\mathrm{d}\phi\,\mathrm{d}\bar{\phi} \\
&\leqslant 2\|s_n - s\|_{C^0}\|\gamma\|_{C^1}|\gamma'(s)|_{H^{1/2}}|s - s_n|_{H^{1/2}}.
\end{aligned}$$

Since the weak convergence of s_n implies that $|s - s_n|_{H^{1/2}} \leqslant C$, also B_3 goes to zero. Finally, due to the regularity of γ,

$$B_2 \geqslant c_\gamma^2|s - s_n|^2_{H^{1/2}},$$

and the statement follows. □

We are finally able to prove the following proposition.

PROPOSITION 3.7 Following the notation used so far, suppose that the maps h_n satisfy the condition of cohesion and $\|h_n\|_{H^1(C_{\lambda_n})} \leqslant C$ for all $n \in \mathbb{N}$. Then there exists a monotone $s \in \mathcal{T}$ and $\lambda \in (0, \infty)$ such that (by passing to a subsequence)

$$s_n \to s \text{ strongly in } H^{1/2}, \quad s_n \rightrightarrows s \text{ uniformly}, \quad \lambda_n \to \lambda,$$

and

$$(s, \lambda) \text{ is stationary for } E.$$

Proof. The condition of cohesion together with the assumption that $E(s_n, \lambda_n) \leqslant C$ implies the uniform convergence of the maps s_n and the existence of a $\lambda \in (0, \infty)$ such that, by passing to a subsequence, $\lambda_n \to \lambda$. Hence the first statement follows from Lemma 3.6.

It remains to check the second statement. Let us work on the stationarity for s_n as expressed in (27). Consider the first term:

$$\begin{aligned}
&\int_{C_1} \frac{\partial}{\partial x}(\Phi(\gamma \circ s_n) \circ \sigma_{\lambda_n}) \frac{\partial}{\partial x}(\Phi(\gamma' \circ s_n \xi) \circ \sigma_{\lambda_n}) \\
&\quad = \int_{C_1} \frac{\partial}{\partial x}(\Phi(\gamma \circ s_n) \circ \sigma_{\lambda_n}) \frac{\partial}{\partial x}(\Phi(\gamma' \circ s_n \xi) \circ \sigma_{\lambda_n} - \Phi(\gamma' \circ s_n \xi) \circ \sigma_\lambda) \\
&\qquad + \int_{C_1} \frac{\partial}{\partial x}(\Phi(\gamma \circ s_n) \circ \sigma_{\lambda_n}) \frac{\partial}{\partial x}(\Phi(\gamma' \circ s_n \xi) \circ \sigma_\lambda - \Phi(\gamma' \circ s \xi) \circ \sigma_\lambda) \\
&\qquad + \int_{C_1} \frac{\partial}{\partial x}(\Phi(\gamma \circ s_n) \circ \sigma_{\lambda_n} - \Phi(\gamma \circ s) \circ \sigma_\lambda) \frac{\partial}{\partial x}(\Phi(\gamma' \circ s \xi) \circ \sigma_\lambda) \\
&\qquad + \int_{C_1} \frac{\partial}{\partial x}(\Phi(\gamma \circ s) \circ \sigma_\lambda) \frac{\partial}{\partial x}(\Phi(\gamma' \circ s \xi) \circ \sigma_\lambda) \\
&\quad = I_1 + I_2 + I_3 + \int_{C_1} \frac{\partial}{\partial x}(\Phi(\gamma \circ s) \circ \sigma_\lambda) \frac{\partial}{\partial x}(\Phi(\gamma' \circ s \xi) \circ \sigma_\lambda).
\end{aligned}$$

Now, I_1 goes to zero by Lemma 3.2 and the boundedness of the maps h_n. Lemma 3.5 implies that I_3 goes to zero. Finally,

$$\begin{aligned}
|I_2| &\leqslant C(\lambda) \|\Phi(\gamma' \circ s_n \xi) \circ \sigma_\lambda - \Phi(\gamma' \circ s \xi) \circ \sigma_\lambda\|_{H^1(C_1)} \\
&\leqslant C(\lambda) \|\gamma' \circ s_n \xi - \gamma' \circ s \xi\|_{H^{1/2}} \leqslant C(\lambda) \|\gamma' \circ s_n - \gamma' \circ s\|_T \|\xi\|_T,
\end{aligned}$$

which also goes to zero by Lemma 3.6. Applying the same arguments to the second term in (27) and letting $\lambda_n \to \lambda$ we obtain

$$\begin{aligned}
&\frac{1}{\lambda} \int_{C_1} \frac{\partial}{\partial x}(\Phi(\gamma \circ s) \circ \sigma_\lambda) \frac{\partial}{\partial x}(\Phi(\gamma' \circ s \xi) \circ \sigma_\lambda) \\
&\qquad + \lambda \int_{C_1} \frac{\partial}{\partial \theta}(\Phi(\gamma \circ s) \circ \sigma_\lambda) \frac{\partial}{\partial \theta}(\Phi(\gamma' \circ s \xi) \circ \sigma_\lambda) = 0 \quad \forall \xi \in T,
\end{aligned}$$

i.e. s is stationary for $E(\cdot, \lambda)$. Now let us consider equation (28). The left hand side can be written as

$$\begin{aligned}
&\int_{C_{\lambda_{n+1}}} \left| \frac{\partial}{\partial \tilde{x}}(h_n \circ k_{n,n+1}) \right|^2 \mathrm{d}\tilde{x}\, \mathrm{d}\tilde{\theta} = \frac{1}{\lambda_{n+1}} \int_{C_1} \left| \frac{\partial}{\partial x}(h_n \circ \sigma_{\lambda_n}) \right|^2 \mathrm{d}x\, \mathrm{d}\theta \\
&\quad = \frac{1}{\lambda_{n+1}} \int_{C_1} \bigg(\frac{\partial}{\partial x}(h_n \circ \sigma_{\lambda_n}) \frac{\partial}{\partial x}(\Phi(\gamma \circ s_n) \circ \sigma_{\lambda_n} - \Phi(\gamma \circ s_n) \circ \sigma_\lambda) \\
&\qquad + \frac{\partial}{\partial x}(h_n \circ \sigma_{\lambda_n}) \frac{\partial}{\partial x}(\Phi(\gamma \circ s_n) \circ \sigma_\lambda - \Phi(\gamma \circ s) \circ \sigma_\lambda) + \frac{\partial}{\partial x}(h_n \circ \sigma_{\lambda_n}) \frac{\partial}{\partial x}\big(\Phi(\gamma \circ s) \circ \sigma_\lambda\big) \bigg) \\
&\quad = \frac{1}{\lambda_{n+1}} I_1 + \frac{1}{\lambda_{n+1}} I_2 + \frac{1}{\lambda_{n+1}} I_3.
\end{aligned}$$

We have

- $I_1 \to 0$ by Lemma 3.2 and $\|h_n \circ \sigma_{\lambda_n}\|_{H^1(C_1)} \leqslant C(\lambda)$,

- $I_2 \leqslant C(\lambda)\,\|\gamma \circ s_n - \gamma \circ s\|_{H^{1/2}(\partial C)} \to 0$ by Lemma 3.6,
- $I_3 \to \int_{C_1} |\frac{\partial}{\partial x}(\Phi(\gamma \circ s) \circ \sigma_\lambda)|^2$ by Lemma 3.5.

Therefore as $n \to \infty$,

$$\int_{C_{\lambda_{n+1}}} \left|\frac{\partial}{\partial \tilde{x}}(h_n \circ k_{n,n+1})\right|^2 \mathrm{d}\tilde{x}\,\mathrm{d}\tilde{\theta} \to \int_{C_\lambda} \left|\frac{\partial h}{\partial \tilde{x}}\right|^2 \mathrm{d}\tilde{x}\,\mathrm{d}\tilde{\theta}.$$

Applying the same arguments to the right hand side of (28) we get

$$\int_{C_\lambda} \left|\frac{\partial h}{\partial \tilde{x}}\right|^2 \mathrm{d}\tilde{x}\,\mathrm{d}\tilde{\theta} = \int_{C_\lambda} \left|\frac{\partial h}{\partial \tilde{\theta}}\right|^2 \mathrm{d}\tilde{x}\,\mathrm{d}\tilde{\theta}.$$

This concludes the proof. □

REMARKS. Proposition 3.7 gives an alternative proof for the existence of an annulus-like minimal surface. Note that it does not necessarily yield the existence of an area minimizer as opposed to Theorem 2.1 (for a proof see [8]).

In [9, §§556–566] another existence proof of an area minimizer for the Douglas problem is given. The proof is quite similar to that of Theorem 2.1, the main difference being that the Douglas condition (see (3)) is replaced by the condition of cohesion. It is also shown that the latter is a weaker assumption (i.e. if the Douglas condition is satisfied, so is the condition of cohesion): in practice the Douglas condition is usually preferred since the condition of cohesion is rather hard to verify.

4. The discrete Douglas problem

4.1 *Discrete function spaces*

It is well known that every cylinder C_λ is locally isometric to a rectangle on the plane with sides of length 2π and λ, where the two sides of length λ are identified. In the attempt to discretize the problem, the identification of C_λ with a flat figure in the real plane turns out to be very useful. Thus we will use the latter as domain of parametrisation. Note that the two sides of length λ do not count as boundary, and that functions are identified with periodic functions.

Let $\mathcal{G}_{\lambda h}$ be a quasi-uniform triangulation of C_λ controlled by h, i.e. each triangle $G \in \mathcal{G}_{\lambda h}$ has diameter at most h and at least σh for some $\sigma > 0$ independent of h, and has angles bounded away from zero independently of h. We can consider $\mathcal{G}_{\lambda h}$ as a one-parameter family of triangulations corresponding to the one-parameter family of domains C_λ.

Define

$$\mathcal{L}_{\lambda h} = \bigcup\{E_j \mid E_j \text{ a boundary interval}\},$$
$$\mathcal{B}_{\lambda h} = \{\phi_1, \ldots, \phi_M\} \text{ is the set of boundary nodes},$$
$$\mathcal{N}_{\lambda h} = \{v_1, \ldots, v_N\} \text{ is the set of all nodes, where } v_j = \phi_j \text{ for } j = 1, \ldots, M.$$

Suppose $f \in C^0(\partial C_\lambda, \mathbb{R}^n)$, $f = (f_1, f_2)$, $f_i : \partial D \to \mathbb{R}^n$ for $i = 1, 2$. Then the continuous and piecewise linear interpolant $I_h f$ is defined on ∂C_λ by $I_h f = (I_h f_1, I_h f_2)$, where

$$I_h f_i(e^{i((1-t)\phi_j + t\phi_{j+1})}) = (1-t) f_i(e^{i\phi_j}) + t f_i(e^{i\phi_{j+1}})$$

for $i = 1, 2$ and ϕ_j, ϕ_{j+1} are consecutive nodes on ∂D. Note that the image of $I_h(\gamma \circ s)$ is a polygonal approximation to Γ.

As in the smooth case, instead of working directly with maps $f : \partial C_\lambda \to \Gamma$, we work with the corresponding maps $s : \partial D \dot{\cup} \partial D \to S^1 \dot{\cup} S^1$, where $f = \gamma \circ s$.

Before introducing some discrete function spaces, let us make the following important remark.

If we take a quasi-uniform triangulation $\mathcal{G}_h$ on the unit cylinder C_1 and then rescale it for various values of λ, the triangles degenerate very easily (and hence the quasi-uniformity is lost). So in general, for the same parameter h and different lengths λ and σ, $\mathcal{G}_{\lambda h}$ and $\mathcal{G}_{\sigma h}$ will not be obtained from each other by a rescaling process, but will be generated independently. However, if $|\lambda - \sigma|$ is sufficiently small, one grid can be rescaled to generate the other and the significant properties of the triangulation are not destroyed. In this case, the main advantage is that the triangulations of the boundaries ∂C_λ and ∂C_σ coincide.

DEFINITION 4.1 Suppose that a fixed set of boundary nodes on ∂C has been given (with the size of the boundary intervals controlled by h). Then we can define

$$H_h = \{\xi_h \in C^0(\partial C, \mathbb{R} \dot{\cup} \mathbb{R}) \mid \xi_h \in P_1(E_j)\ \forall j\},$$
$$\mathcal{H}_h = \{s_h \in C^0(\partial D \dot{\cup} \partial D, S^1 \dot{\cup} S^1) \mid s_h = \mathrm{id} + \sigma_h \text{ for some } \sigma_h \in H_h\}.$$

Here we intentionally omit the length of the cylinder because of the case in which we are looking at a family of cylinders that have equal triangulation of the boundaries (this is the case for example if the triangulations of the cylinders can be obtained from one another by rescaling as discussed above).

Note that $H_h \subset T \subset H$ and H_h is an M-dimensional vector space. Moreover, $\mathcal{H}_h \subset \mathcal{T} \subset \mathcal{H}$, $\mathcal{H}_h$ is an affine space of dimension M, and the space of variations at any $s_h \in \mathcal{H}_h$ is naturally identified with H_h.

Sometimes it is important to stress the choice of domain. We also need some notation for discrete maps which map into $\mathbb{R}^n$.

DEFINITION 4.2

$$X^n_{\lambda h} = \{u_h \in C^0(C_\lambda, \mathbb{R}^n) \mid u_h \in P_1(G) \text{ for } G \in \mathcal{G}_{\lambda h}\},$$
$$x^n_{\lambda h} = \{f_h \in C^0(\partial C_\lambda, \mathbb{R}^n \dot{\cup} \mathbb{R}^n) \mid f_h \in P_1(E_j)\}.$$

Taking $n = 1$ we similarly define $X_{\lambda h}$ and $x_{\lambda h}$.

For $f_h \in x_{\lambda h}$ the *discrete harmonic extension* $\Phi_h f_h \in X_{\lambda h}$ is defined by

$$\Delta_h \Phi_h f_h = 0 \qquad \text{in } C_\lambda, \tag{33}$$
$$\Phi_h f_h = f_h \qquad \text{on } \partial C_\lambda, \tag{34}$$

where Δ_h is the discrete Laplacian. Thus (33) is interpreted in the weak sense, namely

$$\int_{C_\lambda} \nabla(\Phi_h f_h) \nabla \psi_h = 0$$

for all ψ_h in $X_{\lambda h}$ such that $\psi_h = 0$ on $\partial C_{\lambda h}$. If $f_h \in x^n_{\lambda h}$ the discrete harmonic extension $\Phi_h f_h$ is defined componentwise.

4.2 *The discrete energy functional E_h*

DEFINITION 4.3 Given $(s_h, \lambda_h) \in \mathcal{H}_h \times (0, \infty)$, the discrete energy functional is defined by

$$E_h(s_h, \lambda_h) = \frac{1}{2} \int_{C_{\lambda_h}} |\nabla \Phi_h I_h(\gamma \circ s_h)|^2 = \mathcal{D}(\Phi_h I_h(\gamma \circ s_h)). \tag{35}$$

That is, $E_h(s_h, \lambda_h) = \mathcal{D}(u_h)$, where u_h is the discrete harmonic extension of $I_h(\gamma \circ s_h)$ taken over C_{λ_h}. We first apply I_h to $\gamma \circ s_h$ since the latter is not piecewise linear.

Note that for a fixed parametrisation γ and a fixed λ_h, $E_h(s_h, \lambda_h)$ is completely determined by the nodal values $s_h(\phi_j)$. Finally, note that E_h is *not* the restriction of E to $\mathcal{H}_h \times (0, \infty)$.

For later use set

$$u_h = \Phi_h I_h(\gamma \circ s_h), \quad v_h = \Phi_h I_h(\gamma' \circ s_h\, \xi_h), \quad w_h = \Phi_h I_h(\gamma'' \circ s_h\, \xi_h^2), \tag{36}$$

and let $\Psi_h \in X^n_{\lambda_h h}$ with $\Psi_h = 0$ on ∂C_{λ_h} be the discrete solution of

$$\int_{C_{\lambda_h}} \nabla \Psi_h \nabla g_h = \frac{1}{\lambda_h} \int_{C_{\lambda_h}} \left(\frac{\partial u_h}{\partial x} \frac{\partial g_h}{\partial x} - \frac{\partial u_h}{\partial \theta} \frac{\partial g_h}{\partial \theta} \right) \tag{37}$$

for all $g_h \in X^n_{\lambda_h h}$ with $g_h = 0$ on ∂C_{λ_h}. Similarly to the smooth case we compute

$$\langle E_h'(s_h, \lambda_h), (\xi_h, \mu_h) \rangle = \int_{C_{\lambda_h}} \nabla u_h \nabla v_h + \frac{\mu_h}{2\lambda_h} \int_{C_{\lambda_h}} \left(\left| \frac{\partial u_h}{\partial \theta} \right|^2 - \left| \frac{\partial u_h}{\partial x} \right|^2 \right), \tag{38}$$

and

$$\begin{aligned} E''(s_h, \lambda_h)(\xi_h, \mu_h)^2 &= \int_{C_{\lambda_h}} (|\nabla v_h|^2 + \nabla u_h \nabla w_h) \\ &+ \frac{2\mu_h}{\lambda_h} \int_{C_{\lambda_h}} \left(-\frac{\partial u_h}{\partial x} \frac{\partial v_h}{\partial x} + \frac{\partial u_h}{\partial \theta} \frac{\partial v_h}{\partial \theta} \right) + \left(\frac{\mu_h}{\lambda_h} \right)^2 \int_{C_{\lambda_h}} \left| \frac{\partial u_h}{\partial x} \right|^2 - \mu_h^2 \int_{C_{\lambda_h}} |\nabla \Psi_h|^2, \end{aligned} \tag{39}$$

with an analogous expression for $E_h''(s_h, \lambda_h)(\xi_h, \mu_h)(\eta_h, \sigma_h)$ obtained by bilinearity in the case of distinct variations.

We are now ready to give the formulation of the discrete problem.

DEFINITION 4.4 The discrete harmonic function

$$u_h = \Phi_h I_h(\gamma \circ s_h)$$

defined on C_{λ_h} is a *discrete minimal surface spanning Γ* if the pair $(s_h, \lambda_h) \in \mathcal{H}_h \times (0, \infty)$ is stationary for E_h, i.e. if the following two statements are true:

(Eh1) s_h is stationary for $E_h(\cdot, \lambda_h)$ in the sense that

$$\langle E_h'(s_h, \lambda_h), \xi_h \rangle = 0, \quad \forall \xi_h \in H_h,$$

(Eh2) λ_h is such that "equipartition of energy" holds, namely

$$\int_{C_{\lambda_h}} \left|\frac{\partial u_h}{\partial x}\right|^2 \mathrm{d}x\,\mathrm{d}\theta = \int_{C_{\lambda_h}} \left|\frac{\partial u_h}{\partial \theta}\right|^2 \mathrm{d}x\,\mathrm{d}\theta.$$

REMARK. Note that we do not require monotonicity of s_h, as in the case of s in Definition 2.7. Also observe that $(s_h, \lambda_h) \in \mathcal{H}_h \times (0, \infty)$ is a discrete stationary point for E_h if and only if $\langle E_h'(s_h, \lambda_h), (\xi_h, \mu_h)\rangle = 0$ for all $(\xi_h, \mu_h) \in H_h \times \mathbb{R}$.

5. The "discrete sequence"

For the energy functional E we have shown that under suitable conditions it is possible to construct sequences converging to stationary points (see Proposition 3.7). A similar thing can be done in the discrete setting. More precisely, let us define the so called "discrete sequence" in the following way. First choose $\lambda_0 \in (0, \infty)$ and then repeat the following two steps.

Step 1. Given λ_n, find $s_{hn} \in \mathcal{H}_h$ such that s_{hn} is stationary for $E_h(\cdot, \lambda_n)$. In other words, find s_{hn} such that

$$\langle E_h'(s_{hn}, \lambda_n), \xi_h\rangle = 0, \qquad \forall \xi_h \in H_h\ (= x_{\lambda_n h}).$$

Note that for each λ_n a different triangulation (controlled by h) has to be determined. For later use let us denote by

$$h_n := \Phi_h I_h(\gamma \circ s_{hn})$$

the discrete harmonic extension of $I_h(\gamma \circ s_{hn})$ on C_{λ_n}.

Step 2. Given s_{hn}, λ_n, and h_n, find λ_{n+1} such that

$$\int_{C_{\lambda_{n+1}}} \left|\frac{\partial}{\partial \tilde{x}}(h_n \circ k_{n,n+1})\right|^2 \mathrm{d}\tilde{x}\,\mathrm{d}\tilde{\theta} = \int_{C_{\lambda_{n+1}}} \left|\frac{\partial}{\partial \tilde{\theta}}(h_n \circ k_{n,n+1})\right|^2 \mathrm{d}\tilde{x}\,\mathrm{d}\tilde{\theta}, \tag{40}$$

where again we denote by $k_{n,n+1}$ the function $k_{n,n+1} : C_{\lambda_{n+1}} \to C_{\lambda_n}$ which maps $(\tilde{x}, \tilde{\theta})$ to $\left(\frac{\lambda_n}{\lambda_{n+1}}\tilde{x}, \tilde{\theta}\right)$.

PROPOSITION 5.1 Using the notation above, suppose that (by passing to a subsequence) $\lambda_n \to \lambda_h \in (0, \infty)$ for $n \to \infty$ and $\|s_{hn}\|_{C^0} \leqslant C$ for all n sufficiently large. Then a subsequence of $\{(s_{hn}, \lambda_n)\}_{n\in\mathbb{N}}$ converges to a discrete stationary point for E_h.

Proof. Suppose that $\lambda_n \to \lambda_h \in (0, \infty)$. For n sufficiently large, λ_n will be so close to λ_h that we can fix a quasi-uniform triangulation $\mathcal{G}_{\lambda_h h}$ of C_{λ_h} controlled by h and get all other triangulations $\mathcal{G}_{\lambda_n h}$ of C_{λ_n} by rescaling $\mathcal{G}_{\lambda_h h}$. In this situation the triangulations of ∂C_{λ_h} and ∂C_{λ_n} will be the same. Since H_h is a finite-dimensional space and $\|s_{hn}\|_{C^0} \leqslant C$ for all n sufficiently large, there exists $s_h \in \mathcal{H}$ such that, by passing to a subsequence, $s_{hn} \to s_h$ in the C^0 norm (and with respect to every norm that can be defined on the space of piecewise linear functions). It remains to check that (s_h, λ_h) is stationary for E_h. This is done as in the analogous Proposition 3.7, the only differences being that, to evaluate the integrals, only Lemmas 3.2 and 5.2 are used. Note that the fixed cylinder C_1 naturally inherits the (rescaled) triangulation of C_{λ_h}. Also it is necessary to show that the norms $\|\Phi_h I_h(\gamma \circ s_{hn}) \circ \sigma_{\lambda_n}\|_{H^1(C_1)}$ and $\|\Phi_h I_h(\gamma' \circ s_{hn}\xi_h) \circ \sigma_{\lambda_h}\|_{H^1(C_1)}$ are uniformly bounded for all n

sufficiently large and a fixed $\xi_h \in H_h$. This follows from the boundedness of $\|\gamma\|_{C^1}$ and from the fact that when λ_n is sufficiently close to λ_h the stiffness matrices relative to the Poisson problem on C_{λ_n} are comparable to the one relative to the Poisson problem on C_{λ_h}. □

In the next lemma we show that a control on the boundary norm $\|f_h\|_{C^0}$ for $f_h \in H_h$ induces a control on the C^0 norm of the discrete harmonic extension $\Phi_h(f_h)$.

LEMMA 5.2 Suppose that a triangulation $\mathcal{G}_{\lambda h}$ on a cylinder C_λ and a sequence of maps $f_n \in H_h$ are given, with $f_n \to f \in H_h$ in the C^0 norm. Then, by passing to a subsequence, $\Phi_h f_n \to \Phi_h f$ in the C^0 norm (and hence in any other suitable norm).

Proof. Write $\psi_n = \Phi_h(f_n - f)$. Then ψ_n is such that

$$\begin{cases} \Delta_h \psi_n = 0 & \text{in } C_\lambda, \\ \psi_n = f_n - f & \text{on } \partial C_\lambda, \end{cases}$$

and $\|\psi_n\|_{C^0(C_\lambda)} \leqslant C$, since $\|f_n - f\|_{C^0(\partial C_\lambda)}$ is also uniformly bounded.

To prove that $\|\psi_n\|_{C^0(C_\lambda)} \leqslant C$, let us denote by

$$\left(\begin{array}{c|c} A & B \\ \hline B & C \end{array}\right)$$

the stiffness matrix relative to $\mathcal{G}_{\lambda h}$, where A is the block relative to the internal nodes, and B the block relative to the internal/boundary nodes. Then we can write the above PDE in the matrix form

$$A \cdot \psi_{\text{int}} = -B \cdot \psi_{\text{bdry}}, \quad \psi_{\text{bdry}} = f_n - f,$$

where $\psi = (\psi_{\text{int}}, \psi_{\text{bdry}})$ is the vector of components of ψ with respect to the nodal basis. Then $\|\psi_{\text{int}}\|_{\mathbb{R}^p} \leqslant \|A^{-1}\| \, \|B\| \, \|\psi_{\text{bdry}}\|_{\mathbb{R}^m}$, and the statement follows.

Since $X^n_{\lambda h}$ is a finite-dimensional space, passing to a subsequence, $\psi_n \to \psi \in X^n_{\lambda h}$, with ψ discrete harmonic. But $\psi_n \to 0$ on the boundary, hence $\psi = 0$. □

The discrete sequence proves to be interesting because it gives an alternative to the use of the Newton method (as used in [4], [5] for the Plateau problem) to find stationary points for the discrete energy functional. Basically what Proposition 5.1 tells us is that if we implement the discrete sequence and it happens to converge, then what we find is a discrete stationary point for E_h.

6. The numerical algorithm

We now describe the algorithm used for the computation of discrete minimal surfaces. We want to solve the equation

$$E'_h(s_h, \lambda_h) = 0$$

in the discrete space $\mathcal{H}_h \times (0, \infty)$. This is equivalent to computing (s_h, λ_h) such that

$$\langle E'_h(s_h, \lambda_h), (\xi_h, \mu_h)\rangle = 0, \quad \forall (\xi_h, \mu_h) \in H_h \times \mathbb{R}.$$

The algorithm used is based on the idea of the so called “discrete sequence” described in Section 5 (in particular see Proposition 5.1 and the remarks that follow).

We can now sketch the algorithm as follows.

ALGORITHM 6.1 Given tolerances $\epsilon, \epsilon_c > 0$:

1. The user gives an initial λ and h_0, where λ is the length of the first cylinder in the "discrete sequence" and h_0 is the maximum allowed size for a triangle in the various triangulations.
2. A triangulation for the cylinder of length λ is created.
3. A solution s_h of $E_h'(\cdot, \lambda) = 0$ is found and

$$u_h = \Phi_h I_h(\gamma \circ s_h)$$

   is computed on C_λ.
4. If $\|E_h'(s_h, \lambda)\|_{(H_h \times \mathbb{R})'} \leqslant \epsilon$ and

$$\left| \int_{C_\lambda} \left(\left| \frac{\partial u_h}{\partial x} \right|^2 - \left| \frac{\partial u_h}{\partial \theta} \right|^2 \right) \mathrm{d}x \, \mathrm{d}\theta \right| < \epsilon_c$$

   then stop.
5. Compute λ_{new} as described in "The discrete sequence, Step 2" (see Section 5), set $\lambda = \lambda_{\text{new}}$ and go to step 2.

Let us now have a closer look at each step in the algorithm and give a few more details.

Algorithm 6.1, Step 2. Given λ, a triangulation of the cylinder C_λ is created in two steps:

1. A macro triangulation (i.e. an initial coarse grid) is created for C_λ. This is done in accordance with the ratio between λ and 2π. If $\lambda > 2\pi$ (resp. $\lambda \leqslant 2\pi$), and $n = [\lambda/2\pi]$ (resp. $n = [2\pi/\lambda]$), where $[\cdot]$ denotes the greatest integer function, then $2n$ right angled triangles are created. These triangles have the property that the ratio of base to height is close to one (more precisely base/height $\in [1, 2)$ or $(1/2, 1]$, depending on which side of the triangle we take to be the base).
2. The macro triangulation is refined until the diameter of the triangles is less than h_0. The algorithm is based on bisection of triangles. The refinement edges chosen on the macro triangulation prescribe the refinement edges for all simplices created during mesh refinement. For more details see [12, §1.1.1].

For different λ's different triangulations are given and a different number of triangles is created each time, although all triangulations share the property that their triangles' diameters do not exceed h_0. However if $\lambda_i, \lambda_j > 2\pi$ and $[\lambda_i/2\pi] = [\lambda_j/2\pi]$ (or if $\lambda_i, \lambda_j \leqslant 2\pi$ and $[2\pi/\lambda_i] = [2\pi/\lambda_j]$) and $|\lambda_i - \lambda_j|$ is small then the number of triangles is the same and the decompositions of the boundary of the cylinders coincide. This ensures that if the sequence of λ's converges to a $\bar{\lambda}$ (in a monotone way if $\bar{\lambda} = 2k\pi$ for some integer k), then during the last few iterations the triangulation of the boundary of the cylinders will stay the same and Proposition 5.1 applies.

Algorithm 6.1, Step 3. The computation of s_h, a stationary point for $E_h(\cdot, \lambda)$, is done by means of the Newton method as follows.

ALGORITHM 6.2 Given an initial parametrisation $s_h \in \mathcal{H}_h$ and a tolerance $\delta > 0$:

1. Compute $E_h'(s_h, \lambda)$.
2. If $\|E_h'(s_h, \lambda)\|_{H_h'} \leqslant \delta$, then go to step 5 in this algorithm.
3. Solve the linear problem

$$E_h''(s_h, \lambda)(\eta_h, \xi_h) = -\langle E_h'(s_h, \lambda), \xi_h \rangle \quad \forall \xi_h \in H_h.$$

4. Update the solution: $s_h = s_h + \eta_h$ and go to step 1 in this algorithm.
5. Compute the discrete harmonic extension $u_h = \Phi_h I_h(\gamma \circ s_h)$ on C_λ and stop.

For more details see the numerical algorithm developed for the Plateau problem in [4, §5].

Algorithm 6.1, Step 5. If $\|E_h'(s_h, \lambda)\|_{(H_h\times\mathbb{R})'} > \epsilon$ or if $\left|\int_{C_\lambda}\left(\left|\frac{\partial u_h}{\partial x}\right|^2 - \left|\frac{\partial u_h}{\partial\theta}\right|^2\right)\right| > \epsilon_c$, then we compute the next λ in the discrete sequence as described in "The discrete sequence, Step 2" in Section 5 (see also (29)). This amounts to calculating

$$\lambda_{\text{new}}^2 = \lambda^2 \frac{\int_{C_\lambda}\left|\frac{\partial u_h}{\partial x}\right|^2}{\int_{C_\lambda}\left|\frac{\partial u_h}{\partial\theta}\right|^2},$$

where u_h is the piecewise linear function computed in Step 3.

REMARK. Algorithm 6.1 was implemented for the case in which the two given Jordan curves Γ_1, Γ_2 lie in $\mathbb{R}^3$ and are such that $(x, y, z) \in \Gamma_1$ if and only if $(-x, y, z) \in \Gamma_2$. In this particular case it is not hard to prove the existence of a symmetric minimal surface, so that we can assume that for the boundary map $s = (s_1, s_2)$ we have $s_1 = s_2$.

Such a simplification decreased the programming workload and is justified by the fact that the main intention here is to verify the theoretical results rather than to give an exhaustive numerical investigation.

7. Implementation and numerical results

The catenoid is a good test example because here the exact solution(s) for the minimal surface(s) can be computed. Let

$$\Gamma_1 = \{x = d/2,\ y = \sin(\theta),\ z = \cos(\theta) \mid 0 \leqslant \theta \leqslant 2\pi\},$$
$$\Gamma_2 = \{x = -d/2,\ y = \sin(\theta),\ z = \cos(\theta) \mid 0 \leqslant \theta \leqslant 2\pi\}$$

be the two boundary curves.

For d small enough there exist two catenoids, say S_d^1 and S_d^2, with areas $\mathcal{A}(S_d^1) \leqslant \mathcal{A}(S_d^2)$. Precisely we have the following situation (see [9, §515]):

- for $d < d_1 \approx 1.055396$ there exist two minimal surfaces, an absolute minimizer S_d^1 with area $\mathcal{A}(S_d^1) < 2\pi$ and an unstable catenoid S_d^2;
- for $d = d_1$, both catenoids exist and $\mathcal{A}(S_{d_1}^1) = 2\pi$;
- for $d_1 < d < d_2 \approx 1.325487$, both catenoids exist and S_d^1, whose area is now bigger than 2π, represents a strong relative minimum;
- for $d = d_2$, $S_{d_2}^1 = S_{d_2}^2$, i.e. only one unstable solution exists;
- for $d > d_2$, no minimal surface of the topological type of the annulus exists.

Fix $d = 1$, and denote by $\mathcal{D}(u, C_\lambda)$ and $\mathcal{A}(u, C_\lambda)$ the Dirichlet and area energy of the map u defined on C_λ. An easy computation gives the following results.

Stable catenoid: A harmonic and conformal parametrisation is given by

$$G_1 : [-s_1/2, s_1/2] \times [0, 2\pi] \to \mathbb{R}^3,$$
$$G_1(x,\theta) = \left(\frac{x}{s_1}, \frac{1}{s_1}\cosh(x)\sin(\theta), \frac{1}{s_1}\cosh(x)\cos(\theta)\right),$$

where $s_1 \approx 1.178775527$. We have $\mathcal{D}(G_1, C_{s_1}) = \mathcal{A}(G_1, C_{s_1}) \approx 5.991796978 < 2\pi$.

Unstable catenoid: A harmonic and conformal parametrisation is given by

$$G_2 : [-s_2/2, s_2/2] \times [0, 2\pi] \to \mathbb{R}^3,$$
$$G_2(x, \theta) = \left(\frac{x}{s_2}, \frac{1}{s_2} \cosh(x) \sin(\theta), \frac{1}{s_2} \cosh(x) \cos(\theta) \right),$$

where $s_2 \approx 4.253599783$. We have $\mathcal{D}(G_2, C_{s_2}) = \mathcal{A}(G_2, C_{s_2}) \approx 6.845655397$.

The choice of initial $\lambda = 1$ and different h_0 gives the following results.

Stable catenoid, $\epsilon = 10^{-9}$

h_0	(final) h	λ_h	Energy E_h	L^2-error	H^1-error
0.8	0.628318548	1.13947593	5.9664446	0.0514679054	0.624718504
0.6	0.427655043	1.16063973	5.99257619	0.017768278	0.336564961
0.35	0.314159274	1.16756642	5.98581048	0.0126773946	0.313185911
0.3	0.214951172	1.17384821	5.99218485	0.0046370298	0.168648153
0.2	0.157079637	1.17586513	5.99032695	0.00316071535	0.156642544
0.1	0.0785398185	1.17804075	5.99143122	0.000789723004	0.0783249433

We can now display graphically the behaviour of both errors (see Figure 2).

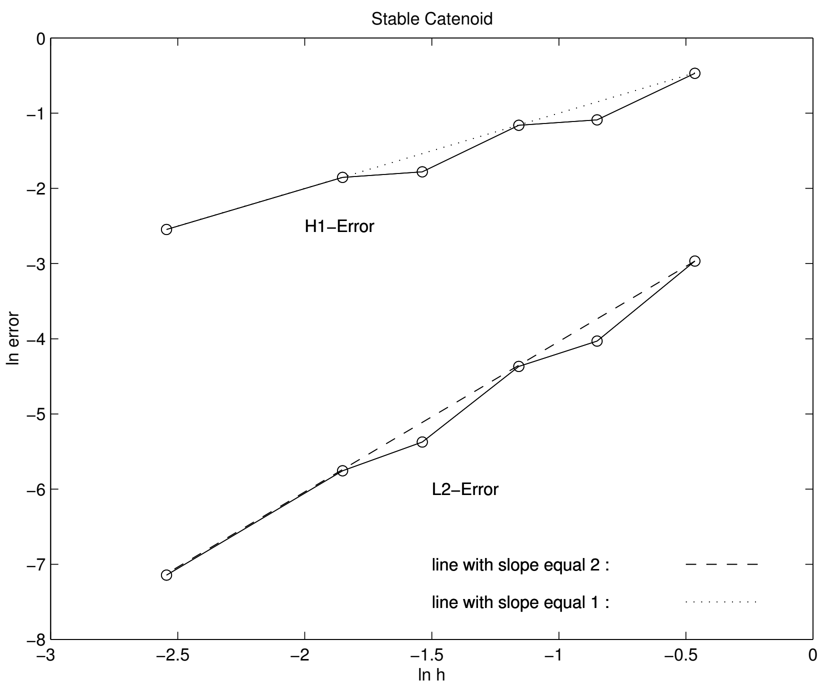

FIG. 2

Denote by e_h the error between the continuous solution and the discrete one. For two successive grids with grid sizes h_1 and h_2 the experimental order of convergence is

$$\mathrm{eoc} = \ln \frac{e_{h_1}}{e_{h_2}} \Big/ \ln \frac{h_1}{h_2}. \tag{41}$$

If, of the previous grid sizes, we consider only those h_i such that $h_{i+1} \approx h_i/2$, then we obtain:

Stable catenoid

h	L^2-eoc	H^1-eoc	L^∞-error
0.628318548	–	–	0.00490793974
0.314159274	2.0214148	0.99618694	0.00150622474
0.157079637	2.0039354	0.99954321	0.000398596735
0.0785398185	2.0008323	0.99993237	0.000101131735

This particular choice of grid sizes is motivated by the fact that it has been observed experimentally that smaller error enters the formula (41) if we start with a grid size h and keep halving it.

As we can see from the displayed tables, these results confirm the accuracy of the convergence rate given in (1) and (2). The convergence rate for λ_h given in (2) can also be readily checked.

Let us now have a look at the case of the unstable catenoid. Here the choice of the initial λ proves to be crucial. Even a very small variation can make the sequence go in the "wrong" direction, either towards the absolute minimum or towards cylinder-like surfaces with increasingly thinner neck. The "good" choices of the initial λ's were made after several trials, starting first the program with the exact $\lambda = s_2 \approx 4.253599783$ and then damping the "λ-step" (i.e. choosing the new lambda to be say 5% or more away from λ in the direction of λ_{new}).

Unstable catenoid, $\epsilon = 10^{-4}$

h_0	(final) h	$\lambda = \lambda_h$	Energy E_h	L^2-error	H^1-error
0.4	0.392699093	4.368	6.77898654	0.0403238796	0.594916311
0.2	0.196349546	4.279	6.82879917	0.00954042252	0.293771424
0.1	0.0981747732	4.259	6.84142998	0.00217344382	0.146426251

With the shown choice of initial λ's, we achieve an accuracy of

$$\left| \int_{C_\lambda} \left(\left| \frac{\partial u_h}{\partial x} \right|^2 - \left| \frac{\partial u_h}{\partial \theta} \right|^2 \right) \mathrm{d}x \, \mathrm{d}\theta \right| < 10^{-3} = \epsilon_c, \tag{42}$$

whereas the accuracy of $\|E_h'(s_h, \lambda)\|_{H_h'}$ and of the boundary map s_h is of the order of 10^{-6}. The latter can also be improved, but no major changes occur in the value of the energy $E_h(s, \lambda)$. An improvement in the accuracy of (42), and a subsequent improvement in the determination of all other variables (i.e. λ_h, E_h, $\|E_h'\|$, etc.), proves however to be very difficult because of the λ's moving away very quickly from the significant region. Also a damped "λ-step" does not seem to help much, unless one is lucky enough to choose exactly the right step. With the discussed accuracy we get the following results.

Unstable catenoid

h	L^2-eoc	H^1-eoc	L^∞-error
0.392699093	–	–	0.017907561
0.196349546	2.0795094	1.018533	0.00430210466
0.0981747732	2.1340708	1.0039795	0.000985368115

Let now

$$\Gamma_1 = \{x = 0.5,\ y = a\cos(\theta),\ z = b\sin(\theta) \mid 0 \leqslant \theta \leqslant 2\pi\},$$
$$\Gamma_2 = \{x = -0.5,\ y = a\cos(\theta),\ z = b\sin(\theta) \mid 0 \leqslant \theta \leqslant 2\pi\}$$

be the two boundary curves, where $a = 0.85$ and $b = 1$. Although the ellipse's eccentricity is close to zero, a few difficulties arise and the discrete sequence does not converge as easily as in the case of the stable catenoid. Experiments show that the sequence of λ's converges smoothly towards the solution. More problems arise instead in the calculation of the boundary map s_h at every λ-iteration: the best accuracy that can be achieved with the Newton method is of order $\delta = 10^{-5}$ (and of order $\delta = 10^{-4}$ for the finest grid). This forces a choice of $\epsilon = 10^{-4}$ ($\epsilon = 10^{-3}$ in the case of the finest grid). As for ϵ_c, it has been found convenient to choose it of the same order of ϵ. To achieve convergence for the boundary map it is also very useful to damp the Newton s-step by 50%.

For initial $\lambda = 1.331$ (or λ around this value) we get:

Elliptic boundary, $\epsilon = 10^{-4}$

h_0	(final) h	λ_h	Energy E_h	L^2-error	H^1-error
0.4	0.392699122	1.3205351	5.47144538	–	–
0.2	0.196349561	1.33184676	5.48461626	0.0179903008	0.36860177
0.1	0.0981747806	1.33490395	5.48787707	0.00455446958	0.18774433
0.05	0.0490873903	1.33553536	5.48869045	0.00128944176	0.0939547043

Note also that since exact smooth solutions are no longer known, the order of convergence is calculated by

$$\mathrm{eoc} = \ln \frac{e_{h_i}}{e_{h_{i+1}}} \Big/ \ln \frac{h_i}{h_{i+1}}. \tag{43}$$

where h_i and h_{i+1} are two consecutive grid sizes, $e_i = \|u_{h_i} - u_{h_{i+1}}\|$ and u_{h_i} denotes the discrete solution calculated on a grid with grid size h_i. Again it is common practice to choose $h_{i+1} \approx h_i/2$. The analysis of the eoc gives:

Elliptic boundary

h	L^2-eoc	H^1-eoc
0.392699122	–	–
0.196349561	–	–
0.0981747806	1.9818645	0.97329366
0.0490873903	1.8205371	0.99873202

We finish this section with a few graphical examples.

In Figure 3 the given boundary curves are two unit circles (the picture on the right is that of an unstable catenoid with 1024 triangles).

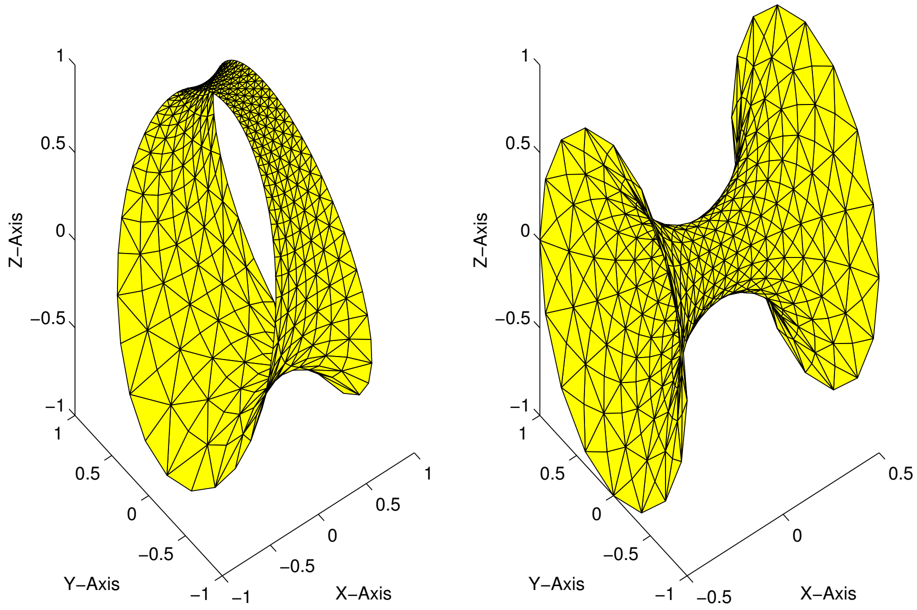
Z−Axis
Y−Axis
X−Axis
1
0.5
0
−0.5
−1
Z−Axis
Y−Axis
X−Axis

FIG. 3

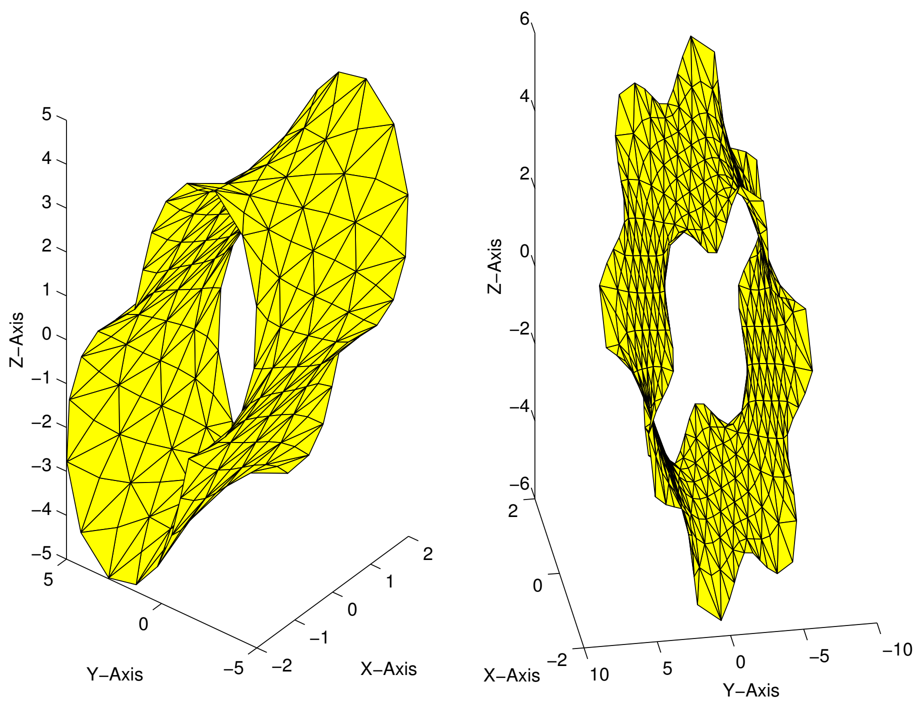
Z−Axis
Y−Axis
X−Axis
Z−Axis
Y−Axis
X−Axis

FIG. 4

In Figure 4 the boundary curves are

$$\Gamma_1 = (2,\ (4 - 2\sin(2\theta))\sin\theta,\ (4 - 2\sin(2\theta))\cos\theta),$$
$$\Gamma_2 = (-2,\ (4 - 2\sin(2\theta))\sin\theta,\ (4 - 2\sin(2\theta))\cos\theta),$$

for the dumb-bell-like minimal surface (left), and

$$\Gamma_1 = (2,\ (5 - \sin(6\theta))\sin\theta,\ (5 - \sin(6\theta))\cos\theta),$$
$$\Gamma_2 = (-2,\ (5 - \sin(6\theta))\sin\theta,\ (5 - \sin(6\theta))\cos\theta),$$

for the "six-leaves catenoid" (right).

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