



**Partial Differential Equations** — *Continuous dependence on the constitutive functions for a class of problems describing fluid flow in porous media*, by IACOPO BORSI, ANGILO FARINA, ROBERTO GIANNI and MARIO PRIMICERIO, communicated on 14 November 2008.

ABSTRACT. — In this paper we consider the PDE describing the fluid flow in a porous medium, focusing on the solution's dependence upon the choice of the saturation curve and the hydraulic conductivity. Basically, we consider two different saturation curves (say  $\theta_1$  and  $\theta_2$ ) and two different hydraulic conductivities ( $\mathcal{K}_1$  and  $\mathcal{K}_2$ ) which are both “close” in the  $L_{loc}^\infty$ -norm. Then we find estimates to prove a constitutive stability for the solutions of the corresponding problems with the same boundary and initial conditions.

KEY WORDS: Flows in porous media; Continuous dependence on parameters.

MSC SUBJECT CLASSIFICATION: 35B30, 76S05.

## 1. INTRODUCTION

Let us consider the well-known equation describing a 1-D Darcyan flow of a fluid through an homogeneous rigid porous medium (see [1]–[2]), i.e.

$$(1.1) \quad \theta_t = [\mathcal{K}(\psi_x + 1)]_x$$

where

- $x \in [0, 1]$  is the dimensionless vertical coordinate pointing upwards.
- $\psi$  is the fluid pressure head (see [1] for more details),

$$\psi = \frac{p}{\rho g},$$

with  $p$  fluid pressure,  $\rho$  liquid density and  $g$  gravity acceleration.

- $\theta$  is the moisture content. In particular,

$$0 \leq \theta \leq \theta_{\max}$$

where  $\theta_{\max}$  coincides with the porosity of the medium.

- $\mathcal{K}$  is the hydraulic conductivity of the medium.

The model is completed:

- by prescribing how  $\mathcal{K}$  depends on  $\theta$ , i.e. giving the so-called *hydraulic conductivity curve* (see e.g. [3]),

$$(1.2) \quad \mathcal{K} = \mathcal{K}(\theta).$$

- By assuming a constitutive relationship linking  $\theta$  and  $\psi$ , namely the so-called *saturation (or retention) curve* (see [4])<sup>1</sup>,

$$(1.3) \quad \theta = \theta(\psi).$$

In particular, both (1.2) and (1.3) are obtained by experimental measurements. It has to be noted, however, that accurate measurements of the unsaturated conductivity and water retention curve is generally cumbersome, costly and very time-consuming. Indeed, in many practical situations experimental data assessing the “precise” shape of the hydraulic functions are not available.

The aim of this paper is to show that “small variations” in the shape of both the saturation curve and the hydraulic conductivity function produce “small variations” of the solutions, i.e. to determine how much changes in the shape of the soil water retention curve and/or conductivity curve affect the prediction of the soil water content.

We note (see Remark 2.1) that two classes of retention curves are used in the literature: one in which  $\theta'(\psi)$  is continuous (we will refer to it as a *degenerate case* for a reason that will be selfexplained later on) and one in which  $\theta'(\psi)$  is discontinuous at  $\psi = 0$  (*non-degenerate case*). The two cases exhibit relevant mathematical differences; on the other hand it is extremely difficult (if not impossible) to discriminate experimentally between the two cases. Therefore, our result of constitutive stability of equation (1.1) seems particularly relevant.

Similar results were found in [6], where the author gives an estimate for a degenerate problem without gravity term and in case of a completely unsaturated domain.

In [7] constitutive stability results are proved (using an homotopy argument) in the particular case of non-degenerate problems.

In [8] the following degenerate diffusion problem is considered

$$\begin{aligned} u_t &= (u^{m-1}u_x)_x, \\ \left(\frac{1}{m}u^m\right)_x \Big|_{x=0,1} &= 0, \\ u|_{t=0} &= u_0(x), \end{aligned}$$

with  $0 < m \leq 1$ , and the author proves an estimate in the  $L^2$ -norm for the continuous dependence of the solution  $u$  on the parameter  $m$ . The proof cannot be extended to the problem we are considering, since it corresponds to a particular choice of the saturation curve  $\theta$  and moreover the gravity term does not appear.

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<sup>1</sup>Prescribing (1.3) means that the equilibrium between pressure and water content is reached instantaneously. A different approach (see [5] for instance) includes dynamical effects expressed by a differential equation linking  $\theta$  and  $\psi$ .

We conclude this section quoting the stability result found in [9] for problems in which the degeneration belongs to a completely different type.

## 2. ASSUMPTIONS AND BASIC EQUATIONS

We consider the following assumptions

$$(H.1) \quad \mathcal{K}(\theta) \in C^1([0, \theta_{\max}]),$$

$$\sup_{\theta \in [0, \theta_{\max}]} \mathcal{K}(\theta) = K_{\text{sat}} < \infty,$$

and

$$(2.1) \quad \mathcal{K}(\theta) \rightarrow 0^+ \quad \text{as } \theta \rightarrow 0^+.$$

$$(H.2) \quad \mathcal{K}'(\theta) \geq 0 \text{ and}$$

$$\sup_{\theta \in [0, \theta_{\max}]} \mathcal{K}'(\theta) = \mathcal{L}_{\mathcal{K}} < \infty.$$

$$(H.3) \quad \mathcal{K}'(\theta) \text{ is uniformly Lipschitz continuous with Lipschitz constant } \mathcal{L}_{\mathcal{K}'}$$

We assume

$$(H.4) \quad \theta \in C(\mathbb{R}) \text{ and it is a strictly increasing function for } \psi < 0 \text{ and } \theta \equiv \theta_{\max} \text{ for } \psi \geq 0.$$

$$(H.5) \quad \theta(\psi) \text{ is uniformly Lipschitz continuous with Lipschitz constant } \mathcal{L}_{\theta}.$$

$$(H.6) \quad \sup_{s \in [-M_0, 0]} \left\{ \left[ \frac{d\mathcal{K}}{d\theta}(\theta(s)) \right]^2 \frac{\theta'(s)}{\mathcal{K}(\theta(s))} \right\} < \infty, \text{ for any fixed } M_0 > 0 \text{ (see also Remark 2.2 below).}$$

REMARK 2.1. For  $\psi = 0$  we shall consider two situations, i.e.

$$(1) \quad \theta'(0^-) = \theta'(0^+) = 0 \text{ (which means that } \theta \in C^1(\mathbb{R})).$$

$$(2) \quad \theta'(0^-) > 0 \text{ while } \theta'(0^+) = 0.$$

Once the saturation curve is given, equation (1.1) takes the following form

$$(2.2) \quad [\theta(\psi)]_t = [K(\psi)(\psi_x + 1)]_x$$

(usually known as the  $\psi$ -form of the Richards' equation), where

$$K(\psi) = \mathcal{K}(\theta(\psi)).$$

REMARK 2.2. In terms of  $K(\psi)$  assumption (H.6) reads as

$$\sup_{s \in [-M_0, 0]} \left\{ \frac{[K'(s)]^2}{K(s)\theta'(s)} \right\} < \infty,$$

for any  $M_0 > 0$ . Even if it seems to be very restrictive, actually the permeability and retention curves commonly used in hydrology fulfill such condition. In particular, the well-known vanGenuchten and Mualem curves satisfy (H.6) (see [3] and [4]).

In particular, after performing the so-called Kirchoff's transformation  $\Gamma$ , defined as

$$u(x, t) = \Gamma(\psi(x, t)) = \int_0^{\psi(x, t)} K(s) ds,$$

equation (2.2) reads as

$$(2.3) \quad [\sigma(u)]_t = [u_x + k(u)]_x,$$

where  $k(u) = K(\Gamma^{-1}(u))$  and the function  $\sigma(u) = \theta(\Gamma^{-1}(u))$  behaves like  $\theta$ .

As a consequence of assumptions (H.1)–(H.6) we have

(F.1)  $\sigma(s) \in C(\mathbb{R})$ ,  $\sigma(s)$  is strictly increasing for  $s < 0$  and  $\sigma \equiv \sigma_s$  for  $s \geq 0$ .  
Moreover  $\sigma$  is uniformly Lipschitz continuous with Lipschitz constant  $\mathcal{L}_\sigma$ .

(F.2) According to Remark 2.1 for  $s = 0$  two options are possible, i.e.

$$(1) \quad \sigma'(0) = 0.$$

$$(2) \quad \sigma'(0) > 0.$$

In case 1 Richards' equation degenerates at  $u = 0$ , while in case 2 equation (2.2) is uniformly parabolic.

(F.3)  $\sup_{s \in [-M_0, 0]} \left\{ \frac{[k'(s)]^2}{\sigma'(s)} \right\} < \infty$ , for any fixed  $M_0 > 0$ .

Concerning the initial datum we consider

$$(2.4) \quad \sigma(u(x, 0)) = v_0(x).$$

with  $v_0(x)$  Lipschitz continuous function. The boundary conditions may be chosen among these:

$$(I) \quad \begin{cases} u(0, t) = p(t), \\ u(1, t) = q(t), \end{cases}$$

$$(II) \quad \begin{cases} u_x(0, t) + k(u(0, t)) = F(t), \\ u_x(1, t) + k(u(1, t)) = N(t), \end{cases}$$

$$(III) \quad \begin{cases} u(0, t) = p(t), \\ u_x(1, t) + k(u(1, t)) = N(t). \end{cases}$$

We shall consider problems (I), (II) and (III) in the domain  $D_T = (0, 1) \times (0, T)$ .

For such problems existence and uniqueness of a solution have been proved (see [10]–[20], for instance). In particular, we can state the existence of a solution  $u$  such that

$$u \in L^2(0, T, H^2(0, 1))$$

and the following estimates hold true

$$(2.5) \quad \|u\|_{\infty} \leq M,$$

$$(2.6) \quad \|u_x\|_{\infty} \leq M_1,$$

$$(2.7) \quad \|u_{xx}\|_{L^2(D_T)} \leq M_2,$$

where the constants depend on the initial and boundary data and coefficients.

**REMARK 2.3.** *Such results imply, in turn, that the function  $\sigma$  is Hölder continuous (see [15] and [20]). Moreover, we note that (2.5) holds true also if we consider problems in an  $n$ -dimension spatial domain, while (2.6)–(2.7) in general are valid only for the 1-D case. We confine our analysis to the latter case.*

**REMARK 2.4.** *It is important to recall that estimate (2.7) does not imply, in general, a similar regularity on  $u_r$ . This is true only if the equation (2.3) is uniformly parabolic, namely only in case 2 of condition (F.2).*

**REMARK 2.5.** *Since a priori estimates like (2.5) ensure that  $\theta$  has a positive lower bound, condition (2.1) in assumption (H1) entails,*

$$\inf_{\theta \in [0, \theta_{\max}]} \mathcal{K}(\theta) = K_{\min} > 0. \quad \square$$

Moreover, we recall the following results

(R.1) In case of problem (III), we may have  $u(x, t) < 0$  in  $D_T$ , for suitable  $N(t) \geq 0$ . We remark, however, that boundary condition (III) should be replaced by a unilateral boundary condition (see [10] and [13] for details).

(R.2) In case of problems (I) and (II), a *saturation region* may appear. In such a case (see [14]–[19]) the following sets could be defined

$$\begin{aligned} \mathcal{D} &= \{(x, t) \in D_T : \sigma(x, t) < 0\} = \{(x, t) \in D_T : u(x, t) < 0\}, \\ \mathcal{P} &= \{(x, t) \in D_T : \sigma(x, t) = 0\} = \{(x, t) \in D_T : u(x, t) \geq 0\}, \end{aligned}$$

corresponding to the *unsaturated* and *saturated region*, respectively, and the interfaces separating the regions can be proved to be Lipschitz continuous.

Hereafter we give three examples concerning the water infiltration through the subsurface that can be described by the problem (I), (II) or (III).

*Example 1: vertical flow through the vadose zone* (see also [10]).

In this case  $x = 0$  represents the so-called water table and  $x = 1$  the ground surface. Problem (I) with  $p(t) = 0$  and  $q(t) \geq 0$  models water infiltration through the unsaturated zone in case of prescribed water pressure at the ground surface. When  $q(t) > 0$  a saturated region appears.

Problem (III) with  $p(t) = 0$  and  $N(t) \geq 0$  describes the same phenomenon in case of flux condition on the ground surface.

*Example 2: vadose zone and phreatic aquifer.*

Such a scenario can be modeled by setting  $x = 0$  at the impervious layer confining the bottom of the aquifer, while  $x = 1$  still represents the ground surface. Possible boundary conditions are the ones of problem (II) with  $F(t) = 0$  (no flux condition on the impervious layer) and  $N(t) \geq 0$  (rain flux condition).

*Example 3: vadose zone and phreatic aquifer in case of evaporation.*

Such a case has been studied in [12]. As before,  $x = 0$  represents the impervious layer confining the bottom of the aquifer and  $x = 1$  the ground surface. Boundary condition of type (II) are used, with  $F(t) = 0$  (no flux condition) and  $N(t) = q = \text{const.}$  where  $q \leq 0$  is the evaporation rate. Actually, as pointed out in [12], the evaporation rate could not be prescribed, since, in general,  $q$  depends on  $u(1, t)$  as well as other physical parameters (e.g. temperature, wind velocity, relative humidity, etc.). However, in some case, e.g. soil surface close to saturation, we may assume a constant evaporation rate.  $\square$

### 3. STABILITY RESULTS

In this section we give an estimate (in the  $L^2$ -norm) for the difference of saturation profiles and the conductivity curves.

Let us take two pairs of constitutive functions characterising the medium, i.e.  $\{\theta_1(\psi); \mathcal{K}_1(\theta)\}$  and  $\{\theta_2(\psi); \mathcal{K}_2(\theta)\}$ , and assume

$$\begin{aligned} \|\theta_1 - \theta_2\|_{L^\infty_{\text{loc}}(\mathbb{R})} &< \varepsilon, \\ \|\mathcal{K}_1 - \mathcal{K}_2\|_{L^\infty([0, \theta_{\max}])} &< \varepsilon. \end{aligned}$$

where  $\varepsilon > 0$  is a constant.

The corresponding  $\{\sigma_1; k_1\}$  and  $\{\sigma_2; k_2\}$  satisfy

$$(3.1) \quad \|\sigma_1 - \sigma_2\|_{L^\infty([-M, M])} \leq \mathcal{C}_\sigma \varepsilon,$$

and

$$(3.2) \quad \|k_1 - k_2\|_{L^\infty([-M, M])} \leq \mathcal{C}_k \varepsilon,$$

where  $\mathcal{C}_\sigma$  and  $\mathcal{C}_k$  are constant depending on  $K_{\min}$ ,  $\mathcal{L}_\theta$  and  $\mathcal{L}_\mathcal{K}$ .

We introduce also the following additional assumption

(F.4) There exists a constant  $N_1 > 0$  such that for all  $w \in [-M, M]$ ,  $v \in [-M, 0]$ , we have

$$\sup_{(v \in [-M, 0]; w \in [-M, M])} \left\{ \frac{|\sigma_2(v) - \sigma_1(v)|}{|\sigma_1(w) - \sigma_1(v)|} |w - v| \right\} \leq N_1.$$

**REMARK 3.1.** *Even if condition (F.4) may seem too artificial, such a requirement is physically reasonable. In particular, it is always fulfilled in case  $\sigma_1$  does not degenerate at 0. In general, (F.4) holds true provided that slight assumptions on the mutual relationship between  $\sigma_1$  and  $\sigma_2$  are satisfied. Details on this point are given in Appendix A.*  $\square$

Now, if  $u_1$  and  $u_2$  are the corresponding solutions to problem (I), or (II) or (III), with the same initial condition and boundary data, we want to estimate  $\|\sigma_1(u_1) - \sigma_2(u_2)\|_{L^2}$  in terms of  $\varepsilon$ .

First we recall a result corresponding to Lemma 1 of [21].

**LEMMA 3.1.** *If assumptions (H.1)–(H.6) and (F.1)–(F.3) are fulfilled, then there exists a constant  $F_0 > 0$  such that*

$$(3.3) \quad [k_i(s_1) - k_i(s_2)]^2 \leq F_0[\sigma_i(s_1) - \sigma_i(s_2)](s_1 - s_2),$$

for any  $s_1, s_2 \in \mathbb{R}$  and with  $i = 1$  or  $2$ .  $\square$

The main result in the paper is the following

**THEOREM 3.1.** *If assumptions (H.1)–(H.6) and (F.1)–(F.4) are fulfilled then*

$$(3.4) \quad \left( \int_0^T \int_0^1 |\sigma_1(u_1(x, t)) - \sigma_2(u_2(x, t))|^2 dx dt \right)^{1/2} \leq C\varepsilon.$$

with  $C$  constant depending on  $\mathcal{L}_\sigma, \mathcal{C}_\sigma, \mathcal{C}_k, M$ , and  $T$ .

**PROOF.** The proof is based on the approach used in [6].

We prove the assertion in case a Dirichlet problem (I) is considered. Slight changes of the proof are necessary to deal with other cases (see Remark 3.2).

The weak form of equation (2.3) reads as

$$(3.5) \quad \iint_{D_T} \{\sigma(u)\phi_t - [u_x + k(u)]\phi_x\} dx dt = \int_0^1 \phi(x, 0)v_0(x) dx,$$

A different form of the expression (3.5) is the following

$$(3.6) \quad \iint_{D_T} \{\sigma(u)\phi_t + u\phi_{xx} - k(u)\phi_x\} dx dt = \int_0^1 \phi(x, 0)v_0(x) dx, \\ \forall \phi \in C^1(\overline{D_T}) \cap L^2(0, T; H_0^2(0, 1)),$$

which is obtained from (3.5) by noting that  $u_x\phi_x = (u\phi_x)_x - u\phi_{xx}$  and  $(u\phi_x) = 0$ , for  $x = 0, 1$ . Now, considering two solutions  $u_1, u_2$  and subtracting the equations corresponding to (3.6), we get

$$(3.7) \quad \iint_{D_T} \{[\sigma_1(u_1) - \sigma_2(u_2)]\phi_t + [u_1 - u_2]\phi_{xx} - [k_1(u_1) - k_2(u_2)]\phi_x\} dx dt = 0,$$

Moreover, adding and subtracting  $\sigma_1(u_2)\phi_t$  and  $k_1(u_2)\phi_x$  in (3.7), we have

$$(3.8) \quad \begin{aligned} & \iint_{D_T} \{[\sigma_1(u_1) - \sigma_1(u_2)]\phi_t + [u_1 - u_2]\phi_{xx} - [k_1(u_1) - k_1(u_2)]\phi_x\} dx dt \\ &= \iint_{D_T} \{[\sigma_2(u_2) - \sigma_1(u_2)]\phi_t + [k_1(u_2) - k_2(u_2)]\phi_x\} dx dt \end{aligned}$$

Let us define

$$A(x, t) = \begin{cases} \frac{\sigma_1(u_1(x, t)) - \sigma_1(u_2(x, t))}{u_1 - u_2}, & \text{if } u_1(x, t) \neq u_2(x, t), \\ 0, & \text{otherwise.} \end{cases}$$

and

$$B(x, t) = \begin{cases} \frac{k_1(u_1(x, t)) - k_1(u_2(x, t))}{u_1 - u_2}, & \text{if } u_1(x, t) \neq u_2(x, t), \\ 0, & \text{otherwise.} \end{cases}$$

which, in general, are non continuous functions. Although, thanks to properties (H.1) and (F.1) we have,

$$0 \leq A(x, t) \leq \mathcal{L}_\sigma, \quad 0 \leq B(x, t) \leq \mathcal{L}_k, \quad \forall (x, t) \in D_T.$$

Now, we rewrite (3.8) as

$$(3.9) \quad \begin{aligned} & \iint_{D_T} (u_1 - u_2)[A(x, t)\phi_t + \phi_{xx} - B(x, t)\phi_x] dx dt \\ &= \iint_{D_T} [\sigma_2(u_2) - \sigma_1(u_2)]\phi_t dx dt + \iint_{D_T} [k_1(u_2) - k_2(u_2)]\phi_x dx dt \end{aligned}$$

Let us consider sequences of functions  $\{\hat{A}_n\} \in C^\infty(\bar{D}_T)$ ,  $\{\hat{B}_n\} \in C^\infty(\bar{D}_T)$ , such that

$$\begin{aligned} 0 \leq \hat{A}_n(x, t) \leq \mathcal{L}_\sigma, \quad \|\hat{A}_n - A\|_{L^2(D_T)} &\leq \frac{1}{n} \text{ as } n \rightarrow \infty \\ 0 \leq \hat{B}_n(x, t) \leq \mathcal{L}_k, \quad \|\hat{B}_n - B\|_{L^2(D_T)} &\leq \frac{1}{n} \text{ as } n \rightarrow \infty \end{aligned}$$

and set

$$A_n = \hat{A}_n + \frac{1}{n}, \quad B_n = \hat{B}_n + \frac{1}{n},$$



so that

$$(3.10) \quad 0 < A_n(x, t) \leq \mathcal{L}_\sigma + 1, \quad \|A_n - A\|_{L^2(D_T)} \rightarrow 0 \text{ as } n \rightarrow \infty,$$

$$(3.11) \quad \left\| \frac{A}{A_n} \right\|_{L^2(D_T)} \text{ is bounded,}$$

$$(3.12) \quad 0 < B_n(x, t) \leq \mathcal{L}_k + 1, \quad \|B_n - B\|_{L^2(D_T)} \rightarrow 0 \text{ as } n \rightarrow \infty,$$

$$(3.13) \quad \frac{B_n^2(x, t)}{|A_n(x, t)|} \text{ is bounded (because of Lemma 3.1).}$$

Moreover, consider a sequence  $\{z_n\} \in C^\infty(\bar{D}_T)$  such that

$$(3.14) \quad \|z_n\|_{L^2(D_T)} \rightarrow \|u_1 - u_2\|_{L^2(D_T)} \text{ as } n \rightarrow \infty.$$

Here and in the sequel  $C_j$ , ( $j = 1, 2, \dots$ ), denotes any constant not dependent on  $n$ .

Now, we look at the following (backward) parabolic problem

$$(3.15) \quad A_n \phi_{n,t} + \phi_{n,xx} - B_n \phi_{n,x} = A_n z_n,$$

$$(3.16) \quad \phi_n(x, T) = 0,$$

$$(3.17) \quad \phi_n(0, t) = 0 = \phi_n(1, t).$$

Problem (3.15)–(3.17) has a unique solution  $\phi_n \in C^{2,1}(\bar{D}_T)$  (see [22], for instance).

**REMARK 3.2.** *Since we are considering a Dirichlet problem, we impose conditions (3.17) so that  $\phi_n$  may be used later on as test function in the weak form of the equation. Due to the regularity of  $\phi_n$ , such conditions imply that  $\phi_{n,t}(1, t) = 0 = \phi_{n,t}(0, t)$  so that*

$$(3.18) \quad \phi_{n,t}(1, t) \phi_{n,x}(1, t) = 0 = \phi_{n,t}(0, t) \phi_{n,x}(0, t),$$

which is a property used in the proof (see below). Although, condition (3.18) is satisfied also in case problem (II) or (III) are considered. As a matter of fact, in such cases instead of (3.17) one should set  $\phi_{n,x}(0, t) = 0 = \phi_{n,x}(1, t)$  or  $\phi_n(0, t) = 0 = \phi_{n,x}(1, t)$ , respectively. In any case property (3.18) is still fulfilled and thus the remaining part of the proof can be applied also to problems (II) and (III).  $\square$

Let us consider  $t_1 \in [0, T)$  and multiply by  $\phi_{n,t}$  both sides of (3.15). Integrating the resulting equation over  $D_{t_1, T} = (0, 1) \times (t_1, T)$  taking into account conditions (3.16)–(3.17), we obtain

$$\begin{aligned} & \iint_{D_{t_1, T}} A_n \phi_{n,t}^2 dx dt + \frac{1}{2} \int_0^1 \phi_{n,x}^2(x, t_1) dx - \iint_{D_{t_1, T}} B_n \phi_{n,x} \phi_{n,t} dx dt \\ & = \iint_{D_{t_1, T}} z_n A_n \phi_{n,t} dx dt, \end{aligned}$$

and so,

$$\begin{aligned}
(3.19) \quad & \iint_{D_{t_1, T}} A_n \phi_{n,t}^2 dx dt + \frac{1}{2} \int_0^1 \phi_{n,x}^2(x, t_1) dx \\
& \leq \iint_{D_{t_1, T}} |B_n| |\phi_{n,x}| |\phi_{n,t}| dx dt + \iint_{D_{t_1, T}} z_n A_n \phi_{n,t} dx dt \\
& \leq \iint_{D_{t_1, T}} \frac{|B_n|}{A_n^{1/2}} A_n^{1/2} |\phi_{n,x}| |\phi_{n,t}| dx dt + \iint_{D_{t_1, T}} z_n A_n \phi_{n,t} dx dt \\
& \leq \frac{1}{4\delta} \iint_{D_{t_1, T}} \frac{B_n^2}{A_n} (A_n \phi_{n,t}^2) dx dt + \delta \iint_{D_{t_1, T}} |\phi_{n,x}|^2 dx dt \\
& \quad + \frac{1}{4\delta} \iint_{D_{t_1, T}} A_n \phi_{n,t}^2 dx dt + \delta \iint_{D_{t_1, T}} A_n |z_n|^2 dx dt,
\end{aligned}$$

where the Cauchy's inequality has been used and  $\delta$  is a positive constant, to be specified later.

Then, recalling (3.13) and choosing

$$(3.20) \quad \delta = \frac{1}{2} [1 + \sup(B_n^2/A_n)],$$

from (3.19) we get  $\forall t_1 \in [0, T)$ ,

$$\begin{aligned}
(3.21) \quad & \frac{1}{2} \iint_{D_{t_1, T}} A_n \phi_{n,t}^2 dx dt + \frac{1}{2} \int_0^1 \phi_{n,x}^2(x, t_1) dx \\
& \leq \delta \iint_{D_{t_1, T}} \phi_{n,x}^2 dx dt + \delta \|A_n^{1/2} z_n\|_{L^2(D_T)}^2.
\end{aligned}$$

In particular, considering the continuous function

$$f(t_1) := \int_0^1 \phi_{n,x}^2(x, t_1) dx,$$

we have

$$(3.22) \quad \forall t_1 \in [0, T), \quad f(t_1) \leq 2\delta \int_{t_1}^T f(t) dt + 2\delta \|A_n^{1/2} z_n\|_{L^2(D_T)}^2.$$

Now, we apply a Gronwall type argument (see Appendix B), obtaining

$$(3.23) \quad \forall t_1 \in [0, T), \quad f(t_1) \leq 2\delta \|A_n^{1/2} z_n\|_{L^2(D_T)}^2 \exp(2\delta T),$$

where  $\delta$  given by (3.20). We can exploit (3.23) to get the following estimate

$$(3.24) \quad \|\phi_{n,x}\|_{L^2(D_T)}^2 \leq C_1 \|A_n^{1/2} z_n\|_{L^2(D_T)}^2.$$

Now, since expression (3.21) holds true for any  $t_1 \in [0, T)$ , we can consider it with  $t_1 = 0$  and use estimate (3.24), obtaining

$$(3.25) \quad \|A_n^{1/2} \phi_{n,t}\|_{L^2(D_T)}^2 \leq C_2 \|A_n^{1/2} z_n\|_{L^2(D_T)}^2.$$

Let us consider  $\phi_n$  as test function in expression (3.8). Adding and subtracting the appropriate terms, we obtain

$$(3.26) \quad \begin{aligned} \iint_{D_T} (u_1 - u_2) A_n z_n \, dx \, dt &= \iint_{D_T} [\sigma_2(u_2) - \sigma_1(u_2)] \phi_{n,t} \, dx \, dt \\ &\quad + \iint_{D_T} [k_1(u_2) - k_2(u_2)] \phi_{n,x} \, dx \, dt \\ &\quad + \iint_{D_T} (u_1 - u_2) [A_n - A] \phi_{n,t} \, dx \, dt \\ &\quad + dx \, dt \iint_{D_T} (u_1 - u_2) [B - B_n] \phi_{n,x} \, dx \, dt \\ &= \mathcal{I}_{1,n} + \mathcal{I}_{2,n} + \mathcal{I}_{3,n} + \mathcal{I}_{4,n} \end{aligned}$$

Before to estimate  $\mathcal{I}_{1,n}$ , we note that in the region  $\{(x, t) \in D_T : u_2(x, t) \geq 0\}$  we have  $\mathcal{I}_{1,n} = 0$ , so that we can confine ourselves to the region

$$\tilde{D}_T = \{(x, t) \in D_T : u_2(x, t) < 0\}.$$

Then, to estimate  $\mathcal{I}_{1,n}$  we use Cauchy's inequality along with conditions (3.1), (F.4), (3.11) and estimate (3.25), i.e.

$$(3.27) \quad \begin{aligned} \mathcal{I}_{1,n} &\leq \iint_{\tilde{D}_T} \frac{[\sigma_2(u_2) - \sigma_1(u_2)]}{A_n^{1/2}} A_n^{1/2} |\phi_{n,t}| \, dx \, dt \\ &\leq \hat{\delta} \iint_{\tilde{D}_T} \frac{|\sigma_2(u_2) - \sigma_1(u_2)|^2}{A_n} \, dx \, dt + \frac{1}{4\hat{\delta}} \iint_{\tilde{D}_T} A_n \phi_{n,t}^2 \, dx \, dt \\ &\leq \hat{\delta} \mathcal{C}_{\sigma} \varepsilon \iint_{\tilde{D}_T} \frac{|\sigma_2(u_2) - \sigma_1(u_2)|}{A} \frac{A}{A_n} \, dx \, dt + \frac{C_2}{4\hat{\delta}} \|A_n^{1/2} z_n\|_{L^2(D_T)}^2 \\ &\leq C_3 \hat{\delta} \varepsilon + \frac{C_2}{4\hat{\delta}} \|A_n^{1/2} z_n\|_{L^2(D_T)}^2, \end{aligned}$$

where  $\hat{\delta}$  is a positive constant to be defined later. For what  $\mathcal{I}_{2,n}$  is concerned, recalling assumption (3.2) and estimate (3.24), we apply again Cauchy's inequality with the constant  $\hat{\delta}$  and get

$$(3.28) \quad \begin{aligned} \mathcal{I}_{2,n} &\leq \iint_{D_T} |k_2(u_2) - k_1(u_2)| |\phi_{n,x}| \, dx \, dt \\ &\leq \hat{\delta} \iint_{D_T} |k_2(u_2) - k_1(u_2)|^2 \, dx \, dt + \frac{1}{4\hat{\delta}} \iint_{D_T} |\phi_{n,x}|^2 \, dx \, dt \\ &\leq C_4 \hat{\delta} \varepsilon^2 + \frac{C_1}{4\hat{\delta}} \|A_n^{1/2} z_n\|_{L^2(D_T)}^2, \end{aligned}$$

Then, using again property (3.11), estimate (3.22) along with property (2.5) on  $|u_i(x, t)|$ , ( $i = 1, 2$ ), we apply Cauchy's and Hölder's inequality to obtain the following estimate

$$\begin{aligned}
 (3.29) \quad \mathcal{I}_{3,n} &\leq \iint_{D_T} |u_1 - u_2| \frac{|A_n - A|}{A_n^{1/2}} A_n^{1/2} \phi_{n,t} dx dt \\
 &\leq M^2 \hat{\delta} \iint_{D_T} \frac{|A_n - A|^2}{A_n} dx dt + \frac{C_8}{4\hat{\delta}} \|A_n^{1/2} z_n\|_{L^2(D_T)}^2 \\
 &\leq M^2 \hat{\delta} \left\| 1 - \frac{A}{A_n} \right\|_{L^2(D_T)} \|A_n - A\|_{L^2(D_T)} + \frac{C_2}{4\hat{\delta}} \|A_n^{1/2} z_n\|_{L^2(D_T)}^2 \\
 &\leq C_5 \hat{\delta} \|A_n - A\|_{L^2(D_T)} + \frac{C_2}{4\hat{\delta}} \|A_n^{1/2} z_n\|_{L^2(D_T)}^2.
 \end{aligned}$$

Finally,

$$\begin{aligned}
 (3.30) \quad \mathcal{I}_{4,n} &\leq \hat{\delta} \iint_{D_T} |u_1 - u_2|^2 |B_n - B|^{1/2} dx dt + \frac{1}{4\hat{\delta}} \iint_{D_T} |\phi_{n,x}|^2 dx dt \\
 &\leq C_6 \hat{\delta} \|B_n - B\|_{L^2(D_T)} + \frac{C_1}{4\hat{\delta}} \|A_n^{1/2} z_n\|_{L^2(D_T)}^2.
 \end{aligned}$$

Let us exploit estimates (3.27)–(3.30) into expression (3.26), i.e.

$$\begin{aligned}
 (3.31) \quad \iint_{D_T} (u_1 - u_2) A_n z_n dx dt &\leq C_3 \hat{\delta} \varepsilon + \frac{C_2}{4\hat{\delta}} \|A_n^{1/2} z_n\|_{L^2(D_T)}^2 \\
 &\quad + C_4 \hat{\delta} \varepsilon^2 + \frac{C_1}{4\hat{\delta}} \|A_n^{1/2} z_n\|_{L^2(D_T)}^2 \\
 &\quad + C_5 \hat{\delta} \|A_n - A\|_{L^2(D_T)} + \frac{C_2}{4\hat{\delta}} \|A_n^{1/2} z_n\|_{L^2(D_T)}^2 \\
 &\quad + C_6 \hat{\delta} \|B_n - B\|_{L^2(D_T)} + \frac{C_1}{4\hat{\delta}} \|A_n^{1/2} z_n\|_{L^2(D_T)}^2,
 \end{aligned}$$

so that, choosing  $\hat{\delta} = (C_1 + C_2)$  we have

$$\begin{aligned}
 (3.32) \quad \iint_{D_T} (u_1 - u_2) A_n z_n dx dt &\leq C_3 \hat{\delta} \varepsilon + C_4 \hat{\delta} \varepsilon^2 + C_5 \hat{\delta} \|A_n - A\|_{L^2(D_T)} \\
 &\quad + C_6 \hat{\delta} \|B_n - B\|_{L^2(D_T)} + \frac{1}{2} \|A_n^{1/2} z_n\|_{L^2(D_T)}^2 \\
 &\leq C_7 \varepsilon + Q(n) + \frac{1}{2} \|A_n^{1/2} z_n\|_{L^2(D_T)}^2,
 \end{aligned}$$

where

$$Q(n) = \hat{\delta} [C_5 \hat{\delta} \|A_n - A\|_{L^2(D_T)} + C_6 \|B_n - B\|_{L^2(D_T)}] \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Finally, passing to the limit in (3.32) with  $n \rightarrow \infty$  we obtain

$$(3.33) \quad \iint_{D_T} (u_1 - u_2)[\sigma_1(u_1) - \sigma_1(u_2)] dx dt \leq C_8 \varepsilon.$$

We note that, because of the monotonicity of  $\sigma_1$ ,

$$[\sigma_1(u_1) - \sigma_1(u_2)](u_1 - u_2) \geq 0.$$

and, recalling assumption (F.1), also the following inequality holds true

$$\frac{1}{\mathcal{L}_\sigma} \iint_{D_T} |\sigma_1(u_1) - \sigma_1(u_2)|^2 dx dt \leq \iint_{D_T} (u_1 - u_2)[\sigma_1(u_1) - \sigma_1(u_2)] dx dt.$$

Exploiting these facts into (3.33), we find

$$(3.34) \quad \iint_{D_T} |\sigma_1(u_1) - \sigma_1(u_2)|^2 dx dt \leq C_9 \varepsilon.$$

The desired estimate easily follows from (3.34). As a matter of fact, it is sufficient to note that

$$|\sigma_1(u_1) - \sigma_2(u_2)| \leq |\sigma_1(u_1) - \sigma_1(u_2)| + |\sigma_2(u_2) - \sigma_1(u_2)|$$

and use Cauchy's inequality to get

$$\begin{aligned} & \iint_{D_T} |\sigma_1(u_1) - \sigma_2(u_2)|^2 dx dt \\ & \leq 2 \left\{ \iint_{D_T} |\sigma_1(u_1) - \sigma_1(u_2)|^2 dx dt + \iint_{D_T} |\sigma_2(u_2) - \sigma_1(u_2)|^2 dx dt \right\}. \end{aligned}$$

Finally, to the r.h.s. we apply estimate (3.33) along with assumption (3.1) and the proof is complete.  $\square$

**REMARK 3.3.** Notice that for problems such that a saturation region never appears (i.e. when it is possible to prove  $u_i(x, t) < 0$  in  $D_T$ ), estimate (3.33) entails an  $L^2$  estimate for  $(u_1 - u_2)$ , since in such cases the curve  $\sigma_1$  is invertible in the whole domain  $D_T$ .

**REMARK 3.4.** In case of no gravity, proving Theorem 3.1 becomes simpler. We report the proof of this particular case in Appendix C.  $\square$

**COROLLARY 3.1.** If assumptions (H.1)–(H.6) and (F.1)–(F.4) are fulfilled then

$$(3.35) \quad \iint_{D_T} |k_1(u_1(x, t)) - k_2(u_2(x, t))|^2 dx dt \leq N_1 \varepsilon.$$

with  $N_1$  constant depending on  $\mathcal{L}_\sigma$ ,  $\mathcal{C}_\sigma$ ,  $\mathcal{C}_k$ ,  $F_0$ ,  $M$ ,  $M_1$  and  $T$ .

PROOF. As above, it is sufficient to note that

$$|k_1(u_1) - k_2(u_2)| \leq |k_1(u_1) - k_1(u_2)| + |k_1(u_2) - k_2(u_2)|,$$

and use Cauchy's inequality to obtain

$$\begin{aligned} & \iint_{D_T} |k_1(u_1) - k_2(u_2)|^2 dx dt \\ & \leq 2 \iint_{D_T} |k_1(u_1) - k_1(u_2)|^2 dx dt + 2 \iint_{D_T} |k_1(u_2) - k_2(u_2)|^2 dx dt. \end{aligned}$$

To the first integral on the r.h.s. we apply Lemma 3.1 along with estimate (3.33). To treat the second integral we simply use assumption (3.2).  $\square$

REMARK 3.5. *The results found so far apply also to the original variables  $\theta$  and  $\psi$ , which are the physical ones. As a matter of fact, let us consider a generalized solution  $u \in L^2(0, T; H^2(0, 1))$  of equation (2.3) and define  $\psi(x, t)$  in the following way,*

(3.36)  $\forall (x, t) \in D_T$ ,  $\psi(x, t)$  is such that

$$\psi(x, t) = \Gamma^{-1}(u(x, t)) \Leftrightarrow u(x, t) = \Gamma(\psi(x, t)) = \int_0^{\psi(x, t)} K(s) ds,$$

where the Kirchoff transformation  $\Gamma$  defined in Section 2 has been used.

By (3.36) the function  $\psi(x, t)$  is uniquely defined (almost everywhere) in  $D_T$ , thanks to the properties of  $K$ . Moreover, let us define

$$\begin{aligned} \theta(x, t) &= \theta(\psi(x, t)) = \theta(\Gamma^{-1}(u(x, t))) = \sigma(u(x, t)), \\ K(x, t) &= K(\psi(x, t)) = K(\Gamma^{-1}(u(x, t))) = k(u(x, t)), \end{aligned}$$

It is easy to check that  $\psi(x, t)$  satisfies

$$(3.37) \quad \iint_{D_T} [\theta(\psi)\phi_t - (K(\psi)(\psi_x + 1))\phi_x] dx dt = \int_0^1 v_0(x)\phi(x, 0) dx.$$

for any test function  $\phi \in C^1(\bar{D}_T) \cap L^2(0, T; H^2(0, 1))$  with  $\phi(0, t) = 0 = \phi(1, t)$  and  $\phi(x, T) = 0$ . Expression (3.37) is the weak form of a Dirichlet problem for equation (2.2) with initial datum  $\theta(x, 0) = v_0(x)$ .

Moreover, if  $u_i$  ( $i = 1, 2$ ) are the generalized solutions corresponding to the pair  $\{\sigma_i; k_i\}$ , we have

$$\begin{aligned} |\sigma_1(u_1(x, t)) - \sigma_2(u_2(x, t))| &= |\theta_1(\Gamma_1^{-1}(u_1(x, t))) - \theta_2(\Gamma_2^{-1}(u_2(x, t)))| \\ &= |\theta_1(\psi_1(x, t)) - \theta_2(\psi_2(x, t))| \end{aligned}$$

and therefore from Theorem 3.1 we get for  $\theta(\psi)$  an estimate of the same type. Similarly, Corollary 3.1 entails an estimate for  $K(\psi)$ .  $\square$

An interesting application of the technique used in Theorem 3.1 lies in the context of unsteady flows exhibiting a variable viscosity. Such type of problems arise from models in which the fluid viscosity is affected by physical properties of the medium (such as temperature) or by concentration of chemical species.

We give more details on this topic in Appendix D.

#### A. REMARKS ON CONDITIONS (F.4)

As stated in Remark 3.1, here we list some sufficient conditions which guarantee that property (F.4) is satisfied.

**CASE A.** First of all, we note that if  $w \in [0, M]$  then  $\sigma_1(w) \equiv \sigma_1(0) = \sigma_s$ . Moreover, for any  $v \in [-M, 0)$  we have

$$|\sigma_2(v) - \sigma_1(v)| \leq \sigma_s - \sigma_1(v),$$

so that

$$\frac{|\sigma_2(v) - \sigma_1(v)|}{\sigma_s - \sigma_1(v)} |v - w| \leq |v - w| \leq 2M,$$

namely (F.4) is satisfied.

**CASE B.** Let us confine ourselves to the case  $w \in [-M, 0)$ . If in addition  $\sigma_1$  does not degenerate, namely  $\sigma_1'(0^-) > 0$ , then (F.4) holds true. Indeed, we know that there exists  $u^* \in (w, v)$  (or, alternately  $u^* \in (v, w)$  if  $v < w$ ), such that

$$\frac{|\sigma_1(w) - \sigma_1(v)|}{|w - v|} = \sigma_1'(u^*) \geq \gamma = \min_{[-M, 0)} \sigma_1' > 0.$$

Hence,

$$\frac{|\sigma_2(v) - \sigma_1(v)|}{|\sigma_1(w) - \sigma_1(v)|} |w - v| \leq |\sigma_2(v) - \sigma_1(v)| \gamma^{-1} \leq 2\sigma_s \gamma^{-1}.$$

Of course, the same argument remains valid if the non-degenerate curve is  $\sigma_2$ . In such a case we exchange the roles of  $\sigma_1$  and  $\sigma_2$  in the proof of Theorem 3.1 so that we require that condition (F.4) is fulfilled by  $\sigma_2$  and we proceed as above.

**CASE C.** In general, the following result holds true,

**PROPOSITION A.1.** *If  $w \in [-M, 0)$  and there exist two constants  $\mu > 0$  and  $N_2 > 0$  such that  $\forall v \in [-\mu, 0)$  the following properties are satisfied*

$$(A.1) \quad \frac{\sigma_1'(v)}{[\sigma_s - \sigma_1(v)]} \geq N_2 > 0,$$

$$(A.2) \quad \sigma_2(v) \geq \sigma_1(v),$$

*then  $\sigma_1$  fulfills property (F.4).*

PROOF. Let us introduce again  $u^* \in [-M, 0)$  such that

$$\sigma'_1(u^*) = \frac{|\sigma_1(w) - \sigma_1(v)|}{|w - v|}.$$

If  $u^* \leq -\mu < 0$  then,

$$\sigma'_1(u^*) \geq \hat{\gamma} = \min_{[-M, -\mu]} \sigma'_1 > 0,$$

and we can proceed as in the non-degenerate case (see Case A).

On the other hand, if  $u^* \in (-\mu, 0)$  then assumption (A.1) is valid and so

$$(A.3) \quad \sigma'_1(u^*) \geq N_2[\sigma_s - \sigma_1(u^*)].$$

Let us assume that  $w < v \Rightarrow u^* < v$ . Hence,  $\sigma_1(u^*) < \sigma_1(v)$  and  $[\sigma_s - \sigma_1(u^*)] > [\sigma_s - \sigma_1(v)]$ , so that (A.3) yields

$$\sigma'_1(u^*) > [\sigma_s - \sigma_1(v)].$$

Therefore,

$$\frac{|\sigma_2(v) - \sigma_1(v)|}{|\sigma_1(w) - \sigma_1(v)|} |w - v| < \frac{|\sigma_s - \sigma_1(v)| + |\sigma_s - \sigma_2(v)|}{N_2[\sigma_s - \sigma_1(v)]} < \frac{1}{N_2} \left[ 1 + \frac{|\sigma_s - \sigma_2(v)|}{\sigma_s - \sigma_1(v)} \right] < \frac{2}{N_2},$$

where last inequality holds true because of  $-\mu < u^* < v$  and so  $\sigma_2(v) > \sigma_1(v)$  due to assumption (A.2).

Finally, if  $v < w$ , then  $u^* \in (v, w)$  and  $\sigma_1(u^*) < \sigma_1(w)$ , so that (A.3) implies

$$[\sigma_s - \sigma_1(u^*)] > [\sigma_s - \sigma_1(w)] \Rightarrow \sigma'_1(u^*) > N_2[\sigma_s - \sigma_1(w)].$$

Thus we have

$$\begin{aligned} & \frac{|\sigma_2(v) - \sigma_1(v)|}{|\sigma_1(w) - \sigma_1(v)|} |w - v| \\ & \leq \left\{ \frac{|\sigma_2(v) - \sigma_2(w)|}{|\sigma_1(w) - \sigma_1(v)|} |w - v| + \frac{|\sigma_2(w) - \sigma_1(w)|}{|\sigma_1(w) - \sigma_1(v)|} |w - v| + |w - v| \right\} \\ & \leq \frac{1}{N_2} \left\{ \frac{\sigma_s - \sigma_2(w)}{\sigma_s - \sigma_1(w)} + \frac{|\sigma_2(w) - \sigma_1(w)|}{\sigma_s - \sigma_1(w)} \right\} + 2M \\ & \leq \frac{1}{N_2} \left\{ \frac{\sigma_s - \sigma_2(w)}{\sigma_s - \sigma_1(w)} + 1 \right\} + 2M \leq \frac{2}{N_2} + 2M, \end{aligned}$$

where in last inequality we have used assumption (A.2) for  $\sigma_1(w)$ , being  $-\mu \leq u^* < w$ .

Therefore, in any case we are able to bound the quantity

$$\frac{|\sigma_2(v) - \sigma_1(v)|}{|\sigma_1(w) - \sigma_1(v)|} |w - v|,$$

namely (F.4) is satisfied. □



**REMARK A.1.** *From a physical point of view, assumptions (A.1), (A.2) are reasonable. As a matter of fact, basically (A.1) requires that as  $v \rightarrow 0^-$  the first derivative of  $\sigma_1$  vanishes less rapidly than  $[\sigma_s - \sigma_1(v)]$ . For instance, such a property is satisfied for any function of the type  $\sigma(v) = \sigma_s - v^p$  ( $p > 0$ ), which is a good approximation near to  $0^-$  for any function representing a saturation curve.*

*Also condition (A.2) is a nonrestrictive assumption. Indeed, one can suppose that both  $\sigma_1$  and  $\sigma_2$  degenerate at  $v = 0$  (otherwise Case B can be applied) so that these functions have to satisfy both properties:  $\sigma_1(0) = \sigma_s = \sigma_2(0)$  and  $\sigma_1'(0) = 0 = \sigma_2'(0)$ . Therefore it is reasonable to assume that (at least in a left neighborhood of 0) they are ordered, namely condition (A.2).*

## B. PROOF OF ESTIMATE (3.23)

Here we prove assertion (3.23).

In particular, let  $f(t)$  a continuous function defined on the interval  $[0, T]$ , with  $f(T) = 0$  and satisfying the integral inequality

$$\forall t \in [0, T), \quad 0 \leq f(t) \leq c_1 \int_t^T f(\tau) d\tau + c_2,$$

where  $c_1 > 0$  and  $c_2 \geq 0$  are given constants. Then,

$$\forall t \in [0, T), \quad f(t) \leq c_2 \exp(c_1 T).$$

**PROOF.** Define  $s = (T - t)$  and

$$(B.1) \quad g(s) = \int_0^s f(T - \eta) d\eta.$$

We have

$$(B.2) \quad g'(s) = f(T - s) = f(t).$$

moreover, performing the change of variable  $\tau = (T - \eta)$  into the integral of (B.1), we easily obtain the following expression

$$(B.3) \quad g(s) = - \int_T^{T-s} f(\tau) d\tau = \int_{T-s}^T f(\tau) d\tau = \int_t^T f(\tau) d\tau.$$

Therefore, (B.3) and (B.2) together with the assumption on  $f(t)$  imply that

$$(B.4) \quad g'(s) \leq c_1 g(s) + c_2.$$

Now, applying the same argument used in the well-known proof of Gronwall's lemma (differential form), we get

$$g(s) \leq \frac{c_2}{c_1} [\exp(c_1 T) - 1].$$

so that

$$f(t) \leq c_1 g(s) + c_2 \leq c_2 \exp(c_1 T),$$

giving the desired estimate on  $f(t)$ .  $\square$

### C. PROBLEMS WITHOUT THE GRAVITY TERM

Let us confine ourselves to the simpler case of equations without gravity term, i.e. flows described by

$$(C.1) \quad [\sigma(u)]_t = u_{xx},$$

instead of (2.3).

**PROPOSITION C.1.** *If all the assumptions listed above are fulfilled, then there exists a positive constant  $A$  such that*

$$(C.2) \quad \int_0^T \int_0^1 |\sigma_i(u_1(x, t)) - \sigma_i(u_2(x, t))|^2 dx dt \leq A\varepsilon.$$

where  $i = 1, 2$ .

**PROOF.** We prove the assertion for  $i = 1$ . The weak form of equation (C.1) is

$$\iint_{D_T} [\sigma(u)\phi_t - u_x\phi_x] dx dt = 0$$

for any test function<sup>2</sup>  $\phi$ . So considering  $\sigma_1$  and  $\sigma_2$ ,

$$(C.3) \quad \iint_{D_T} \{[\sigma_1(u_1) - \sigma_2(u_2)]\phi_t - [u_{1,x} - u_{2,x}]\phi_x\} dx dt = 0$$

In particular, following the technique used in [19], we can take an arbitrary  $t_1 \in (0, T]$  and select the following test function

$$(C.4) \quad \phi(x, t) = \begin{cases} \int_t^{t_1} [u_1(x, s) - u_2(x, s)] ds, & \text{if } 0 < t \leq t_1, \\ 0, & \text{if } t_1 \leq t \leq T. \end{cases}$$

Then expression (C.3) reads as

$$(C.5) \quad \iint_{D_T} [\sigma_1(u_1) - \sigma_2(u_2)](u_1 - u_2) dx dt - \int_0^T \int_0^1 \left[ (u_1 - u_2)_x \int_t^{t_1} [u_1(x, s) - u_2(x, s)]_x ds \right] dx dt = 0.$$

---

<sup>2</sup>The properties to be satisfied by  $\phi$  depend on the type of boundary condition we are dealing with.

We note that

$$(C.6) \quad \begin{aligned} & \frac{1}{2} \frac{d}{dt} \left\{ \int_t^T [(u_1 - u_2)(x, s)]_x ds \right\}^2 \\ &= -[(u_1 - u_2)(x, t)]_x \int_t^T [(u_1 - u_2)(x, s)]_x ds, \end{aligned}$$

and so, if in (C.5) we replace the term in the square brackets with (C.6) and integrate in time between 0 and  $T$ , we obtain

$$\iint_{D_T} [\sigma_1(u_1) - \sigma_2(u_2)](u_1 - u_2) dx dt + \frac{1}{2} \int_0^1 \left\{ \int_t^{t_1} [(u_1 - u_2)(x, s)]_x ds \right\}^2 dx = 0$$

which implies

$$\int_0^T \int_0^1 [\sigma_1(u_1) - \sigma_2(u_2)](u_1 - u_2) dx dt \leq 0.$$

Next, adding and subtracting the term  $\sigma_1(u_2)$  within the integral, we have

$$(C.7) \quad \begin{aligned} & \int_0^T \int_0^1 [\sigma_1(u_1) - \sigma_1(u_2)](u_1 - u_2) dx dt \\ & \leq \int_0^T \int_0^1 [\sigma_1(u_2) - \sigma_2(u_2)](u_2 - u_1) dx dt. \end{aligned}$$

Now, because of the monotonicity of  $\sigma_1$ , we have that

$$[\sigma_1(u_1) - \sigma_1(u_2)](u_1 - u_2) \geq 0,$$

for any pair  $u_1, u_2$ . This implies that the previous expression can be written also as

$$(C.8) \quad \begin{aligned} & \int_0^T \int_0^1 |\sigma_1(u_1) - \sigma_1(u_2)| |u_1 - u_2| dx dt \\ & \leq \int_0^T \int_0^1 [\sigma_1(u_2) - \sigma_2(u_2)](u_2 - u_1) dx dt. \end{aligned}$$

Exploiting (F.1), we have also

$$(C.9) \quad \begin{aligned} & \frac{1}{\mathcal{L}_\sigma} \int_0^T \int_0^1 (\sigma_1(u_1) - \sigma_1(u_2))^2 dx dt \\ & \leq \int_0^T \int_0^1 |\sigma_1(u_1) - \sigma_1(u_2)| |u_1 - u_2| dx dt, \end{aligned}$$

hence, recalling (C.7),

$$\frac{1}{\mathcal{L}_\sigma} \iint_{D_T} [\sigma_1(u_1) - \sigma_1(u_2)]^2 dx dt \leq \sqrt{T} \|\sigma_1 - \sigma_2\|_{L^2(D_T)} \|u_1 - u_2\|_\infty.$$

Finally, from properties (2.5) and (2.6) we have

$$\int_0^T \int_0^1 (\sigma_1(u_1) - \sigma_1(u_2))^2 dx dt \leq A\varepsilon. \quad \square$$

#### D. A PARTICULAR CASE: CONTINUOUS DEPENDENCE ON VISCOSITY

We consider a viscosity dependence on time and space and look at how a solution of Richards' equation is affected by this phenomenon.

Moreover, since the procedure is quite similar to the one presented in Section 3, we do not show every detail of the proofs.

Let us consider in the domain  $D_T = (0, 1) \times (0, T)$  a slight different form of equation (2.2), i.e.

$$(D.1) \quad [\theta(\psi)]_t = \left[ \frac{\mathfrak{K}(\psi)}{\mu(x, t)} (\psi_x + 1) \right]_x$$

where  $\mu$  is the fluid viscosity and  $\mathfrak{K}$  is the relative permeability of medium<sup>3</sup>. After the transformation  $G$ , defined as

$$u(x, t) = G(\psi(x, t)) = \int_0^{\psi(x, t)} \mathfrak{K}(s) ds,$$

equation (2.2) reads as

$$(D.2) \quad [\sigma(u)]_t = \left[ \frac{1}{\mu} (u_x + k(u)) \right]_x,$$

where  $k(u) = \mathfrak{K}(G^{-1}(u))$  and the function  $\sigma(u) = \theta(G^{-1}(u))$  behaves like  $\theta$ .

For  $\theta$ ,  $\mathfrak{K}$ ,  $k$  and  $\sigma$  we stipulate all the assumptions (H.1)–(H.6) and (F.1)–(F.3) made in Section 2. Consider now the Dirichlet problem given by (D.2) endowed with the following conditions

$$(D.3) \quad \sigma(u(x, 0)) = v_0(x),$$

$$(D.4) \quad u(0, t) = f(t),$$

$$(D.5) \quad u(1, t) = g(t),$$

---

<sup>3</sup>Actually, in general we refer as relative permeability to the quantity  $\rho g \mathfrak{K}(\psi)$ , being  $\rho$  and  $g$  the water density and gravity acceleration, respectively. Here we include these constants into the function  $\mathfrak{K}$  to make the notation simpler.

where  $v_0(x)$ ,  $f(t)$  and  $g(t)$  are suitable data, and assume  $\mu$  satisfies the following properties

$$(D.6) \quad \mu \in C^1(D_T), \quad 0 < \alpha \leq \mu(x, t) \leq \beta, \quad \forall (x, t) \in \bar{D}_T.$$

We give the following

**DEFINITION D.1.** *We call weak solution of problem (D.2)–(D.5) a function  $u \in L^2(0, T; H(0, 1))$  such that*

$$(D.7) \quad \iint_{D_T} \left\{ \sigma(u) \phi_t - \left[ \frac{1}{\mu} (u_x + k(u)) \right] \phi_x \right\} dx dt = \int_0^1 v_0(x) \phi(x, 0) dx,$$

for all  $\phi \in C^1(\bar{D}_T)$ .

The following existence and uniqueness result can be proved

**THEOREM D.1.** *If assumptions (D.6) and (H.1)–(H.6), (F.1)–(F.3) of Section 2 are fulfilled, then there exists a unique solution  $u$  of problem (D.2)–(D.5), in the sense of Definition D.1. Moreover, the following estimates hold true*

$$(D.8) \quad \|u\|_\infty \leq M,$$

$$(D.9) \quad \|u_x\|_\infty \leq M_1,$$

where the constants depend on the initial and boundary data and coefficients.

**PROOF (sketch).** To prove the assertion the well-known technique of parabolic regularization can be applied. One can follow the proof given for the classical problem (see [14], for instance) since the presence of the term  $(\frac{1}{\mu})$  in the elliptic part does not entail additional difficulties thanks to assumption (D.6).  $\square$

Now, let us assume there exist two functions,  $\mu_1, \mu_2$  satisfying (D.6) and that there exists a constant  $\varepsilon > 0$  such that

$$(D.10) \quad \|\mu_1 - \mu_2\|_{L^2(D_T)} \leq \varepsilon.$$

Calling  $u_1$  and  $u_2$  the weak solution corresponding to  $\mu_1$  and  $\mu_2$ , respectively, we prove the analogous of Theorem 3.1, i.e.

**THEOREM D.2.** *If assumptions (H.1)–(H.6) and (F.1)–(F.4) of Section 2 are fulfilled then*

$$(D.11) \quad \left( \int_0^T \int_0^1 |\sigma(u_1(x, t)) - \sigma(u_2(x, t))|^2 dx dt \right)^{1/2} \leq C_1 \varepsilon,$$

and

$$(D.12) \quad \left( \int_0^T \int_0^1 |k(u_1(x, t)) - k(u_2(x, t))|^2 dx dt \right)^{1/2} \leq C_2 \varepsilon.$$

where  $C_1$  and  $C_2$  are constant depending on  $\alpha, \beta, \mathcal{L}_\sigma, \mathcal{C}_\sigma, \mathcal{C}_k, M$ , and  $T$ .

PROOF (sketch). If one considers test functions

$$\phi \in C^1(\overline{D_T}) \cap L^2(0, T; H_0^2(0, 1)),$$

the weak form (D.7) can be rewritten as

$$(D.13) \quad \iint_{D_T} \left\{ \sigma(u) \phi_t + \frac{u}{\mu} \phi_{xx} - \frac{k(u)}{\mu} \phi_x \right\} dx dt = \int_0^1 \phi(x, 0) v_0(x) dx.$$

Subtracting the equations for  $u_1$  and  $u_2$  corresponding to (D.13), and adding and subtracting the appropriate terms, we get

$$(D.14) \quad \begin{aligned} & \iint_{D_T} \left\{ [\sigma(u_1) - \sigma(u_2)] \phi_t + \frac{1}{\mu_1} (u_1 - u_2) \phi_{xx} + \left( \frac{\mu_2 - \mu_1}{\mu_1 \mu_2} \right) u_2 \phi_{xx} \right\} dx dt \\ & = \iint_{D_T} \left\{ \frac{1}{\mu_1} [k(u_1) - k(u_2)] \phi_x + \frac{\mu_2 - \mu_1}{\mu_1 \mu_2} (u_1 - u_2) \right\} dx dt. \end{aligned}$$

Introducing the function

$$A(x, t) = \begin{cases} \frac{\sigma(u_1(x, t)) - \sigma(u_2(x, t))}{u_1 - u_2}, & \text{if } u_1(x, t) \neq u_2(x, t), \\ 0, & \text{otherwise.} \end{cases}$$

and

$$B(x, t) = \begin{cases} \frac{k(u_1(x, t)) - k(u_2(x, t))}{u_1 - u_2}, & \text{if } u_1(x, t) \neq u_2(x, t), \\ 0, & \text{otherwise.} \end{cases}$$

we rewrite (D.14) as

$$(D.15) \quad \begin{aligned} & \iint_{D_T} (u_1 - u_2) \left[ A \phi_t + \frac{1}{\mu_1} \phi_{xx} - \frac{1}{\mu_1} B \phi_x \right] dx dt \\ & = \iint_{D_T} \left( \frac{\mu_2 - \mu_1}{\mu_1 \mu_2} \right) (u_2 \phi_{xx} - k(u_2) \phi_x) dx dt. \end{aligned}$$

Then we select as test function the solution of the regularized backward parabolic equation and we proceed as in Theorem 3.1, getting appropriate estimates. We omit further details of the proof.  $\square$

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