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Hypersurfaces of constant curvature in hyperbolic space II

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Abstract. This is the second of a series of papers in which we investigate the problem of finding, in hyperbolic space, complete hypersurfaces of constant curvature with a prescribed asymptotic boundary at infinity for a general class of curvature functions. In this paper we focus on graphs over a domain with nonnegative mean curvature.

Keywords. Hyperbolic space, hypersurfaces of constant curvature, asymptotic boundary, fully nonlinear elliptic equations

1. Introduction

In this paper we continue our study of complete hypersurfaces in hyperbolic space $\mathbb{H}^{n+1}$ of constant curvature with a prescribed asymptotic boundary at infinity. Given $\Gamma \subset \partial_\infty \mathbb{H}^{n+1}$ and a smooth symmetric function f of n variables, we seek a complete hypersurface Σ in $\mathbb{H}^{n+1}$ satisfying

$$f(\kappa[\Sigma]) = \sigma, \tag{1.1}$$

$$\partial\Sigma = \Gamma, \tag{1.2}$$

where $\kappa[\Sigma] = (\kappa_1, \dots, \kappa_n)$ denotes the hyperbolic principal curvatures of Σ and $\sigma \in (0, 1)$ is a constant.

We will use the half-space model,

$$\mathbb{H}^{n+1} = \{(x, x_{n+1}) \in \mathbb{R}^{n+1} : x_{n+1} > 0\}$$

equipped with the hyperbolic metric

$$ds^2 = \frac{1}{x_{n+1}^2} \sum_{i=1}^{n+1} dx_i^2 \,. \tag{1.3}$$

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Thus $\partial_\infty \mathbb{H}^{n+1}$ is naturally identified with $\mathbb{R}^n = \mathbb{R}^n \times \{0\} \subset \mathbb{R}^{n+1}$ and (1.2) may be understood in the Euclidean sense.

As in our earlier work [11, 9, 5, 7], we will take $\Gamma = \partial\Omega$ where $\Omega \subset \mathbb{R}^n$ is a smooth domain and seek Σ as the graph of a function $u(x)$ over Ω, i.e.

$$\Sigma = \{(x, x_{n+1}) : x \in \Omega,\ x_{n+1} = u(x)\}.$$

Then the coordinate vector fields and upper unit normal are given by

$$X_i = e_i + u_i e_{n+1}, \qquad \mathbf{n} = u\nu = u\frac{-u_i e_i + e_{n+1}}{w},$$

where $w = \sqrt{1 + |\nabla u|^2}$. The first fundamental form g_{ij} is then given by

$$g_{ij} = \langle X_i, X_j\rangle = \frac{1}{u^2}(\delta_{ij} + u_i u_j) = \frac{g^e_{ij}}{u^2}. \tag{1.4}$$

To compute the second fundamental form h_{ij} we use

$$\Gamma^k_{ij} = \frac{1}{x_{n+1}}\{-\delta_{jk}\delta_{i\,n+1} - \delta_{ik}\delta_{j\,n+1} + \delta_{ij}\delta_{k\,n+1}\} \tag{1.5}$$

to obtain

$$\nabla_{X_i} X_j = \left(\frac{\delta_{ij}}{x_{n+1}} + u_{ij} - \frac{u_i u_j}{x_{n+1}}\right) e_{n+1} - \frac{u_j e_i + u_i e_j}{x_{n+1}}. \tag{1.6}$$

Then

$$\begin{aligned} h_{ij} = \langle \nabla_{X_i} X_j, u\nu\rangle &= \frac{1}{uw}\left(\frac{\delta_{ij}}{u} + u_{ij} - \frac{u_i u_j}{u} + 2\frac{u_i u_j}{u}\right) \\ &= \frac{1}{u^2 w}(\delta_{ij} + u_i u_j + u u_{ij}) = \frac{h^e_{ij}}{u} + \frac{g^e_{ij}}{u^2 w}. \end{aligned} \tag{1.7}$$

The hyperbolic principal curvatures κ_i of Σ are the roots of the characteristic equation

$$\det(h_{ij} - \kappa g_{ij}) = u^{-n} \det\left(h^e_{ij} - \frac{1}{u}\left(\kappa - \frac{1}{w}\right) g^e_{ij}\right) = 0.$$

Therefore,

$$\kappa_i = u\kappa^e_i + \frac{1}{w}. \tag{1.8}$$

We will present other more explicit and useful expressions for the κ_i in Section 2.

The function f is assumed to satisfy the fundamental structure conditions:

$$f_i(\lambda) \equiv \frac{\partial f(\lambda)}{\partial \lambda_i} > 0 \quad \text{in } K,\ 1 \le i \le n, \tag{1.9}$$

$$f \text{ is a concave function in } K, \tag{1.10}$$

and

$$f > 0 \quad \text{in } K, \qquad f = 0 \quad \text{on } \partial K, \tag{1.11}$$

where $K \subset \mathbb{R}^n$ is an open symmetric convex cone such that

$$K_n^+ := \{\lambda \in \mathbb{R}^n : \text{each component } \lambda_i > 0\} \subset K. \tag{1.12}$$

In addition, we shall assume that f is normalized,

$$f(1, \ldots, 1) = 1 \tag{1.13}$$

and

$$f \text{ is homogeneous of degree one.} \tag{1.14}$$

Since f is symmetric, by (1.10), (1.13) and (1.14) we have

$$f(\lambda) \leq f(\mathbf{1}) + \sum f_i(\mathbf{1})(\lambda_i - 1) = \sum f_i(\mathbf{1})\lambda_i = \frac{1}{n}\sum \lambda_i \quad \text{in } K \tag{1.15}$$

and

$$\sum f_i(\lambda) = f(\lambda) + \sum f_i(\lambda)(1 - \lambda_i) \geq f(\mathbf{1}) = 1 \quad \text{in } K. \tag{1.16}$$

Lemma 1.1. *Suppose f satisfies* (1.9)–(1.14). *Then*

$$\sum_{i \neq r} f_i \lambda_i^2 \geq \frac{1}{n-1}(2f|\lambda_r| + f_r \lambda_r^2) \quad \textit{if } \lambda_r < 0 \tag{1.17}$$

and so

$$\sum_{i \neq r} f_i \lambda_i^2 \geq \frac{1}{n} \sum f_i \lambda_i^2 \quad \textit{if } \lambda_r < 0. \tag{1.18}$$

Proof. Suppose $\lambda_r < 0$ and order the eigenvalues with $\lambda_1 > 0$ the largest and $\lambda_n < 0$ the smallest. Then as a consequence of the concavity condition (1.10) we have

$$f_n \geq f_i \quad \text{for all } i \text{ and so} \quad f_n \lambda_n^2 \geq f_r \lambda_r^2. \tag{1.19}$$

By (1.14),

$$\sum_{i \neq n} f_i \lambda_i = f + f_n |\lambda_n|.$$

By the Schwarz inequality and (1.19),

$$f^2 + 2ff_n|\lambda_n| + f_n^2 \lambda_n^2 \leq \sum_{i \neq n} f_i \sum_{i \neq n} f_i \lambda_i^2 \leq (n-1) f_n \sum_{i \neq n} f_i \lambda_i^2.$$

Therefore,

$$\sum_{i \neq n} f_i \lambda_i^2 \geq \frac{1}{n-1}(2f|\lambda_n| + f_n \lambda_n^2).$$

Using (1.19) this implies

$$\sum_{i \neq r} f_i \lambda_i^2 \geq \sum_{i \neq n} f_i \lambda_i^2 \geq \frac{1}{n-1}(2f|\lambda_n| + f_n \lambda_n^2) \geq \frac{1}{n-1}(2f|\lambda_r| + f_r \lambda_r^2), \tag{1.20}$$

completing the proof. □

All of the above assumptions (1.9)–(1.14) are fairly standard. In the present work, the following more technical assumption is important:

$$\lim_{R\to\infty} f(\lambda_1,\dots,\lambda_{n-1},\lambda_n+R) \geq 1+\varepsilon_0 \quad \text{uniformly in } B_{\delta_0}(\mathbf{1}) \tag{1.21}$$

for some fixed $\varepsilon_0 > 0$ and $\delta_0 > 0$, where $B_{\delta_0}(\mathbf{1})$ is the ball of radius δ_0 centered at $\mathbf{1} = (1,\dots,1) \in \mathbb{R}^n$.

The assumption (1.21) is fairly mild. For $f = H_k^{1/k}$ corresponding to the "higher order mean curvatures", where H_k is the k-th normalized elementary function,

$$\lim_{R\to\infty} f(\mathbf{1}+O(\varepsilon)+Re_n) = \infty$$

while for $f = (H_{k,l})^{1/(k-l)} = (H_k/H_l)^{1/(k-l)}$, $k > l$, the class of curvature quotients,

$$\lim_{R\to\infty} f(\mathbf{1}+O(\varepsilon)+Re_n) = (1+O(\varepsilon))(k/l)^{1/(k-l)}.$$

Problem (1.1)–(1.2) reduces to the Dirichlet problem for a fully nonlinear second order equation which we shall write in the form

$$G(D^2u, Du, u) = \sigma, \quad u > 0 \quad \text{in } \Omega \subset \mathbb{R}^n \tag{1.22}$$

with the boundary condition

$$u = 0 \quad \text{on } \partial\Omega. \tag{1.23}$$

The exact formula for G will be given in Section 2.

We seek solutions of the Dirichlet problem (1.22)–(1.23) satisfying $\kappa[u] \equiv \kappa[\mathrm{graph}(u)] \in K$. Following the literature we define the class of *admissible* functions

$$\mathcal{A}(\Omega) = \{u \in C^2(\Omega) : \kappa[u] \in K\}.$$

Our main result may be stated as follows.

Theorem 1.2. *Let $\Gamma = \partial\Omega \times \{0\} \subset \mathbb{R}^{n+1}$ where Ω is a bounded smooth domain in $\mathbb{R}^n$. Suppose that the Euclidean mean curvature $\mathcal{H}_{\partial\Omega}$ is nonnegative and $\sigma \in (0,1)$ satisfies $\sigma > \sigma_0$, where σ_0 is the unique zero in $(0,1)$ of*

$$\phi(a) := \frac{8}{3}a + \frac{22}{27}a^3 - \frac{5}{27}(a^2+3)^{3/2}. \tag{1.24}$$

(Numerical calculations show $0.3703 < \sigma_0 < 0.3704$.) Under conditions (1.9)–(1.14) and (1.21), there exists a complete hypersurface Σ in $\mathbb{H}^{n+1}$ satisfying (1.1)–(1.2) with uniformly bounded principal curvatures

$$|\kappa[\Sigma]| \leq C \quad \text{on } \Sigma. \tag{1.25}$$

Moreover, Σ is the graph of a unique admissible solution $u \in C^\infty(\Omega) \cap C^1(\bar{\Omega})$ of the Dirichlet problem (1.22)–(1.23). Furthermore, $u^2 \in C^\infty(\Omega) \cap C^{1,1}(\overline{\Omega})$ and

$$\begin{aligned} &\sqrt{1+|Du|^2} \leq 1/\sigma, \quad u|D^2u| \leq C \quad \text{in } \Omega, \\ &\sqrt{1+|Du|^2} = 1/\sigma \quad \text{on } \partial\Omega. \end{aligned} \tag{1.26}$$

Theorem 1.2 holds for a large family of functions f such as

$$f = N^{-1} \sum_{l=1}^{N} (f_{l1} \cdots f_{lN_l})^{1/N_l}$$

where each f_{lk} satisfies (1.9)–(1.14) and at least one of $f_{l1}, \dots, f_{lN_l}$ (for every l) satisfies (1.21).

By [2] condition (1.9) implies that equation (1.22) is elliptic for admissible solutions. As we shall see in Section 2, equation (1.22) is degenerate where $u = 0$. It is therefore natural to approximate the boundary condition (1.23) by

$$u = \varepsilon > 0 \quad \text{on } \partial\Omega. \tag{1.27}$$

When ε is sufficiently small, the Dirichlet problem (1.22), (1.27) is solvable for all $\sigma \in (0, 1)$.

Theorem 1.3. *Let Ω be a bounded smooth domain in $\mathbb{R}^n$ with $\mathcal{H}_{\partial\Omega} \geq 0$ and suppose f satisfies* (1.9)–(1.14) *and* (1.21). *Then for any $\sigma \in (0, 1)$ and $\varepsilon > 0$ sufficiently small, there exists a unique admissible solution $u^\varepsilon \in C^\infty(\bar{\Omega})$ of the Dirichlet problem* (1.22), (1.27). *Moreover, u^ε satisfies the a priori estimates*

$$\sqrt{1 + |Du^\varepsilon|^2} \leq 1/\sigma \qquad \text{in } \Omega, \tag{1.28}$$

$$u^\varepsilon |D^2 u^\varepsilon| \leq C/\varepsilon^2 \quad \text{in } \Omega, \tag{1.29}$$

where C is independent of ε.

We shall use the continuity method to reduce the proof of Theorem 1.3 to obtaining C^2 a priori estimates for admissible solutions. This approach critically depends on the sharp global gradient estimate (1.28), which is carried out in Section 3 under the assumption $\mathcal{H}_{\partial\Omega} \geq 0$. It implies that the linearized operator of equation (1.22) is invertible for all $\varepsilon \in (0, 1]$, a crucial condition for the continuity method. The centerpiece of this paper is the boundary second derivative estimate, which we derive in Section 5. Here we make use of Lemma 1.1 and a careful analysis of the linearized operator to derive the mixed normal-tangential estimate. Again the sharp global gradient estimate (1.28) enters into the proof in an essential way. We then use assumption (1.21) to establish a pure normal second derivative estimate. In order to use Theorem 1.3 to obtain Theorem 1.2 (see the end of Section 4 for a more detailed explanation), we need a uniform (in ε) estimate for the hyperbolic principal curvatures of the graph u^ε. Therefore in Section 6 we prove a maximum principle for the maximal hyperbolic principal curvature using a method derived in our earlier paper [7]. It is here that we have had to restrict the allowable range of $\sigma \in (0, 1)$. Otherwise our approach is completely general and we expect Theorem 1.2 is valid for all $\sigma \in (0, 1)$. In Section 2 we summarize the basic information about vertical graphs and the linearized operator that we will need, and in Section 3 we review some important barrier arguments using equidistant sphere solutions.

2. Vertical graphs and the linearized operator

Suppose Σ is locally represented as the graph of a function $u \in C^2(\Omega)$, $u > 0$, in a domain $\Omega \subset \mathbb{R}^n$:

$$\Sigma = \{(x, u(x)) \in \mathbb{R}^{n+1} : x \in \Omega\}.$$

oriented by the upward (Euclidean) unit normal vector field ν to Σ:

$$\nu = \left(\frac{-Du}{w}, \frac{1}{w}\right), \qquad w = \sqrt{1 + |Du|^2}.$$

The Euclidean metric and second fundamental form of Σ are given respectively by

$$g^e_{ij} = \delta_{ij} + u_i u_j, \qquad h^e_{ij} = u_{ij}/w.$$

According to [3], the Euclidean principal curvatures $\kappa^e[\Sigma]$ are the eigenvalues of the symmetric matrix $A^e[u] = \{a^e_{ij}\}$:

$$a^e_{ij} := \frac{1}{w}\gamma^{ik} u_{kl} \gamma^{lj}, \tag{2.1}$$

where

$$\gamma^{ij} = \delta_{ij} - \frac{u_i u_j}{w(1 + w)}. \tag{2.2}$$

Note that the matrix $\{\gamma^{ij}\}$ is invertible with inverse

$$\gamma_{ij} = \delta_{ij} + \frac{u_i u_j}{1 + w} \tag{2.3}$$

which is the square root of $\{g^e_{ij}\}$, i.e., $\gamma_{ik}\gamma_{kj} = g^e_{ij}$. By (1.8) the hyperbolic principal curvatures $\kappa[u]$ of Σ are the eigenvalues of the matrix $A[u] = \{a_{ij}[u]\}$:

$$a_{ij}[u] := \frac{1}{w}(\delta_{ij} + u\gamma^{ik} u_{kl} \gamma^{lj}). \tag{2.4}$$

Let $\mathcal{S}$ be the vector space of $n \times n$ symmetric matrices and

$$\mathcal{S}_K = \{A \in \mathcal{S} : \lambda(A) \in K\},$$

where $\lambda(A) = (\lambda_1, \ldots, \lambda_n)$ denotes the eigenvalues of A. Define a function F by

$$F(A) = f(\lambda(A)), \qquad A \in \mathcal{S}_K. \tag{2.5}$$

Throughout the paper we denote

$$F^{ij}(A) = \frac{\partial F}{\partial a_{ij}}(A), \qquad F^{ij,kl}(A) = \frac{\partial^2 F}{\partial a_{ij} \partial a_{kl}}(A). \tag{2.6}$$

The matrix $\{F^{ij}(A)\}$, which is symmetric, has eigenvalues $f_1, \ldots, f_n$, and therefore is positive definite for $A \in \mathcal{S}_K$ if f satisfies (1.9), while (1.10) implies that F is concave for $A \in \mathcal{S}_K$ (see [2]), that is,

$$F^{ij,kl}(A)\xi_{ij}\xi_{kl} \le 0, \quad \forall\{\xi_{ij}\} \in \mathcal{S},\ A \in \mathcal{S}_K. \tag{2.7}$$

We have

$$F^{ij}(A)a_{ij} = \sum f_i(\lambda(A))\lambda_i, \tag{2.8}$$

$$F^{ij}(A)a_{ik}a_{jk} = \sum f_i(\lambda(A))\lambda_i^2. \tag{2.9}$$

The function G in equation (1.22) is determined by

$$G(D^2u, Du, u) = F(A[u]) \tag{2.10}$$

where $A[u] = \{a_{ij}[u]\}$ is given by (2.4). Let

$$\mathcal{L} = G^{st}\partial_s\partial_t + G^s\partial_s + G_u \tag{2.11}$$

be the linearized operator of G at u, where

$$G^{st} = \frac{\partial G}{\partial u_{st}}, \quad G^s = \frac{\partial G}{\partial u_s}, \quad G_u = \frac{\partial G}{\partial u}. \tag{2.12}$$

We shall give the exact formula for G^s later but note that

$$G^{st} = \frac{u}{w}F^{ij}\gamma^{is}\gamma^{jt}, \quad G^{st}u_{st} = uG_u = F^{ij}a_{ij} - \frac{1}{w}\sum F^{ii}, \tag{2.13}$$

and

$$G^{pq,st} := \frac{\partial^2 G}{\partial u_{pq}\partial u_{st}} = \frac{u^2}{w^2}F^{ij,kl}\gamma^{is}\gamma^{tj}\gamma^{kp}\gamma^{ql} \tag{2.14}$$

where $F^{ij} = F^{ij}(A[u])$, etc. It follows that, under condition (1.9), equation (1.22) is elliptic for u if $A[u] \in \mathcal{S}_K$, while (1.10) implies that $G(D^2u, Du, u)$ is concave with respect to D^2u.

For later use, the eigenvalues of $\{G^{ij}\}$ and $\{F^{ij}\}$ (which are the f_i) are related by

Lemma 2.1. *Let* $0 < \mu_1 \le \cdots \le \mu_n$ *and* $0 < f_1 \le \cdots \le f_n$ *denote the eigenvalues of* $\{G^{ij}\}$ *and* $\{F^{ij}\}$ *respectively. Then*

$$w\mu_k \le uf_k \le w^3\mu_k, \quad 1 \le k \le n. \tag{2.15}$$

Proof. For any $\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n$ we have from (2.13)

$$uF^{ij}\xi_i\xi_j = wG^{kl}\gamma_{ik}\gamma_{lj}\xi_i\xi_j = wG^{kl}\xi'_k\xi'_l$$

where

$$\xi'_i = \gamma_{ik}\xi_k = \xi_i + \frac{(\xi \cdot Du)u_i}{1+w}.$$

Note that

$$|\xi|^2 \le |\xi'|^2 = |\xi|^2 + |\xi \cdot Du|^2 \le w^2|\xi|^2$$

where $\xi' = (\xi'_1, \ldots, \xi'_n)$. Since both $\{G^{ij}\}$ and $\{F^{ij}\}$ are positive, (2.15) follows from the minimax characterization of eigenvalues. □

3. Height estimates and the asymptotic angle condition

In this section let Σ be a hypersurface in $\mathbb{H}^{n+1}$ with $\partial\Sigma \subset P(\varepsilon) := \{x_{n+1} = \varepsilon\}$ so Σ separates $\{x_{n+1} \geq \varepsilon\}$ into an inside (bounded) region and an outside (unbounded) one. Let Ω be the region in $\mathbb{R}^n \times \{0\}$ such that its vertical lift Ω^ε to $P(\varepsilon)$ is bounded by $\partial\Sigma$ (and $\mathbb{R}^n \setminus \Omega$ is connected and unbounded). It is allowable that Ω has several connected components. Suppose $\kappa[\Sigma] \in K$ and $f(\kappa) = \sigma \in (0, 1)$ with respect to the outer normal.

Let $B_1 = B_R(a)$ be a ball of radius R centered at $a = (a', -\sigma R) \in \mathbb{R}^{n+1}$ where $\sigma \in (0, 1)$ and $S_1 = \partial B_1 \cap \mathbb{H}^{n+1}$. Then $\kappa_i[S_1] = \sigma$ for all $1 \leq i \leq n$ with respect to its outward normal. Similarly, let $B_2 = B_R(b)$ be a ball of radius R centered at $b = (b', \sigma R) \in \mathbb{R}^{n+1}$ with $S_2 = \partial B_2 \cap \mathbb{H}^{n+1}$. Then $\kappa_i[S_2] = \sigma$ for all $1 \leq i \leq n$ with respect to its inward normal.

These so called equidistant spheres serve as useful barriers.

Lemma 3.1.

$$
\begin{aligned}
&\text{(i) } \Sigma \cap \{x_{n+1} < \varepsilon\} = \emptyset.\\
&\text{(ii) } \textit{If } \partial\Sigma \subset B_1, \textit{ then } \Sigma \subset B_1.\\
&\text{(iii) } \textit{If } B_1 \cap P(\varepsilon) \subset \Omega^\varepsilon, \textit{ then } B_1 \cap \Sigma = \emptyset.\\
&\text{(iv) } \textit{If } B_2 \cap \Omega^\varepsilon = \emptyset, \textit{ then } B_2 \cap \Sigma = \emptyset.
\end{aligned}
\tag{3.1}
$$

Proof. For (i) let $c = \min_{x\in\Sigma} x_{n+1}$ and suppose $0 < c < \varepsilon$. Then the horosphere $P(c)$ satisfies $f(\kappa) = 1$ with respect to the upward normal, lies below Σ and has an interior contact violating the maximum principle. Thus $c = \varepsilon$. For (ii), (iii), (iv) we perform homothetic dilations from $(a', 0)$ and $(b', 0)$ respectively which are hyperbolic isometries and use the maximum principle. For (ii), expand B_1 continuously until it contains Σ and then reverse the process. Since the curvatures of Σ and S_1 are calculated with respect to their outward normals and both hypersurfaces satisfy $f(\kappa) = \sigma$, there cannot be a first contact. For (iii) and (iv) we shrink B_1 and B_2 until they are respectively inside and outside Σ. When we expand B_1 there cannot be a first contact as above. Now shrink B_2 until it lies below $P(\varepsilon)$ and so is disjoint (outside) from Σ. Now reverse the process and suppose there is a first interior contact. Then the outward normal to Σ at this contact point is the inward normal to S_2. Since the curvatures of S_2 are calculated with respect to its inner normal and it satisfies $f(\kappa) = \sigma$, this contradicts the maximum principle. □

Lemma 3.2. *Suppose f satisfies* (1.9), (1.11) *and* (1.14). *Assume that $\partial\Sigma \in C^2$ and let u denote the height function of Σ. Then for $\varepsilon > 0$ sufficiently small,*

$$
-\frac{\varepsilon\sqrt{1-\sigma^2}}{r_2} - \frac{\varepsilon^2(1+\sigma)}{r_2^2} < \nu^{n+1} - \sigma < \frac{\varepsilon\sqrt{1-\sigma^2}}{r_1} + \frac{\varepsilon^2(1-\sigma)}{r_1^2} \quad \textit{on } \partial\Sigma \tag{3.2}
$$

where r_2 and r_1 are the maximal radii of exterior and interior spheres to $\partial\Omega$, respectively. In particular, $\nu^{n+1} \to \sigma$ on $\partial\Sigma$ as $\varepsilon \to 0$.

Proof. Assume first $r_2 < \infty$. Fix a point $x_0 \in \partial\Omega$ and let e_1 be the outward pointing unit normal to $\partial\Omega$ at x_0. Let B_1, B_2 be balls in $\mathbb{R}^{n+1}$ with centers $a_1 = (x_0 - r_1 e_1, -R_1\sigma$, $a_2 = (x_0 + r_2 e_1, R_2\sigma)$ and radii R_1, R_2 respectively satisfying

$$R_1^2 = r_1^2 + (R_1\sigma + \varepsilon)^2, \quad R_2^2 = r_2^2 + (R_2\sigma - \varepsilon)^2. \tag{3.3}$$

Then $B_1 \cap P(\varepsilon)$ is an n-ball of radius r_1 internally tangent to $\partial\Omega^\varepsilon$ at x_0 while $B_2 \cap P(\varepsilon)$ is an n-ball of radius r_2 externally tangent to $\partial\Omega^\varepsilon$ at x_0. By Lemma 3.1(iii) & (iv), $B_i \cap \Sigma = \emptyset$, $i = 1, 2$. Hence,

$$-\frac{u - \sigma R_2}{R_2} < \nu^{n+1} < \frac{u + \sigma R_1}{R_1} \quad \text{at } x_0.$$

That is,

$$-\frac{\varepsilon}{R_2} < \nu^{n+1} - \sigma < \frac{\varepsilon}{R_1} \quad \text{at } x_0. \tag{3.4}$$

From (3.3),

$$\frac{1}{R_1} = \frac{\sqrt{(1-\sigma^2)r_1^2 + \varepsilon^2} - \varepsilon\sigma}{r_1^2 + \varepsilon^2} < \frac{\sqrt{1-\sigma^2}}{r_1} + \frac{\varepsilon(1-\sigma)}{r_1^2},$$
$$\frac{1}{R_2} = \frac{\sqrt{(1-\sigma^2)r_2^2 + \varepsilon^2} + \varepsilon\sigma}{r_2^2 + \varepsilon^2} < \frac{\sqrt{1-\sigma^2}}{r_2} + \frac{\varepsilon(1+\sigma)}{r_2^2}.$$

These estimates and (3.4) give (3.2), completing the proof of the lemma. □

4. The approximating problems and the continuity method

We study the approximating Dirichlet problem

$$\begin{aligned} G(D^2u, Du, u) &= \sigma \quad \text{in } \Omega, \\ u &= \varepsilon \quad \text{on } \partial\Omega, \end{aligned} \tag{4.1}$$

using the continuity method.

Consider for $0 \le t \le 1$ the family of Dirichlet problems

$$\begin{aligned} G(D^2u^t, Du^t, u^t) &= \sigma^t := t\sigma + (1-t) \quad \text{in } \Omega, \\ u^t &= \varepsilon \quad \text{on } \partial\Omega, \\ u^0 &\equiv \varepsilon. \end{aligned} \tag{4.2}$$

For Ω a $C^{2+\alpha}$ domain, we find (starting from $u^0 \equiv \varepsilon$) a smooth family of solutions u^t, $0 \le t \le 2t_0$, by the implicit function theorem since $G_u|_{u^0} \equiv 0$. We shall show in a moment that these solutions are unique. By elliptic regularity it is now well understood that if we can find uniform estimates in C^2 for $0 < t_0 \le t \le 1$ then we can solve (4.1).

By Lemma 3.1, we obtain the C^0 estimate

$$\varepsilon \le u^t \le C \quad \text{in } \Omega. \tag{4.3}$$

4.1. The C^1 estimate

The following proposition shows that we have uniform C^1 estimates in the continuity method and that the linearized operator $\mathcal{L}$ satisfies the maximum principle.

Proposition 4.1. *Let $u^t \in C^{2+\alpha}(\overline{\Omega})$ be a family of admissible solutions of* (4.2) *for* $0 \le t \le t^*$. *Suppose $\mathcal{H}_{\partial\Omega} \ge 0$. Then $G_u|_{u^t} \le 0$ so we have uniqueness. Hence w^t assumes its maximum on $\partial\Omega$ and $w^t \le 1/\sigma^t$ on $\overline{\Omega}$ for all* $0 \le t \le t^*$.

Proof. We (usually) suppress the t dependence for convenience. By (2.13) and (1.16),

$$uG_u = \sigma^t - \frac{1}{w^t}\sum f_i \le \sigma^t - \frac{1}{w^t}.$$

For $t = 0$, we have $\sigma^0 = 1$, $u^0 \equiv \varepsilon$, $\kappa_i = 1$, $f_i = 1/n$ and so $uG_u \equiv 0$. Note also that $\frac{d}{dt}(\sigma^t - 1/w^t)|_{t=0} = \sigma - 1 < 0$. Hence for $t > 0$ sufficiently small, $uG_u < 0$ so the operator $\mathcal{L}$ given by (2.11) satisfies the maximum principle. But $\mathcal{L}u_k = 0$ so each derivative u_k achieves its maximum on $\partial\Omega$. In particular, w assumes its maximum on $\partial\Omega$. Let $0 \in \partial\Omega$ be a point where w assumes its maximum. Choose coordinates $(x_1, \dots, x_n)$ at 0 with x_n the inner normal direction for $\partial\Omega$. Then at 0,

$$u_\alpha = 0, \quad 1 \le \alpha < n, \quad u_n > 0, \quad u_{nn} \le 0,$$

and

$$\sum_{\alpha<n} u_{\alpha\alpha} = -u_n(n-1)\mathcal{H}_{\partial\Omega} \le 0.$$

Note that by (1.15), the hyperbolic mean curvature of graph(u) is at least σ. Therefore,

$$\frac{n}{\varepsilon}\left(\sigma - \frac{1}{w}\right) \le \frac{1}{w}\left(\sum_{\alpha<n} u_{\alpha\alpha} + \frac{u_{nn}}{w^2}\right) \le -(n-1)\frac{u_n}{w}\mathcal{H}_{\partial\Omega} \le 0.$$

Hence $\sigma - 1/w \le 0$, or $w \le 1/\sigma$. Thus $G_u \le 0$ so $\mathcal{L}$ satisfies the maximum principle. Consequently, the same estimates must continue to hold as we increase t up to t^*. □

In Section 5, we will make use of Proposition 4.1 to complete the proof of the C^2 estimates (see Theorem 5.1 and Corollary 5.8). Since the linearized operator is invertible, we have unique smooth solvability all the way to $t = 1$, completing the proof of Theorem 1.3. Using the global maximum principle, Theorem 6.1 of Section 6 and Theorem 5.1, we obtain uniform estimates for the hyperbolic principal curvatures. Note also that by Lemma 3.1(iii), we have a positive lower bound (uniform in ε) on each compact subdomain of Ω for the solutions u^ε obtained in Theorem 1.3. This allows us to obtain uniform $C^{2+\alpha}$ estimates for u^ε on compact subdomains of Ω by the interior estimates of Evans–Krylov. We can now let ε tend to zero to obtain Theorem 1.2.

5. Boundary estimates for second derivatives

In this section we establish boundary estimates for second derivatives of admissible solutions to the Dirichlet problem (4.2) for all $t_0 \le t \le 1$. Clearly it suffices to consider the case $t = 1$. Throughout this section let Ω be a bounded smooth domain in $\mathbb{R}^n$ with $\mathcal{H}_{\partial\Omega} \ge 0$, and $u \in C^3(\bar{\Omega})$ an admissible solution of the Dirichlet problem

$$\begin{cases} G(D^2u, Du, u) = \sigma & \text{on } \overline{\Omega}, \\ u = \varepsilon & \text{on } \partial\Omega, \end{cases} \tag{5.1}$$

where G is defined in (2.10).

Theorem 5.1. *Suppose that f satisfies* (1.9)–(1.14) *and* (1.21). *If ε is sufficiently small, then*

$$u|D^2u| \le C \quad \text{on } \partial\Omega \tag{5.2}$$

where C is independent of ε.

Recall that in Section 4, we proved the global gradient estimate $w \le 1/\sigma$. In particular, $\varepsilon \le u \le (1 + 1/\sigma)\varepsilon$ in an ε-neighborhood of $\partial\Omega$, This will be used repeatedly in the proof of Theorem 5.1 without comment.

The notation of this section follows that of Section 2. Let $\mathcal{L}'$ denote the partial linearized operator of G at u:

$$\mathcal{L}' = \mathcal{L} - G_u = G^{st}\partial_s\partial_t + G^s\partial_s$$

where G^{st}, G_u are defined in (2.12) and

$$G^s := \frac{\partial G}{\partial u_s} = -\frac{u_s}{w^2}F^{ij}a_{ij} - \frac{2}{w}F^{ij}a_{ik}\left(\frac{wu_k\gamma^{sj} + u_j\gamma^{ks}}{1+w}\right) + \frac{2}{w^2}F^{ij}u_i\gamma^{sj} \tag{5.3}$$

by the formula (2.21) in [6], where $F^{ij} = F^{ij}(A[u])$ and $a_{ij} = a_{ij}[u]$.

Since $F = \{F^{ij}\}$ and $A = \{a_{ij}\}$ are simultaneously diagonalizable by an orthogonal matrix P, we have

$$|F^{ij}a_{ik}| = (FA)_{jk} = |(P(P^TFP)(P^TAP)P^T)_{jk}| = \left|\sum P_{jr}f_r\kappa_r P_{kr}\right| \le \sum f_r|\kappa_r|. \tag{5.4}$$

Hence from (5.3) and (5.4), we obtain

Lemma 5.2. *Suppose that f satisfies* (1.9), (1.10), (1.13) *and* (1.14). *Then*

$$|G^s| \le \frac{\sigma}{w} + \frac{2}{w}\sum F^{ii} + 2\sum f_i|\kappa_i|. \tag{5.5}$$

Since $\gamma^{sj}u_s = u_j/w$,

$$G^s u_s = \left(\frac{1}{w^2} - 1\right)F^{ij}a_{ij} - \frac{2}{w^2}F^{ij}a_{ik}u_k u_j + \frac{2}{w^3}F^{ij}u_i u_j. \tag{5.6}$$

It follows from (2.6), (2.8) and (2.13) that

$$\mathcal{L}'u = \frac{1}{w^2}F^{ij}a_{ij} - \frac{1}{w}\sum F^{ii} - \frac{2}{w^2}F^{ij}a_{ik}u_k u_j + \frac{2}{w^3}F^{ij}u_i u_j. \tag{5.7}$$

Lemma 5.3. *Suppose that f satisfies* (1.9), (1.10), (1.13) *and* (1.14). *Then*

$$\mathcal{L}\left(1 - \frac{\varepsilon}{u}\right) \leq -\frac{(1-\sigma)\varepsilon}{u^2 w}\sum F^{ii} - \frac{2\varepsilon}{u^2 w^2}F^{ij}a_{ik}u_k u_j \quad \text{in } \Omega. \tag{5.8}$$

Proof. By (5.7), (2.13) and (1.14),

$$\begin{aligned}\mathcal{L}\left(1 - \frac{\varepsilon}{u}\right) &= \frac{\varepsilon}{u^2}\mathcal{L}'u - \frac{2\varepsilon}{u^3}G^{st}u_s u_t + G_u\left(1 - \frac{\varepsilon}{u}\right) \\ &= \frac{\varepsilon}{u^2}\left(\frac{\sigma}{w^2} - \frac{1}{w}\sum F^{ii}\right) + G_u\left(1 - \frac{\varepsilon}{u}\right) - \frac{2\varepsilon}{u^2 w^2}F^{ij}a_{ik}u_k u_j. \end{aligned} \tag{5.9}$$

Since $G_u \leq 0$ by Proposition 4.1, (5.8) now follows from (1.16). □

We now refine Lemma 5.3. For the symmetric matrix $A = A[u]$ we can uniquely define the symmetric matrices (see [10])

$$|A| = \{AA^T\}^{1/2}, \quad A^+ = \frac{1}{2}(|A| + A), \quad A^- = \frac{1}{2}(|A| - A), \tag{5.10}$$

which all commute and satisfy $A^+A^- = 0$. Moreover, $F = \{F^{ij}\}$ commutes with $|A|$, $A^\pm$ and so all are simultaneously diagonalizable. Write $A^\pm = \{a^\pm_{ij}\}$ and define

$$L = \mathcal{L} - \frac{2}{w^2}F^{ij}a^-_{ik}u_k\partial_j. \tag{5.11}$$

Lemma 5.4. *Suppose that f satisfies* (1.9), (1.10), (1.13) *and* (1.14). *Then*

$$L\left(1 - \frac{\varepsilon}{u}\right) \leq -\frac{(1-\sigma)\varepsilon}{u^2 w}\sum F^{ii} \quad \text{in } \Omega. \tag{5.12}$$

Proof. Since $\{F^{ij}\}$ is positive definite and simultaneously diagonalizable with $A^\pm$,

$$F^{ij}a^\pm_{ik}\xi_j\xi_k \geq 0, \quad \forall \xi \in \mathbb{R}^n.$$

Therefore,

$$F^{ij}a_{ik}u_k u_j = F^{ij}(a^+_{ik} - a^-_{ik})u_k u_j \geq -F^{ij}a^-_{ik}u_k u_j. \tag{5.13}$$

Combining (5.13) and Lemma 5.3 we obtain (5.12). □

The following lemma is stated in [4]; it applies to our situation since horizontal rotations are hyperbolic isometries. For completeness we sketch the proof.

Lemma 5.5. *Suppose that f satisfies* (1.9)*,* (1.10)*,* (1.13) *and* (1.14)*. Then*

$$\mathcal{L}(x_i u_j - x_j u_i) = 0, \quad \mathcal{L}u_i = 0, \quad 1 \le i, j \le n. \tag{5.14}$$

Proof. Without loss of generality we may assume $i = 2$, $j = 1$. Let $R(\theta)$ be the orthogonal $n \times n$ matrix with entries $r_{11} = r_{22} = \cos\theta$, $r_{12} = -r_{21} = -\sin\theta$, $r_{kl} = \delta_{kl}$ for k or $l \le 3$. Let $y = Rx$ and $v(y) = u(x)$. Then since rotations in $x_1, \ldots, x_n$ are hyperbolic isometries, $v(y)$ satisfies

$$G(D^2 v(y), Dv(y), v(y)) = \sigma, \tag{5.15}$$

where

$$v(y) = u(R^T y), \quad Dv(y) = RDu(R^T y), \quad D^2 v(y) = R(D^2 u(R^T y))R^T. \tag{5.16}$$

We differentiate (5.15) with respect to θ and evaluate at $\theta = 0$. With the obvious notation, we obtain

$$G^{kl}\dot{v}_{kl} + G^s \dot{v}_s + G_u \dot{v} = 0. \tag{5.17}$$

Using (5.16) and the definition of R, we compute

$$\begin{aligned}
\dot{v} &= u_i \frac{\partial x_i}{\partial \theta}\bigg|_{\theta=0} = u_i \dot{r}_{pi}(0)x_p = x_2 u_1 - x_1 u_2,\\
\dot{v}_s &= \dot{r}_{si}(0)u_i + r_{si}(0)u_{ij}\dot{r}_{pj}(0)x_p = x_2 u_{1s} - x_1 u_{2s} + u_1 \delta_{s2} - u_2 \delta_{s1}\\
&= (x_2 u_1 - x_1 u_2)_s,\\
\dot{v}_{kl} &= \delta_{ki}\delta_{lj} u_{ijm}\dot{r}_{nm}(0)x_n + (u_{il}\dot{r}_{ki}(0) + u_{kj}\dot{r}_{lj}(0)) = (x_2 u_1 - x_1 u_2)_{kl}.
\end{aligned}$$

Hence $\mathcal{L}(\dot{v}) = 0$ as stated. The statement $\mathcal{L}(u_i) = 0$ is left to the reader. □

Proof of Theorem 5.1. Consider an arbitrary point on $\partial\Omega$, which we may assume to be the origin of $\mathbb{R}^n$ and choose the coordinates so that the positive x_n axis is the interior normal to $\partial\Omega$ at the origin. There exists a uniform constant $r > 0$ such that $\partial\Omega \cap B_r(0)$ can be represented as a graph

$$x_n = \rho(x') = \frac{1}{2}\sum_{\alpha,\beta<n} B_{\alpha\beta} x_\alpha x_\beta + O(|x'|^3), \quad x' = (x_1, \ldots, x_{n-1}). \tag{5.18}$$

We shall assume $\varepsilon \le r$ below. Since $u = \varepsilon$ on $\partial\Omega$, we see that $u(x', \rho(x')) = \varepsilon$ and

$$u_{\alpha\beta}(0) = -u_n \rho_{\alpha\beta}, \quad \alpha, \beta < n. \tag{5.19}$$

Consequently,

$$|u_{\alpha\beta}(0)| \le C|Du(0)|, \quad \alpha, \beta < n, \tag{5.20}$$

where C depends only on the (Euclidean maximal principal) curvature of $\partial\Omega$.

As in [1], we consider for fixed $\alpha < n$ the operator

$$T_\alpha = \partial_\alpha + \sum_{\beta<n} B_{\alpha\beta}(x_\beta \partial_n - x_n \partial_\beta). \tag{5.21}$$

Using Lemma 5.5 and the boundary condition $u = \varepsilon$ on $\partial\Omega$ we have

$$\begin{aligned}
&\mathcal{L}T_\alpha u = 0,\\
&|T_\alpha u| + \frac{1}{2}\sum_{l<n} u_l^2 \le C \qquad \text{in } \Omega \cap B_\varepsilon(0),\\
&|T_\alpha u| + \frac{1}{2}\sum_{l<n} u_l^2 \le C|x|^2 \quad \text{on } \partial\Omega \cap B_\varepsilon(0).
\end{aligned} \tag{5.22}$$

Now define

$$\phi = \pm T_\alpha u + \frac{1}{2}\sum_{l<n} u_l^2 - \frac{C}{\varepsilon^2}|x|^2$$

where C is chosen large enough (independent of ε) so that $\phi \le 0$ on $\partial(\Omega \cap B_\varepsilon(0))$. This is possible by (5.22).

By (5.5), (5.22), (2.13) and Lemma 2.1 (recall $u \le c\varepsilon$ in $B_\varepsilon(0)$ by virtue of the C^1 estimates),

$$\mathcal{L}\phi \ge \sum_{l<n} G^{ij} u_{li} u_{lj} - \frac{C}{\varepsilon}\Big(\sum f_i + \sum f_i |\kappa_i|\Big) \quad \text{in } \Omega \cap B_\varepsilon(0). \tag{5.23}$$

Following Ivochkina, Lin and Trudinger [8] we have

Proposition 5.6. *At each point in $\Omega \cap B_\varepsilon(0)$ there is an index r such that*

$$\sum_{l<n} G^{ij} u_{li} u_{lj} \ge c_0 u \sum_{i \ne r} f_i (\kappa_i^e)^2 \ge \frac{c_0}{2u}\Big(\sum_{i\ne r} f_i \kappa_i^2 - \frac{2}{w^2}\sum f_i\Big). \tag{5.24}$$

Proof. Let P be an orthogonal matrix that simultaneously diagonalizes $\{F^{ij}\}$ and A^e. By (2.13) and (2.1),

$$\begin{aligned}
\sum_{l<n} G^{ij} u_{li} u_{lj} &= \frac{u}{w}\sum_{l<n} F^{st}\gamma^{is}\gamma^{jt} u_{li} u_{lj} = uw \sum_{l<n} F^{st} a^e_{sq} a^e_{pt} \gamma_{pl}\gamma_{ql}\\
&= uw \sum_{l<n} f_i (\kappa_i^e)^2 P_{pi}\gamma_{pl} P_{qi}\gamma_{ql} = uw \sum_{l<n} f_i (\kappa_i^e)^2 b_{il}^2,
\end{aligned} \tag{5.25}$$

where $B = \{b_{rs}\} = \{P_{ir}\gamma_{is}\}$ and $\det B = \det(B^T) = w$.

Suppose for some i, say $i = 1$, that

$$\sum_{l<n} b_{l1}^2 < \theta^2.$$

Expanding $\det B$ by cofactors along the first column gives

$$1 \le w = \det B = b_{11}C^{11} + \cdots + b_{n-1\,1}C^{1\,n-1} + b_{n1}\det M \le c_1\theta + c_2 \det M,$$

where the C^{1j} are cofactors and M is the $n-1$ by $n-1$ matrix

$$M = \begin{bmatrix} b_{12} & \dots & b_{n-1\,2} \\ \vdots & \ddots & \vdots \\ b_{1n} & \dots & b_{n-1\,n} \end{bmatrix}. \tag{5.26}$$

So $\det M \geq (1 - c_1\theta)/c_2$. Now expanding $\det M$ by cofactors along row $r \geq 2$ gives

$$\det M \leq c_3 \Big(\sum_{l<n} b_{lr}^2 \Big)^{1/2}$$

by the Schwarz inequality. Hence

$$\sum_{l<n} b_{lr}^2 \geq \left(\frac{1 - c_1\theta}{c_2 c_3} \right)^2. \tag{5.27}$$

If we choose $\theta < 1/(2c_1)$, (5.27) and (5.25) imply

$$\sum_{l<n} G^{ij} u_{li} u_{lj} \geq c_0 u \sum_{i \neq r} f_i (\kappa_i^e)^2 \quad \text{for some } r.$$

Finally using $\kappa_i^e = \frac{1}{u}(\kappa_i - 1/w)$ yields (5.24). □

Proposition 5.7. *Let L be defined by* (5.11). *Then*

$$L\phi \geq -C_1 \Big(G^{ij} \phi_i \phi_j + \frac{1}{\varepsilon} \sum f_i \Big)$$

for a controlled constant C_1 independent of ε.

Proof. By the generalized Schwarz inequality,

$$\begin{aligned} \frac{2}{w^2} |F^{ij} a_{jk}^- u_i \phi_k| &\leq 2(u F^{ij} \phi_i \phi_j)^{1/2} \Big(\frac{1}{u} F^{ij} a_{il}^- a_{kj}^- \frac{u_k u_l}{w^2} \Big)^{1/2} \\ &\leq \frac{c_0}{8nu} \sum_{\kappa_i < 0} f_i \kappa_i^2 + C G^{ij} \phi_i \phi_j \end{aligned} \tag{5.28}$$

where we have used Lemma 2.1 to compare $u F^{ij} \phi_i \phi_j$ to $G^{ij} \phi_i \phi_j$.

Since (recall (1.14))

$$\sum f_i |\kappa_i| = \sigma + 2 \sum_{\kappa_i < 0} f_i |\kappa_i|,$$

using (5.28), (5.23), Proposition 5.6 and Lemma 1.1 we have

$$\begin{aligned} L\phi &\geq \frac{c_0}{2u} \sum_{i \neq r} f_i \kappa_i^2 - \frac{c_0}{4nu} \sum_{\kappa_i < 0} f_i \kappa_i^2 - C \Big(G^{ij} \phi_i \phi_j + \frac{1}{\varepsilon} \sum f_i \Big) \\ &\geq -C_1 \Big(G^{ij} \phi_i \phi_j + \frac{1}{\varepsilon} \sum f_i \Big) \end{aligned} \tag{5.29}$$

for a controlled constant C_1 independent of ε. □

Let $h = (e^{C_1\phi} - 1) - A(1 - \varepsilon/u)$ with C_1 the constant in Proposition 5.7 and A large compared to C_1. By Proposition 5.7 and Lemma 5.4 (here again we use $u \le c\varepsilon$ in $B_\varepsilon(0)$ by virtue of the C^1 estimates),

$$h \le 0 \quad \text{on } \partial(\Omega \cap B_\varepsilon(0)), \qquad Lh \ge 0 \quad \text{in } \Omega \cap B_\varepsilon(0).$$

By the maximum principle $h \le 0$ in $\Omega \cap B_\varepsilon(0)$. Since $h(0) = 0$, we have $h_n(0) \le 0$, which gives

$$|u_{\alpha n}(0)| \le \frac{A}{C_1\varepsilon} u_n(0). \tag{5.30}$$

Finally, $|u_{nn}(0)|$ can be estimated as in [7] using hypothesis (1.21). For completeness we include the argument here. We may assume $[u_{\alpha\beta}(0)]$, $1 \le \alpha, \beta < n$, to be diagonal. Note that $u_\alpha(0) = 0$ for $\alpha < n$. We have, at $x = 0$,

$$A[u] = \frac{1}{w}\begin{bmatrix} 1 + uu_{11} & 0 & \dots & uu_{1n}/w \\ 0 & 1 + uu_{22} & \dots & uu_{2n}/w \\ \vdots & \vdots & \ddots & \vdots \\ uu_{n1}/w & uu_{n2}/w & \dots & 1 + uu_{nn}/w^2 \end{bmatrix}.$$

By Lemma 1.2 in [2], if $\varepsilon u_{nn}(0)$ is very large, the eigenvalues $\lambda_1, \dots, \lambda_n$ of $A[u]$ are asymptotically given by

$$\begin{aligned} \lambda_\alpha &= \frac{1}{w}(1 + \varepsilon u_{\alpha\alpha}(0)) + o(1), \qquad \alpha < n, \\ \lambda_n &= \frac{\varepsilon u_{nn}(0)}{w^3}\left(1 + O\left(\frac{1}{\varepsilon u_{nn}(0)}\right)\right). \end{aligned} \tag{5.31}$$

By (5.20) and assumptions (1.14), (1.21), for all $\varepsilon > 0$ sufficiently small,

$$\sigma = \frac{1}{w} F(wA[u](0)) \ge \frac{1}{w}\left(1 + \frac{\varepsilon_0}{2}\right)$$

if $\varepsilon u_{nn}(0) \ge R$ where R is a uniform constant. By the hypothesis (1.21) and Proposition 4.1 however,

$$\sigma \ge \frac{1}{w}\left(1 + \frac{\varepsilon_0}{2}\right) \ge \sigma\left(1 + \frac{\varepsilon_0}{2}\right) > \sigma,$$

which is a contradiction. Therefore $\varepsilon|u_{nn}(0)| \le R$ and the proof is complete. □

Applying the maximum principle for the largest principal curvature $\kappa_{\max}$ obtained in Theorem 5.2 of [7] we obtain

Corollary 5.8. *Let Ω be a bounded smooth domain in $\mathbb{R}^n$ with $\mathcal{H}_{\partial\Omega} \ge 0$, and $u \in C^3(\bar{\Omega}) \cap C^4(\Omega)$ an admissible solution of problem* (5.1). *Suppose that f satisfies* (1.9)–(1.14) *and* (1.21). *Then, if ε is sufficiently small,*

$$u|D^2u| \le C/\varepsilon^2 \quad \text{in } \overline{\Omega} \tag{5.32}$$

where C is independent of ε.

Note that Corollary 5.8 suffices to complete the proof of Theorem 1.3 but we cannot use it to pass to the limit as $\varepsilon \to 0$. In the following section we address this problem.

6. Global estimates for second derivatives

In this section we prove a maximum principle for the largest hyperbolic principal curvature $\kappa_{\max}(x)$ of solutions of $f(\kappa[u]) = \sigma$. We make extensive use of our previous calculations in Section 5 of [7].

Let Σ be the graph of u. For $x \in \Omega$ let $\kappa_{\max}(x)$ be the largest principal curvature of Σ at the point $X = (x, u(x)) \in \Sigma$. We define, as in [4],

$$M_0 = \max_{x\in\overline{\Omega}} \frac{\kappa_{\max}(x)}{\eta - a},$$

where $\eta = \nu^{n+1} = \mathbf{e} \cdot \nu$ and $0 < a < \sigma \leq \inf \eta$. Here $\mathbf{e}$ is the vertical Euclidean unit vector and as before ν is the Euclidean upward unit normal to Σ.

Theorem 6.1. *Suppose that f satisfies* (1.9)–(1.14) *and $\sigma \in (0, 1)$ satisfies $\sigma > \sigma_0$, where σ_0 is the unique zero in* (0, 1) *of*

$$\phi(a) := \frac{8}{3}a + \frac{22}{27}a^3 - \frac{5}{27}(a^2 + 3)^{3/2}. \tag{6.1}$$

Let $u \in C^4(\Omega)$ be an admissible solution of (5.1) *such that $\nu^{n+1} = 1/w \geq \sigma$. Then at an interior maximum of M_0,*

$$\kappa_{\max} \leq C/(\sigma - \sigma_0)^2$$

where C is independent of ε. Numerical calculations show $0.3703 < \sigma_0 < 0.3704$.

Proof. Suppose M_0 is attained at an interior point $x_0 \in \Omega$ and let $X_0 = (x_0, u(x_0))$. After a horizontal translation of the origin in $\mathbb{R}^{n+1}$, we may write Σ locally near X_0 as a radial graph

$$X = e^v \mathbf{z}, \quad \mathbf{z} \in \mathbb{S}^n_+ \subset \mathbb{R}^{n+1}, \tag{6.2}$$

with $X_0 = e^{v(\mathbf{z}_0)}\mathbf{z}_0$, $\mathbf{z}_0 \in \mathbb{S}^n_+$, such that $\nu(X_0) = \mathbf{z}_0$. Note that the height function $u = ye^v$, and the upward unit (Euclidean) normal is $\nu = (\mathbf{z} - \nabla v)/w$ where $y = \mathbf{e} \cdot \mathbf{z}$ and $w = (1 + |\nabla v|^2)^{1/2}$. Hence $\eta = (y - \mathbf{e} \cdot \nabla v)/w$.

We choose an orthonormal local frame $\tau_1, \ldots, \tau_n$ around $\mathbf{z}_0$ on $\mathbb{S}^n_+$ such that $v_{ij} = \nabla_{\tau_j} \nabla_{\tau_i} v$ is diagonal at $\mathbf{z}_0$, where ∇ denotes the Levi-Civita connection on $\mathbb{S}^n$. As shown in Section 2.2 of [7], the hyperbolic principal curvatures of the radial graph X are the eigenvalues of the matrix $A^{\mathbf{s}}[v] = \{a^{\mathbf{s}}_{ij}[v]\}$:

$$a^{\mathbf{s}}_{ij}[v] := \frac{1}{w}(y\gamma^{ik} v_{kl} \gamma^{lj} - \mathbf{e} \cdot \nabla v \delta_{ij}) \tag{6.3}$$

where

$$\gamma^{ij} = \delta_{ij} - \frac{v_i v_j}{w(1 + w)} .$$

We can then rewrite equation (5.1) in the form

$$F(A^{\mathbf{s}}[v]) = \sigma. \tag{6.4}$$

Henceforth we write $A[v] = A^{\mathbf{s}}[v]$ and $a_{ij} = a^{\mathbf{s}}_{ij}[v]$.

Since $\nu(X_0) = \mathbf{z}_0$, $\nabla v(\mathbf{z}_0) = 0$ and hence

$$a_{ij} = y v_{ij} = \kappa_i \delta_{ij} \tag{6.5}$$

at $\mathbf{z}_0$ by (6.3), where $\kappa_1, \ldots, \kappa_n$ are the principal curvatures of Σ at X_0.

We note that, at $\mathbf{z}_0$,

$$\begin{aligned}
&y_i = \nabla_i(\mathbf{e}\cdot\mathbf{z}) = \mathbf{e}\cdot\nabla_i\mathbf{z} = \mathbf{e}\cdot\tau_i,\\
&(\mathbf{e}\cdot\nabla v)_k = \mathbf{e}\cdot v_{ik}\tau_i = y_i v_{ik} = y_k v_{kk},\\
&\eta_i = \left(\frac{y - \mathbf{e}\cdot\nabla v}{w}\right)_i = y_i(1 - v_{ii}),\\
&a_{ij,k} = y v_{ijk} + y_k(v_{ii} - v_{kk})\delta_{ij},\\
&v_{ijk} = v_{ikj} = v_{kij} \quad \text{(since } \nabla v(\mathbf{z}_0) = 0\text{)},\\
&y(a_{i1,1} - a_{11,i}) = y_i(\kappa_i - \kappa_1).
\end{aligned} \tag{6.6}$$

We may assume

$$\kappa_1 = \kappa_{\max}(X_0). \tag{6.7}$$

The function $a_{11}/(\eta - a)$, which is defined locally near $\mathbf{z}_0$, then achieves its maximum at $\mathbf{z}_0$. Therefore at $\mathbf{z}_0$,

$$\left(\frac{a_{11}}{\eta - a}\right)_i = 0, \quad 1 \le i \le n, \tag{6.8}$$

and

$$y^2(y-a)F^{ii}a_{11,ii} - y^2\kappa_1 F^{ii}\eta_{ii} = y^2(y-a)^2 F^{ii}\left(\frac{a_{11}}{\eta - a}\right)_{ii} \le 0. \tag{6.9}$$

The left hand side of (6.9) is exactly calculated (these calculations are long) in Proposition 5.3 and Lemma 5.4 of [7] (with $\phi = \eta$) and yield

$$\begin{aligned}
&\sigma(y-a)\kappa_1^2 + a\kappa_1 \sum f_i\kappa_i^2 + (a - 2(1-y^2)(y-a))\kappa_1 \sum f_i\\
&\quad \le 2\sigma\kappa_1 + \frac{2a\kappa_1}{\alpha}\sum f_i(\kappa_i - \alpha)\, y_i^2 - \frac{2a^2\kappa_1^2}{\alpha^2(y-a)}\sum_{i=2}^n \frac{f_i - f_1}{\kappa_1 - \kappa_i}(\kappa_i - \alpha)^2 y_i^2
\end{aligned} \tag{6.10}$$

where $\alpha = a\kappa_1/(\kappa_1 - (y-a))$. We note only that differentiation of equation (6.4) twice gives

$$y^2(y-a)F^{ii}a_{ii,11} = -y^2(y-a)F^{ij,rs}a_{ij,1}a_{kl,1} \tag{6.11}$$

and the last term in (6.10) comes from this "concavity term"

$$-y^2(y-a)F^{ij,kl}a_{ij,1}a_{kl,1} \ge 2(y-a)\sum_{i=2}^n \frac{f_i - f_1}{\kappa_1 - \kappa_i}(y a_{i1,1})^2 \tag{6.12}$$

where, since $(\frac{a_{11}}{\eta - a})_i = 0$, that is, $a_{11,i} = \frac{\kappa_1}{y-a}\eta_i$, we find, using (6.6),

$$y a_{i1,1} = y a_{11,i} + (\kappa_i - \kappa_1)y_i = (a\kappa_1 - (\kappa_1 - (y-a))\kappa_i)\frac{y_i}{y-a} = -\frac{a\kappa_1(\kappa_i - \alpha)y_i}{\alpha(y-a)}.$$

We also recall the identity

$$\sum y_i^2 = 1 - y^2$$

which has been used in (6.10), which follows from

$$y_i = \nabla_i(\mathbf{e}\cdot\mathbf{z}) = \mathbf{e}\cdot\tau_i \quad \text{and} \quad \mathbf{e} = \sum(\mathbf{e}\cdot\tau_i)\tau_i + y\mathbf{z}.$$

It was shown in Section 6 of [7] that the coefficient $\gamma(y)$ of $\kappa_1 \sum f_i$ in (6.10),

$$\gamma(y) = a - 2(1-y^2)(y-a), \tag{6.13}$$

satisfies

$$\gamma(y) > \frac{7}{3}a - \frac{4}{27}a^3 - \frac{4}{27}(a^2+3)^{3/2} > 0 \quad \text{on } (a,1) \tag{6.14}$$

if $a^2 > 1/8$. Therefore the terms on the left hand side of (6.10) are all positive and we have one term of order κ_1^2. The only "dangerous" term on the right hand side of (6.10) is the second one and we may throw away those terms in that sum where $\kappa_i \le \alpha$. Thus we need only concern ourselves with

$$I = \{i : \kappa_i > \alpha > a\}.$$

Fix $\theta \in (0,1)$ to be chosen later and let

$$J = \{i \in I : f_1 \le \theta f_i\}, \qquad K = \{i \in I : f_1 > \theta f_i\}.$$

Then

$$a\kappa_1 \sum_{i\in J} f_i\kappa_i^2 > a^3\kappa_1 \sum_{i\in J} f_i \tag{6.15}$$

and

$$\frac{2a\kappa_1}{\alpha}\sum_{i\in K} f_i(\kappa_i-\alpha)\,y_i^2 - a\kappa_1^3 f_1 \le \kappa_1^2\left(\frac{2}{\theta} - a\kappa_1\right)f_1 < 0, \tag{6.16}$$

provided $\kappa_1 > 2/(a\theta)$. On the other hand by the Cauchy–Schwarz inequality (or completing the square),

$$\begin{aligned}
\sum_{i\in J} f_i(\kappa_i-\alpha)\,y_i^2 &- \frac{a\kappa_1}{\alpha(y-a)}\sum_{i\in J}\frac{f_i-f_1}{\kappa_1-\kappa_i}(\kappa_i-\alpha)^2 y_i^2 \\
&\le \sum_{i\in J} f_i y_i^2\left((\kappa_i-\alpha) - \frac{(1-\theta)a}{\alpha(y-a)}(\kappa_i-\alpha)^2\right) \\
&\le \frac{\alpha(y-a)(1-y^2)}{4(1-\theta)a}\sum_{i\in J} f_i = \frac{\alpha(a-\gamma(y))}{8(1-\theta)a}\sum_{i\in J} f_i.
\end{aligned} \tag{6.17}$$

Combining (6.10), (6.15), (6.16) and (6.17) we obtain

$$\sigma(y-a)\kappa_1^2 + \phi_\theta(y)\kappa_1\sum_{i\in J} f_i \le 2\sigma\kappa_1 \tag{6.18}$$

where the coefficient of $\kappa_1 \sum_{i\in J} f_i$ in (6.18) is

$$\phi_\theta(y) = \gamma(y) - \frac{a-\gamma(y)}{4(1-\theta)} + a^3.$$

Note that by (6.14),

$$\begin{aligned}\phi_0(y) &= \frac{5}{4}\left\{\gamma(y) + \frac{4}{5}a^3 - \frac{a}{5}\right\}\\ &> \frac{5}{4}\left\{\frac{7}{3}a - \frac{4}{27}a^3 - \frac{4}{27}(a^2+3)^{3/2} + \frac{4}{5}a^3 - \frac{a}{5}\right\}\\ &= \frac{8}{3}a + \frac{22}{27}a^3 - \frac{5}{27}(a^2+3)^{3/2} =: \phi(a). \qquad (6.19)\end{aligned}$$

For $a \in (0, 1)$ it is easily checked that $\phi'(a) > 0,\ \phi(0) < 0,\ \phi(1) > 0$. Let σ_0 be the unique zero of $\phi(a)$ in $(0, 1)$. Numerical calculations show that $0.3703 < \sigma_0 < 0.3704$.

Now assume that $2\varepsilon_0 := \sigma - \sigma_0 > 0$ and choose $a = \sigma_0 + \varepsilon_0$. Then $\phi_\theta(y) > 0$ on $(a, 1)$ if $\theta > 0$ is chosen sufficiently small. By Proposition 4.1, $y - a \geq \sigma - a \geq \varepsilon_0$ at $\mathbf{z}_0$, so by (6.18) (assuming $\kappa_1 > 2/(a\theta)$) we obtain $\varepsilon_0\kappa_1^2 \leq 2\kappa_1$. Hence

$$\kappa_1 \leq 2\max\left\{\frac{1}{a\theta}, \frac{1}{\varepsilon_0}\right\} = 4\max\left\{\frac{1}{\theta(\sigma+\sigma_0)}, \frac{1}{\sigma-\sigma_0}\right\}$$

and so (since $\eta - a < 1$)

$$\max_{x\in\overline{\Omega}} \kappa_{\max}(x) \leq \frac{\kappa_1(\mathbf{z}_0)}{\varepsilon_0} \leq 8\max\left\{\frac{1}{\theta(\sigma^2-\sigma_0^2)}, \frac{1}{(\sigma-\sigma_0)^2}\right\},$$

completing the proof of Theorem 6.1. □

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