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Capacitary estimates of positive solutions of semilinear elliptic equations with absorption

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Abstract. Let Ω be a bounded domain of class C^2 in \mathbb{R}^N and let K be a compact subset of $\partial\Omega$. Assume that $q \geq (N+1)/(N-1)$ and denote by U_K the maximal solution of $-\Delta u + u^q = 0$ in Ω which vanishes on $\partial\Omega \setminus K$. We obtain sharp upper and lower estimates for U_K in terms of the Bessel capacity $C_{2/q, q'}$ and prove that U_K is σ -moderate. In addition we describe the precise asymptotic behavior of U_K at points $\sigma \in K$, which depends on the “density” of K at σ , measured in terms of the capacity $C_{2/q, q'}$.

1. Introduction

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain whose boundary is of class C^2 and let $q > 1$. If μ is a Radon measure on $\Sigma := \partial\Omega$, the problem

$$-\Delta u + u^q = 0 \quad \text{in } \Omega, \quad u = \mu \quad \text{on } \Sigma, \quad (1.1)$$

has a solution if and only if μ vanishes on sets of $C_{2/q, q'}$ capacity zero, which is equivalent to $\mu \in W_+^{-2/q, q}(\Sigma)$ (see [15]). The solution is unique and will be denoted by u_μ . Following Dynkin and Kuznetsov [7], a positive solution $u \in C^2(\Omega)$ of the equation

$$-\Delta u + u^q = 0 \quad \text{in } \Omega \quad (1.2)$$

is called σ -moderate if there exists an increasing sequence of positive Radon measures μ_n on Σ such that the sequence $\{u_{\mu_n}\}$ converges to u . Similarly a measure $\mu \in (\mathcal{B}_r(\Sigma))_+$ (\mathcal{B}_r = space of regular Borel measures, not necessarily bounded) is called σ -moderate if it is the limit of an increasing sequence of measures belonging to $W_+^{-2/q, q}(\Sigma)$. The space of σ -moderate measures will be denoted by $\mathcal{B}_+^q(\Sigma)$. If $1 < q < q_c = (N+1)/(N-1)$ a singleton has positive $C_{2/q, q'}$ capacity so that every bounded Borel measure is in $W_+^{-2/q, q}(\Sigma)$. If $q \geq q_c$ a singleton has $C_{2/q, q'}$ capacity zero. The number q_c is called the *critical value of the exponent* for (1.1).

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A central question in the study of (1.2) is the behavior of solutions in the neighborhood of boundary singularities. Indeed this pertains to the essence of the boundary value problem (1.1) which naturally includes singular boundary data. To make the notion of singularity more precise we introduce the following notation. Let

$$\mathcal{K}_q(\partial\Omega) = \{K \subset \partial\Omega : K \text{ compact, } C_{2/q, q'}(K) > 0\}.$$

For every $K \in \mathcal{K}_q(\partial\Omega)$ we denote by U_K the maximal solution of (1.2) which vanishes outside K . It is known that U_K is not the trivial solution. Let u be a positive solution of (1.2). If there exists a set $K \in \mathcal{K}_q(\partial\Omega)$ such that $U_K \leq u$ we say that u is *singular* on K .

In the subcritical case, there exists a minimal singularity from which all other singularities can be constructed, namely, the point singularity. The asymptotic behavior of solutions at a point singularity on the boundary was investigated by Gmira and Véron [9]. Several years later, these estimates constituted one of the main ingredients in the study of the general boundary value problem (1.1) in the subcritical case (see Marcus and Véron [12] and [13]). In the later work it was shown that every positive solution has a boundary trace in $(\mathcal{B}_r(\Sigma))_+$, and conversely, for every positive $\mu \in \mathcal{B}_r(\Sigma)$ there exists a unique solution of the equation whose boundary trace is μ . In the special case $q = N = 2$ this result was previously established by Le Gall [11] by probabilistic methods.

In the supercritical case, such a minimal singularity does not exist, and this makes the situation enormously more complicated. In order to relate singular solutions to boundary singular sets, let us consider the case $\Omega = B_1(0)$. We say that two sets $K_1, K_2 \in \mathcal{K}_q(\partial\Omega)$ are *q-related* if there exists a rotation T such that $T(K_1) \cap K_2 \in \mathcal{K}_q(\partial\Omega)$. Equivalently, the singular solutions U_{K_1}, U_{K_2} are *q-related* if there exists a rotation T and a positive solution u such that $u \leq \min(U_{T(K_1)}, U_{K_2})$.

The characterization of the boundary data and its interpretation, i.e. the sense in which the boundary data is attained, is another problem whose treatment is more complicated in the supercritical case. The space $(\mathcal{B}_r)_+$ is no longer sufficient in this case. Although it is still true that problem (1.1) has a solution for every $\mu \in (\mathcal{B}_r(\Sigma))_+$, uniqueness fails. This fact was observed by Le Gall in 1997. Following this observation Dynkin and Kuznetsov [7] introduced the class of σ -moderate solutions and showed that, in this class, problem (1.1) has a unique solution. In a remarkable paper, Mselati [17] proved that, if $q = 2$, all solutions are σ -moderate. A related result, applying to arbitrary $q > 1$, was recently established by Marcus and Véron [16]: *For every compact set $K \in \mathcal{K}_q(\partial\Omega)$ the maximal solution U_K is σ -moderate.*

In the case $q = 2$ this result was previously established by Mselati, in the paper quoted above, and played a key role in the derivation of his main result. His proof used mainly probabilistic techniques and in particular the notion of Brownian snake introduced by Le Gall. Our proof employs purely analytic methods. (The proof was sketched in [16] and it is given in detail in the present paper.) Following this, Dynkin [3]–[5] announced the extension of Mselati's main result to the range $1 < q \leq 2$ by employing probabilistic methods developed in joint work with Kuznetsov (see [8] and references therein) and the above mentioned result of [16]. He also provided an alternative, probabilistic proof of the latter result which, however, applies only to the range $1 < q \leq 2$. These results are presented in a forthcoming book [6].

We turn to the description of our main results. The first of these provides *sharp* estimates, from above and below, for maximal solutions U_K of (1.2). Besides U_K we also consider the solution u_K defined as the supremum of all solutions of (1.2) whose boundary trace is a bounded measure supported in K . This is in fact the maximal σ -moderate solution vanishing on $\partial\Omega \setminus K$.

Theorem 1.1. *Let $q \geq q_c$ and $K \in \mathcal{K}_q(\partial\Omega)$. Then there exist positive constants c_1, c_2 depending only on q, N and Ω such that the following inequalities hold for every $x \in \Omega$:*

$$U_K(x) \leq c_1 \rho(x) \rho_K(x)^{-1-2/(q-1)} C_{2/q, q'}(K/\rho_K(x)), \tag{1.3}$$

$$U_K(x) \geq u_K(x) \geq c_2 \rho(x) \sum_{m=-\ell(\Omega)}^{\infty} r_m^{-1-2/(q-1)} C_{2/q, q'}(K_m(x)/r_m), \tag{1.4}$$

where $\ell(\Omega)$ is a number which depends only on $\text{diam } \Omega$ and

$$\begin{aligned} \rho(x) &:= \text{dist}(x, \partial\Omega), & \rho_K(x) &:= \text{dist}(x, K), \\ K_m(x) &:= \{y \in K : r_{m+1} \leq |x - y| \leq r_m\}, & r_m &:= 2^{-m} \quad \forall m \in \mathbb{Z}. \end{aligned}$$

It is easy to see that (1.3) implies

$$U_K(x) \leq c_3 \rho(x) \sum_{m=-\ell(\Omega)}^{\infty} r_m^{-1-2/(q-1)} C_{2/q, q'}(K_m(x)/r_m). \tag{1.5}$$

Therefore the inequalities are sharp. In the case $q = 2$, (1.3) was obtained by Mselati [17] employing a mixture of analytic and probabilistic techniques which apply only to this case. Our proof, which is purely analytic, is based mainly on two techniques. The first involves a linear lifting from $C(\partial\Omega)$ to $C^{0,1}(\Omega)$, somewhat similar to the lifting introduced by the authors in [15]. Using this lifting we obtain an *integral* capacity estimate from above. The second technique involves an (apparently) new estimate for Poisson kernels (see Appendix C) and is used in order to pass from this integral estimate to the pointwise estimate (1.3). The lower estimate obtained by Mselati [17], for $q = 2$, was expressed in probabilistic terms and it is not directly comparable to (1.4). In the case of solutions with *interior* singularities, i.e. singularities in the domain Ω , capacity estimates were obtained by Labutin [10] for all $q \geq N/(N - 2)$, the latter being the critical value of the exponent for the equation $-\Delta u + u^q = \mu$. Those estimates are sharp for $q > N/(N - 2)$, but not in the critical case. Our proof of the lower estimate (1.4) is inspired by some of the techniques of [10].

Next we present several applications of Theorem 1.1. As a first application we derive the identity

$$U_K = u_K, \tag{1.6}$$

which shows that U_K is σ -moderate.

The second application is an estimate of $U_K(x)$ as $x \in \Omega$ tends (non-tangentially) to a point $\sigma \in K$. Here the behavior of U_K depends on the ‘‘capacity distribution’’ of K in the neighborhood of σ , which we define by

$$\theta_K(t; \sigma) := C_{2/q, q'} \left(\frac{1}{t} (K \cap B_t(\sigma)) \right), \quad 0 < t < 1.$$

Now, for every $a \geq 1$, we have

$$\begin{aligned} \frac{1}{c(a)} \int_s^1 t^{-1-2/(q-1)} \theta_K(t; \sigma) \frac{dt}{t} &\leq \frac{U_K(x)}{|x - \sigma|} \\ &\leq c(a) \int_s^1 t^{-1-2/(q-1)} \theta_K(t; \sigma) \frac{dt}{t} + O(1) \end{aligned} \tag{1.7}$$

for all $x \in \Omega$ with $s = |x - \sigma| \leq a\rho(x)$. The constant $c(a)$ depends only on a, q, N, Ω . Evidently these too are sharp estimates. Further we show that if K is $(2/q, q')$ -thick at σ (for the definition of this term see [1] or Section 4 below) then

$$\int_0^1 t^{-1-2/(q-1)} \theta_K(t; \sigma) \frac{dt}{t} = \infty. \tag{1.8}$$

Therefore at such a point

$$U_K(x)/|x - \sigma| \rightarrow \infty \quad \text{as } x \rightarrow \sigma \text{ non-tangentially.}$$

It is known that K is $(2/q, q')$ -thick everywhere in K , with the possible exception of a set of $C_{2/q, q'}$ -capacity zero (or briefly $(2/q, q')$ -a.e. in K). For every $\alpha \in [0, 1 + 2/(q - 1)]$ and $\sigma_0 \in \partial\Omega$ one can construct a set $K \in \mathcal{K}_q(\partial\Omega)$ such that $\theta_K(t; \sigma_0) \approx t^\alpha$. For such K , (1.8) holds at σ_0 and we have

$$U_K(x) \approx |x - \sigma_0|^{\alpha-2/(q-1)} \quad \text{as } x \rightarrow \sigma_0 \text{ non-tangentially.}$$

Thus $U_K(x)$ may remain bounded or blow up at any rate not larger than $s^{-2/(q-1)}$ as x tends non-tangentially to a point where (1.8) holds.

On the other hand, we show that at every $(2/q, q')$ -thick point of K , U_K blows up in an integral sense, at the maximal rate. More precisely we prove:

If K is $(2/q, q')$ -thick at σ then

$$\int_0^1 U_K(\Gamma(t))^{q-1} t \, dt = \infty \tag{1.9}$$

for every curve $\Gamma \in \text{Lip}([0, 1], \Omega \cup \{\sigma\})$ such that $\Gamma(0) = \sigma$ and $0 < |\Gamma(t) - \sigma| \leq a\rho(\gamma(t))$ for some $a \geq 1$ and every $t \in (0, 1]$.

Incidentally, this fact and the previous remarks imply that (1.8) may hold at a point σ where K is not $(2/q, q')$ -thick.

The plan of the paper is as follows: Section 2 is devoted to the proof of the upper estimate (1.3). In Section 3 we present the proof of the lower estimate (1.4). Section 4 is devoted to the proof of (1.6), (1.7), (1.9) and related results. In Appendix A and B we derive some elementary estimates involving capacity which are used throughout the paper. Appendix C is devoted to an estimate of Poisson kernels which is used in Section 2. This estimate may be of interest in itself.

2. Boundary singularities: estimates from above

We start with some basic notations. $C_{\alpha, p}$ ($0 < \alpha, 1 < p < \infty$) denotes Bessel capacity in \mathbb{R}^{N-1} or alternatively on a smooth manifold such as $\Sigma = \partial\Omega$. If $A \subset \Sigma$ then $C_{\alpha, p}(A)$

denotes the capacity of A relative to Σ and, if $\gamma > 0$, $C_{\alpha,p}(\gamma A)$ denotes the capacity of γA relative to $\gamma \Sigma$. Further, for any set $E \subset \mathbb{R}^N$ and $\beta > 0$, we write

$$\rho_E(x) := \text{dist}(x, E), \quad (E)^\beta := \{x \in \mathbb{R}^N : \rho_E(x) < \beta\}. \tag{2.1}$$

For any domain D and $\beta_1, \beta_2 > 0$, we define

$$\dot{\rho}_D(x) := \begin{cases} \rho_{\partial D}(x) & \text{if } x \in D, \\ -\rho_{\partial D}(x) & \text{if } x \in D^c, \end{cases} \tag{2.2}$$

and

$$D_{\beta_1, \beta_2} := \{x \in \mathbb{R}^N : \beta_1 < \dot{\rho}_D(x) < \beta_2\}, \quad D_\beta := D_{0, \beta}, \quad D_\beta^e := D_{-\beta, 0}. \tag{2.3}$$

Since $\Sigma = \partial\Omega$ is a closed, compact manifold of class C^2 there exists a positive number $\beta^i(\Omega)$ (resp $\beta^e(\Omega)$) such that for every x in the closure of Ω_{β^i} (resp. $\Omega_{\beta^e}^e$) there exists a unique point $\sigma_\Sigma(x) \in \Sigma$ for which $|x - \sigma_\Sigma(x)| = \rho_\Sigma(x)$. Furthermore the mapping

$$\Pi : \Omega_{-\beta^e, \beta^i} \rightarrow (-\beta^e, \beta^i) \times \Sigma \quad \text{given by} \quad \Pi(x) = (\dot{\rho}_\Omega(x), \sigma_\Sigma(x)) \tag{2.4}$$

is a C^2 diffeomorphism. We set $\beta_0(\Omega) := \min(\beta^i(\Omega), \beta^e(\Omega))$. For every $x \in \Omega_{-\beta^e, \beta^i}$, denote by $\nu_\Omega(x)$ the outward unit normal (with respect to Ω) at $\sigma_\Sigma(x)$. Clearly $\nu_\Omega \in C^1$. To simplify the notation, we drop the subscript in ρ_Σ and σ_Σ .

If $E \subset \Sigma$ and $a \geq 1$, put

$$C_a^\Omega(E) = \{x \in \Omega : \rho_E(x) \leq a \rho(x)\} \tag{2.5}$$

and $C_a^\Omega(\xi) := C_a^\Omega(E)$ if $E = \{\xi\}$ is a singleton.

Throughout this paper c denotes a positive constant which depends only on q, N, Ω and the choice of β_0 . The value of the constant may change from one occurrence to another. The notation $X \approx Y$ means $(1/c)X \leq Y \leq cX$ for some constant c . If this constant depends on an additional parameter, say a , we write $X \stackrel{a}{\approx} Y$. Finally, we shall assume, without further mention, that $q \geq q_c = (N + 1)/(N - 1)$ and K is a compact subset of Σ .

Our upper estimate is presented in the following theorem.

Theorem 2.1. *Let u be a positive solution of (1.2) which vanishes on $K^c := \Sigma \setminus K$. Then there exists a constant c , independent of u and K , such that*

$$u(x) \leq c \rho(x) \rho_K(x)^{-1-2/(q-1)} C_{2/q, q'}(K/\rho_K(x)) \tag{2.6}$$

for every $x \in \Omega$.

Remark. For $\xi \in \mathbb{R}^N$ and $r > 0$ put

$$[K/r]_\xi := \frac{K - \xi}{r} + \xi = \frac{K}{r} + \frac{r - 1}{r} \xi.$$

Clearly $C_{2/q, q'}([K/r]_\xi) = C_{2/q, q'}(K/r)$.

A key element in the proof of the theorem is a lifting from $C(\partial\Omega)$ into $C(\bar{\Omega}) \cap C^{0,1}(\Omega)$ which is a modification of the lifting used in [15]. It is defined as follows.

Definition 2.2. Let $\eta \in C^2(\Sigma)$ and let H_η be the solution of the initial value problem

$$\begin{aligned} \frac{\partial H}{\partial \rho} &= \Delta_\Sigma H && \text{in } \mathbb{R}_+ \times \Sigma, \\ H(0, \cdot) &= \eta(\cdot) && \text{in } \Sigma. \end{aligned} \tag{2.7}$$

Let φ be the solution of the problem

$$-\Delta\varphi = 1 \quad \text{in } \Omega, \quad \varphi = 0 \quad \text{on } \partial\Omega.$$

Let $R = R_\eta$ be the lifting given by

$$R_\eta(x) = \begin{cases} H_\eta(\varphi(x)^2, \sigma(x)) & \forall x \in \bar{\Omega}_{\beta_0}, \\ V_\eta & \forall x \in \Omega \setminus \Omega_{\beta_0}, \end{cases} \tag{2.8}$$

where V_η is the harmonic function in Ω'_{β_0} such that

$$V_\eta(x) = H_\eta(\varphi(x)^2, \sigma(x)) \quad \forall x \in \Sigma_{\beta_0}.$$

We observe that the lifting is positive and satisfies

$$R_{1-\eta} = 1 - R_\eta. \tag{2.9}$$

If $\eta \in C^1(\Sigma)$ then $R_\eta \in C^{0,1}(\bar{\Omega})$.

The first step in the proof of the theorem is the following integral estimate.

Lemma 2.3. Let K and u be as in the statement of the theorem. Assume that $\text{diam } K < \beta_1 := \beta_0(\Omega)/4$ and let x_0 be a point on $\partial\Omega$ such that $K \subset B_{\beta_1}(x_0)$. Let $\eta \in C^2(\partial\Omega)$ be a function such that

$$0 \leq \eta \leq 1, \quad \eta = \begin{cases} 0 & \text{in a neighborhood } A_\eta \text{ of } K, \ A_\eta \subset B_{\beta_1}(x_0), \\ 1 & \text{in } \partial\Omega \setminus B_{2\beta_1}(x_0). \end{cases} \tag{2.10}$$

Put $\zeta := \varphi R_\eta^{2q'}$. Then

$$\int_\Omega (u^q \varphi + u) R_\eta^{2q'} dx \leq c \|1 - \eta\|_{W^{2/q, q'}(\Sigma)}^{q'}, \tag{2.11}$$

where c depends only on q, N, Ω and the choice of β_0 .

Proof. If $F \in C(\Omega) \cap C^2(\Omega \setminus \Sigma_{\beta_0})$ put

$$\begin{aligned} \tilde{\Delta}F &= \chi_{\Omega_{\beta_0}} \Delta F + \chi_{\Omega'_{\beta_0}} \Delta F, & \tilde{\nabla}F &= \chi_{\Omega_{\beta_0}} \nabla F + \chi_{\Omega'_{\beta_0}} \nabla F, \\ F^e &= \chi_{\Omega_{\beta_0}} F, & F^i &= \chi_{\Omega'_{\beta_0}} F, \end{aligned}$$

where χ_E denotes the characteristic function of the set E . Note that R_η^e (resp. R_η^i) can be extended as a function in $C^1(\bar{\Omega}_{\beta_0})$ (resp. $C^1(\bar{\Omega}'_{\beta_0})$).

We observe that

$$\int_{\Omega_{\beta_0}} (-u\Delta\zeta + u^q\zeta) dx = \int_{\Sigma_{\beta_0}} \left(\zeta \frac{\partial u}{\partial \bar{v}} - u \frac{\partial \zeta^e}{\partial \bar{v}} \right) dS, \tag{2.12}$$

where \bar{v} is the unit normal on Σ_{β_0} pointing outward with respect to Ω_{β_0} . This is verified in the same way as in [15, Thm. 3.3] (see proof of (3.11) there). In addition

$$\int_{\Omega'_{\beta_0}} (-u\Delta\zeta + u^q\zeta) dx = - \int_{\Sigma_{\beta_0}} \left(\zeta \frac{\partial u}{\partial \bar{v}} - u \frac{\partial \zeta^i}{\partial \bar{v}} \right) dS. \tag{2.13}$$

Hence

$$\int_{\Omega} (-u\tilde{\Delta}\zeta + u^q\zeta) dx = - \int_{\Sigma_{\beta_0}} u \left(\frac{\partial \zeta^e}{\partial \bar{v}} - \frac{\partial \zeta^i}{\partial \bar{v}} \right) dS. \tag{2.14}$$

By a straightforward computation

$$\begin{aligned} \tilde{\Delta}\zeta &= -R_\eta^{2q'} + \varphi\tilde{\Delta}R_\eta^{2q'} + 2\nabla\varphi \cdot \tilde{\nabla}R_\eta^{2q'} \\ &= -R_\eta^{2q'} + 2q'\varphi R_\eta^{2q'-1}\tilde{\Delta}R_\eta + 2q'(2q'-1)\varphi R_\eta^{2q'-2}|\tilde{\nabla}R_\eta|^2 \\ &\quad + 4q'R_\eta^{2q'-1}\nabla\varphi \cdot \tilde{\nabla}R_\eta. \end{aligned}$$

Therefore, by (2.14),

$$\int_{\Omega} (R_\eta^{2q'}u + u^q\zeta) dx = 2q' \int_{\Omega} uR_\eta^{2q'-2}G_\eta dx - \int_{\Sigma_{\beta_0}} u \left(\frac{\partial \zeta^e}{\partial \bar{v}} - \frac{\partial \zeta^i}{\partial \bar{v}} \right) dS \tag{2.15}$$

where

$$G_\eta = \varphi R_\eta \tilde{\Delta}R_\eta + (2q'-1)\varphi|\tilde{\nabla}R_\eta|^2 + 2R_\eta\nabla\varphi \cdot \tilde{\nabla}R_\eta.$$

Further we obtain

$$\frac{\partial \zeta^e}{\partial \bar{v}} = \frac{\partial \varphi}{\partial \bar{v}} R_\eta^{2q'-1} + 2q'\varphi R_\eta^{2q'-1} \frac{\partial R_\eta^e}{\partial \bar{v}}, \quad \frac{\partial \zeta^i}{\partial \bar{v}} = \frac{\partial \varphi}{\partial \bar{v}} R_\eta^{2q'-1} + 2q'\varphi R_\eta^{2q'-1} \frac{\partial R_\eta^i}{\partial \bar{v}},$$

and hence

$$\left| \frac{\partial \zeta^e}{\partial \bar{v}} - \frac{\partial \zeta^i}{\partial \bar{v}} \right| \leq 2q'\varphi R_\eta^{2q'-1} \left(\left| \frac{\partial R_\eta^e}{\partial \bar{v}} \right| + \left| \frac{\partial R_\eta^i}{\partial \bar{v}} \right| \right).$$

Put

$$M_\eta = \varphi^{1/q'} (|\tilde{\nabla}R_\eta|^2 + |\tilde{\Delta}R_\eta|) + \varphi^{-1/q} |\nabla\varphi \cdot \tilde{\nabla}R_\eta|, \tag{2.16}$$

$$N_\eta = \varphi^{1/q'} \left(\left| \frac{\partial R_\eta^e}{\partial \bar{v}} \right| + \left| \frac{\partial R_\eta^i}{\partial \bar{v}} \right| \right). \tag{2.17}$$

Since $0 < R_\eta \leq 1$ it follows that

$$|G_\eta| \leq 2q'\varphi^{1/q} M_\eta, \quad \left| \frac{\partial \zeta^e}{\partial \bar{v}} - \frac{\partial \zeta^i}{\partial \bar{v}} \right| \leq 2q'\zeta^{1/q} N_\eta.$$

Hence

$$\begin{aligned} \int_{\Omega} u R_{\eta}^{2q'-2} |G_{\eta}| dx &\leq 2q' \int_{\Omega} u \zeta^{1/q} M_{\eta} dx \\ &\leq 2q' \left(\int_{\Omega} u^q \zeta dx \right)^{1/q} \left(\int_{\Omega} M_{\eta}^{q'} dx \right)^{1/q'}, \end{aligned} \tag{2.18}$$

$$\begin{aligned} \int_{\Sigma_{\beta_0}} u \left| \frac{\partial \zeta^e}{\partial \bar{\nu}} - \frac{\partial \zeta^i}{\partial \bar{\nu}} \right| dS &\leq 2q' \int_{\Sigma_{\beta_0}} u \zeta^{1/q} N_{\eta} dS \\ &\leq 2q' \left(\int_{\Sigma_{\beta_0}} u^q \zeta dS \right)^{1/q} \left(\int_{\Sigma_{\beta_0}} N_{\eta}^{q'} dS \right)^{1/q'}. \end{aligned} \tag{2.19}$$

Next we show that

$$\int_{\Sigma_{\beta_0}} u^q \zeta dS \leq c \int_{\Omega} (R_{\eta}^{2q'} u + u^q \zeta) dx. \tag{2.20}$$

In view of the Keller–Osserman inequality

$$\int_{\Sigma_{\beta_0}} u^q \zeta dS \leq c_1(\beta_0) \int_{\Sigma_{\beta_0}} u \zeta dS. \tag{2.21}$$

Let φ' be the solution of

$$-\Delta \varphi' = 1 \quad \text{in } \Omega'_{\beta_0}, \quad \varphi' = 0 \quad \text{on } \Sigma_{\beta_0} = \partial \Omega'_{\beta_0}.$$

Put

$$c_2(\beta_0) := \max_{\Sigma_{\beta_0}} |\partial_{\mathbf{n}'} \varphi'|, \quad c_3(\beta_0) := \max_{\Omega'_{\beta_0}} \varphi,$$

where $\partial_{\mathbf{n}'}$ denotes the derivative in the direction of the outward normal on $\Sigma_{\beta_0} = \partial \Omega'_{\beta_0}$. Then

$$\begin{aligned} \int_{\Omega'_{\beta_0}} (u + u^q \varphi') dx &= - \int_{\Sigma_{\beta_0}} u \partial_{\mathbf{n}'} \varphi' dS \\ &\geq c_2(\beta_0) \int_{\Sigma_{\beta_0}} u dS \geq \frac{c_2(\beta_0)}{c_3(\beta_0)} \int_{\Sigma_{\beta_0}} u \zeta dS. \end{aligned} \tag{2.22}$$

Let ω denote the characteristic function of $\Sigma \setminus B_{2\beta_1}(x_0)$ and put

$$c_4(\beta_0) = \inf_{\Omega'_{\beta_0}} R_{\omega}^{2q'}.$$

By (2.10), $\eta \geq \omega$; hence

$$R_{\eta}^{2q'} \geq c_4(\beta_0) \quad \forall x \in \Omega'_{\beta_0}. \tag{2.23}$$

Using this inequality and the fact that $\varphi' < \varphi$ we obtain

$$\int_{\Omega'_{\beta_0}} (u + u^q \varphi') dx \leq \frac{1}{c_4(\beta_0)} \int_{\Omega'_{\beta_0}} (u + u^q \varphi) R_{\eta}^{2q'} dx. \tag{2.24}$$

Inequalities (2.21)–(2.24) imply (2.20). Further (2.15)–(2.20) imply

$$\int_{\Omega} (u + u^q \varphi) R_{\eta}^{2q'} dx \leq c \left(\int_{\Omega} M_{\eta}^{q'} dx + \int_{\Sigma_{\beta_0}} N_{\eta}^{q'} dS \right). \tag{2.25}$$

Put $\eta^* = 1 - \eta$ and note that

$$M_{\eta} = M_{\eta^*}, \quad N_{\eta} = N_{\eta^*}.$$

We turn to the estimate of the terms on the right hand side of (2.25). By [15, Lemma 1.2],

$$\|\varphi^{1/q'} |\Delta R_{\eta^*}| + \varphi^{-1/q} |\nabla \varphi \cdot \nabla R_{\eta^*}|\|_{L^{q'}(\Omega_{\beta_0})} \leq c \|\eta^*\|_{W^{2/q, q'}(\Sigma)}. \tag{2.26}$$

In order to estimate the norm of $\varphi^{1/q'} (|\nabla R_{\eta^*}|^2)$ in $L^{q'}(\Omega_{\beta_0})$ we proceed as in the proof of [15, Th. 3.3] (see (3.24) and the argument following it). First we obtain

$$\begin{aligned} & \int_{\Omega_{\beta_0}} \varphi (|\nabla R_{\eta^*}|^{2q'}) dx \\ & \leq c \int_{\Sigma} \int_0^{\rho_0} \left(\rho^{q'} \left| \frac{\partial H_{\eta^*}(\rho, \sigma)}{\partial \rho} \right|^{2q'} + |\nabla_{\sigma} H_{\eta^*}(\rho, \sigma)|^{2q'} \right) d\rho d\sigma, \end{aligned} \tag{2.27}$$

where $\rho_0 = \sup_{\sigma \in \Sigma} \varphi^2(\beta_0, \sigma)$, and then, estimating the terms on the right hand side by employing interpolation inequalities, we obtain

$$\int_{\Omega_{\beta_0}} \varphi (|\tilde{\nabla} R_{\eta^*}|^{2q'}) dx \leq c \|\eta^*\|_{W^{2/q, q'}(\Sigma)}^{q'}. \tag{2.28}$$

Inequalities (2.26) and (2.28) yield

$$\|M_{\eta^*}\|_{L^{q'}(\Omega_{\beta_0})} \leq c \|\eta^*\|_{W^{2/q, q'}(\Sigma)}. \tag{2.29}$$

Next we estimate the norm of M in $L^{q'}(\Omega'_{\beta_0})$. Recall that, in Ω'_{β_0} , $R_{\eta^*} = V_{\eta^*}$ (see Definition 2.2); since V_{η^*} is harmonic,

$$\begin{aligned} M_{\eta^*} &= \varphi^{1/q'} |\nabla V_{\eta^*}|^2 + \varphi^{-1/q} |\nabla \varphi \cdot \nabla V_{\eta^*}|, \\ &\leq c (|\nabla V_{\eta^*}|^2 + |\nabla V_{\eta^*}|) \quad \text{in } \Omega'_{\beta_0}. \end{aligned} \tag{2.30}$$

Since H_{η} is a solution of (2.7) it follows that, for every $\beta \in (0, \beta_0)$,

$$\|H_{\eta^*}\|_{C^2([\beta, \beta_0] \times \Sigma)} \leq c(\beta) \|\eta^*\|_{L^{q'}(\Sigma)}.$$

Let v be the function given by

$$v(x) := H_{\eta^*}(\varphi(x)^2, \sigma(x)) \quad \forall x \in \bar{\Omega}_{\beta_0}.$$

Then

$$\|v\|_{C^2(\Omega_{\beta_0} \setminus \Omega_{\beta})} \leq c'(\beta) \|\eta^*\|_{L^{q'}(\Sigma)}. \tag{2.31}$$

Since V_{η^*} is the Poisson potential of $v|_{\Sigma_{\beta_0}}$ in Ω'_{β_0} , it follows that

$$\|V_{\eta^*}\|_{C^1(\bar{\Omega}'_{\beta_0})} \leq c'(\beta_0)\|\eta^*\|_{L^{q'}(\Sigma)}. \tag{2.32}$$

Therefore, in view of (2.30) and (2.10),

$$\|M_{\eta^*}\|_{L^{q'}(\Omega'_{\beta_0})} \leq c\|\eta^*\|_{L^{q'}(\Sigma)}. \tag{2.33}$$

By (2.29) and (2.33),

$$\|M_{\eta^*}\|_{L^{q'}(\Omega)} \leq c\|\eta^*\|_{W^{2/q,q'}(\Sigma)}. \tag{2.34}$$

Since $v = R_{\eta^*}^e$, $V_{\eta^*} = R_{\eta^*}^i$, (2.31) and (2.32) imply

$$\|N_{\eta^*}\|_{L^{q'}(\Sigma_{\beta_0})} \leq c\|\eta^*\|_{L^{q'}(\Sigma)}. \tag{2.35}$$

Finally, (2.34), (2.35) and (2.25) imply (2.11). □

At this point we need some additional notations. If V is an open set in \mathbb{R}^d we denote by $C_{\alpha,p}^V(K)$ the $C_{\alpha,p}$ -capacity of a compact set $K \subset V$ relative to V , i.e., relative to test functions with compact support in V . It is known that

$$C_{\alpha,p}^V(K) = r^{d-\alpha p} C_{\alpha,p}^{V/r}(K/r) \quad \forall K \subset V, K \text{ compact}. \tag{2.36}$$

Furthermore, for every positive θ , there exists a constant c_θ such that, for every compact set $K \subset V$ which satisfies $\text{dist}(K, \partial V) > \theta$,

$$C_{\alpha,p}(K) \leq C_{\alpha,p}^V(K) \leq c_\theta C_{\alpha,p}(K). \tag{2.37}$$

If V is a relatively open subset of Σ , F is a compact subset of V and $C_{\alpha,p}$ denotes capacity on Σ then

$$C_{\alpha,p}^V(K) \approx r^{N-1-\alpha p} C_{\alpha,p}^{V/r}(K/r) \quad \forall K \subset F, K \text{ compact}, \tag{2.38}$$

and (2.37) remains valid. Finally, for every $x_0 \in \Sigma$, we define

$$\Gamma_s(x_0) := B_s(x_0) \cap \Sigma, \quad D_s(x_0) := \Omega \setminus B_s(x_0). \tag{2.39}$$

Lemma 2.4. *Let K and u be as in the statement of the theorem. Assume that $\text{diam } K < \gamma < \beta_1$ and let $x_0 \in \Sigma$ be a point such that $K \subset \Gamma_\gamma(x_0)$. Then there exists a constant c , depending only on q, N, Ω and the choice of β_0 , such that*

$$\int_{D_{3\gamma}(x_0)} (u^q \varphi + u) dx \leq c\gamma^{N-1-2/(q-1)} C_{2/q,q'}(K/\gamma) \tag{2.40}$$

where φ is as in Definition 2.2.

Proof. Let $\eta \in C^2(\partial\Omega)$ be a function such that

$$0 \leq \eta \leq 1, \quad \eta = \begin{cases} 0 & \text{in a neighborhood } A_\eta \text{ of } K, A_\eta \subset B_\gamma(x_0), \\ 1 & \text{in } \partial\Omega \setminus B_{2\gamma}(x_0). \end{cases} \tag{2.41}$$

By Lemma 2.3 inequality (2.11) holds.

If ω_γ denotes the characteristic function of $\Sigma \setminus \Gamma_{2\gamma}$ then

$$R_\eta \geq R_{\omega_\gamma}.$$

In addition there exists $c_0 > 0$, depending only on Ω and β_0 , such that

$$\inf_{D_{3\gamma}(x_0)} R_{\omega_\gamma} > c_0 \tag{2.42}$$

for every $\gamma \in (0, \beta_0/4)$ and every $x_0 \in \Sigma$. This follows in a straightforward manner from Definition 2.2. Consequently, if η satisfies (2.41),

$$R_\eta \geq c \quad \text{in } D_{3\gamma}(x_0). \tag{2.43}$$

This fact and (2.11) imply that

$$\int_{\Omega \setminus B_{3\gamma}(x_0)} (u^q \varphi + u) \, dx \leq c \|1 - \eta\|_{W^{2/q, q'}(\Sigma)}^{q'} \tag{2.44}$$

Taking the infimum over all such functions η we obtain

$$\int_{D_{3\gamma}(x_0)} (u^q \varphi + u) \, dx \leq c C_{2/q, q'}^{\Gamma_{2\gamma}(x_0)}(K).$$

By (2.37) and (2.38),

$$C_{2/q, q'}^{\Gamma_{2\gamma}(x_0)}(K) \approx \gamma^{N-1-2/(q-1)} C_{2/q, q'}^{\Gamma_2(x_0)}(K/\gamma) \leq c \gamma^{N-1-2/(q-1)} C_{2/q, q'}(K/\gamma).$$

The last two inequalities imply (2.40). □

Lemma 2.5. *Let K and u be as in the statement of the theorem. Pick γ such that $\text{diam } K < \gamma < \beta_0/16$. Let $x_0 \in \Sigma$ be a point such that $K \subset \Gamma_{3\gamma/2}(x_0)$. Then there exists a constant c , depending only on q, N, Ω and the choice of β_0 , such that*

$$\int_{\Omega \cap \partial B_{4\gamma}(x_0)} u \rho \, dS \leq c \gamma^{N-2/(q-1)} C_{2/q, q'}(K/\gamma). \tag{2.45}$$

Proof. Put

$$h_\tau(x; x_0) = \begin{cases} \frac{1}{N-2} \left(1 - \frac{\tau^{N-2}}{|x-x_0|^{N-2}} \right) & \text{if } N \geq 3, \\ 1 - \frac{\tau}{|x-x_0|} & \text{if } N = 2, \end{cases} \tag{2.46}$$

and

$$\psi_\tau(x; x_0) = \phi(x) h_\tau(x; x_0) \quad \forall x \in \Omega, \tag{2.47}$$

where $d(\Omega) = \text{diam } \Omega$ and ϕ is the eigenfunction corresponding to the first eigenvalue of $-\Delta$ in Ω , normalized so that $\max \phi = 1$. Assuming $\tau \leq \beta_0/4$, we get

$$\int_{D_\tau(x_0)} u^q \psi_\tau \, dx = \int_{D_\tau(x_0)} u \Delta \psi_\tau \, dx - \int_{\Omega \cap \partial B_\tau(x_0)} u \partial_{\mathbf{n}} \psi_\tau \, dS, \tag{2.48}$$

where $\partial_{\mathbf{n}}$ denotes the directional derivative in the direction of the outward unit normal on $D_\tau(x_0)$. Note that

$$-\partial_{\mathbf{n}} \psi_\tau(x) = \phi(x)/\tau \geq c \rho(x)/\tau \quad \forall x \in \Omega \cap \partial B_\tau(x_0). \tag{2.49}$$

$$\Delta\psi_\tau = \begin{cases} -\lambda_1\phi h_\tau + 2\frac{\tau^{N-2}\langle x-x_0, \nabla\phi(x)\rangle}{|x-x_0|^N} & \text{if } N > 2, \\ -\lambda_1\phi h_\tau - 2\frac{\tau\phi}{|x-x_0|^3} + 2\frac{\tau\langle x-x_0, \nabla\phi(x)\rangle}{|x-x_0|^2} & \text{if } N = 2. \end{cases} \tag{2.50}$$

We claim that there exists a constant $b(\Omega) > 0$ such that

$$\inf_{\substack{x \in \Omega \\ x_0 \in \Sigma}} \frac{\langle x-x_0, \nabla\phi \rangle}{|x-x_0|^2} \geq -b(\Omega). \tag{2.51}$$

Since $|\nabla\phi(x) - \nabla\phi(x_0)|/|x-x_0|$ is bounded it is sufficient to verify that there exists a constant c such that

$$I(x, x_0) := \frac{\langle x-x_0, \nabla\phi(x_0)\rangle}{|x-x_0|^2} > -c \tag{2.52}$$

for all $x_0 \in \Sigma$ and $x \in \Omega$. Put $R = \beta_0/2$ and

$$|x-x_0| = r, \quad \bar{x}_0 = x_0 + \nu(x_0)R, \quad z = \cos\left(\frac{x-x_0}{|x-x_0|}, \nu(x_0)\right).$$

(Recall that $\nu(x_0)$ denotes the outward unit normal on $\partial\Omega$ at x_0 .) Then the ball $B_R(\bar{x}_0)$ lies outside Ω and touches $\partial\Omega$ only at the point x_0 .

For every $x \in \Omega$,

$$R^2 \leq |x-\bar{x}_0|^2 = r^2 + R^2 - 2rRz.$$

Hence $z \leq r/2R$. Since $\nabla\phi(x_0) = -|\nabla\phi(x_0)|\nu(x_0)$ this implies (2.52) with

$$c = \frac{1}{\beta_0} \inf_{\Sigma} |\nabla\phi(x_0)|.$$

Thus (2.51) is verified. Using (2.52) we also obtain, for $x \in D_\tau(x_0)$,

$$\frac{\tau\phi(x)}{|x-x_0|^3} = \frac{\tau\nabla\phi(x_0) \cdot (x-x_0)}{|x-x_0|^3} + O(1) \geq I(x, x_0) + O(1) \geq -c_1, \tag{2.53}$$

where c_1 is a constant depending only on N and Ω .

Note that $\sup_{D_\tau(x_0)} h_\tau(\cdot; x_0)$ is bounded by a constant independent of τ or x_0 . (Recall that $0 < \tau < \beta_0/4$.) Therefore, (2.50), (2.51) and (2.53) imply

$$\Delta\psi_\tau \geq -A \quad \text{in } D_\tau(x_0), \tag{2.54}$$

where A is a constant depending only on N and Ω . By (2.48), (2.49) and (2.54),

$$\int_{D_\tau(x_0)} (u^q\phi + u) dx \geq \frac{A}{\tau} \int_{\Omega \cap \partial B_\tau(x_0)} u \rho dS \tag{2.55}$$

for every $\tau \in (0, \beta_0/4)$ and $x_0 \in \Sigma$. Finally, (2.40) and (2.55) imply (2.45). □

Remark. The constant in inequality (2.45) is independent of γ but depends on Ω . However the inequality is invariant with respect to dilation: if Ω and K are replaced by

$\Omega^\kappa := \kappa(\Omega - x_0)$ and $K^\kappa = \kappa(K - x_0)$ respectively, then (2.45) remains valid with the same constant c . To simplify the notation assume that $x_0 = 0$. If U is a solution of our equation in Ω^κ which vanishes on $\partial\Omega^\kappa \setminus K^\kappa$ then $u(x) = \kappa^{2/(q-1)}U(\kappa x)$ is a solution in Ω which vanishes on $\partial\Omega \setminus K$. Therefore (2.45) holds with respect to u and we obtain

$$\kappa^{2/(q-1)} \int_{\Omega \cap \partial B_{3\gamma}(0)} U(\kappa x) \rho(x) dS(x) \leq c\gamma^{N-2/(q-1)} C_{2/q,q'}(K/\gamma).$$

Put $\xi := \kappa x$. Then $\rho_{\partial\Omega^\kappa}(\xi) = \kappa\rho(x)$ and we obtain

$$\int_{\Omega^\kappa \cap \partial B_{3\kappa\gamma}(0)} U(\xi) \rho_{\partial\Omega^\kappa}(\xi) dS(\xi) \leq c(\kappa\gamma)^{N-2/(q-1)} C_{2/q,q'}(K^\kappa/(\kappa\gamma)).$$

Lemma 2.6. *Let K, u be as in the statement of the theorem. Then there exists $\gamma_0 = \gamma_0(\Omega) < 10^{-3}\beta_0$ such that the following statement holds.*

Assume that $\text{diam}(K) < \gamma < \gamma_0$. Then there exists a constant c , depending only on q, N, Ω and the choice of β_0 , such that

$$\int_{D_{9\gamma/2,5\gamma}(x_0)} u\rho dx \leq c\gamma^{1+N-2/(q-1)} C_{2/q,q'}(K/\gamma) \tag{2.56}$$

for every $\delta \in [0, \gamma/4)$ and $x_0 \in \Sigma_\delta$ such that $K \subset \Gamma_{5\gamma/4}(x_0)$. Here

$$D_{r,r'}(x_0) := \{x \in \Omega : r < |x - x_0| < r'\}.$$

Proof. If $z = (z_1, 0, \dots, 0)$ with $0 \leq z_1 < 1$ then

$$\{x \in \mathbb{R}_+^N : 17/4 < |x - z| < 5\} \subset \{x \in \mathbb{R}_+^N : 4 < |x| < 6\}.$$

Therefore, as Ω is bounded of class C^2 ,

$$\{x \in \Omega : 9\gamma/2 < |x - x_0| < 5\gamma\} \subset \{x \in \Omega : 4\gamma < |x - \sigma(x_0)| < 7\gamma\} \tag{2.57}$$

for every $x_0 \in \Sigma_\delta$, provided that $0 \leq \delta < \gamma < \gamma_0$ and γ_0 is sufficiently small. The value of γ_0 depends only on the C^2 “norm” of $\partial\Omega$.

By Lemma 2.5, if $K \subset \Gamma_{3\gamma/2}(\sigma(x_0))$,

$$\int_{\Omega \cap \partial B_{4\gamma}(\sigma(x_0))} u\rho dS \leq c\gamma^{N-2/(q-1)} C_{2/q,q'}(K/\gamma).$$

The assumptions of the present lemma imply that the above condition on K is satisfied. Hence,

$$\int_{\Omega \cap \partial B_{t\gamma}(\sigma(x_0))} u\rho dS \leq c\gamma^{N-2/(q-1)} C_{2/q,q'}(K/\gamma) \tag{2.58}$$

for every $t \in [4, 7]$, with c independent of t . Integrating over t we obtain

$$\int_{D_{4\gamma,7\gamma}(\sigma(x_0))} u\rho dx \leq c\gamma^{1+N-2/(q-1)} C_{2/q,q'}(K/\gamma). \tag{2.59}$$

This inequality and (2.57) imply (2.56). □

Proof of Theorem 2.1. Let $\gamma_0(\Omega)$ and $m(\Omega)$ be as in Lemma 2.6 and Theorem C.1 respectively. Assume that $\text{diam } K < \theta := \min(\gamma_0, \beta_0/40m)$ and pick $\gamma \in (\text{diam } K, \theta)$. Let $x_0 \in \Omega$ be a point such that

$$\rho_{\partial\Omega}(x_0) = \rho_K(x_0) = \delta := \gamma/4. \tag{2.60}$$

Then $K \subset B_{5\gamma/4}(x_0)$ and Lemma 2.6 applies, so that inequality (2.56) holds.

Denote by $\mathcal{P}_\tau^{x_0}$ the Poisson kernel of $D_\tau(x_0)$. Since u is subharmonic,

$$u(x) \leq v(x) := \int_{y \in \Omega: |y-x_0|=\tau} P_\tau(x, y) u(y) dS(y) \tag{2.61}$$

for every $\tau > 5\gamma/4$. Assume, as we may, that $m(\Omega) > 20$. Then

$$\tau \in [9\gamma/2, 5\gamma] \implies \tau < \min(\beta_0/8m, m\delta). \tag{2.62}$$

Assuming that τ satisfies this condition, it follows, by Theorem C.1, that there exists a constant c , independent of τ and x_0 , such that

$$\mathcal{P}_\tau^{x_0}(x, y) \leq c \frac{\rho(x)\rho(y)}{\tau|x-y|^N} \tag{2.63}$$

in the set

$$E_\Omega(\tau, x_0) = \{(x, y) \in \Omega \times \Omega : |y-x_0| = \tau, |x-x_0| \geq 4\tau\}. \tag{2.64}$$

Note that, in this set, $|x-y| \approx |x-x_0|$. Therefore, if $|x-x_0| \geq 4\tau$,

$$u(x) \leq c \frac{\rho(x)}{\tau|x-x_0|^N} \int_{y \in \Omega: |y-x_0|=\tau} \rho(y) u(y) dS(y). \tag{2.65}$$

This inequality holds for every $\tau \in [9\gamma/2, 5\gamma]$ and $x \in D_{4\tau}(x_0)$. Therefore, integrating over τ in this interval yields

$$u(x)\gamma \leq c \frac{\rho(x)}{\gamma|x-x_0|^N} \int_{D_{9\gamma/2, 5\gamma}(x_0)} \rho(y) u(y) dy \tag{2.66}$$

for every $x \in D_{20\gamma}(x_0)$, $\gamma \in (\text{diam } K, \theta)$ and x_0 satisfying (2.60). Hence, by (2.56),

$$u(x)\gamma \leq c \frac{\rho(x)}{\gamma|x-x_0|^N} \gamma^{1+N-2/(q-1)} C_{2/q, q'}(K/\gamma) \tag{2.67}$$

and consequently

$$u(x) \leq c\rho(x)\gamma^{-1-2/(q-1)} C_{2/q, q'}(K/\gamma), \tag{2.68}$$

for every $x \in \Omega$ and $\gamma \in (\text{diam } K, \theta)$ such that $\rho_K(x) > 25\gamma$. This inequality and Lemma B.2 imply that

$$u(\xi) \leq cC_{2/q, q'}(K/\theta) \leq cC_{2/q, q'}(K/\beta_0) \quad \forall \xi \in \Omega : \rho_K(\xi) = \beta_0.$$

By the maximum principle, applied in $D_{\beta_0}^K = \{x \in \Omega : \rho_K(x) > \beta_0\}$,

$$u(\xi) \leq cC_{2/q, q'}(K/\beta_0) \quad \forall \xi \in D_{\beta_0}^K,$$

and by Lemma B.2,

$$C_{2/q,q'}(K/\beta_0) \leq cC_{2/q,q'}(K/d(\Omega)) \leq cC_{2/q,q'}(K/\alpha) \quad \forall \alpha \in (0, d(\Omega)),$$

where $d(\Omega) := \text{diam } \Omega$. Consequently, (2.6) holds for every $\xi \in D_{\beta_0}^K$.

Next we show that, for every compact set $K \subset \Sigma$ and every integer $n > d(\Omega)/20\theta$, (2.6) holds (with a constant depending on n) for all points $\xi \in \Omega$ such that

$$\frac{\text{diam } \Omega}{n} \text{diam } K \leq \rho_K(\xi). \quad (2.69)$$

There is a number M_n such that $\partial\Omega$ can be covered by M_n balls of diameter $\frac{\text{diam } \Omega}{20n} \text{diam } K$. Consequently, $K = \bigcup_{m=1}^{M_n} K_m$, where K_m is compact and

$$\text{diam } K_m < \theta, \quad m = 1, \dots, M_n.$$

Furthermore, if ξ satisfies (2.69),

$$20 \text{diam } K_m \leq \rho_{K_m}(\xi), \quad m = 1, \dots, M_n.$$

Hence

$$U_{K_m}(\xi) \leq c\rho(\xi)\rho_{K_m}(\xi)^{-1-2/(q-1)}C_{2/q,q'}(K_m/\rho_{K_m}(\xi)), \quad m = 1, \dots, M_n.$$

However $u \leq \sum_{m=1}^{M_n} U_{K_m}$ and $\rho_K(\xi) \leq \rho_{K_m}(\xi)$. Therefore

$$u(\xi) \leq cM_n\rho(x)\rho_K(\xi)^{-1-2/(q-1)}C_{2/q,q'}(K/\rho_K(\xi)).$$

Thus (2.6) holds, with a constant c depending on n , for every $\xi \in \Omega$ satisfying (2.69).

Finally we show that (2.6) holds for every $\xi \in \Omega$ such that

$$\frac{1}{4} \text{diam } K \geq \rho_K(\xi). \quad (2.70)$$

Let ξ be such a point and put $\delta = \frac{1}{2}\rho_K(\xi)$. Let

$$F_j(\xi) := \{x \in K : 2^{j-1}\delta \leq |x - \xi| \leq 2^j\delta\}.$$

Let $J(\xi) = \max\{j \in \mathbb{N} : C_{2/q,q'}(F_j) \neq 0\}$. Then

$$K = \bigcup_{j=1}^J F_j(\xi), \quad \text{diam } F_j \leq 2^{j+1}\delta \leq 4\rho_{F_j}(\xi).$$

Consequently,

$$U_{F_j}(\xi) \leq c\rho(\xi)\rho_{F_j}(\xi)^{-1-2/(q-1)}C_{2/q,q'}(F_j/\rho_{F_j}(\xi)), \quad (2.71)$$

with c independent of j . By [1] (see Remark below), taking into account the fact that $\rho_{F_j}(\xi) \geq 2^{j-1}\delta$, it follows that

$$C_{2/q,q'}(F_j/\rho_{F_j}(\xi)) \leq C_{2/q,q'}(F_j/2^{j-1}\delta) \leq c2^{-(j-1)(N-1-2/(q-1))}C_{2/q,q'}(F_j/\delta).$$

Therefore, by (2.71),

$$\begin{aligned}
 u(\xi) &\leq \sum_{j=1}^J U_{F_j}(\xi) \\
 &\leq c\rho(\xi) \sum_{j=1}^J 2^{-(j-1)(1+2/(q-1))} \delta^{-1-2/(q-1)} 2^{-(j-1)(N-1-2/(q-1))} C_{2/q,q'}(K/\delta) \\
 &\leq c\rho(\xi) \delta^{-1-2/(q-1)} C_{2/q,q'}(K/\delta) \sum_{j=1}^{\infty} 2^{-(j-1)N} \\
 &\leq c\rho(\xi) \rho_K(\xi)^{-1-2/(q-1)} C_{2/q,q'}(K/\rho_K(\xi)). \quad \square
 \end{aligned}$$

Remark. By the argument employed in the first part of the proof of [1, Thm. 5.2.1] (see in particular inequality (5.2.3)),

$$C_{2/q,q'}(tE) \leq At^{N-1-2/(q-1)} C_{2/q,q'}(E) \quad \forall t \in (0, 1], \tag{2.72}$$

for every $q \geq q_c = (N + 1)/(N - 1)$ and every compact set $E \subset \partial\Omega$, with the constant A depending only on $q, N, \partial\Omega$. (Recall that, for $q \geq q_c$, $2q'/q = 2/(q - 1) \leq N - 1$.) In addition, for every positive number τ ,

$$C_{2/q,q'}(\alpha E) \leq AC_{2/q,q'}(E) \quad \forall \alpha \in (0, \tau], \tag{2.73}$$

where A depends only on $q, N, \partial\Omega$ and τ . For the proof of this assertion see Lemma B.2.

3. Boundary singularities: estimates from below

Given a point $\xi \in \mathbb{R}^N$, a set $K \subset \mathbb{R}^N$ and an integer m , put

$$\begin{aligned}
 S_m(\xi) &:= \{x \in \mathbb{R}^N : r_{m+1} \leq |x - \xi| \leq r_m\}, \quad r_m = 2^{-m}, \\
 K_m(\xi) &:= K \cap S_m(\xi).
 \end{aligned} \tag{3.1}$$

In addition we define

$$u_K = \sup\{u_\mu : \mu \in W_+^{2/q,q'}(\Sigma), \mu(\Sigma \setminus K) = 0\}. \tag{3.2}$$

Theorem 3.1. Assume that $q \geq q_c = (N + 1)/(N - 1)$ and let K be a compact subset of $\partial\Omega$. Then there exists a constant $c = c(\Omega, N, q)$, independent of K , such that

$$u_K(\xi) \geq c\rho(\xi) \sum_{m=-\ell(\Omega)}^{\infty} r_m^{-1-2/(q-1)} C_{2/q,q'}(K_m(\xi)/r_m) \tag{3.3}$$

for every point $\xi \in \Omega$. Here $\ell(\Omega)$ is an integer such that

$$\text{diam } \Omega \leq 2^{\ell(\Omega)} < 2 \text{ diam } \Omega.$$

Proof. Inequality (3.3) is invariant with respect to dilations. Therefore, without loss of generality, we assume that $\beta_0 = 1$.

It is sufficient to prove (3.3) under the following additional conditions:

$$(i) \text{ diam } K \leq 1/8, \quad (ii) \rho_K(\xi) \leq 1/8. \quad (3.4)$$

We verify this assertion in two steps.

First, assuming that (3.3) holds for K and ξ satisfying (3.4), we show that it holds for all $\xi \in \Omega$. Put

$$\Omega_\tau^K = \{x \in \Omega : \rho_K(x) > \tau\}. \quad (3.5)$$

In $\Omega_{1/8}^K$ inequality (3.3) is equivalent to

$$u_K(\xi) \geq c\rho(\xi)C_{2/q,q'}(K). \quad (3.6)$$

Clearly, for $\xi \in \Omega_{1/8}^K$, the right hand side of (3.3) is bounded above by $c\rho(\xi)C_{2/q,q'}(K)$ so that (3.6) implies (3.3). On the other hand, (3.3) implies that for every $\xi \in \Omega_{1/8}^K$,

$$\begin{aligned} u_K(\xi) &\geq c\rho(\xi) \sum_{m=-\ell(\Omega)}^3 r_m^{-1-2/(q-1)} C_{2/q,q'}(K_m(\xi)/r_m) \\ &\geq c\rho(\xi) \sum_{m=-\ell(\Omega)}^3 C_{2/q,q'}(K_m(\xi)/r_m) \geq c\rho(\xi)C_{2/q,q'}(K). \end{aligned} \quad (3.7)$$

By Theorem 2.1,

$$u_K(\xi) \leq c(\tau)\rho(\xi)C_{2/q,q'}(K) \quad \forall \xi \in D_\tau^K. \quad (3.8)$$

Therefore, setting $\eta = u_K^{q-1}$, we have

$$-\Delta u_K + \eta u_K = 0 \quad \text{in } \Omega_{1/16}^K, \quad (3.9)$$

where $\sup_{\Omega_{1/16}^K} |\eta| \leq c$. By Hopf's lemma, $-\partial_n u_K(x) > 0$ for every $x \in \partial\Omega$ such that $\rho_K(x) > 1/16$. Therefore $\inf_{\Omega \cap \Omega_{1/8}^K} u_K/\rho > 0$. By (3.7) and our assumption,

$$u_K(\xi) \approx \rho(\xi)C_{2/q,q'}(K) \quad \text{on } \Omega \cap \partial\Omega_{1/8}^K.$$

Consequently, there exists a constant c depending only on Ω , q such that (3.6) holds in $\Omega_{1/8}^K$.

Next we show that if (3.3) holds for K such that $\text{diam } K \leq 1/8$ then it holds for every compact set $K \subset \partial\Omega$. There exists a natural number n_0 such that Σ can be covered by n_0 balls of diameter $1/16$. Therefore every compact set K can be written as a union $\bigcup_{i=1}^{n_0} F_i$ of compact sets, each of which has diameter smaller than $1/8$. Since

$$u_K \geq \frac{1}{n_0} \sum_{i=1}^{n_0} u_{F_i}$$

we conclude that

$$\begin{aligned}
 u_K(\xi) &\geq \frac{c}{n_0} \rho(\xi) \sum_{i=1}^{n_0} \sum_{m=-\ell(\Omega)}^{\infty} r_m^{-1-2/(q-1)} C_{2/q,q'}((F_i)_m(\xi)/r_m) \\
 &\geq c\rho(\xi) \sum_{m=-\ell(\Omega)}^{\infty} r_m^{-1-2/(q-1)} C_{2/q,q'}(K_m(\xi)/r_m)
 \end{aligned} \tag{3.10}$$

for every $\xi \in \Omega$.

Assume that $\text{diam } K \leq 1/8$ and put

$$A_K := \{x \in \Omega : \rho_K(x) \leq 1/8\}.$$

Then

$$\text{diam } A_K \leq 3/8, \quad K \subset B_{1/4}(\xi), \quad A_K \subset B_{3/8}(\xi) \quad \forall \xi \in A_K. \tag{3.11}$$

Pick a point $x_0 \in A_K$ and let $Q = Q(x_0)$ be a subdomain of Ω of class C^2 such that

$$\Omega \cap \bar{B}_{3/8}(x_0) \subset Q(x_0) \subset \Omega \cap B_{1/2}(x_0).$$

By (3.11),

$$A_K \subset Q(x_0) \subset B_{7/8}(\xi) \quad \forall \xi \in A_K. \tag{3.12}$$

Let u_K^0 be defined as in (3.2) with Ω replaced by $Q(x_0)$. Then

$$u_K^0 \leq u_K, \quad \rho_{\partial Q}(x) = \rho_{\partial \Omega}(x) \quad \forall x \in A_K. \tag{3.13}$$

The first assertion is obvious and the second is easily verified. Indeed, if it does not hold, there exists $x' \in A_K$ such that $\rho_{\partial Q}(x') < \rho_{\partial \Omega}(x')$. Let $y' \in \partial Q$ be a point such that $\rho_{\partial Q}(x') = |x' - y'|$. Then $y' \notin \partial \Omega$ and therefore $|y' - x_0| \geq 1/2$. On the other hand, by (3.11), $|x' - x_0| \leq 3/8$, so that

$$\rho_{\partial Q}(x') \geq |y' - x_0| - |x' - x_0| \geq 1/8.$$

This is impossible because $\rho_{\partial \Omega}(x') \leq 1/8$.

In the remaining part of the proof we establish the following:

Assertion 1. *For every compact set $K \subset \partial \Omega$ and $x_0 \in A_K$, if*

$$\text{diam } K < 1/8, \tag{3.14}$$

then inequality (3.3) is valid in $Q(x_0)$ (as previously defined), with u_K replaced by u_K^0 and with a constant independent of K and $Q(x_0)$.

First we verify that the assertion implies the statement of the theorem. Assumption (3.14) implies that $K \subset B_{1/4}(x_0)$ and $A_K \subset Q$. In view of (3.13), the assertion implies that (3.3) holds in A_K for every compact set $K \subset \partial \Omega$ satisfying (3.14). Therefore, by the first part of the proof, it implies the statement of the theorem.

Proof of Assertion 1. To simplify the notation we write $Q = Q(x_0)$. By the first part of the proof it is sufficient to show that (3.3) holds for $\xi \in A_K$. Let ξ be such a point and denote by $M = M(\xi)$ an integer such that

$$r_{M+1} < \rho_K(\xi) \leq r_M.$$

Put

$$Q_m(\xi) := Q \cap S_m(\xi), \quad m = 0, \dots, M(\xi), \quad Q_{M+1} = Q \cap B_{r_{M+1}}(\xi).$$

By (3.11), (3.12),

$$K = \bigcup_{m=1}^M K_m(\xi), \quad Q = \bigcup_{m=0}^{M+1} Q_m(\xi).$$

By translation we may locate the point ξ at the origin. Therefore in what follows we assume that $\xi = 0$ and write $Q_m = Q_m(0)$, $K_m = K_m(0)$, etc.

Let $\nu_m \in W_+^{2/q, q'}(\partial\Omega)$ be a measure with support in K_m and put $\nu = \sum_{m=1}^M \nu_m$ and $\nu_\epsilon = \epsilon\nu$ (for $\epsilon > 0$). Consider the problem

$$-\Delta u + u^q = 0 \quad \text{in } Q, \quad u = \nu_\epsilon \quad \text{on } \partial Q. \tag{3.15}$$

Let u be the (unique) solution of this problem and let v be the harmonic function with the same boundary data. Then $u < v$ so that

$$-\Delta u + v^q > 0 \quad \text{in } Q, \quad u = \nu_\epsilon \quad \text{on } \partial Q.$$

Hence, by the maximum principle,

$$u(x) \geq \int_K P(x, y) d\nu_\epsilon(y) - \int_Q G(x, z) \left(\int_K P(z, y) d\nu_\epsilon(y) \right)^q dz, \tag{3.16}$$

where P and G denote the Poisson and Green kernels in Q . Since u vanishes on $\partial Q \setminus K$,

$$u < u_K^0 \quad \text{in } Q. \tag{3.17}$$

From (3.16) we obtain

$$\begin{aligned} u(0) &\geq \epsilon \sum_{m=1}^M \int_{K_m} P(0, y) d\nu_m(y) - \epsilon^q \int_Q G(0, z) \left(\sum_{m=1}^M \int_{K_m} P(z, y) d\nu_m(y) \right)^q dz \\ &= \epsilon I_1 - \epsilon^q I_2. \end{aligned} \tag{3.18}$$

Recall that c or c_j denote positive constants which may vary from one formula to another, and which depend only on q, N, Q . (In particular, the constants do not depend on ξ .) To simplify notation we write $\rho'(x) = \rho_{\partial Q}(x)$.

First we observe that

$$P(x, y) \approx \rho'(x)|x - y|^{-N} \quad \forall x \in Q, \forall y \in K. \tag{3.19}$$

Hence,

$$I_1 = \sum_{m=1}^M \int_{K_m} P(0, y) d\nu_m(y) \geq c\rho'(0) \sum_{m=1}^M r_m^{-N} \nu_m(K_m). \tag{3.20}$$

In order to estimate I_2 from above we use the following well-known estimate of the Green kernel:

$$G(x, z) \approx \frac{\rho'(x)\rho'(z)|x - z|^{2-N}}{|x - z|^2 + \rho'(x)^2 + \rho'(z)^2} \quad \forall x, z \in Q. \tag{3.21}$$

As a consequence we find that, for every $z \in B_{r_j}(\Xi)$,

$$G(0, z) \leq c\rho'(0)\rho'(z)r_j^{-N}. \tag{3.22}$$

Hence

$$\begin{aligned} I_2 &= \int_Q G(0, z) \left(\sum_{m=1}^M \int_{K_m} P(z, y) dv_m(y) \right)^q dz \\ &= \sum_{j=0}^{M+1} \int_{Q_j} G(0, z) \left(\sum_{m=1}^M \int_{K_m} P(z, y) dv_m(y) \right)^q dz \\ &\leq c\rho'(0) \sum_{j=0}^{M+1} r_j^{-N} \int_{Q_j} \left(\sum_{m=1}^M \int_{K_m} P(z, y) dv_m(y) \right)^q \rho'(z) dz. \end{aligned} \tag{3.23}$$

Up to this point the measures v_m have not been specified; we define them below. Set

$$D_m := \frac{1}{r_{m-1}}(Q \cap B_{r_{m-2}}(0)) = (Q/r_{m-1}) \cap B_2(0), \tag{3.24}$$

$$F_m := K_m/r_{m-1} = (K/r_{m-1}) \cap S_1(0), \tag{3.25}$$

$$F_m^* := (K/r_{m-1}) \cap B_{1/2}(0), \quad m = 1, \dots, M. \tag{3.26}$$

If μ_m is the capacitary measure of F_m relative to the capacity $C_{2/q,q'}$ on ∂D_m (see [1, Sec. 2.2]) we define v_m as follows:

$$v_m(A) := r_{m-1}^{N-1-2/(q-1)} \mu_m(A/r_{m-1}) \tag{3.27}$$

for any Borel set $A \subset K_m$. By [1, Thm. 2.2.7], μ_m is a positive measure in $W^{-2/q,q}(\partial D_m)$ supported in F_m and

$$C_{2/q,q'}(F_m) = \mu_m(F_m) = \|\mu_m\|_{W^{-2/q,q}(\partial D_m)}^q. \tag{3.28}$$

By (3.27) and (3.28),

$$v_m(K_m) = r_{m-1}^{N-1-2/(q-1)} \mu_m(F_m) = r_{m-1}^{N-1-2/(q-1)} C_{2/q,q'}(F_m). \tag{3.29}$$

By (3.20) and (3.29),

$$I_1 \geq c\rho'(0) \sum_{m=1}^M r_{m-1}^{-1-2/(q-1)} C_{2/q,q'}(F_m). \tag{3.30}$$

In addition we have the following inequality:

$$\sum_{m=1}^M r_{m-1}^{-1-2/(q-1)} C_{2/q,q'}(F_m) \geq c \sum_{m=1}^M r_{m-1}^{-1-2/(q-1)} C_{2/q,q'}(F_m^*). \tag{3.31}$$

For its proof see Appendix A.

In order to estimate I_2 we need the following result [15]: If D is a C^2 bounded domain then, for every measure $\mu \in W^{-2/q,q}(\partial D)$,

$$\|\mathbb{P}_D(\mu)\|_{L^q(D, \rho_{\partial D} dx)} \approx \|\mu\|_{W^{-2/q,q}(\partial D)}. \tag{3.32}$$

In general, the constants involved in the relation \approx depend on q, N and D . However, a careful examination of the proof of Theorem 2.2 of [15] shows that, for some families of domains, the estimate is uniform. In particular we have:

Assertion 2. *Let $\xi \in A_K$ and $D_m := (Q/r_{m-1}) \cap B_2(\xi)$ for $1 \leq m \leq M(\xi)$. Then there exists a constant c , independent of ξ and m , such that*

$$c^{-1} \|\tau\|_{W^{-2/q,q}} \leq \|\mathbb{P}_{D_m}(\tau)\|_{L^q(D_m, \rho_m dx)} \leq c \|\tau\|_{W^{-2/q,q}} \tag{3.33}$$

for any measure $\tau \in W_+^{-2/q,q}(\partial D_m)$ with $\text{supp } \tau \subset F_m$. Here $\rho_m := \rho_{\partial D_m}$.

We continue the proof of Assertion 1, using Assertion 2, whose proof is deferred to the end of this section.

By (3.23),

$$\begin{aligned} I_2 &\leq c\rho'(0) \sum_{j=0}^{M+1} r_j^{-N} \int_{Q_j} \left(\sum_{m=1}^M \int_{K_m} P(z, y) dv_m(y) \right)^q \rho'(z) dz, \\ &\leq c\rho'(0) \sum_{j=0}^{M+1} r_j^{-N} \int_{Q_j} \sum_{m=j-1}^{j+1} \left(\int_{K_m} P(z, y) dv_m(y) \right)^q \rho'(z) dz \\ &\quad + c\rho'(0) \sum_{j=0}^{M+1} r_j^{-N} \left(\sum_{m=1}^{j-2} \sup_{z \in S_j} \int_{K_m} P(z, y) dv_m(y) \rho'(z)^{1/q} \right)^q |S_j| \\ &\quad + c\rho'(0) \sum_{j=0}^{M+1} r_j^{-N} \left(\sum_{m=j+2}^M \sup_{z \in S_j} \int_{K_m} P(z, y) dv_m(y) \rho'(z)^{1/q} \right)^q |S_j| \\ &= c\rho'(0)(I_{2,1} + I_{2,2} + I_{2,3}) \end{aligned} \tag{3.34}$$

where we put $v_m = 0$ for $m < 1$ or $m > M$. We proceed to estimate each of the three terms on the right hand side of this inequality.

Estimate of $I_{2,1}$. If $1 \leq m \leq M$ and $j - 1 \leq m \leq j + 1$ then

$$Q_j/r_{m-1} \subset D_m, \tag{3.35a}$$

$$P(z, y) \leq cr_{m-1}^{1-N} P_{D_m}(z', y') \quad \forall z \in Q_j, \forall y \in K_m, \tag{3.35b}$$

where $y' := y/r_{m-1}, z' := z/r_{m-1}$ and c is independent of j, m . The first relation follows directly from the definition of the domains. To verify the second relation observe that

$$\rho'(z) = \text{dist}(z, \partial Q) = r_{m-1} \text{dist}(z', r_{m-1}^{-1} \partial Q) \leq 4r_{m-1} \rho_m(z'),$$

where $\rho_m = \rho_{\partial D_m}$. Consequently, for every $z \in Q_j$ and $y \in K_m$,

$$P(z, y) \leq c_1 \rho'(z) |z - y|^{-N} \leq c_2 r_{m-1}^{1-N} \rho_{\partial D_m}(z') |z' - y'|^{-N} \leq c_3 r_{m-1}^{1-N} P_{D_m}(z', y').$$

Using (3.35) and (3.27) we obtain

$$\begin{aligned}
& \int_{Q_j} \left(\int_{K_m} P(z, y) dv_m(y) \right)^q \rho'(z) dz \\
& \leq c \int_{D_m} \left(\int_{F_m} r_{m-1}^{1-N} P_{D_m}(z', y') r_{m-1}^{N-1-2/(q-1)} d\mu_m(y') \right)^q r_{m-1}^{N+1} \rho_m(z') dz' \\
& \leq c r_{m-1}^{N+1-2q/(q-1)} \int_{D_m} \left(\int_{F_m} P_{D_m}(z', y') d\mu_m(y') \right)^q \rho_m(z') dz' \\
& \leq c r_{m-1}^{N-1-2/(q-1)} C_{2/q, q'}(F_m). \tag{3.36}
\end{aligned}$$

By (3.28), (3.33) and (3.36),

$$\begin{aligned}
I_{2,1} &= \sum_{j=0}^{M+1} r_j^{-N} \int_{Q_j} \sum_{m=j-1}^{j+1} \left(\int_{K_m} P(z, y) dv_m(y) \right)^q \rho'(z) dz \\
&\leq c \sum_{m=1}^M r_{m-1}^{-1-2/(q-1)} C_{2/q, q'}(F_m). \tag{3.37}
\end{aligned}$$

Estimate of $I_{2,2}$. Here $z \in S_j$, $y \in K_m$ and $m \leq j-2$. Thus $\rho'(z) \approx r_j$, $|y-z| \approx r_m$ and

$$P(z, y) \leq c \rho'(z) |y-z|^{-N} \leq c r_m^{-N} r_j.$$

Hence,

$$\begin{aligned}
I_{2,2} &= \sum_{j=3}^{M+1} r_j^{-N} \left(\sum_{m=1}^{j-2} \sup_{z \in S_j} \int_{K_m} P(z, y) dv_m(y) \rho'(z)^{1/q} \right)^q |S_j| \\
&\leq c \sum_{j=3}^{M+1} \left(\sum_{m=1}^{j-2} \sup_{z \in S_j} \int_{K_m} P(z, y) dv_m(y) \rho'(z)^{1/q} \right)^q \\
&\leq c \sum_{j=3}^{M+1} r_j^{1+q} \left(\sum_{m=1}^{j-2} r_m^{-N} v_m(K_m) \right)^q. \tag{3.38}
\end{aligned}$$

Consequently, by (3.29),

$$\begin{aligned}
I_{2,2} &\leq c \sum_{j=3}^{M+1} r_j^{1+q} \left(\sum_{m=1}^{j-2} r_{m-1}^{-1-2/(q-1)} C_{2/q, q'}(F_m) \right)^q \\
&\leq c \sum_{j=3}^{M+1} r_j^{1+q} \left(\sum_{m=1}^{j-2} r_{m-1}^{-1-2/(q-1)} C_{2/q, q'}(F_m^*) \right)^q. \tag{3.39}
\end{aligned}$$

Furthermore,

$$\sum_{j=3}^{M+1} r_j^{1+q} \left(\sum_{m=1}^{j-2} r_{m-1}^{-1-2/(q-1)} C_{2/q, q'}(F_m^*) \right)^q \leq c \sum_{m=1}^M r_{m-1}^{-1-2/(q-1)} C_{2/q, q'}(F_m^*). \tag{3.40}$$

For the proof of this inequality see Appendix B.

Estimate of $I_{2,3}$. Here $z \in S_j$, $y \in K_m$ and $m \geq j + 2$. Thus $\rho'(z) \approx r_j$, $|y - z| \approx r_j$ and

$$P(z, y) \leq c\rho'(z)|y - z|^{-N} \leq cr_j^{1-N}.$$

Hence

$$\begin{aligned} I_{2,3} &= \sum_{j=0}^{M-2} r_j^{-N} \left(\sum_{m=j+2}^M \sup_{z \in S_j} \int_{K_m} P(z, y) dv_m(y) \rho'(z)^{1/q} \right)^q |S_j| \\ &\leq c \sum_{j=0}^{M-2} \left(\sum_{m=j+2}^M \sup_{z \in S_j} \int_{K_m} P(z, y) dv_m(y) \rho'(z)^{1/q} \right)^q \\ &\leq c \sum_{j=0}^{M-2} r_j \left(\sum_{m=j+2}^M r_j^{1-N} v_m(K_m) \right)^q \\ &\leq c \sum_{j=0}^{M-2} r_j^{1-q(N-1)} \left(\sum_{m=j+2}^M r_m^{N-1-2/(q-1)} C_{2/q, q'}(F_m^*) \right)^q. \end{aligned} \quad (3.41)$$

Put

$$\theta(r) = \int_0^r t^{N-2-2/(q-1)} C_{2/q, q'} \left(\frac{2}{t} (K \cap B_t) \right) dt.$$

By Lemmas B.1 and B.2 we obtain

$$\begin{aligned} \sum_{j=0}^{M-2} r_j^{1-q(N-1)} \left(\sum_{m=j+2}^M r_m^{N-1-2/(q-1)} C_{2/q, q'}(F_m^*) \right)^q \\ \leq c \sum_{j=0}^{M-2} r_j^{1-q(N-1)} \theta(r_j)^q \leq c \int_0^\infty t^{-q(N-1)} \theta(t)^q dt. \end{aligned} \quad (3.42)$$

Further, by Hardy's inequality,

$$\begin{aligned} \int_0^\infty t^{-q(N-1)} \theta(t)^q dt &\leq c \int_0^\infty t^{-q(N-2)} \theta'(t)^q dt \\ &\leq c \int_0^\infty t^{-2q/(q-1)} C_{2/q, q'} \left(\frac{2}{t} (K \cap B_t) \right)^q dt. \end{aligned} \quad (3.43)$$

Combining (3.42) with (3.43) and applying once again Lemmas B.1 and B.2 we obtain

$$\begin{aligned} \sum_{j=0}^{M-2} r_j^{1-q(N-1)} \left(\sum_{m=j+2}^M r_m^{N-1-2/(q-1)} C_{2/q, q'}(F_m^*) \right)^q \\ \leq c \sum_{m=1}^M r_m^{1-2q/(q-1)} C_{2/q, q'}(F_m^*)^q \leq c' \sum_{m=1}^M r_m^{1-2q/(q-1)} C_{2/q, q'}(F_m^*). \end{aligned} \quad (3.44)$$

For the last inequality we used the fact that $C_{2/q,q'}(F_m^*)$ is bounded (see (3.26)). By (3.41) and (3.44),

$$I_{2,3} \leq c \sum_{m=1}^M r_{m-1}^{-1-2/(q-1)} C_{2/q,q'}(F_m^*). \tag{3.45}$$

Combining (3.34), (3.37), (3.39), (3.40) and (3.45) we obtain

$$I_2 \leq c \sum_{m=1}^M r_{m-1}^{-1-2/(q-1)} C_{2/q,q'}(F_m^*). \tag{3.46}$$

Finally, by (3.18), (3.30), (3.31) and (3.46),

$$u(0) \geq \rho'(0)(c_1\epsilon - c_2\epsilon^q) \sum_{m=0}^{M+1} r_{m-1}^{-1-2/(q-1)} C_{2/q,q'}(F_m^*). \tag{3.47}$$

Thus, choosing $\epsilon > 0$ sufficiently small we obtain

$$u(0) \geq c\rho'(0) \sum_{m=1}^M r_{m-1}^{-1-2/(q-1)} C_{2/q,q'}(F_m^*). \quad \square \tag{3.48}$$

Proof of Assertion 2. By translation we locate the origin at ξ . By an additional rotation we locate the point $(0, \dots, 0, \rho'(0))$ at $\sigma(0)$ (= the nearest point to the origin on ∂Q). By the definition of $M = M(\xi)$,

$$2^{M-1} \rho'(0) \leq 2^{M-1} \rho_K(0) \leq 1/2. \tag{3.49}$$

Therefore, in view of (3.24) and the regularity of ∂Q , if $m_0 \in \mathbb{N}$ is sufficiently large (depending only on Q) then, for $m_0 < M$, the domains D_m with $m_0 \leq m \leq M$ “approach” the domain $B_2(0) \cap \{x : x_N < 2^{m-1} \rho'(0)\}$. To make this statement more precise we need some notation:

$$O_m^r := B_r(0) \cap \{x : x_N < 2^{m-1} \rho'(0)\}, \tag{3.50}$$

$$E_m := \partial D_m \cap \left(\frac{1}{r_{m-1}} \partial Q \right). \tag{3.51}$$

For every $\alpha > 0$ there exists m_α such that for every $M > m_\alpha$ and every integer $m \in [m_\alpha, M]$, there exists an open neighborhood U_m of \bar{D}_m , a C^2 diffeomorphism $T_m : U_m \rightarrow U'_m$ and a number $d \in (1, 1 + \alpha)$, independent of m , such that

$$\sup_{m_\alpha \leq m \leq M} \|T_m\|_{C^2(U_m)} \leq c, \quad \sup_{m_\alpha \leq m \leq M} \|T_m^{-1}\|_{C^2(U'_m)} \leq c, \tag{3.52a}$$

$$d^{-1}|x| \leq |T_m(x)| \leq d|x|, \tag{3.52b}$$

$$E'_m := T_m(E_m) \subset \{x : x_N = a_m\}, \tag{3.52c}$$

where

$$\rho'(0)/d \leq r_{m-1} a_m \leq \rho'(0)d.$$

Since $F_m \subset E_m \cap S_1(0)$ we have

$$F'_m := T_m(F_m) \subset \{x : x_N = a_m, 1/4d \leq |x| \leq d/2\}. \tag{3.53}$$

Let L_m be the elliptic operator in D'_m corresponding to the Laplacian in D_m by the transformation T_m . Let \tilde{T}_m denote the mapping induced by T_m on the space of Borel measures $\mathfrak{M}(\partial D_m)$ onto $\mathfrak{M}(\partial D'_m)$. Then \tilde{T}_m maps $W^{-2/q,q}(\partial D_m)$ continuously onto $W^{-2/q,q}(\partial D'_m)$ and has a continuous inverse. Let P_{D_m} (resp. $P_{D'_m}$) denote the Poisson kernel for Δ in D_m (resp. in D'_m) and let $P_{D'_m}^{L_m}$ denote the Poisson kernel for L_m in D'_m . If $\tau \in W^{-2/q,q}(\partial D_m)$ and $\tilde{\tau} := \tilde{T}_m(\tau)$, then

$$\tilde{u}_m := \mathbb{P}_{D'_m}^{L_m}(\tilde{\tau}), \quad u_m := \mathbb{P}_{D_m}(\tau) \implies u_m = \tilde{u}_m \circ \tilde{T}_m \tag{3.54}$$

and

$$\mathbb{P}_{D'_m}^{L_m}(\tilde{\tau}) \approx \mathbb{P}_{D'_m}(\tilde{\tau}). \tag{3.55}$$

The last relation follows from the equivalence of the Poisson kernels for L_m and Δ in D'_m . In view of the uniform bounds on T_m , this equivalence relation is uniform with respect to m and ξ .

Put

$$B_r^-(0) := \{x \in B_r(0) : x_N < 0\}, \quad B_r^0 := \{x \in B_r(0) : x_N = 0\}, \\ S_m := (0, 0, \dots, 0, a_m), \quad A^1 := B_{3/2}^-(0), \quad A^2 := B_3^-(0).$$

Then, by (3.52) and (3.49), assuming that $0 < \alpha < 10^{-1}$,

$$S_m + A^1 \subset O_m^{2/d} \subset D'_m \subset O_m^{2d} \subset S_m + A^2. \tag{3.56}$$

By (3.53),

$$F'_m = T_m(F_m) \subset S_m + B_1^0 \subset S_m + \partial A^i, \quad i = 1, 2. \tag{3.57}$$

Hence, by (3.56),

$$P_{S_m+A^1}(x, y) \leq P_{D'_m}(x, y) \quad \forall x \in S_m + A^1, \tag{3.58a}$$

$$P_{S_m+A^2}(x, y) \geq P_{D'_m}(x, y) \quad \forall x \in D'_m, \tag{3.58b}$$

for every $y \in S_m + B_1^0$.

If τ is a bounded Borel measure with support in F_m , let $\tilde{\tau}_m := \tilde{T}_m(\tau)$ and denote by τ_m^* the measure on $x_N = 0$ given by $\tau_m^*(B) = \tilde{\tau}_m(B + S_m)$ for every Borel set B . Then, in view of (3.57),

$$\int_{F'_m} P_{S_m+A^i}(x, y) d\tilde{\tau}(y) = \int_{B_1^0} P_{A^i}(x', y') d\tau^*(y') \quad \forall x \in S_m + A^i, \tag{3.59}$$

for $i = 1, 2$ and $x' = x + S_m$. By (3.32), if $\tau \in W^{-2/q,q}(\partial D_m)$ and $\text{supp } \tau \subset F_m$,

$$\|\tau^*\|_{W^{-2/q,q}(\partial A^i)} \approx \left\| \int_{B_1^0} P_{A^i}(\cdot, y) d\tau^*(y) \right\|_{L_{bw}^q(A^i)}, \quad i = 1, 2. \tag{3.60}$$

Here for any open set $O \subset \mathbb{R}^N$, $L^q_{bw}(O)$ denotes the weighted Lebesgue space L^q with weight $\text{dist}(x, \partial O)$.

Combining (3.58), (3.59) and (3.60) we obtain

$$\begin{aligned} c_1 \|\tilde{\tau}\|_{W^{-2/q,q}(S_m+\partial A^1)} &= c_1 \|\tau^*\|_{W^{-2/q,q}(\partial A^1)} \\ &\leq \|\mathbb{P}_{A^1}(\tau^*)\|_{L^q_{bw}(A^1)} \leq \|\mathbb{P}_{D'_m}(\tilde{\tau})\|_{L^q_{bw}(D'_m)} \leq \|\mathbb{P}_{A^2}(\tau^*)\|_{L^q_{bw}(A^2)} \\ &\leq c_2 \|\tau^*\|_{W^{-2/q,q}(\partial A^2)} = c_2 \|\tilde{\tau}\|_{W^{-2/q,q}(S_m+\partial A^2)}. \end{aligned} \tag{3.61}$$

Obviously the constants depend only on A_i . For measures $\tau \in W^{-2/q,q}(\partial D_m)$ with support in F_m ,

$$\|\tilde{\tau}\|_{W^{-2/q,q}(S_m+\partial A^i)} \approx \|\tilde{\tau}\|_{W^{-2/q,q}(S_m+B^0_{5/4})} \approx \|\tau\|_{W^{-2/q,q}(E_m)}$$

and, in view of (3.52), the equivalence relations hold with constants independent of m or ξ . Hence (3.61) implies (3.33). \square

4. A uniqueness result and asymptotic behavior of solutions

The inequalities established in the previous section are fundamental in the study of positive solutions of the equation (1.2). In this section we present some of the first consequences of these estimates. Further applications, including the full characterization of such solutions in terms of their boundary trace will be presented in a follow-up paper.

Theorem 4.1. *Let Ω be a bounded domain of class C^2 and K be a compact subset of $\partial\Omega$. As in Section 3 denote by u_K the maximal σ -moderate solution of (1.2) (see (3.2)) and by U_K the maximal solution of this equation. Then $U_K = u_K$.*

Proof. By Theorem 3.1, for every point $\xi \in \Omega$,

$$u_K(\xi) \geq c\rho(\xi) \sum_{m=-\ell(\Omega)}^{\infty} r_m^{-1-2/(q-1)} C_{2/q,q'}(K_m(\xi)/r_m), \tag{4.1}$$

with $K_m(\xi)$ as in (3.1). The constant c depends only on Ω , N and q .

For every $\xi \in \Omega$,

$$K = \bigcup_{m=-\ell(\Omega)}^M K_m(\xi), \quad M = M(\xi) := \sup\{m \in \mathbb{Z} : K_m(\xi) \neq \emptyset\}, \tag{4.2}$$

and consequently,

$$U_K(x) \leq \sum_{m=-\ell(\Omega)}^{\infty} U_{K_m(\xi)}(x) \quad \forall x \in \Omega. \tag{4.3}$$

To verify this inequality, keep ξ fixed and let V_m denote a relatively open neighborhood of $K_m(\xi)$ on $\partial\Omega$. Put $V := \partial\Omega \setminus \bigcup_{m=-\ell(\Omega)}^M V_m$ and

$$V_m^\beta = \{x \in \Sigma_\beta : \sigma(x) \in V_m\}, \quad V^\beta = \{x \in \Sigma_\beta : \sigma(x) \in V\}$$

for $\beta \in (0, \beta_0)$ and $-\ell(\Omega) \leq m \leq M(\xi)$. Denote by v_m^β and v^β the solutions of the boundary value problem

$$-\Delta v + v^q = 0 \quad \text{in } \Omega'_\beta, \quad v = u_K \chi_W \quad \text{on } \Sigma_\beta \tag{4.4}$$

with $W = V_m^\beta$ and $W = V^\beta$ respectively. Clearly $\lim_{\beta \rightarrow 0} v^\beta = 0$ and there exists a sequence $\{\beta_j\}$ decreasing to zero such that $\{v_m^{\beta_j}\}$ converges to a solution v_m of (1.2) in Ω , for every integer $m \in [-\ell(\Omega), M(\xi)]$. Furthermore v_m vanishes on $\partial\Omega \setminus \bar{V}_m$ so that $v_m \leq u_{\bar{V}_m}$. In addition, for every $\beta \in (0, \beta_0)$,

$$U_K \leq \sum_{m=-\ell(\Omega)}^M v_m^\beta + v^\beta,$$

so that

$$U_K \leq \sum_{m=-\ell(\Omega)}^M v_m \leq \sum_{m=-\ell(\Omega)}^M U_{\bar{V}_m}.$$

Since $U_{K_m} = \inf_{K_m \subset F} U_F$ (F compact subsets of $\partial\Omega$), the last inequality implies (4.3).

By Theorem 2.1,

$$U_{K_m(\xi)}(x) \leq c \rho_{\partial\Omega}(x) \rho_m(x)^{-1-2/(q-1)} C_{2/q,q'}(K_m(\xi)/\rho_m(x)) \quad \forall x \in \Omega, \tag{4.5}$$

where $\rho_m(x) = \rho_{K_m(\xi)}(x)$ and c is a constant depending only on Ω, N and q . Hence, by (4.3),

$$U_K(x) \leq c \sum_{m=-\ell(\Omega)}^M \rho_{\partial\Omega}(x) \rho_m(x)^{-1-2/(q-1)} C_{2/q,q'}(K_m(\xi)/\rho_m(x)) \quad \forall x \in \Omega. \tag{4.6}$$

Since $\rho_m(\xi) \in [r_{m+1}, r_m]$ it follows that (see Lemma B.2)

$$C_{2/q,q'}(K_m(\xi)/\rho_m(\xi)) \approx C_{2/q,q'}(K_m(\xi)/r_m).$$

Therefore, by (4.6)

$$U_K(\xi) \leq c \rho_{\partial\Omega}(\xi) \sum_{m=-\ell(\Omega)}^M r_m^{-1-2/(q-1)} C_{2/q,q'}(K_m(\xi)/r_m). \tag{4.7}$$

Recall that ξ was an arbitrary point in Ω . Inequalities (4.1) and (4.7) imply that there exists a constant C depending only on q, N, Ω such that

$$U_K(x) \leq C u_K(x) \quad \forall x \in \Omega. \tag{4.8}$$

Clearly $u_K \leq U_K$. If the two solutions are not identical we have

$$u(x) < U(x) \quad \forall x \in \Omega. \tag{4.9}$$

Let $\alpha = 1/2C$ and put $v = (1 + \alpha)u_K - \alpha U_K$. Then $\alpha u_K < v < u_K$. As in [13] we find that v is a supersolution of (1.2). Since αu_K is a subsolution it follows that there exists a solution w such that $\alpha u_K < w < u_K$. If μ is a positive bounded Borel measure on $\partial\Omega$

then $u_{\alpha\mu}$ is the smallest solution which dominates the subsolution αu_μ . Hence, if μ is supported in K , $u_{\alpha\mu} < w$. Therefore

$$u_K = \sup\{u_{\alpha\mu} : \mu \in \mathcal{M}_+(\partial\Omega)\} \leq w.$$

This contradiction completes the proof. \square

Theorem 4.2. *For every $a \geq 1$ there exists a constant $c(a) > 0$, depending also on q, N, Ω , such that, for every $\sigma \in \partial\Omega$,*

$$\begin{aligned} \frac{1}{c(a)} \int_s^{\beta_0} t^{-1-2/(q-1)} C_{2/q,q'} \left(\frac{1}{t} (K \cap \bar{B}_t(\sigma)) \right) \frac{dt}{t} &\leq \frac{U_K(x)}{|x - \sigma|} \\ &\leq c(a) \int_s^{\beta_0} t^{-1-2/(q-1)} C_{2/q,q'} \left(\frac{1}{t} (K \cap \bar{B}_t(\sigma)) \right) \frac{dt}{t} + O(1) \end{aligned} \quad (4.10)$$

for all $x \in \Omega$ with $s = |x - \sigma| \leq a\rho(x)$. Note that if $\sigma \notin K$ then the integral is bounded.

Proof. The inequality is trivial for x such that $\rho(x) \geq \beta_0$. Therefore we assume $\rho(x) < \beta_0$. Let $\sigma \in K$. It is sufficient to prove (4.10) (with a constant depending only on q, N, Ω) for $x = \sigma - s\nu(\sigma)$, $0 < s < \beta_0$. (Recall that $\nu(\sigma)$ is the outward unit normal at σ .) Indeed, once it is proved in this case, (4.10) follows, by Harnack's inequality, for arbitrary $x \in \Omega$ such that $s = |x - \sigma| \leq a\rho(x)$.

By (4.1), (4.7) and (3.31), if $x = \sigma - s\nu(\sigma)$, $0 < s < \beta_0$,

$$\begin{aligned} U_K(x) &\approx s \sum_{m=-\ell(\Omega)}^M r_m^{-1-2/(q-1)} C_{2/q,q'}(K_m(x)/r_m) \\ &\approx s \sum_{m=-\ell(\Omega)}^M r_m^{-1-2/(q-1)} C_{2/q,q'} \left(\frac{1}{r_m} (K \cap B_{r_{m+1}}(x)) \right), \end{aligned} \quad (4.11)$$

with $M = M(x)$ as in (4.2). These relations together with Corollary B.3 imply

$$U_K(x) \approx s \int_{2s}^{\text{diam } \Omega} t^{-1-2/(q-1)} C_{2/q,q'} \left(\frac{1}{t} (K \cap B_{t/2}(x)) \right) \frac{dt}{t}. \quad (4.12)$$

Finally, since $B_{t-s}(\sigma) \subset B_t(x) \subset B_{3(t-s)}(\sigma)$, a simple computation (using Lemma B.2) yields (4.10) for x as above. \square

Remark. By [1, Sec. 5.2] (see in particular Cor. 5.2.3 and the first part of the proof of Thm. 5.2.1).

$$\begin{aligned} C_{2/q,q'}(\gamma E) &\approx c\gamma^{N-1-2/(q-1)} C_{2/q,q'}(E) \quad \forall \gamma > 0, \quad \text{if } q > q_c, \\ C_{2/q,q'}(\gamma E) &\leq c\gamma^{N-1-2/(q-1)} C_{2/q,q'}(E) \quad \forall \gamma \in (0, 1), \quad \text{if } q = q_c. \end{aligned} \quad (4.13)$$

A set K is said to be $(2/q, q')$ -thick at a point $\sigma \in \Sigma$ if

$$J_q(\sigma; K) := \int_0^{\beta_0} \left(\frac{C_{2/q,q'}(K \cap \bar{B}_t(\sigma))}{t^{N-1-2/(q-1)}} \right)^{q-1} \frac{dt}{t} = \infty. \quad (4.14)$$

If $J_q(\sigma; K) < \infty$ we say that K is $(2/q, q')$ -thin at σ (see [1]). By (4.13),

$$\begin{aligned} J_q(\sigma; K) &\approx \int_0^{\beta_0} C_{2/q, q'} \left(\frac{1}{t} (K \cap \bar{B}_t(\sigma)) \right)^{q-1} \frac{dt}{t} \quad \text{if } q > q_c, \\ J_q(\sigma; K) &\leq \int_0^{\beta_0} C_{2/q, q'} \left(\frac{1}{t} (K \cap \bar{B}_t(\sigma)) \right)^{q-1} \frac{dt}{t} \quad \text{if } q = q_c. \end{aligned} \tag{4.15}$$

It is well known that K is $(2/q, q')$ -thick, $(2/q, q')$ -a.e. in K , i.e., everywhere except for a subset of $C_{2/q, q'}$ -capacity zero.

In the next definition we introduce a related notion:

Definition 4.3. A point $\sigma \in \Sigma$ is a $(2/q, q')$ -concentration point of K if

$$J_q^*(\sigma; K) := \int_0^{\beta_0} t^{-1-2/(q-1)} C_{2/q, q'} \left(\frac{1}{t} (K \cap \bar{B}_t(\sigma)) \right) \frac{dt}{t} = \infty. \tag{4.16}$$

If the integral is finite we say that K is $(2/q, q')$ -sparse at σ .

The function $\theta_K(\cdot; \sigma)$ defined by

$$\theta_K(t; \sigma) := C_{2/q, q'} \left(\frac{1}{t} (K \cap \bar{B}_t(\sigma)) \right), \quad 0 < t \leq 1,$$

will be called the capacity distribution of K at σ .

It is clear that if K is closed (as in our case) then every point at which it is $(2/q, q')$ -thick (resp. $(2/q, q')$ -concentrated) belongs to K . The following blow-up criterion is an immediate consequence of Theorem 4.2.

Corollary 4.4. For every $\sigma \in \Sigma$, either

$$\lim_{\substack{x \rightarrow \sigma \\ x \in C_a^\sigma(\Omega)}} U_K(x) |x - \sigma|^{-1} = \infty \quad \text{for every } a \geq 1,$$

or

$$\sup_{x \in C_a^\sigma(\Omega)} U_K(x) |x - \sigma|^{-1} < \infty \quad \text{for every } a \geq 1.$$

The first case occurs iff σ is a $(2/q, q')$ -concentration point of K .

Our next result provides more detailed information on the rate of blow up at points of concentration of K .

Theorem 4.5. (a) If K is $(2/q, q')$ -thick at σ then

$$\int_0^1 U_K(\Gamma(t))^{q-1} t \, dt = \infty \tag{4.17}$$

for every curve $\Gamma \in \text{Lip}([0, 1], \Omega \cup \{\sigma\})$ such that $\Gamma(0) = \sigma$ and $0 < |\Gamma(t) - \sigma| \leq a\rho(\gamma(t))$ for some $a \geq 1$ and every $t \in (0, 1]$. Thus (4.17) holds $(2/q, q')$ -a.e. in K . Obviously the integral is finite everywhere outside K .

(b) If K is $(2/q, q')$ -thick at σ then it is $(2/q, q')$ -concentrated at σ .

(c) For every $\sigma \in \Sigma$,

$$\frac{1}{c(a)} s^{-2/(q-1)} \theta_K(s; \sigma) \leq U_K(x) \quad \forall x \in C_a^\Omega(\sigma), \quad (4.18)$$

where $0 < s := |x - \sigma| < 1$.

(d) The following asymptotic estimate holds:

$$\limsup_{\substack{x \rightarrow \sigma \\ x \in C_a^\Omega(\sigma)}} |x - \sigma|^{2/(q-1)} U_K(x) \approx \limsup_{s \rightarrow 0} \theta_K(s; \sigma). \quad (4.19)$$

If, in addition, $\theta_K(\cdot; \sigma)$ has a limit as $s \rightarrow 0$, which we denote by $\theta_K^0(\sigma)$, then

$$\begin{aligned} \frac{1}{c(a)} \theta_K^0(\sigma) &\leq \liminf_{\substack{x \rightarrow \sigma \\ x \in C_a^\Omega(\sigma)}} |x - \sigma|^{2/(q-1)} U_K(x) \\ &\leq \limsup_{\substack{x \rightarrow \sigma \\ x \in C_a^\Omega(\sigma)}} |x - \sigma|^{2/(q-1)} U_K(x) \leq c(a) \theta_K^0(\sigma) \end{aligned} \quad (4.20)$$

for every $a \geq 1$.

Proof. By (4.10), for every $\sigma \in \Sigma$,

$$\frac{s}{c(a)} \int_s^{2s} t^{-1-2/(q-1)} \theta_K(t; \sigma) \frac{dt}{t} \leq U_K(x)$$

for every $x \in C_a^\Omega(\sigma)$, $s = |x - \sigma| < 1/2$. By Corollary B.3,

$$\theta_K(t; \sigma) \approx \theta_K(s; \sigma), \quad s < t < 2s.$$

This implies (4.18). This inequality can be rewritten in the form

$$\frac{1}{c(a)} \theta_K(s; \sigma)^{q-1} \leq U_K(x)^{q-1} s^2. \quad (4.21)$$

By (4.21) and (4.15), if $x_s = \sigma - s\nu(\sigma)$,

$$J_q(\sigma; K) \leq c(a) \int_0^{\beta_0} U_K(x_s)^{q-1} s \, ds. \quad (4.22)$$

This proves (4.17) in the case $a = 1$. Assertion (a) in the general case follows from (4.22) and the Harnack inequality.

If K is $(2/q, q')$ -thick at σ then, by (4.17), for every $\epsilon > 0$,

$$\limsup_{s \rightarrow 0} U_K(x_s) s^{2/(q-1)-\epsilon} = \infty.$$

By Corollary 4.4 it follows that $\lim U_K(x_s)/s = \infty$ and K is $(2/q, q')$ -concentrated at σ .

We turn to the proof of the last assertion. Put

$$V(s) := \int_s^1 t^{-1-2/(q-1)} \theta_K(t; \sigma) \frac{dt}{t}.$$

For every $k \geq 1$,

$$0 \leq V(s) - V(ks) = \int_s^{ks} t^{-1-2/(q-1)} \theta_K(t; \sigma) \frac{dt}{t} \quad (4.23)$$

and by Lemma B.2,

$$\theta_K(t; \sigma) \stackrel{k}{\approx} \theta_K(s; \sigma), \quad s < t < ks.$$

(The notation $\stackrel{k}{\approx}$ indicates that the constants involved in the equivalence relation depend on k .) Hence

$$0 \leq V(s) - V(ks) \stackrel{k}{\approx} s^{-1-2/(q-1)} \theta_K(s; \sigma). \quad (4.24)$$

Let $k \geq 1$ and put $x_s = \sigma - s\nu(\sigma)$. Then, by (4.10),

$$\frac{1}{c} s V(s) \leq U_K(x_s) \leq cs(V(s) + O(1)). \quad (4.25)$$

By (4.24) and (4.25),

$$\begin{aligned} \frac{1}{cs} U_K(x_s) &\leq V(s) + O(1) \\ &\leq V(ks) + c_k s^{-1-2/(q-1)} \theta_K(s; \sigma) + O(1) \\ &\leq \frac{c}{ks} U_K(x_{ks}) + c_k s^{-1-2/(q-1)} \theta_K(s; \sigma) + O(1). \end{aligned} \quad (4.26)$$

Therefore,

$$U_K(x_s) \leq \frac{c^2}{k} U_K(x_{ks}) + c_k s^{-2/(q-1)} \theta_K(s; \sigma) + O(s). \quad (4.27)$$

Applying this inequality with $k^{1+2/(q-1)} = 2c^2$ we obtain

$$\limsup_{s \rightarrow 0} s^{2/(q-1)} U_K(x_s) \leq c \limsup_{s \rightarrow 0} \theta_K(s; \sigma). \quad (4.28)$$

Combining (4.18) and (4.28) we obtain (4.19) in the case $a = 1$. In the general case the inequality follows by an application of Harnack's inequality. Finally, (4.20) is a simple consequence of (4.18) and (4.19). \square

The following is an immediate consequence of the theorem.

Corollary 4.6. *If*

$$\liminf_{s \rightarrow 0} \theta_K(s; \sigma) > 0, \quad (4.29)$$

then, for every $a \geq 1$,

$$\frac{1}{c(a)} |x - \sigma|^{-2/(q-1)} \leq U_k(x) \leq c(a) |x - \sigma|^{-2/(q-1)} \quad \text{in } C_a^\Omega(\sigma). \quad (4.30)$$

If

$$\lim_{s \rightarrow 0} \theta_K(s; \sigma) = 0 \quad (4.31)$$

then

$$|x - \sigma|^{2/(q-1)} U_K(x) \rightarrow 0 \quad \text{as } x \rightarrow \sigma \text{ non-tangentially.} \quad (4.32)$$

Remark. Obviously, condition (4.31) may occur at points where K is $(2/q, q')$ -thick. Therefore U_K may blow up at a rate weaker than the maximal rate even at such points.

Appendix A. Proof of inequality (3.31)

Let $k \in \mathbb{N}$ be a number to be determined later on. Noting that

$$F_m^* = (r_k F_{m+k}^*) \cup \bigcup_{j=0}^{k-1} (r_j F_{m+j}),$$

we obtain

$$\sum_{j=0}^{k-1} C_{2/q, q'}(r_j F_{m+j}) \geq C_{2/q, q'}(F_m^*) - C_{2/q, q'}(r_k F_{m+k}^*). \quad (\text{A.1})$$

By the argument employed in the first part of the proof of [1, Thm. 5.2.1] (see in particular inequality (5.2.3))

$$C_{2/q, q'}(tE) \leq At^{N-1-2/(q-1)} C_{2/q, q'}(E) \quad \forall t \in (0, 1], \quad (\text{A.2})$$

for every $q \geq q_c = (N+1)/(N-1)$ and every compact set $E \subset \partial Q$, with the constant A depending only on $q, N, \partial Q$. (Recall that, for $q \geq q_c$, $2q'/q = 2/(q-1) \leq N-1$.) Consequently, by (A.1),

$$\begin{aligned} C_{2/q, q'}(F_m^*) - Ar_k^{N-1-2/(q-1)} C_{2/q, q'}(F_{m+k}^*) \\ \leq A \sum_{j=0}^{k-1} r_j^{N-1-2/(q-1)} C_{2/q, q'}(F_{m+j}) \quad \text{for } m = 1, \dots, M. \end{aligned} \quad (\text{A.3})$$

Now choose k sufficiently large so that $a := Ar_k^{N-1} < 1$. Recall that $F_{M+j} = F_{M+j}^* = \emptyset$ for $j \geq 1$. Clearly

$$\begin{aligned} (1-a) \sum_{m=1}^M r_{m-1}^{-1-2/(q-1)} C_{2/q, q'}(F_m^*) \\ \leq \sum_{m=1}^M (r_{m-1}^{-1-2/(q-1)} C_{2/q, q'}(F_m^*) - ar_{m+k-1}^{-1-2/(q-1)} C_{2/q, q'}(F_{m+k}^*)). \end{aligned} \quad (\text{A.4})$$

Further, by (A.3) and (A.4),

$$\begin{aligned} \sum_{m=1}^M r_{m-1}^{-1-2/(q-1)} C_{2/q, q'}(F_m^*) &\leq A' \sum_{m=1}^M r_{m-1}^{-1-2/(q-1)} \sum_{j=0}^{k-1} r_j^{N-1-2/(q-1)} C_{2/q, q'}(F_{m+j}) \\ &= A' \sum_{j=0}^{k-1} r_j^N \sum_{m=1}^M r_{m+j-1}^{-1-2/(q-1)} C_{2/q, q'}(F_{m+j}) \leq kA' \sum_{m=1}^M r_{m-1}^{-1-2/(q-1)} C_{2/q, q'}(F_m), \end{aligned} \quad (\text{A.5})$$

where $A' = A/(1-a)$. This proves (3.31).

Appendix B. Proof of inequality (3.40)

We start with the following elementary lemma.

Lemma B.1. *Let $\varphi_i : (0, \infty) \rightarrow [0, \infty)$, $i = 1, 2$, be measurable functions satisfying the following conditions. There exists a constant A such that, for every $\alpha \in [1/2, 2]$ and every $t > 0$,*

$$A^{-1}\varphi_2(t) \leq \varphi_2(\alpha t) \leq A\varphi_2(t), \quad (\text{B.1})$$

and either

$$A^{-1}\varphi_1(t/2) \leq \varphi_1(\alpha t) \leq A\varphi_1(2t), \quad (\text{B.2a})$$

or

$$A\varphi_1(t/2) \geq \varphi_1(\alpha t) \geq A^{-1}\varphi_1(2t), \quad (\text{B.2b})$$

Then there exists a constant $c > 0$ such that, for every $i, k \in \mathbb{N}$, $i < k$, the function $\varphi := \varphi_1\varphi_2$ satisfies either

$$c^{-1} \sum_{j=i}^k \varphi(r_{j+1}) \leq \int_{r_{k+1}}^{r_i} \varphi(t) \frac{dt}{t} \leq c \sum_{j=i}^k \varphi(r_j), \quad (\text{B.3a})$$

or

$$c \sum_{j=i}^k \varphi(r_{j+1}) \geq \int_{r_{k+1}}^{r_i} \varphi(t) \frac{dt}{t} \geq c^{-1} \sum_{j=i}^k \varphi(r_j), \quad (\text{B.3b})$$

according to which of the conditions (B.2) holds.

Proof. By (B.1), (B.2a), and the mean value theorem there exists $\tau_j \in [r_{j+1}, r_j]$ such that

$$\frac{1}{2A^2}\varphi(r_{j+1}) \leq \int_{r_{j+1}}^{r_j} \varphi(t) \frac{dt}{t} = r_{j+1}\varphi(\tau_j)/\tau_j \leq A^2\varphi(r_j). \quad (\text{B.4})$$

This implies (B.3a). The second case is similarly verified. \square

Lemma B.2. *For every compact set $F \subset \partial Q$ and every positive number τ ,*

$$C_{2/q, q'}(\alpha F) \leq A(\tau)C_{2/q, q'}(F) \quad \forall \alpha \in (0, \tau], \quad (\text{B.5})$$

where the constant $A(\tau)$ depends only on q , N , ∂Q and τ if $q = q_c$, and $A(\tau) = A_0\tau^{N-1-2/(q-1)}$, with A_0 independent of τ , if $q > q_c$.

Remark. Recall that $C_{2/q, q'}(F)$ denotes capacity relative to ∂Q while $C_{2/q, q'}(\alpha F)$ denotes capacity relative to $\alpha\partial Q$.

Proof. Since ∂Q can be covered by a finite number of coordinate patches it is sufficient to prove the inequality in the case that F is contained in a coordinate patch. Thus there exists a C^2 diffeomorphism of a relative neighborhood U of F onto an open set $U' \subset \mathbb{R}^{N-1}$. Denote this diffeomorphism by T and let T_α denote the induced mapping on αU :

$$T_\alpha(\alpha x) = \alpha T(x) \quad \forall x \in U.$$

Thus $T_\alpha(\alpha F) = \alpha T(F)$ and the Lipschitz constant for T_α is the same as that for T . Since

$$C_{2/q,q'}(F) \approx C_{2/q,q'}(T(F)), \quad C_{2/q,q'}(\alpha F) \approx C_{2/q,q'}(T_\alpha(\alpha F)),$$

it is sufficient to prove (B.5) in the case that $F \subset B_1 \cap \mathbb{R}^{N-1}$ and the capacity is relative to \mathbb{R}^{N-1} . If $q > q_c$ the inequality follows immediately from [1, Cor. 5.2.3]. If $q = q_c$ we first apply [1, Thm. 5.2.1] to obtain

$$C_{2/q,q'}(\tau F) \leq C_\tau C_{2/q,q'}(F),$$

where C_τ depends only on N and τ . Next we observe that the first part of the proof of [1, Thm. 5.2.1] and in particular inequality (5.2.3) implies that, for every compact set $E \subset \mathbb{R}^{N-1}$,

$$C_{2/q,q'}(\gamma E) \leq C' \gamma^{N-2/(q-1)} C_{2/q,q'}(E) \quad \forall \gamma \in (0, 1],$$

where C' is a constant depending only on q , N . This is valid for $1 < q \leq q_c$. Combining the last two inequalities we obtain (B.5). \square

Corollary B.3. *Let K be a compact subset of ∂Q and put*

$$\phi(t) = C_{2/q,q'}\left(\frac{1}{t}(K \cap B_{t/2})\right) = C_{2/q,q'}\left(\frac{1}{t}K \cap B_{1/2}\right) \quad \forall t > 0. \quad (\text{B.6})$$

Then ϕ satisfies (B.2a) and, for every $a \in \mathbb{R}$ and $i, k \in \mathbb{N}$, $i < k$,

$$\frac{1}{c} \sum_{m=i+1}^k r_m^a \phi(r_m) \leq \int_{r_k}^{r_i} t^a \phi(t) \frac{dt}{t} \leq c \sum_{m=i+1}^k r_{m-1}^a \phi(r_{m-1}), \quad (\text{B.7})$$

where c is a constant depending on a , q , Q .

Proof. If $\alpha \in [1/2, 2]$ then, by Lemma B.2,

$$\begin{aligned} \phi(\alpha t) &\geq c C_{2/q,q'}\left(\frac{1}{t}(K \cap B_{\alpha t/2})\right) \\ &\geq c C_{2/q,q'}\left(\frac{1}{t}(K \cap B_{t/4})\right) \geq c C_{2/q,q'}\left(\frac{2}{t}(K \cap B_{t/4})\right) = c\phi(t/2), \end{aligned} \quad (\text{B.8})$$

$$\begin{aligned} \phi(\alpha t) &\leq c C_{2/q,q'}\left(\frac{1}{t}(K \cap B_{\alpha t/2})\right) \\ &\leq c C_{2/q,q'}\left(\frac{1}{t}(K \cap B_t)\right) \leq c C_{2/q,q'}\left(\frac{1}{2t}(K \cap B_t)\right) = c\phi(2t). \end{aligned} \quad (\text{B.9})$$

Thus the function ϕ satisfies (B.2a), and consequently, Lemma B.1 implies (B.7). \square

Proof of (3.40). Let ϕ be defined as in (B.6). With this notation

$$\phi(r_{m-1}) = C_{2/q,q'}(F_m^*).$$

Put

$$\varphi(r, s) := \int_r^s t^{-1-2/(q-1)} \phi(t) \frac{dt}{t}, \quad 0 < r < s. \quad (\text{B.10})$$

By Corollary B.3,

$$\begin{aligned} \frac{1}{c} \sum_{m=i+1}^k r_m^{-1-2/(q-1)} C_{2/q,q'}(F_{m+1}^*) &\leq \varphi(r_k, r_i) \\ &\leq c \sum_{m=i+1}^k r_{m-1}^{-1-2/(q-1)} C_{2/q,q'}(F_m^*) \end{aligned} \quad (\text{B.11})$$

for every $i, k \in \mathbb{N}$, $i < k$. The constant c depends only on q , N , Q . Hence

$$\varphi(0, r_i) := \lim_{r \downarrow 0} \varphi(r, r_i) \leq c \sum_{m=i+1}^M r_{m-1}^{-1-2/(q-1)} C_{2/q,q'}(F_m^*). \quad (\text{B.12})$$

Since ϕ is bounded,

$$\varphi(r, \infty) \leq cr^{-1-2/(q-1)}, \quad r > 0. \quad (\text{B.13})$$

Further, by (B.11) and (B.10),

$$\begin{aligned} \sum_{j=3}^{M+1} r_j^{1+q} \left(\sum_{m=1}^{j-2} r_{m-1}^{-1-2/(q-1)} C_{2/q,q'}(F_m^*) \right)^q \\ \leq c \sum_{j=3}^{M+1} r_j^{1+q} \left(\int_{r_{j-3}}^2 t^{-1-2/(q-1)} \phi(t) \frac{dt}{t} \right)^q = \sum_{j=3}^{M+1} r_j^{1+q} \varphi(r_{j-3}, 2)^q. \end{aligned} \quad (\text{B.14})$$

By Lemma B.1 (since ϕ is non-increasing, it satisfies (B.2b)),

$$\sum_{j=3}^{M+1} r_j^{1+q} \varphi(r_{j-3}, 2)^q \leq c \int_{r_{M-1}}^2 t^{1+q} \varphi(t, 2)^q \frac{dt}{t} = c \int_0^2 (t\varphi(t, 2))^q dt. \quad (\text{B.15})$$

By (B.13) and (B.12),

$$\begin{aligned} \int_0^2 (t\varphi(t, 2))^q dt &= -\frac{q}{q+1} \int_0^2 t^{q+1} \varphi^{q-1} \varphi' dt \\ &\leq -c \int_0^2 \varphi' dt = c\varphi(0, 2) \leq c \sum_{m=1}^M r_{m-1}^{-1-2/(q-1)} C_{2/q,q'}(F_m^*). \end{aligned} \quad (\text{B.16})$$

Finally (B.14)–(B.16) imply (3.40). \square

Appendix C. Estimates of Poisson kernels

This appendix is devoted to the derivation of the following theorem.

Theorem C.1. *Given $x_0 \in \mathbb{R}^N$ and $0 < r$ put*

$$D_r(x_0) := \Omega \setminus \bar{B}_r(x_0), \quad \Sigma_r := \{x \in \mathbb{R}^N : \dot{\rho}_{\partial\Omega}(x) = r\}$$

and let \mathcal{P}_r denote the Poisson kernel of $-\Delta$ in D_r . Then there exists a number $m = m(\Omega) \geq 2$ such that, for every $r_1 \in (0, \beta_0/8m)$, $\gamma \in I := [r_1/m, r_1/2)$ and $x_0 \in \Sigma_\gamma$, the Poisson kernel of $-\Delta$ in $D_{r_1}(x_0)$, to be denoted by $\mathcal{P}_{r_1}^{x_0}$, satisfies the inequality

$$\mathcal{P}_{r_1}^{x_0}(x, y) \leq c \frac{\rho_{\partial\Omega}(x)\rho_{\partial\Omega}(y)}{r_1|x-y|^N} \quad \forall x \in D_{4r_1}(x_0), y \in \partial B_{r_1}(x_0) \cap \Omega, \quad (\text{C.1})$$

where c, m depend only on N and the C^2 “norm” of $\partial\Omega$.

We begin with a related estimate whose proof is based on the Harnack inequality up to the boundary (see [2]).

Lemma C.2. *Let G be a domain in \mathbb{R}^N with compact boundary of class C^2 and let $\beta^i(G), \beta^e(G)$ be as in (2.4). Put $\beta_0 = \min(\beta^i(G), \beta^e(G))$. For every $\gamma \in (0, \beta^e)$ and $\sigma \in \partial G$ put*

$$M_\gamma(G) := \{\xi \in \mathbb{R}^N : \dot{\rho}_{\partial G}(\xi) = -\gamma\}, \quad Q_\gamma(\sigma) := \sigma + \gamma\nu(\sigma). \quad (\text{C.2})$$

Further set $G_r(Q) := G \cap B_r(Q)$. Then:

(a) *For every $\sigma \in \partial G$ there exists $\bar{\beta} \in (0, \beta_0/4)$ and an increasing function*

$$R^\sigma : I \mapsto (0, \infty), \quad I := (0, \bar{\beta}),$$

such that

$$2\gamma < R^\sigma(\gamma) < \min(\beta^i + \gamma, \beta^e - \gamma)$$

and, for every $\gamma \in I$,

$$-\nu(x) \cdot (x - Q) \geq 0 \quad \forall x \in G_{R^\sigma(Q)}, \quad Q = Q_\gamma(\sigma) \quad (\text{C.3})$$

(Recall that $\nu(x)$ denotes the outward unit normal on ∂G at $\sigma(x)$.)

(b) *There exists a constant C , depending only on N, β^e and the Lipschitz constant of ∂G , such that, for every $\sigma \in \partial G$ and $\gamma \in I$, the Poisson kernel of $-\Delta$ in $G_R(Q)$, where $Q = Q^\sigma(\gamma)$, $2\gamma < R \leq R^\sigma(\gamma)$, satisfies*

$$P_{G_R(Q)}(x, y) \leq \frac{C}{\gamma} \frac{\rho_{\partial G}(x)\rho_{\partial G}(y)}{|x-y|^N} \quad (\text{C.4})$$

for every $x \in G \cap B_{3R/4}(Q)$ and $y \in G \cap \partial B_R(Q)$.

Proof. To verify statement (a) it is enough to observe that, if

$$\bar{\beta} := -(\beta_0 + 1/2) + \sqrt{(\beta_0 + 1/2)^2 + \beta_0}, \quad R(\gamma) := \sqrt{2\beta_0\gamma + \gamma^2},$$

then the function $R(\cdot)$ on I has all the properties mentioned in (a), for every $\sigma \in \partial G$. However, in general, R^σ need not tend to zero as $\gamma \rightarrow 0$. For instance, if G is the complement of a ball of radius β_0 and $\bar{\beta} = \beta_0/4$ then the function given by $R(\gamma) := \beta_0 - \gamma$ satisfies all the conditions stated in (a).

We turn to the proof of (b). We assume (as we may) that Q is the origin and write B_r, G_r for $B_r(0), G_r(0)$. The conditions on x, y imply that $R/4 \leq |x - y| < R$. Therefore (C.4) is equivalent to

$$P_{G_R}(x, y) \leq \frac{C}{\gamma R^N} \rho_{\partial G}(x) \rho_{\partial G}(y), \tag{C.5}$$

where $\gamma = \text{dist}(0, G)$.

For any bounded C^2 domain D with compact C^2 boundary and any positive α ,

$$P_D(x, y) = \alpha^{N-1} P_{\alpha D}(\alpha x, \alpha y). \tag{C.6}$$

Let $\gamma \in I$ and $R > 2\gamma$ be given. In order to prove that (C.5) holds in G_R with $\text{dist}(0, G) = \gamma$, it is sufficient to prove that it holds in $(\frac{1}{R}G)_1$ with $\text{dist}(\frac{1}{R}G, 0) = \gamma/R$:

$$P_{(\frac{1}{R}G)_1}(X, Y) \leq \frac{CR}{\gamma} \rho_{\partial G}(X) \rho_{\partial G}(Y) \tag{C.7}$$

for every $X \in (\frac{1}{R}G) \cap B_{3/4}$ and $Y \in (\frac{1}{R}G) \cap \partial B_1$. Indeed, for $x = RX, y = RY$,

$$\begin{aligned} P_{G_R}(x, y) &= R^{1-N} P_{(\frac{1}{R}G)_1}(x/R, y/R) \\ &\leq \frac{C}{\gamma} R^{2-N} \rho_{\partial G}(x/R) \rho_{\partial G}(y/R) \\ &= \frac{C}{\gamma} R^{-N} \rho_{\partial G}(x) \rho_{\partial G}(y). \end{aligned} \tag{C.8}$$

Note that if (C.3) holds in G_R , relative to $Q = 0$, then it also holds in $(\frac{1}{R}G)_1$ relative to the origin.

We turn to the proof of (C.5) assuming that $\gamma = \text{dist}(0, G) \in (0, 1/2)$, $R = 1$ and $\beta_0 > 2$.

We start with the construction of a certain superharmonic function in $G_1 = G \cap B_1$, vanishing on the boundary, which will be used in our estimate of the Poisson kernel. Let $h, \eta(\cdot; M, \gamma)$ be solutions of

$$-\Delta h = 1 \quad \text{in } B_1, \quad h = 0 \quad \text{on } \partial B_1, \tag{C.9}$$

and

$$-\Delta \eta = 1 \quad \text{in } B_M \setminus B_\gamma, \quad \eta = 0 \quad \text{on } \partial B_\gamma, \quad \eta = 1 \quad \text{on } \partial B_M, \tag{C.10}$$

where $M > \gamma$. Then both h and η are radially symmetric, the first decreasing and the

second increasing with respect to $r = |x|$. Clearly, for every $\alpha > 0$, $\eta(x; M, \gamma) = \eta(\alpha x; \alpha M, \alpha \gamma)$. In addition, if $M' > M$ then $\eta(\cdot; M', \gamma) < 1$ on ∂B_M and consequently

$$\eta(x; M', \gamma) \leq \eta(x; M, \gamma).$$

Therefore, if $\gamma \in (0, 1)$,

$$\eta(x; M, \gamma) = \eta(x/\gamma; M/\gamma, 1) \quad \forall x \in B_M \setminus \bar{B}_\gamma \tag{C.11}$$

$$\partial_r \eta(\cdot; M, \gamma)|_{\partial B_\gamma} = \frac{1}{\gamma} \partial_r \eta(\cdot; M/\gamma, 1)|_{\partial B_1} \leq \frac{1}{\gamma} \partial_r \eta(\cdot; M, 1)|_{\partial B_1}, \tag{C.12}$$

where ∂_r denotes differentiation in the radial direction on the indicated sphere. For $|x|=r$, let $h_0(r) := h(x)$, $\eta_\gamma(r) := \eta(x; 4, \gamma)$.

For every $x \in G_1$, put

$$\tilde{\eta}(x) := \inf\{\eta(x - P) : P \in M_\gamma(G), \sigma(P) \in \partial G \cap B_1\}. \tag{C.13}$$

Then

$$-\Delta \tilde{\eta} \geq 1 \quad \text{in } G_1 = G \cap B_1, \quad \tilde{\eta} = 0 \quad \text{on } \partial G \cap B_1, \tag{C.14}$$

and

$$\tilde{\eta}(x) = \eta(x - P_x) \quad \text{with } P_x = \sigma(x) + \gamma v(x) \quad \forall x \in G_1, \tag{C.15}$$

where $v(x) = v_G(x)$ denotes the outward unit normal on ∂G at the point $\sigma(x)$. The first statement is obvious since, for every $y \in \partial G \cap B_1$, $P_y \in M_\gamma(G)$ and $\eta(y - P_y) = 0$. The second statement follows from the fact that $|x - P_x| \leq |x - P|$ for every $P \in M_\gamma(G)$ such that $\sigma(P) \in \partial G \cap B_1$.

Since $\sigma(\cdot)$ and $v(\cdot)$ are in $C^1((\partial G)^{\beta_0})$, the mapping $x \mapsto P_x$ is in $C^1(G_1)$. Hence $\tilde{\eta} \in C^1(G_1)$ and

$$\begin{aligned} \partial_j \tilde{\eta}(x) &= \eta'_0(|x - P_x|) \frac{x - P_x}{|x - P_x|} \cdot \partial_j(x - P_x) \\ &= \frac{\eta'_0(|x - P_x|)}{2|x - P_x|} \partial_j |x - P_x|^2 = \eta'_0(|x - P_x|) \partial_j |x - P_x|. \end{aligned} \tag{C.16}$$

By (C.15), $|x - P_x| = \gamma + \rho_{\partial G}(x)$. Hence

$$\nabla \tilde{\eta}(x) = \eta'_0(|x - P_x|) \nabla \rho_{\partial G}(x). \tag{C.17}$$

If $v := h\tilde{\eta}$,

$$\begin{aligned} -\Delta v &= -(\Delta h)\tilde{\eta} - h\Delta \tilde{\eta} - 2\nabla h \cdot \nabla \tilde{\eta} \\ &\geq \tilde{\eta} + h - 2h'_0(|x|)\eta'_\gamma(|x - P_x|)(x/|x|) \cdot \nabla \rho_{\partial G}(x). \end{aligned} \tag{C.18}$$

Now $h'_0 \leq 0$, $\eta'_\gamma \geq 0$ and $\nabla \rho_{\partial G}(x)$ is in the direction of $-v(x)$. Therefore, by (C.3),

$$-\Delta v \geq 0 \quad \text{in } G_1, \quad v = 0 \quad \text{on } \partial G_1. \tag{C.19}$$

The vanishing on the boundary follows from (C.14) and (C.9).

We turn to the proof of (C.5) with $R = 1$. By the Harnack inequality up to the boundary (see [2, p. 623]), there exists a constant C which depends only on N and the Lipschitz constant of $\partial G \cap B_1$ such that, if w_1, w_2 are positive harmonic functions in G_1 , vanishing on $\partial G \cap B_1$,

$$w_1(x)/w_2(x) \leq C w_1(\xi)/w_2(\xi) \quad \forall x, \xi \in \bar{G}_{3/4}. \tag{C.20}$$

Pick $a, y \in G \cap \partial B_1$ and $\xi \in G \cap \partial B_{3/4}$ such that $\sigma(a) = \sigma(\xi)$. Applying (C.20) with $w_1 := P_{G_1}(\cdot, y), w_2 := P_{G_1}(\cdot, a)$ we have

$$P_{G_1}(x, y) \leq C P_{G_1}(x, a) \frac{P_{G_1}(\xi, y)}{P_{G_1}(\xi, a)} \quad \forall x \in \bar{G}_{3/4}. \tag{C.21}$$

We estimate each of the factors on the right hand side.

Since $|a - \xi| = 1/4$ and $B_{1/4}(\xi) \subset G_1$ it follows that

$$P_{G_1}(\xi, a) \geq P_{B_{1/4}}(0, a - \xi) = c_1, \tag{C.22}$$

where c_1 is a positive number depending only on N .

Let T_a denote the half space tangent to B_1 at a , which contains B_1 . Then $G_1 \subset T_a \setminus B_r(Q^\sigma(r)) := T_{a,r}^\sigma, r := \min(\beta^e/2, 2^{-4})$, for every $\sigma \in \partial G \cap B_1$. Consequently,

$$P_{T_{a,r}^\sigma}(x, a) \geq P_{G_1}(x, a) \quad \forall x \in G_1.$$

By Hopf's lemma, there exists a constant c_2 depending only on N and β^e such that

$$P_{T_{a,r}^\sigma}(x, a) \leq c_2|x - \sigma| \quad \forall x \in G \cap B_{3/4}.$$

Hence,

$$P_{G_1}(x, a) \leq P_{T_{a,r}^\sigma}(x, a) \leq c_2\rho_{\partial G}(x) \quad \forall x \in B_{3/4}. \tag{C.23}$$

For the estimate of $P_{G_1}(\xi, y), y \in G \cap \partial B_1$, we use the superharmonic function previously constructed. Since $B_1 \subset B_4(\xi)$ it follows that, for every $z \in G_1 \setminus \{\xi\}$,

$$\mathcal{G}_{G_1}(\xi, z) \leq \mathcal{G}_{B_4(\xi)}(\xi, z) = \mathcal{G}_{B_4}(0, z - \xi),$$

where \mathcal{G}_D denotes the Green function of $-\Delta$ in D . For $\epsilon \in (0, 1/8)$, put

$$A_\epsilon := \max_{|\zeta|=\epsilon} \mathcal{G}_{B_4}(0, \zeta), \quad A'_\epsilon = \min_{|\zeta|=\epsilon} v(\xi + \zeta). \tag{C.24}$$

Since $0 < v < 1, A'_\epsilon \leq 1$. Let $\epsilon_0 := \max\{\epsilon \in (0, 1/8) : A_\epsilon \geq 1\}$ and put $c_3 := A_{\epsilon_0}/A'_{\epsilon_0}$. Clearly ϵ_0 depends only on N and

$$\mathcal{G}_{G_1}(\xi, z) \leq c_3v(z) \quad \forall z \in \partial B_{\epsilon_0}(\xi).$$

The function c_3v is superharmonic in G_1 ; therefore, by the maximum principle (recall that $v = 0$ on ∂G_1),

$$\mathcal{G}_{G_1}(\xi, z) \leq c_3v(z) \quad \text{in } G_1 \setminus B_{\epsilon_0}(\xi). \tag{C.25}$$

This implies

$$P_{G_1}(\xi, y) = -\partial_{\mathbf{n}}\mathcal{G}_{G_1}(\xi, y) \leq -c_3\partial_{\mathbf{n}}v(y) \quad \forall y \in G \cap \partial B_1, \tag{C.26}$$

where $\mathbf{n} = y/|y|$. Using (C.17) we obtain

$$\begin{aligned} -\partial_{\mathbf{n}}v(y) &= -h'_0(|y|)\tilde{\eta}(y) - h(y)\partial_{\mathbf{n}}\tilde{\eta}(y) \\ &= -h'_0(1)\eta(y - P_y) - h(y)\eta'_\gamma(|y - P_y|)\mathbf{n} \cdot \nabla\rho_{\partial G}(y) \\ &\leq -h'_0(1)\eta(y - P_y). \end{aligned} \tag{C.27}$$

The last inequality follows from the fact that $\eta'_\gamma \geq 0$ and (C.3) which implies that $\mathbf{n} \cdot \nabla\rho_{\partial G}(y) \geq 0$. Since $\eta_\gamma(\gamma) = 0$ and η_γ is increasing and concave,

$$\eta_\gamma(|y - P_y|) = \eta_\gamma(\gamma + \rho_{\partial G}(y)) \leq \eta'_\gamma(\gamma)\rho_{\partial G}(y). \tag{C.28}$$

By (C.26)–(C.28) and (C.12),

$$P_{G_1}(\xi, y) \leq \frac{c_4}{\gamma}\rho_{\partial G}(y), \tag{C.29}$$

where c_4 depends only on N . Finally, (C.21)–(C.23) and (C.29) imply (C.4). \square

Remark. From the proof of the lemma it is clear that the result is of a local nature. In addition, if the C^2 condition on the boundary is satisfied in a relatively open neighborhood of a point $S \in \partial G$, say $\Gamma_1(S)$, then (C.4) holds for all points $Q \in \bar{G}^c \cap B_{1/2}(S)$ sufficiently close to the boundary of G . In this case the relevant values of β^e and β^i are those associated with the C^2 “norm” of $\Gamma_{3/4}(S)$.

The following is an immediate consequence of the lemma:

Corollary C.3. *If G is a bounded convex domain with C^2 boundary then (C.4) holds for every $\gamma \in (0, \beta_0(G)/3)$ and $2\gamma < R < \gamma + \beta_0$ and the constant depends only on N and the Lipschitz constant of G . In particular if G is a ball of radius β_0 then the constant depends only on N . If G is a half space then (C.4) holds for every $\gamma > 0$ and $R > 2\gamma$, the constant depending only on N .*

Proof. If G is a bounded convex domain with C^2 boundary then $\beta^e = \infty$ and condition (C.3) holds for every $Q \notin \bar{G}$. This implies the statements of the lemma. \square

Lemma C.4. *Let $\Omega = \{X \in \mathbb{R}^N : X_1 > 0\}$, $\gamma^* > 0$ and $R^* \geq 3\gamma^*$. Then the Poisson kernel of the domain*

$$D^* := D_{R^*}(Z) = \Omega \setminus B_{R^*}(Z), \quad Z = (\gamma^*, 0, \dots, 0),$$

satisfies the inequality

$$\mathcal{P}_{D^*}(X, Y) \leq c \frac{X_1 Y_1}{\gamma^* |X - Y|^N} \tag{C.30}$$

for every $X, Y \in \Omega$ such that $|X - Z| > 4R^/3$ and $|Y - Z| = R^*$, with the constant c depending only on N .*

Proof. Put

$$\gamma = \frac{\gamma^*}{(R^*)^2 - (\gamma^*)^2}, \quad R := \frac{R^*}{(R^*)^2 - (\gamma^*)^2}, \quad Q_\gamma^0 = (-\gamma, 0, \dots, 0).$$

By scaling (see (C.6)) we may assume that $R^* = 1$ and $\gamma^* < 1/3$. In this case $0 < \gamma < 3/8$ and $1 < R < 9/8$. The inversion T relative to the sphere $|x| = 1$ maps $B_R(Q_\gamma^0)$ onto the exterior of the ball $B_{R^*}(Z)$ and Ω onto itself. Consequently, $B_R(Q_\gamma^0) \cap \Omega$ is transformed to D^* . By Corollary C.3, the Poisson kernel of $D = B_R(Q_\gamma^0) \cap \Omega$ satisfies inequality (C.5), which in this case is equivalent to (C.4). By a straightforward computation, if Q is a bounded domain of class C^1 then

$$P_Q(x, y) = |x|^{2-N} P_{T(Q)}(X, Y), \quad X = x/|x|^2, \quad Y = y/|y|^2. \quad (C.31)$$

Applying this relation to $D = B_R(Q_\gamma^0) \cap \Omega$ and $D^* = T(D) = D_{R^*}(Z)$ and using (C.5), we obtain

$$P_{D^*}(X, Y) \leq |X|^{2-N} \frac{C}{\gamma} x_1 y_1$$

for every X, Y such that $|x - Q_\gamma^0| < 3R/4$ and $|y - Q_\gamma^0| = 1$. Since $x_1 = X_1/|X|^2$ we obtain

$$P_{D^*}(X, Y) \leq \frac{C}{\gamma} |X|^{-N} X_1 Y_1 |Y|^{-2}.$$

Since $R^* = 1$, $1 < R < 9/8$ and $R/\gamma = R^*/\gamma^*$ it follows that $1 < \gamma/\gamma^* < 9/8$. If $|X - Z| > 4/3$ and $|Y - Z| = 1$ then $|X - Y| \approx |X|$, $|Y| \approx 1$. Hence

$$P_{D^*}(X, Y) \leq \frac{C}{\gamma^*} |X - Y|^{-N} X_1 Y_1. \quad \square$$

Lemma C.5. *Let R, r_1, γ_1 be given such that*

$$R/r_1 := m > 4, \quad 1/m < \gamma_1/r_1 < 1.$$

Denote by \mathbf{e} the unit vector in the direction of the positive x_1 -axis, put $\tau_1 := 2R + \gamma_1$ and

$$\Omega := \{x \in \mathbb{R}^N : |x + R\mathbf{e}| > R\}, \quad D^* := \Omega \setminus B_{r_1}(-\tau_1\mathbf{e}).$$

Then the Poisson kernel of D^ satisfies the inequality*

$$P_{D^*}(x, y) \leq cmR^{N-1} \frac{\rho_{\partial\Omega}(x)\rho_{\partial\Omega}(y)}{r_1|x-y|^N} (R + \rho_{\partial\Omega}(x)) \quad (C.32)$$

for every $x, y \in \Omega$ such that $|x + \tau_1\mathbf{e}| > 4r_1$ and $|y + \tau_1\mathbf{e}| = r_1$, with the constant c depending only on N .

Proof. Put $a := 1/2R$, $\mu := \tau_1/r_1 = (2R + \gamma_1)/r_1$ and compute γ_0 and r_0 from the relations

$$r_1 = \frac{r_0}{(a - \gamma_0)^2 - r_0^2}, \quad \tau_1 = \frac{a - \gamma_0}{(a - \gamma_0)^2 - r_0^2}. \quad (C.33)$$

This yields

$$r_0 = \frac{1}{r_1(\mu^2 - 1)} = \frac{\mu}{\tau_1(\mu^2 - 1)}, \quad \gamma_0 = a - \mu r_0. \quad (C.34)$$

We note that

$$\gamma_0 \geq \frac{1}{16m^3 r_1} = \frac{r_1}{16mR^2}. \quad (C.35)$$

Indeed, by (C.34),

$$\gamma_0 = \frac{1}{\tau_1 - \gamma_1} - \frac{\tau_1}{\tau_1^2 - r_1^2} = \frac{\gamma_1 \tau_1 - r_1^2}{(\tau_1 - \gamma_1)(\tau_1^2 - r_1^2)}$$

and, by assumption, $\tau_1/r_1 = 2m$ so that $\gamma_1 \tau_1 > 2r_1^2$. Hence

$$\gamma_0 \geq \frac{r_1^2}{(\tau_1 - \gamma_1)(\tau_1^2 - r_1^2)} = \frac{1}{r_1(\mu^2 - 1)(\mu - 1)}, \quad (\text{C.36})$$

which implies (C.35).

Now consider the domain

$$D_0 := \{X \in \mathbb{R}^N : X_1 > -a\} \setminus B_{r_0}((-a + \gamma_0)\mathbf{e}).$$

Let T denote the inversion relative to the unit sphere centered at the origin: $TX = X/|X|^2$. Then

$$T(E) = \begin{cases} \Omega & \text{if } E = \{X \in \mathbb{R}^N : X_1 > -a\}, \\ B_{r_1}(-\tau_1\mathbf{e}) & \text{if } E = B_{r_0}((-a + \gamma_0)\mathbf{e}), \\ B_{r_2}(-\tau_2\mathbf{e}) & \text{if } E = B_{2r_0}((-a + \gamma_0)\mathbf{e}), \end{cases} \quad (\text{C.37})$$

where

$$r_2 = \frac{2r_0}{(a - \gamma_0)^2 - 4r_0^2}, \quad \tau_2 = \frac{a - \gamma_0}{(a - \gamma_0)^2 - 4r_0^2}.$$

Therefore $T(D_0) = D^*$ and

$$B_{r_2}(-\tau_2\mathbf{e}) \subset B_{4r_1}(-\tau_1\mathbf{e}). \quad (\text{C.38})$$

Indeed, by (C.34), $\mu = (a - \gamma_0)/r_0$ and consequently

$$\frac{\tau_2 - \tau_1}{r_1} = \frac{3r_0(a - \gamma_0)}{(a - \gamma_0)^2 - 4r_0^2} = \frac{3\mu}{\mu^2 - 4}.$$

If $X = -\tau_1\mathbf{e} + 4r_1\bar{h}$, where \bar{h} is a unit vector, it follows that

$$|X + \tau_2\mathbf{e}| \geq 4r_1 - (\tau_2 - \tau_1) = r_1 \left(4 - \frac{3\mu}{\mu^2 - 4} \right),$$

and, since $\mu = \tau_1/r_1 > 2m \geq 8$,

$$r_1 \left(4 - \frac{3\mu}{\mu^2 - 4} \right) \geq r_2 = \frac{2}{r_0(\mu^2 - 4)} = \frac{2r_1(\mu^2 - 1)}{\mu^2 - 4},$$

which proves (C.38).

By Lemma C.4,

$$P_{D_0}(X, Y) \leq c \frac{(X_1 + a)(Y_1 + a)}{\gamma_0 |X - Y|^N} \quad (\text{C.39})$$

for every $X, Y \in \{X \in \mathbb{R}^N : X_1 > -a\}$ with $|X + (a - \gamma_0)\mathbf{e}| > 2r_0$ and $|Y + (a - \gamma_0)\mathbf{e}| = r_0$. The constant depends only on N . By (C.31), as $T(D_0) = D^*$,

$$P_{D^*}(x, y) = |x|^{2-N} P_{D_0}(X, Y), \quad X = x/|x|^2, \quad Y = y/|y|^2. \tag{C.40}$$

From (C.38)–(C.40) we obtain

$$P_{D^*}(x, y) \leq c \frac{(X_1 + a)(Y_1 + a)}{\gamma_0 |X - Y|^N} |x|^{2-N} \tag{C.41}$$

for every $x, y \in \Omega$ such that $|x + \tau_1 \mathbf{e}| > 4r_1$ and $|y + \tau_1 \mathbf{e}| = r_1$. It is easily verified that

$$|Tx - Ty| = \frac{|x - y|}{|x||y|}. \tag{C.42}$$

Therefore, by (C.41),

$$P_{D^*}(x, y) \leq c \frac{(2Rx_1 + |x|^2)(2Ry_1 + |y|^2)}{\gamma_0 R^2 |x - y|^N |y|^{2-N}}, \tag{C.43}$$

for x, y as before. A simple computation yields the identity

$$|x|^2 + 2Rx_1 = \delta_R(x)(2R + \delta_R(x)), \quad \delta_R(x) := \text{dist}(x, \partial B_R(-R\mathbf{e})).$$

Hence (C.43) implies

$$P_{D^*}(x, y) \leq c \frac{\delta_R(x)\delta_R(y)(2R + \delta_R(x))(2R + \delta_R(y))}{\gamma_0 R^2 |x - y|^N |y|^{2-N}}.$$

If $|y + \tau_1 \mathbf{e}| = r_1$ then $\delta_R(y) \leq 2r_1 = 2R/m$ and $R \leq |y| \leq 3R$. Therefore, by (C.43) and (C.35),

$$\begin{aligned} P_{D^*}(x, y) &\leq c \frac{\delta_R(x)\delta_R(y)(2R + \delta_r(x))}{\gamma_0 R^{3-N} |x - y|^N} \\ &\leq cmR^{N-1} \frac{\delta_R(x)\delta_R(y)}{r_1 |x - y|^N} (R + (\delta_R(x))), \end{aligned} \tag{C.44}$$

with the constant c depending only on N . □

Proof of the theorem. Choose m large enough so that if $r_1 \in (0, \beta_0/8m)$ and $x_0 \in \Omega$ satisfies $\rho_{\partial\Omega}(x_0) = \gamma \in [r_1/m, r_1/8)$ then

$$v(x) \cdot (x - x_0) \geq 0 \quad \forall x \in \partial B_{r_1}(x_0) \cap \partial\Omega. \tag{C.45}$$

Such a choice of m exists because Ω is bounded of class C^2 .

Let r_1 and x_0 be as above and let ξ be a point in $\partial B_{r_1}(x_0) \cap \partial\Omega$. Put $R = mr_1$ and $\xi_0 = Rv(\xi)$. Define

$$B'_R(\xi_0) := \{x \in \mathbb{R}^N : |x - \xi_0| > R\}, \quad D^*_{r_1}(x_0; \xi_0) := B'_R(\xi_0) \setminus \bar{B}_{r_1}(x_0).$$

Then

$$\bar{\Omega} \cap \bar{B}'_R(\xi_0) = \{\xi\}, \quad D_{r_1}(x_0) \subset D^*_{r_1}(x_0, \xi_0).$$

Therefore

$$\mathcal{P}_{r_1}(x, y) \leq P_{D_{r_1}^*}(x, y) \quad \forall x \in D_{r_1}(x_0), y \in \partial B_{r_1}(x_0) \cap \Omega. \quad (\text{C.46})$$

A simple computation shows that

$$\gamma \leq \text{dist}(x_0, \partial B'_R(\xi_0)) \leq \gamma + \frac{c}{R}r_1^2 = \gamma + cr_1/m,$$

where c is a constant depending only on Ω and β_0 . Therefore, by choosing m sufficiently large, we guarantee that $\gamma \leq \text{dist}(x_0, \partial B'_R(\xi_0)) \leq 2\gamma$. By Lemma C.5,

$$P_{D_{r_1}^*}(x, y) \leq C \frac{\rho_{E_R}(x)\rho_{E_R}(y)}{r_1|x-y|^N} \leq C \frac{\rho_{E_R}(y)}{r_1|x-y|^{N-1}}, \quad E_R = \partial B'_R(\xi_0), \quad (\text{C.47})$$

for every $x, y \in \Omega$ such that $|x-x_0| > 4r_1$, $|y-x_0| = r_1$, with $C = cm(1+\text{diam } \Omega)\beta_0^{N-1}$. The second inequality follows from the fact that, for any x, y as above,

$$\rho_{E_R}(x) \leq \rho_{D_{r_1}^*}(x) + 2r_1 \leq 2|x-y|.$$

If $y \in \partial B_{r_1}(x_0) \cap \Omega$ is a point such that

$$|y-\xi| = \inf_{\eta \in \partial B_{r_1}(x_0) \cap \partial \Omega} |y-\eta|$$

then $\rho_{E_R}(y) \approx |y-\xi| \approx \rho_{\partial \Omega}(y)$ and consequently, by (C.46) and (C.47),

$$\mathcal{P}_{r_1}(x, y) \leq C \frac{\rho_{\partial \Omega}(y)}{r_1|x-y|^{N-1}}$$

for every $x \in \Omega$ such that $4r_1 < |x-x_0|$. For every $y \in \partial B_{r_1}(x_0) \cap \Omega$ there exists a nearest point $\xi \in \partial B_{r_1}(x_0) \cap \partial \Omega$ (although ξ may not be unique). Since C is independent of ξ , it follows that

$$\mathcal{P}_{r_1}(x, y) \leq C \frac{\rho_{\partial \Omega}(y)}{r_1|x-y|^{N-1}} \quad (\text{C.48})$$

for every $y \in \partial B_{r_1}(x_0) \cap \Omega$ and every $x \in \Omega$ such that $4r_1 < |x-x_0|$.

By a standard argument, (C.48) implies (C.1). Obviously (C.1) holds for pairs of points (x, y) such that $|x-y| \leq 8\rho_{\partial \Omega}(x)$. Therefore we consider pairs (x', y') such that

$$y' \in \partial B_{r_1}(x_0) \cap \Omega, \quad x' \in D_{4r_1}(x_0), \quad \rho_{\partial \Omega}(x') \leq 1/8|x'-y'|.$$

Put

$$E' := \{x \in \Omega : |x-\sigma(x')| < \alpha/2\}, \quad \alpha := |x'-y'|.$$

For $x \in E'$, we have $|x-y'| \leq \alpha$. Therefore

$$\mathcal{P}_{r_1}(x, y') \leq C \frac{\rho_{\partial \Omega}(y')}{\alpha^{N-1}r_1} := A \quad \forall x \in E'.$$

The function $\mathcal{P}_{r_1}(\cdot, y')$ is harmonic and vanishes on $\partial \Omega$. By Hopf's lemma,

$$\mathcal{P}_{r_1}(x, y') \leq A\rho_{\partial \Omega}(x)/\alpha \quad \forall x \in \Omega : |x-\sigma(x')| < \frac{1}{4}|x'-y'|.$$

In particular

$$\mathcal{P}_{r_1}(x', y') \leq C \frac{\rho_{\partial \Omega}(x')\rho_{\partial \Omega}(y')}{\alpha^N r_1}. \quad \square$$

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