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Cycles on algebraic models of smooth manifolds

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Abstract. Every compact smooth manifold M is diffeomorphic to a nonsingular real algebraic set, called an algebraic model of M. We study modulo 2 homology classes represented by algebraic subsets of X, as X runs through the class of all algebraic models of M. Our main result concerns the case where M is a spin manifold.

Keywords. Real algebraic sets, algebraic cohomology classes, algebraic models

1. Introduction

Let X be a compact nonsingular real algebraic set (in $\mathbb{R}^n$ for some n). A cohomology class in $H^k(X, \mathbb{Z}/2)$ is said to be *algebraic* if the homology class Poincaré dual to it can be represented by an algebraic subset of X. The set $H^k_{\text{alg}}(X, \mathbb{Z}/2)$ of all algebraic cohomology classes in $H^k(X, \mathbb{Z}/2)$ is a subgroup, while the direct sum $H^*_{\text{alg}}(X, \mathbb{Z}/2)$ of the $H^k_{\text{alg}}(X, \mathbb{Z}/2)$, for $k \geq 0$, forms a subring of the cohomology ring $H^*(X, \mathbb{Z}/2)$. Early papers dealing with algebraic cohomology (or homology) classes provided examples of X with $H^*_{\text{alg}}(X, \mathbb{Z}/2) \neq H^*(X, \mathbb{Z}/2)$ (cf. [1, 5, 6, 14, 19, 20]). The reader can find a survey of properties and applications of $H^*_{\text{alg}}(-, \mathbb{Z}/2)$ in [11].

Every compact smooth (of class $\mathcal{C}^\infty$) manifold M is diffeomorphic to a nonsingular real algebraic set, called an *algebraic model* of M (cf. [23]; see also [7, Theorem 14.1.10] and, for a weaker but influential result, [18]). The following question is a challenging problem: *How does the ring $H^*_{\text{alg}}(X, \mathbb{Z}/2)$ vary as X runs through the class of algebraic models of M?* This paper provides partial answers. Due to technical difficulties it is easier to describe how the group $H^k_{\text{alg}}(X, \mathbb{Z}/2)$ varies for a fixed k. Results of this type are in [8] for $k = 1$, in [10] for $k = 2$, and in [16] for $k \geq 3$. If $k \geq 2$ and especially if $k \geq 3$ they are far from complete.

We say that a subring A of $H^*(M, \mathbb{Z}/2)$ is *algebraically realizable* if there exist an algebraic model X of M and a smooth diffeomorphism $\varphi : X \to M$ with $\varphi^*(A) \subseteq H^*_{\text{alg}}(X, \mathbb{Z}/2)$. The original goal of several researchers was to show that the

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whole ring $H^*(M, \mathbb{Z}/2)$ is algebraically realizable, that is, M has an algebraic model X with $H^*_{\mathrm{alg}}(X, \mathbb{Z}/2) = H^*(X, \mathbb{Z}/2)$ (such a conjecture, motivated by far-reaching potential applications, was explicitly stated in [1]). However, since the publication of [3] it has been known that for some manifolds M this is impossible. An important algebraically realizable subring of $H^*(M, \mathbb{Z}/2)$ is identified in [4, Theorem 4, Remark 8]. It is the subring $A(M)$ generated by the Stiefel–Whitney classes of all real vector bundles on M together with the cohomology classes Poincaré dual to the homology classes represented by all smooth submanifolds of M. A conjecture proposed in [3], and still open at the present time, suggests that every algebraically realizable subring of $H^*(M, \mathbb{Z}/2)$ is contained in $A(M)$.

For us, certain subrings of $A(M)$ will play a crucial role. We say that a subring A of $H^*(M, \mathbb{Z}/2)$ is *admissible* if it is generated by the Stiefel–Whitney classes of some real vector bundles on M and the cohomology classes Poincaré dual to the homology classes represented by some smooth submanifolds of M. Thus $A(M)$ is the largest admissible subring of $H^*(M, \mathbb{Z}/2)$. However, in general, not every subring of $A(M)$ is admissible. Given any subring A of $H^*(M, \mathbb{Z}/2)$, we set $A^k = A \cap H^k(M, \mathbb{Z}/2)$. As usual, we denote by $w_i(M)$ the ith Stiefel–Whitney class of M. Recall that M is called a *spin manifold* if $w_1(M) = 0$ and $w_2(M) = 0$.

Theorem 1.1. *Let M be a compact connected spin manifold. Assume that* $\dim M \geq 7$ *and the group $H_i(M, \mathbb{Z})$ has no 2-torsion for $i = 1, 2$. Then for any admissible subring A of $H^*(M, \mathbb{Z}/2)$, there exist an algebraic model X of M and a smooth diffeomorphism $\varphi : X \to M$ satisfying*

$$\varphi^*(A) \subseteq H^*_{\mathrm{alg}}(X, \mathbb{Z}/2) \quad \textit{and} \quad \varphi^*(A^k) = H^k_{\mathrm{alg}}(X, \mathbb{Z}/2) \quad \textit{for } k = 0, 1, 2, 3.$$

As we mentioned above, some results of this type have already been known. More precisely, for M and A as in Theorem 1.1, given $k = 1$ or $k = 2$, one can find an algebraic model X_k and a smooth diffeomorphism $\phi_k : X_k \to M$ with $\phi_k^*(A^k) = H^k_{\mathrm{alg}}(X_k, \mathbb{Z}/2)$ (cf. [8, 10]; see also [16] for $k = 3$, but with different, somewhat artificial, assumptions). Thus the main contribution of Theorem 1.1 is the existence, under natural assumptions, of X and ϕ satisfying $\phi^*(A^k) = H^k_{\mathrm{alg}}(X, \mathbb{Z}/2)$ simultaneously for $k = 1, 2, 3$ ($k = 0$ being trivial). Our more general result, Theorem 2.4 in Section 2, concerns arbitrary k, but requires rather technical conditions on M and A. In view of Lemma 2.5, these technical conditions disappear for $k \leq 3$, and thus we get Theorem 1.1. It seems, however, that a completely new idea is needed in order to obtain interesting results for $k > 3$.

Theorem 1.1 is particularly nice in dimension 7, 8 or 9.

Corollary 1.2. *Let M be a compact connected spin manifold of dimension m, where $m = 7, 8$, or 9. Assume that the group $H_i(M, \mathbb{Z})$ has no 2-torsion for $i = 1, \dots, m-5$. Then for any subring A of $H^*(M, \mathbb{Z}/2)$, there exist an algebraic model X of M and a smooth diffeomorphism $\varphi : X \to M$ satisfying*

$$\varphi^*(A) \subseteq H^*_{\mathrm{alg}}(X, \mathbb{Z}/2) \quad \textit{and} \quad \varphi^*(A^k) = H^k_{\mathrm{alg}}(X, \mathbb{Z}/2) \quad \textit{for } k = 0, 1, 2, 3.$$

It suffices to prove that under the assumptions of Corollary 1.2, every subring of $H^*(M, \mathbb{Z}/2)$ is admissible. The latter fact easily follows from known results (see the next section). One can also drop the assumption about the dimension of M in Corollary 1.2, provided that the topology of M is not too complicated (cf. Example 2.6).

For manifolds which are not necessarily spin, we have the following result.

Theorem 1.3. *Let M be a compact connected smooth manifold. Assume that* $\dim M = m \geq 5$ *and the group $H_{m-2}(M, \mathbb{Z})$ has no 2-torsion. Then for any admissible subring A of $H^*(M, \mathbb{Z}/2)$, the following conditions are equivalent:*

(a) *There exist an algebraic model X of M and a smooth diffeomorphism $\varphi : X \to M$ satisfying*

$$\varphi^*(A) \subseteq H^*_{\text{alg}}(X, \mathbb{Z}/2) \quad \textit{and} \quad \varphi^*(A^k) = H^k_{\text{alg}}(X, \mathbb{Z}/2) \quad \textit{for } k = 0, 1, 2.$$

(b) *$w_i(M)$ is in A^i for $i = 1, 2$.*

If $\dim M = 5$, then every homology class in $H_d(M, \mathbb{Z}/2)$, $d \geq 0$, can be represented by a smooth submanifold [22, Théorème II.26], and hence every subring of $H^*(M, \mathbb{Z}/2)$ is admissible.

In order to compare the assumptions in Theorems 1.1 and 1.3, let us note that for any orientable compact smooth manifold M of dimension m, the groups $H_1(M, \mathbb{Z})$ and $H_{m-2}(M, \mathbb{Z})$ have isomorphic torsion subgroups. Indeed, this follows from the Poincaré duality and the universal coefficient theorem for cohomology.

Theorems 1.1, 1.3 and Corollary 1.2 are proved in Section 2.

2. Proofs and further results

We will need some constructions from real algebraic geometry. Throughout this paper the term *real algebraic variety* designates a locally ringed space isomorphic to an algebraic subset of $\mathbb{R}^n$, for some n, endowed with the Zariski topology and the sheaf of $\mathbb{R}$-valued regular functions. Morphisms between real algebraic varieties will be called *regular maps*. Background material on real algebraic varieties and regular maps can be found in [7]. Every real algebraic variety carries also the Euclidean topology, which is determined by the usual metric topology on $\mathbb{R}$. Unless explicitly stated otherwise, all topological notions relating to real algebraic varieties will refer to the Euclidean topology.

The Grassmannian $\mathbb{G}_{n,r}$ of r-dimensional vector subspaces of $\mathbb{R}^n$ is endowed with a canonical structure sheaf which makes it into a real algebraic variety in the sense of this paper [7, Theorem 3.4.4] (an affine real algebraic variety according to the terminology used in [7]). Moreover, $\mathbb{G}_{n,r}$ is nonsingular and

$$H^*_{\text{alg}}(\mathbb{G}_{n,r}, \mathbb{Z}/2) = H^*(\mathbb{G}_{n,r}, \mathbb{Z}/2)$$

(cf. [7, Propositions 3.4.3 and 11.3.3]). The universal vector bundle $\gamma_{n,r}$ on $\mathbb{G}_{n,r}$ is algebraic. If ξ is an algebraic vector bundle of rank r on a real algebraic variety X and if

n is a sufficiently large integer, then there is a regular map $f : X \to \mathbb{G}_{n,r}$ with $f^*\gamma_{n,r}$ algebraically isomorphic to ξ (cf. [7, Theorem 12.1.7]). Here referring to algebraic vector bundles we follow [7], while in [4, 5, 6, 8, 9, 10] such bundles are called strongly algebraic.

Given a compact nonsingular real algebraic variety X, we define $\mathrm{Alg}^k(X)$ to be the set of all elements u of $H^k(X, \mathbb{Z}/2)$ for which there exist a compact nonsingular irreducible real algebraic variety T (depending on u), two points t_0 and t_1 in T and a cohomology class z in $H^k_{\mathrm{alg}}(X \times T, \mathbb{Z}/2)$ such that

$$u = i^*_{t_1}(z) - i^*_{t_0}(z),$$

where for any t in T, we let $i_t : X \to X \times T$ denote the map $i_t(x) = (x, t)$ for all x in X. An equivalent description of $\mathrm{Alg}^k(X)$, which immediately implies that $\mathrm{Alg}^k(X)$ is a subgroup of $H^k_{\mathrm{alg}}(X, \mathbb{Z}/2)$, is given in [15, 16]. The groups $H^k_{\mathrm{alg}}(-, \mathbb{Z}/2)$ and $\mathrm{Alg}^k(-)$ have the expected functorial properties. If $f : X \to Y$ is a regular map between compact nonsingular real algebraic varieties, then the induced homomorphism $f^* : H^*(Y, \mathbb{Z}/2) \to H^*(X, \mathbb{Z}/2)$ satisfies

$$f^*(H^k_{\mathrm{alg}}(Y, \mathbb{Z}/2)) \subseteq H^k_{\mathrm{alg}}(X, \mathbb{Z}/2) \quad \text{and} \quad f^*(\mathrm{Alg}^k(Y)) \subseteq \mathrm{Alg}^k(X)$$

(cf. [12, Section 5] or [6] for the former inclusion and [16] for the latter).

The following fact will be very useful.

Theorem 2.1 (cf. [15, Theorem 2.1]). *Let X be a compact nonsingular real algebraic variety. Then $\langle u \cup v, [X] \rangle = 0$ for all u in $\mathrm{Alg}^k(X)$ and v in $H^\ell_{\mathrm{alg}}(X, \mathbb{Z}/2)$, where $k + \ell = \dim X$.*

As usual $\cup$ and $\langle\ ,\ \rangle$ denote the cup product and scalar (Kronecker) product, while $[X]$ stands for the fundamental class of X in $H_d(X, \mathbb{Z}/2)$, $d = \dim X$.

We will also need some properties of $\mathrm{Alg}^k(-)$ for very specific real algebraic varieties. Let B^n be a nonsingular irreducible real algebraic variety with precisely two connected components B^n_0 and B^n_1, each diffeomorphic to the unit n-sphere, $n \geq 1$. For example, one can take

$$B^n = \{(x_0, \dots, x_n) \in \mathbb{R}^{n+1} \mid x_0^4 - 4x_0^2 + 1 + x_1^2 + \cdots + x_n^2 = 0\}.$$

Let $B^n(d) = B^n \times \cdots \times B^n$ and $B^n_0(d) = B^n_0 \times \cdots \times B^n_0$ be the d-fold products, and let $\delta : B^n_0(d) \hookrightarrow B^n(d)$ be the inclusion map. Then according to [16, Example 4.5],

$$H^q(B^n_0(d), \mathbb{Z}/2) = \delta^*(H^q(B^n(d), \mathbb{Z}/2)) = \delta^*(\mathrm{Alg}^q(B^n(d))) \tag{2.2}$$

for all $q \geq 0$.

We now recall an important result from differential topology. All manifolds that appear here are without boundary.

Theorem 2.3 ([13, (17.3)]). *Let P be a smooth manifold. Two smooth maps $f : M \to P$ and $g : N \to P$, where M and N are compact smooth manifolds of dimension m, represent the same bordism class in the unoriented bordism group $\mathcal{N}_*(P)$ if and only if for every nonnegative integer q and every cohomology class v in $H^q(P, \mathbb{Z}/2)$, one has*

$$\langle w_{i_1}(M) \cup \cdots \cup w_{i_r}(M) \cup f^*(v), [M]\rangle = \langle w_{i_1}(N) \cup \cdots \cup w_{i_r}(N) \cup g^*(v), [N]\rangle$$

for all nonnegative integers $i_1, \ldots, i_r$ with $i_1 + \cdots + i_r = m - q$.

Let M be a compact smooth manifold. For any positive integer k, we define $G^k(M)$ to be the subgroup of $H^k(M, \mathbb{Z}/2)$ consisting of the cohomology classes u satisfying

$$\langle w_{i_1}(M) \cup \cdots \cup w_{i_r}(M) \cup u, [M]\rangle = 0$$

for all nonnegative integers $i_1, \ldots, i_r$ with $i_1 + \cdots + i_r = m - k$.

A cohomology class v in $H^k(M, \mathbb{Z}/2)$, $k \geq 1$, is said to be *spherical* provided $v = f^*(c)$, where $f : M \to S^k$ is a continuous (or equivalently smooth) map from M into the unit k-sphere S^k and c is the unique generator of the group $H^k(S^k, \mathbb{Z}/2) \cong \mathbb{Z}/2$. It is well known that v is spherical if and only if the homology class Poincaré dual to v can be represented by a smooth submanifold of M with trivial normal vector bundle (cf. [22, Théorème II.2]). Denote by $S^k(M)$ the set of all spherical cohomology classes in $H^k(M, \mathbb{Z}/2)$. It readily follows from the characterization of spherical cohomology classes recalled above that $S^k(M)$ is a subgroup of $H^k(M, \mathbb{Z}/2)$ if $2k \geq \dim M + 1$.

For any smooth submanifold N of M of codimension k, we denote by $[N]^M$ the cohomology class in $H^k(M, \mathbb{Z}/2)$ Poincaré dual to the homology class represented by N. As usual, if ξ is a real vector bundle on M, then $w(\xi)$ and $w_k(\xi)$ will stand for, respectively, its total and kth Stiefel–Whitney class. The total Stiefel–Whitney class of M will be denoted by $w(M)$.

Given a collection $\mathcal{F}$ of real vector bundles on M and a collection $\mathcal{G}$ of smooth submanifolds of M, we denote by $A(\mathcal{F}, \mathcal{G})$ the subring of $H^*(M, \mathbb{Z}/2)$ generated by $w_k(\xi)$ and $[N]^M$ for all ξ in $\mathcal{F}$, $k \geq 0$, and N in $\mathcal{G}$. Since $H^*(M, \mathbb{Z}/2)$ is a finite set, we may assume without loss of generality that the collections $\mathcal{F}$ and $\mathcal{G}$ are finite. By definition, any admissible subring of $H^*(M, \mathbb{Z}/2)$ is of the form $A(\mathcal{F}, \mathcal{G})$.

Theorem 2.4. *Let M be a compact connected smooth manifold of dimension m. Let $\mathcal{F}$ be a collection of real vector bundles on M and let $\mathcal{G}$ be a collection of smooth submanifolds of M. Assume that there is an integer $k_0 \geq 2$ such that $2k_0 + 1 \leq m$ and $\operatorname{codim}_M N \geq k_0$ for all N in $\mathcal{G}$. Then for the subring $A = A(\mathcal{F}, \mathcal{G})$ of $H^*(M, \mathbb{Z}/2)$, the following conditions are equivalent:*

(a) *There exist an algebraic model X of M and a smooth diffeomorphism $\varphi : X \to M$ satisfying*

$$\varphi^*(A) \subseteq H^*_{\mathrm{alg}}(X, \mathbb{Z}/2)$$

and

$$\varphi^*(A^k) = H^k_{\mathrm{alg}}(X, \mathbb{Z}/2) \quad \text{for all } k \text{ with } k \leq k_0 \text{ and } G^{m-k}(M) \subseteq S^{m-k}(M).$$

(b) *$w(M)$ is in A.*

Proof. If Y is a compact nonsingular real algebraic variety, then $w(Y)$ is in $H^*_{\mathrm{alg}}(Y, \mathbb{Z}/2)$ (cf. [6, 11, 12]), and hence (a) implies (b).

Assume that (b) holds. Let $\mathcal{F} = \{\xi_1, \dots, \xi_a\}$ and $\mathcal{G} = \{N_1, \dots, N_b\}$. For the use in a latter part of the proof, we modify each submanifold N_j, without affecting the cohomology class $[N_j]^M$, so as to obtain a new N_j connected and nonorientable. This is possible since M is connected and $\operatorname{codim}_M N_j \geq 2$. Indeed, the last inequality implies that if U is an open subset of M diffeomorphic to $\mathbb{R}^m$, then there is a smooth connected nonorientable submanifold P_j of M contained in U and with $\dim P_j = \dim N_j$. Joining P_j and the connected components of N_j with tubes, we get the required modification of N_j.

By transversality, the submanifolds $N_1, \dots, N_b$ can be chosen in general position. Hence in view of [4, Theorem 4, Remark 8], we may assume that M is a nonsingular real algebraic variety, $N_1, \dots, N_b$ are nonsingular Zariski closed subvarieties of M, and every topological real vector bundle on M is isomorphic to an algebraic vector bundle. In particular, we may assume that $\xi_1, \dots, \xi_a$ are algebraic vector bundles. Setting $r_i = \operatorname{rank} \xi_i$ and choosing a sufficiently large integer n, we can find a regular map $f_i : M \to \mathbb{G}_{n,r_i}$ such that ξ_i is isomorphic to $f_i^* \gamma_{n,r_i}$, and hence $w(\xi_i) = f_i^*(w(\gamma_{n,r_i}))$. Therefore

$$A \text{ is generated by } f_i^*(w_k(\gamma_{n,r_i})) \text{ and } [N_j]^M,\ 1 \leq i \leq a,\ 1 \leq j \leq b,\ k \geq 0. \tag{1}$$

Setting

$$G = \mathbb{G}_{n,r_1} \times \cdots \times \mathbb{G}_{n,r_a} \quad \text{and} \quad f = (f_1, \dots, f_a) : M \to G,$$

and making use of Künneth's theorem, we obtain

$$f^*(H^*(G, \mathbb{Z}/2)) \subseteq A. \tag{2}$$

Let $k_1, \dots, k_s$ be all the integers such that $k_0 \geq k_1 > \cdots > k_s \geq 1$ and $G^{m-k_\ell}(M) \subseteq S^{m-k_\ell}(M)$ for $\ell = 1, \dots, s$. Clearly,

$$\Gamma_\ell := \{v \in H^{m-k_\ell}(M, \mathbb{Z}/2) \mid \langle u \cup v, [M] \rangle = 0 \text{ for all } u \in A^{k_\ell}\} \tag{3}$$

is a subgroup of $G^{m-k_\ell}(M)$. Choose an integer d with $\dim_{\mathbb{Z}/2} \Gamma_\ell \leq d$ for $\ell = 1, \dots, s$. Let

$$B^{m-k_\ell}(d) = B^{m-k_\ell} \times \cdots \times B^{m-k_\ell} \quad \text{and} \quad B_0^{m-k_\ell} = B_0^{m-k_\ell} \times \cdots \times B_0^{m-k_\ell}$$

be as in (2.2) (with $n = m - k_\ell$). Since every cohomology class in Γ_ℓ is spherical, there exists a smooth map $g_\ell = (g_{\ell 1}, \dots, g_{\ell d}) : M \to B^{m-k_\ell}(d)$ satisfying

$$g_\ell(M) \subseteq B_0^{m-k_\ell}(d) \quad \text{and} \quad \Gamma_\ell = g_\ell^*(H^{m-k_\ell}(B^{m-k_\ell}(d), \mathbb{Z}/2)). \tag{4}$$

Set

$$B = B^{m-k_1}(d) \times \cdots \times B^{m-k_s}(d), \qquad B_0 = B_0^{m-k_1}(d) \times \cdots \times B_0^{m-k_s}(d),$$
$$g = (g_1, \dots, g_s) : M \to B.$$

Making use of Künneth's theorem and the inequalities $2(m - k_\ell) \geq 2(m - k_0) \geq m + 1$ for $\ell = 1, \ldots, s$, we get

$$H^q(B, \mathbb{Z}/2) = 0 \quad \text{for } 0 < q \leq m,\ q \notin \{m - k_1, \ldots, m - k_s\}. \tag{5}$$

Künneth's theorem also implies

$$\Gamma_\ell = g^*(H^{m-k_\ell}(B, \mathbb{Z}/2)) \quad \text{for } 1 \leq \ell \leq s. \tag{6}$$

Assertion 1. *The restriction $g|N : N \to B$, where $N := N_1 \cup \cdots \cup N_b$, is null homotopic.*

Clearly, it suffices to prove that for each pair of integers (ℓ, e) with $1 \leq \ell \leq s$ and $1 \leq e \leq d$, the map $h_{\ell e}|N : N \to B_0^{m-k_\ell}$ is null homotopic, where $h_{\ell e} : M \to B^{m-k_\ell}$ is defined by $h_{\ell e}(x) = g_{\ell e}(x)$ for all x in M. Recall that $B_0^{m-k_\ell}$ is diffeomorphic to S^{m-k_ℓ}. Let σ be a generator of $H^{m-k_\ell}(B_0^{m-k_\ell}, \mathbb{Z}) \cong \mathbb{Z}$. Since $\dim N_j \leq m - k_\ell$ for $j = 1, \ldots, b$, it follows from Hopf's classification theorem that $h_{\ell e}|N$ is null homotopic if and only if $(h_{\ell e}|N)^*(\sigma) = 0$ in $H^{m-k_\ell}(N, \mathbb{Z})$. By the Mayer–Vietoris exact sequence, the last condition is equivalent to $(h_{\ell e}|N_j)^*(\sigma) = 0$ in $H^{m-k_\ell}(N_j, \mathbb{Z})$ for all $j = 1, \ldots, b$.

If $\dim N_j < m - k_\ell$, then trivially $(h_{\ell e}|N_j)^*(\sigma) = 0$.

Suppose that $\dim N_j = m - k_\ell$. In that case necessarily $\ell = 1$ and $k_1 = k_0$. In order to ease notation, set $h = h_{1e}$. Since N_j is connected and nonorientable, $(h|N_j)^*(\sigma) = 0$ in $H^{m-k_1}(N_j, \mathbb{Z})$ if and only if $(h|N_j)^*(\bar{\sigma}) = 0$ in $H^{m-k_1}(N_j, \mathbb{Z}/2)$ where $\bar{\sigma}$ in $H^{m-k_1}(B_0^{m-k_1}, \mathbb{Z}/2)$ is the reduction modulo 2 of σ. It follows from (4) that $h^*(\bar{\sigma})$ is in Γ_1, and hence (3) implies

$$\langle h^*(\bar{\sigma}) \cup [N_j]^M, [M] \rangle = 0.$$

Therefore denoting by $\epsilon : N_j \hookrightarrow M$ the inclusion map, we have

$$\begin{aligned}\langle (h|N_j)^*(\bar{\sigma}), [N_j] \rangle &= \langle \epsilon^*(h^*(\bar{\sigma})), [N_j] \rangle = \langle h^*(\bar{\sigma}), \epsilon_*([N_j]) \rangle = \langle h^*(\bar{\sigma}), [N_j]^M \cap [M] \rangle \\ &= \langle h^*(\bar{\sigma}) \cup [N_j]^M, [M] \rangle = 0.\end{aligned}$$

Since N_j is connected, we get $(h|N_j)^*(\bar{\sigma}) = 0$, as required. Assertion 1 is proved.

Choose a compact subset K of M such that N is contained in the interior of K and N is a deformation retract of K, while (M, K) is a polyhedral pair. Then $g|K : K \to B$ is null homotopic and, by the homotopy extension theorem [21, p. 118, Corollary 5], there exists a continuous map $g' : M \to B$ which is homotopic to g and $g'|K$ is a constant map. Thus there is a smooth map $g'' : M \to B$ homotopic to g' and equal to g' on N. Replacing, if necessary, g by g'', we may assume that

$$g : M \to B \text{ is constant on } N = N_1 \cup \cdots \cup N_b, \tag{7}$$

while (4) and (6) still hold.

Let $c : M \to B$ be a constant map sending M to a point in B_0.

Assertion 2. *The maps $(f, g) : M \to G \times B$ and $(f, c) : M \to G \times B$ represent the same bordism class in the unoriented bordism group $\mathcal{N}_*(G \times B)$.*

In view of Theorem 2.3 and Künneth's theorem, it suffices to prove that for every pair (p, q) of nonnegative integers and all cohomology classes α in $H^p(G, \mathbb{Z}/2)$ and β in $H^q(B, \mathbb{Z}/2)$, we have

$$\langle w_{i_1}(M) \cup \cdots \cup w_{i_r}(M) \cup (f, g)^*(\alpha \times \beta), [M]\rangle \\ = \langle w_{i_1}(M) \cup \cdots \cup w_{i_r}(M) \cup (f, c)^*(\alpha \times \beta), [M]\rangle \quad (8)$$

for all nonnegative integers $i_1, \dots, i_r$ with $i_1 + \cdots + i_r = m - (p + q)$. Note that $(f, g)^*(\alpha \times \beta) = f^*(\alpha) \cup g^*(\beta)$ and $(f, c)^*(\alpha \times \beta) = f^*(\alpha) \cup c^*(\beta)$.

If $q = 0$, then $g^*(\beta) = c^*(\beta)$, and hence (8) holds.

Suppose now $0 < q \leq m$. Then $c^*(\beta) = 0$ and (8) is equivalent to

$$\langle w_{i_1}(M) \cup \cdots \cup w_{i_r}(M) \cup f^*(\alpha) \cup g^*(\beta), [M]\rangle = 0. \quad (9)$$

If $q \notin \{m - k_1, \dots, m - k_s\}$, then $\beta = 0$ according to (4), and hence (9) holds. If $q = m - k_\ell$ for some ℓ, then $g^*(\beta)$ is in Γ_ℓ in view of (5). Since (b) is satisfied, (2) implies that $w_{i_1}(M) \cup \cdots \cup w_{i_r}(M) \cup f^*(\alpha)$ is in A^{k_ℓ}. Thus (9) holds in view of (3). Assertion 2 is proved.

The proof of Theorem 2.4 can be completed as follows. We may assume that M is a Zariski closed nonsingular subvariety of $\mathbb{R}^\mu$ for some μ. Then N, being a union of finitely many Zariski closed nonsingular subvarieties of $\mathbb{R}^\mu$, is a nice set, equivalently, a quasi-regular subvariety, in the terminology used in [2] and [24], respectively (cf. [24, p. 75]). Since (f, c) is a regular map, and by (7) the restriction $(f, g)|N$ is also regular, it follows from Assertion 2 that [2, Theorem 2.8.4] is applicable. Hence there exist a nonnegative integer ν, a Zariski closed nonsingular subvariety X of $\mathbb{R}^{\mu+\nu}$, a smooth diffeomorphism $\varphi : X \to M$, and a regular map $(\bar{f}, \bar{g}) : X \to G \times B$ such that identifying $\mathbb{R}^\mu$ with $\mathbb{R}^\mu \times \{0\} \subseteq \mathbb{R}^{\mu+\nu}$, we have $N \subseteq X$, $\varphi(x) = x$ for all x in N, and $(\bar{f}, \bar{g})$ is homotopic to $(f, g) \circ \varphi = (f \circ \varphi, g \circ \varphi)$. In particular, setting

$$\bar{f} = (\bar{f}_1, \dots, \bar{f}_a) : X \to G = \mathbb{G}_{n,r_1} \times \cdots \times \mathbb{G}_{n,r_a},$$
$$\bar{g} = (\bar{g}_1, \dots, \bar{g}_s) : X \to B = B^{m-k_1}(d) \times \cdots \times B^{m-k_s}(d),$$

we obtain $\bar{f}_i^* = \varphi^* \circ f_i^*$ and $\bar{g}_\ell^* = \varphi^* \circ g_\ell^*$ in cohomology for $1 \leq i \leq a$ and $1 \leq \ell \leq s$.

The cohomology class

$$\varphi^*(f_i^*(w(\gamma_{n,r_i}))) = \bar{f}_i^*(w(\gamma_{n,r_i}))$$

is in $H^*_{\text{alg}}(X, \mathbb{Z}/2)$, the map $\bar{f}_i$ being regular. Clearly,

$$\varphi^*([N_j]^M) = [N_j]^X$$

is also in $H^*_{\text{alg}}(X, \mathbb{Z}/2)$. Hence (1) implies

$$\varphi^*(A) \subseteq H^*_{\text{alg}}(X, \mathbb{Z}/2).$$

In particular,

$$\varphi^*(A^{k_\ell}) \subseteq H^{k_\ell}_{\mathrm{alg}}(X, \mathbb{Z}/2) \quad \text{for } \ell = 1, \dots, s. \tag{10}$$

It remains to prove that the inclusion in (10) is actually an equality. By (2.2) and (4),

$$\Gamma_\ell = g_\ell^*(\mathrm{Alg}^{m-k_\ell}(B^{m-k_\ell}(d))),$$

and hence

$$\varphi(\Gamma_\ell) = \varphi^*(g_\ell^*(\mathrm{Alg}^{m-k_\ell}(B^{m-k_\ell}(d)))) = \bar{g}_\ell^*(\mathrm{Alg}^{m-k_\ell}(B^{m-k_\ell}(d))).$$

Consequently,

$$\varphi^*(\Gamma_\ell) \subseteq \mathrm{Alg}^{m-k_\ell}(X), \tag{11}$$

the map $\bar{g}_\ell : X \to B^{m-k_\ell}(d)$ being regular. By the Poincaré duality,

$$H^{k_\ell}(M, \mathbb{Z}/2) \times H^{m-k_\ell}(M, \mathbb{Z}/2) \to \mathbb{Z}/2, \quad (u, v) \mapsto \langle u \cup v, [M] \rangle$$

is a dual pairing, and therefore (3), (10), (11) and Theorem 2.1 taken together imply

$$\varphi^*(A^{k_\ell}) = H^{k_\ell}_{\mathrm{alg}}(X, \mathbb{Z}/2) \quad \text{for } \ell = 1, \dots, s,$$

as required. The proof is complete. □

We will need the following, purely technical, observation.

Lemma 2.5. *Let M be a compact connected smooth manifold of dimension m. Then:*

(i) $G^{m-1}(M) \subseteq S^{m-1}(M)$ *provided* $m \geq 2$.
(ii) $G^{m-2}(M) \subseteq S^{m-2}(M)$ *provided* $m \geq 5$ *and* $H_{m-2}(M, \mathbb{Z})$ *has no* 2*-torsion.*
(iii) $G^{m-2}(M) \subseteq S^{m-2}(M)$ *provided* $m \geq 5$, *M is orientable, and* $H_1(M, \mathbb{Z})$ *has no* 2*-torsion.*
(iv) $H^{m-3}(M, \mathbb{Z}/2) = S^{m-3}(M)$ *provided* $m \geq 7$, *M is a spin manifold, and* $H_2(M, \mathbb{Z})$ *has no* 2*-torsion.*

Proof. Given a smooth manifold P, we denote by τ_P its tangent bundle. The normal bundle of a smooth submanifold N of M will be denoted by ν_N. Recall that ν_N is a trivial vector bundle if and only if $[N]^M$ is in $S^k(M)$, $k = \mathrm{codim}_M N$.

(i) Let u be in $G^{m-1}(M)$, that is, $\langle w_1(M) \cup u, [M] \rangle = 0$. Since M is connected, we have

$$w_1(M) \cup u = 0.$$

Choose a smooth connected curve C in M with $u = [C]^M$. It suffices to prove that the normal bundle ν_C is trivial or, equivalently, $w_1(\nu_C) = 0$. Since $\tau_C \oplus \nu_C = \tau_M|C$ and τ_C is trivial, we have

$$w_1(\nu_C) = w_1(\tau_M|C) = e^*(w_1(M)),$$

where $e : C \hookrightarrow M$ is the inclusion map. A simple computation yields

$$e_*(e^*(w_1(M)) \cap [C]) = w_1(M) \cap e_*([C]) = w_1(M) \cap ([C]^M \cap [M])$$
$$= (w_1(M) \cup [C]^M) \cap [M] = (w_1(M) \cup u) \cap [M] = 0.$$

Since C is connected, we get $e^*(w_1(M)) \cap [C] = 0$, and hence $e^*(w_1(M)) = 0$. Thus $w_1(\nu_C) = 0$, as required.

(ii) By the universal coefficient theorem, the torsion subgroups of $H_{m-2}(M, \mathbb{Z})$ and $H^{m-1}(M, \mathbb{Z})$ are isomorphic, and hence $H^{m-1}(M, \mathbb{Z})$ has no 2-torsion. It follows from another version of the universal coefficient theorem that the reduction modulo 2 homomorphism $\rho : H^{m-2}(M, \mathbb{Z}) \to H^{m-2}(M, \mathbb{Z}/2)$ is surjective.

By Wu's theorem [17, Theorem 11.14], the second Wu class of M is equal to $w_1(M) \cup w_1(M) + w_2(M)$, and consequently the Steenrod square

$$\mathrm{Sq}^2 : H^{m-2}(M, \mathbb{Z}/2) \to H^m(M, \mathbb{Z}/2)$$

is given by $\mathrm{Sq}^2(u) = (w_1(M) \cup w_1(M) + w_2(M)) \cup u$. Therefore for u in $G^{m-2}(M)$, we have $\langle \mathrm{Sq}^2(u), [M] \rangle = 0$, which implies $\mathrm{Sq}^2(u) = 0$, the manifold M being connected. Since ρ is surjective, Steenrod's classification theorem [21, p. 460, Theorem 15] implies that the cohomology class u is spherical. Thus u is in $S^{m-2}(M)$, and the proof of (ii) is complete.

(iii) By the universal coefficient theorem, the torsion subgroups of $H^2(M, \mathbb{Z})$ and $H_1(M, \mathbb{Z})$ are isomorphic. The Poincaré duality implies $H^2(M, \mathbb{Z}) \cong H_{m-2}(M, \mathbb{Z})$, and hence (iii) follows from (ii).

(iv) Since $H_2(M, \mathbb{Z})$ has no 2-torsion, the reduction modulo 2 homomorphism $H_3(M, \mathbb{Z}) \to H_3(M, \mathbb{Z}/2)$ is surjective. Hence by Thom's theorem [22, Théorème II.27] each homology class in $H_3(M, \mathbb{Z}/2)$ can be represented by an orientable smooth submanifold of M. It remains to prove that if N is an orientable smooth submanifold of M of dimension 3, then the normal bundle ν_N is trivial. The orientability of N implies $w_i(N) = 0$ for $i = 1, 2$. Since $\tau_N \oplus \nu_N = \tau_M|N$ and M is a spin manifold, we get $w_i(\nu_N) = 0$ for $i = 1, 2$. It follows from the last equality that ν_N is stably trivial (cf. for example [9, Lemma 1.2]). Finally, ν_N is trivial, since $\operatorname{rank} \nu_N \geq 4 > 3 = \dim N$.

We are now ready to prove the results announced in Section 1.

Proof of Theorem 1.1. Every element of $H^1(M, \mathbb{Z}/2)$ is of the form $w_1(\lambda)$ for some real line bundle λ on M. Clearly

$$w(\lambda) = 1 + w_1(\lambda). \qquad (*)$$

We claim that every element of $H^2(M, \mathbb{Z}/2)$ is of the form $w_2(\xi)$ for some rank 2 real vector bundle ξ on M with $w_1(\xi) = 0$. Indeed, by the universal coefficient theorem, the torsion subgroups of $H_2(M, \mathbb{Z})$ and $H^3(M, \mathbb{Z})$ are isomorphic. Hence $H^3(M, \mathbb{Z})$ has no 2-torsion, which implies that the reduction modulo 2 homomorphism $\rho : H^2(M, \mathbb{Z}) \to H^2(M, \mathbb{Z}/2)$ is surjective. Every element of $H^2(M, \mathbb{Z})$ is the first Chern class $c_1(\xi)$ of some complex line bundle ξ on M. Regarding ξ as a rank 2 real vector bundle, we get $w_2(\xi) = \rho(c_1(\xi))$ and $w_1(\xi) = 0$, which proves the claim. Note that

$$w(\xi) = 1 + w_2(\xi). \tag{$**$}$$

Since M is a spin manifold, we have $w_i(M) = 0$ for $i = 1, 2, 3$ (cf. [17, Problem 8-B]). Let B be the subring of $H^*(M, \mathbb{Z}/2)$ generated by A and $w_j(M)$ for $j \geq 0$. Then B is an admissible subring with

$$A \subseteq B \quad \text{and} \quad A^k = B^k \quad \text{for } k = 0, 1, 2, 3.$$

In view of $(*)$ and $(**)$, one can find a collection $\mathcal{F}$ of real vector bundles on M and a collection $\mathcal{G}$ of smooth submanifolds of M such that $B = A(\mathcal{F}, \mathcal{G})$ and $\operatorname{codim}_M N \geq 3$ for all N in $\mathcal{G}$. By Theorem 2.4 and Lemma 2.5(i), (iii), (iv), there exist an algebraic model X of M and a smooth diffeomorphism $\varphi : X \to M$ satisfying

$$\varphi^*(B) \subseteq H^*_{\mathrm{alg}}(X, \mathbb{Z}/2) \quad \text{and} \quad \varphi^*(B^k) = H^k_{\mathrm{alg}}(X, \mathbb{Z}/2) \quad \text{for } k = 0, 1, 2, 3.$$

The proof is complete. □

Proof of Corollary 1.2. We first recall some results due to Thom [22]. Let N be a compact n-dimensional manifold. By [22, Théorème II.26], every homology class in $H_k(N, \mathbb{Z}/2)$ can be represented by a smooth submanifold, provided $2k \leq n$ or $k = n - 1$ or $(n, k) = (7, 4)$. If N is orientable and $n \leq 9$, then according to [22, Corollaire II.28], every homology class in $H_\ell(N, \mathbb{Z})$, $\ell \geq 0$, can be represented by an oriented smooth submanifold.

We can now easily complete the proof. By the Poincaré duality and the universal coefficient theorem, the reduction modulo 2 homomorphism $H_p(M, \mathbb{Z}) \to H_p(M, \mathbb{Z}/2)$ is surjective in either of the following two cases:

(i) $m = 7$ and $p = 5$,
(ii) $m = 8$ or 9 and $m/2 < p \leq m - 2$.

Hence Thom's results recalled above imply that every homology class in $H_k(M, \mathbb{Z}/2)$, $k \geq 0$, can be represented by a smooth submanifold. In particular, every subring of $H^*(M, \mathbb{Z}/2)$ is admissible. The proof is complete in view of Theorem 1.1. □

Proof of Theorem 1.3. We already recalled in the proof of Theorem 2.4 that $w(Y)$ is in $H^*(Y, \mathbb{Z}/2)$ for every compact nonsingular real algebraic variety Y. Hence (a) implies (b).

Assume that (b) holds. By Lemma 2.5, $G^{m-k}(M) \subseteq S^{m-k}(M)$ for $k = 1, 2$. Since every element of $H^1(M, \mathbb{Z}/2)$ is of the form $w_1(\lambda)$ for some real line bundle λ on M and since $w(\lambda) = 1 + w_1(\lambda)$, we have $A = A(\mathcal{F}, \mathcal{G})$, where $\mathcal{F}$ is a collection of real vector bundles on M and $\mathcal{G}$ is a collection of smooth submanifolds of M with $\operatorname{codim}_M N \geq 2$ for all N in $\mathcal{G}$. It follows from Theorem 2.4 that (a) is satisfied. □

We conclude this paper by examining consequences of Theorems 1.1 and 2.4 for the n-fold product $T^n = S^1 \times \cdots \times S^1$. The interested reader will notice that there are other examples of a similar type.

Example 2.6. Every homology class in $H_p(T^n, \mathbb{Z}/2)$, $p \geq 0$, can be represented by a smooth submanifold, and hence every subring A of $H^*(T^n, \mathbb{Z}/2)$ is admissible. By Theorem 1.1, if $n \geq 7$, then there exist an algebraic model X of T^n and a smooth diffeomorphism $\varphi : X \to T^n$ satisfying

$$\varphi^*(A) \subseteq H^*_{\mathrm{alg}}(X, \mathbb{Z}/2) \quad \text{and} \quad \varphi^*(A^k) = H^k_{\mathrm{alg}}(X, \mathbb{Z}/2) \quad \text{for } k = 0, 1, 2, 3.$$

Furthermore, for any $n \geq 1$, if A is generated by 1 and some cohomology classes in $H^i(T^n, \mathbb{Z}/2)$, $i = 1, 2$, then X and φ can be chosen in such a way that

$$\varphi^*(A) \subseteq H^*_{\mathrm{alg}}(X, \mathbb{Z}/2) \quad \text{and} \quad \varphi^*(A^k) = H^k_{\mathrm{alg}}(X, \mathbb{Z}/2) \quad \text{for } 2k + 1 \leq n.$$

Indeed, one readily checks that $A = A(\mathcal{F})$, where $\mathcal{F}$ is a collection of real vector bundles on T^n. Since $H^{m-k}(T^n, \mathbb{Z}/2) = S^{m-k}(T^n)$ for all k with $2k + 1 \leq n$, the existence of X and φ satisfying the required properties is guaranteed by Theorem 2.4.

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