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$W^{2,p}$ and $W^{1,p}$-Estimates at the Boundary for Solutions of Fully Nonlinear, Uniformly Elliptic Equations

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Abstract. In this paper we extend Caffarelli's result on interior $W^{2,p}$-estimates for viscosity solutions of uniformly elliptic equations and prove $W^{2,p}$-estimates at a flat boundary. Moreover we extend a result of A. Świech and prove $W^{1,p}$-estimates at the boundary. Thereafter we combine these results and prove global $W^{2,p}$-estimates for equations with dependence on Du and u. Finally, we show that the previous estimates lead to an existence result for $W^{2,p}$-strong solutions.

Keywords. Viscosity solutions, uniformly elliptic equations, $W^{2,p}$-regularity, $W^{1,p}$-regularity

Mathematics Subject Classification (2000). 35J15, 35J25

Introduction

We consider viscosity solutions $u \in C^0(\overline{\Omega})$ of the Dirichlet problem

$$\begin{cases} F(D^2u, Du, u, x) = f & \text{in } \Omega \\ u = \varphi & \text{on } \partial\Omega \end{cases} \tag{1}$$

on a bounded domain $\Omega \subset \mathbb{R}^n$. Let $F : S(n) \times \mathbb{R}^n \times \mathbb{R} \times \Omega \to \mathbb{R}$ where $S(n)$ is the set of symmetric $n \times n$ matrices, equipped with its usual order: For $M, N \in S(n)$ we write $M \le N$ if and only if the matrix $N - M$ is positive semi-definite. Throughout this paper we deal with uniformly elliptic equations, i.e., there exist constants $0 < \lambda < \Lambda < \infty$ such that

$$\lambda\|N\| \le F(M + N, p, r, x) - F(M, p, r, x) \le \Lambda\|N\|$$

holds for $M, N \in S(n)$, $N \ge 0$, $p \in \mathbb{R}^n$, $r \in \mathbb{R}$ and $x \in \Omega$, where the matrix-norm $\|\cdot\|$ is defined by $\|M\| := \sup_{|x|=1} |Mx|$.

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L. Caffarelli proved in [1] that viscosity solutions of

$$F(D^2u, x) = f(x) \quad \text{in } B_1 = B_1(0) \tag{2}$$

satisfy $u \in W^{2,p}(B_{1/2})$ and the interior estimate

$$\|u\|_{W^{2,p}(B_{1/2})} \leq C \left(\|u\|_{L^\infty(B_1)} + \|f\|_{L^p(B_1)} \right).$$

This result was proved under the following assumptions: F is continuous, $f \in L^p(B_1) \cap C^0(B_1)$ for $n < p < \infty$, F is uniformly elliptic and satisfies additional assumptions on the oscillation in x and on the existence of $C^{1,1}$-estimates for solutions of the equation without dependence on x; see [1, Theorem 1] for the precise statement.

In the present paper we are going to show that a similar result holds at a flat boundary. More precisely, denoting $\Omega^+ := \Omega \cap \{x_n > 0\}$, we prove that viscosity solutions of

$$\begin{cases} F(D^2u, x) = f & \text{in } B_1^+ \\ u = 0 & \text{on } B_1 \cap \{x_n = 0\} \end{cases} \tag{3}$$

satisfy $u \in W^{2,p}(B_{1/2})$ and that the corresponding estimate holds; see Theorem 2.2. The method of proof is similar to that of Caffarelli: First we show how to obtain estimates for paraboloids (i.e., polynomials of degree 2) at the boundary. Then we iterate these estimates to prove the theorem. Note that a result of this type has already been stated by L. Wang in the parabolic case, see [13, Theorem 5.8], but without a proof. It is possible to extend Theorem 2.2 with L. Escauriaza's method [5] to the range $p > n - \epsilon_0$ where ϵ_0 depends only on $\frac{\Lambda}{\lambda}$ and n.

Thereafter we consider equations with measurable ingredients, allowing $p > n - \epsilon_0$ and prove $W^{1,p}$-estimates at the boundary for uniformly elliptic equations with dependence on Du and u, see Theorem 3.1. This result is a generalisation of a theorem due to A. Świech, see [11] and also [1] and [2].

In the last section we extend the boundary estimates of Section 2 to equations without the continuity assumption on f and F in x. First we consider (2) and (3) with $F(D^2u, x)$ replaced by $F(D^2u, Du, u, x)$ and prove $W^{2,p}$-estimates similar to those above, see Theorems 4.2 and 4.3. Combining these estimates we obtain global $W^{2,p}$-estimates for viscosity solutions of Dirchlet problem (1). Finally we use the previous results to derive an existence result for $W^{2,p}$-strong solutions of Dirichlet problem (1).

Recall that a strong solution of

$$F(D^2u, Du, u, x) = f \quad \text{in } \Omega \tag{4}$$

is a function $u \in W^{2,p}_{loc}(\Omega)$ such that the equation is satisfied almost everywhere in Ω after inserting the weak derivatives. The definition of subsolutions and supersolutions is similar.

For the reader's convenience we have collected preparatory material in the first section: We recall the notion of viscosity solutions and some basic properties. We will introduce Pucci's extremal operators to characterise viscosity solutions of an important class of fully nonlinear equations. Moreover we recall the Alexandroff maximum principle, the Harnack inequality and prove the weak Harnack inequality at the boundary. From these results one deduces global Hölder regularity in the standard way.

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1. Preliminaries

1.1. Definitions and basic properties. For the reader's convenience we recall the definition of viscosity solutions of fully nonlinear equations and provide a brief collection of basic results related to the notion of viscosity solution.

Definition 1.1. Let f, F be continuous in all variables. A upper (lower) semicontinuous function u is a C^2*-viscosity subsolution* (*supersolution*) of (4), if, for all $\varphi \in C^2(B_r(x_0))$ whenever $B_r(x_0) \subset \Omega$, $\epsilon > 0$ and

$$F(D^2\varphi(x_0), D\varphi(x_0), u(x_0), x_0) \leq f(x_0) - \epsilon$$
$$(F(D^2\varphi(x_0), D\varphi(x_0), u(x_0), x_0) \geq f(x_0) + \epsilon),$$

$u - \varphi$ can not attain a local maximum (minimum) at x_0. u is called a C^2*-viscosity solution* of (4), if u is both a subsolution and a supersolution.

Without the continuity assumption on f we consider

Definition 1.2. Let F be continuous in M, p, r, measurable in x and we assume $f \in L^p_{loc}(\Omega)$ for $p > \frac{n}{2}$. A continuous function u is a $W^{2,p}$*-viscosity subsolution* (*supersolution*) of (4), if, for all $\varphi \in W^{2,p}(B_r(x_0))$ where $B_r(x_0) \subset \Omega$, $\epsilon > 0$ and

$$F(D^2\varphi(x), D\varphi(x), u(x), x) \leq f(x) - \epsilon$$
$$(F(D^2\varphi(x), D\varphi(x), u(x), x) \geq f(x) + \epsilon)$$

almost everywhere in $B_r(x_0)$, then $u - \varphi$ can not attain a local maximum (minimum) at x_0. u is called a $W^{2,p}$*-viscosity solution*, if u is both a subsolution and a supersolution.

We say that $F(D^2u, Du, u, x) \geq (\leq, =) f$ in Ω in the C^2 or $W^{2,p}$-viscosity sense whenever u is a C^2 or $W^{2,p}$-viscosity subsolution (-supersolution,-solution). We refer to [2, Chapter 2] for basic properties of C^2-viscosity solutions.

In order to define the set of viscosity solutions of a certain class of uniformly elliptic equations we introduce Pucci's operators: Let $0 < \lambda \leq \Lambda$ be given constants. For $M \in S(n)$ we define:

$$\mathcal{M}^+(\lambda, \Lambda, M) := \Lambda \sum_{e_i > 0} e_i + \lambda \sum_{e_i < 0} e_i, \quad \mathcal{M}^-(\lambda, \Lambda, M) := \lambda \sum_{e_i > 0} e_i + \Lambda \sum_{e_i < 0} e_i\,,$$

where e_i are the eigenvalues of M. We will write $\mathcal{M}^\pm(M) = \mathcal{M}^\pm(\lambda, \Lambda, M)$ when the choice of λ, Λ is clear. Again we refer to [2] for the properties of Pucci's operators. Consider

$$L^\pm(\lambda, \Lambda, b, u) := \mathcal{M}^\pm(\lambda, \Lambda, D^2u) \pm b|Du|$$

to define the class S:

Definition 1.3. Let $b \geq 0$, $0 < \lambda \leq \Lambda$ be given constants. We define the classes $\underline{S}(\lambda, \Lambda, b, f)$ and $\overline{S}(\lambda, \Lambda, b, f)$ to be the set of all continuous functions u that satisfy $L^+u \geq f$, respectively $L^-u \leq f$ in the C^2 or $W^{2,p}$-viscosity sense in Ω. We define

$$\begin{aligned} S(\lambda, \Lambda, b, f) &:= \overline{S}(\lambda, \Lambda, b, f) \cap \underline{S}(\lambda, \Lambda, b, f) \\ S^*(\lambda, \Lambda, b, f) &:= \overline{S}(\lambda, \Lambda, b, |f|) \cap \underline{S}(\lambda, \Lambda, b, -|f|). \end{aligned}$$

The notation of the class S is independent of the type of viscosity solution. To make the notation clear we emphasise that in the continuous case we always deal with C^2-viscosity solutions. In this case C^2-viscosity solutions are $W^{2,p}$-viscosity solutions, see [3] for further details.

Continuous functions $u \in \overline{S}, \underline{S}$ and S are called supersolutions, subsolutions and solutions, respectively. We write $\overline{S}, \underline{S}, S(\lambda, \Lambda, b, f) = \overline{S}, \underline{S}, S(b, f)$ when the choice of the ellipticity constants is understood. The following Proposition is a direct consequence of the previous definitions.

Proposition 1.4. *Let $F(M, p, r, x)$ be uniformly elliptic with ellipticity constants λ, Λ and let u be a C^2 or $W^{2,p}$-viscosity subsolution (supersolution) of (4). We assume that F satisfies the following structure conditions:*

$$\begin{aligned} &F(M, p, r, x) - F(N, q, s, x) \leq \mathcal{M}^+(M - N) + b|p - q| \\ &\big(F(M, p, r, x) - F(N, q, s, x) \geq \mathcal{M}^-(M - N) - b|p - q|\big), \end{aligned}$$

where $b \geq 0$ is a constant. Then

$$\begin{aligned} &\mathcal{M}^+(D^2u) + b|Du| + F(0, 0, u, x) \geq f \\ &\big(\mathcal{M}^-(D^2u) - b|Du| + F(0, 0, u, x) \leq f\big) \end{aligned}$$

in the C^2 or $W^{2,p}$-viscosity sense.

We introduce another structure condition that will be frequently used in this paper:

$$\begin{aligned}\mathcal{M}^-(\lambda,\Lambda,M-N) - b|p-q| - c|r-s| \\ \le F(M,p,r,x) - F(N,q,s,x) \\ \le \mathcal{M}^+(\lambda,\Lambda,M-N) + b|p-q| + c|r-s|\end{aligned} \tag{5}$$

for all $M,N \in S(n)$, $p,q \in \mathbb{R}^n$, $r,s \in \mathbb{R}$, $x \in \Omega$, and constants $b,c \in \mathbb{R}_+$. Clearly, for $p=q, r=s$, condition (5) implies that F is uniformly elliptic. Note that whenever F is assumed to be merely measurable in x we understand (5) to hold for a.e. $x \in \Omega$ only.

We state a stability result for $W^{2,p}$-viscosity solutions. Except for some straightforward modifications the proof of the following lemma is the same as the proof of [3, Theorem 3.8].

Proposition 1.5. *For $k \in \mathbb{N}$ let $\Omega_k \subset \Omega_{k+1}$ be an increasing sequence of domains and $\Omega := \bigcup_{k\ge 1}\Omega_k$. Let $p > n - \epsilon_0\big(\frac{\Lambda}{\lambda}, n, b, \operatorname{diam}(\Omega)\big)$ and F, F_k be measurable in x and satisfy structure condition (5). Assume $f \in L^p(\Omega)$, $f_k \in L^p(\Omega_k)$ and that $u_k \in C^0(\Omega_k)$ are $W^{2,p}$-viscosity subsolutions (supersolutions) of $F_k(D^2u_k, Du_k, u_k, x) = f_k$ in Ω_k. Suppose that $u_k \to u$ locally uniformly in Ω and that for $B_r(x_0) \subset \Omega$ and $\varphi \in W^{2,p}(B_r(x_0))$ we have*

$$\|(g-g_k)^+\|_{L^p(B_r(x_0))} \to 0 \qquad \big(\|(g-g_k)^-\|_{L^p(B_r(x_0))} \to 0\big)\,, \tag{6}$$

where $g(x) := F(D^2\varphi, D\varphi, u, x) - f(x)$ and $g_k(x) := F_k(D^2\varphi, D\varphi, u_k, x) - f_k(x)$. Then u is an $W^{2,p}$-viscosity subsolution (supersolution) of $F(D^2u, Du, u, x) = f$ in Ω. Moreover, if F, f are continuous, then u is an C^2-viscosity subsolution (supersolution) provided that (6) holds for $\varphi \in C^2(B_r(x_0))$.

We proceed with a first existence result for C^2-viscosity solutions. A more general version will be proven at the end of this section.

Proposition 1.6. *Let $\Omega \subset\subset \mathbb{R}^n$ be open and $\partial\Omega$ satisfy a uniform exterior sphere condition, i.e., there exists a radius $r_0 > 0$, such that for every $x_0 \in \partial\Omega$ there exists a ball $B_{r_0}(z_0)$ such that $\overline{\Omega} \cap \overline{B}_{r_0}(z_0) = \{x_0\}$. Suppose that $f \in C^0(\Omega)$ is bounded, $\varphi \in C^0(\partial\Omega)$ and that $F = F(M,p,r,x)$ is Lipschitz continuous in x, and satisfies $F(0,0,0,x) = 0$, (5), and*

$$d(r-s) \le F(M,p,s,x) - F(M,p,r,x) \tag{7}$$

for all $x \in \Omega$, $p, \in \mathbb{R}^n$, $r,s \in \mathbb{R}$, $r \ge s$, $M \in S(n)$, and a constant $d \in \mathbb{R}_+$. Then there exists a C^2-viscosity solution u of Dirichlet problem (1).

Proof. In order to apply Perron's method we need a comparison result and the existence of subsolutions and supersolutions of (1). We show how to construct supersolutions first. Consider functions

$$v_{x_0,\epsilon}(x) := \varphi(x_0) + \epsilon + C_\epsilon w(|x - z_0|)$$

for any $x_0 \in \partial\Omega$, $\epsilon > 0$ and $w(r) := \tau\left(r_0^{-\sigma} - r^{-\sigma}\right)$, where σ, τ are positive constants and r_0 is the radius of the exterior sphere condition. A straightforward computation yields $F(D^2 v_{x_0,\epsilon}, Dv_{x_0,\epsilon}, v_{x_0,\epsilon}, x) \le -\tau\theta < 0$ in Ω for a constant $\theta > 0$, provided that σ is chosen large enough. Set $\tau = \sup_\Omega \frac{f^-}{\theta}$ to obtain $F(D^2 v_{x_0,\epsilon}, Dv_{x_0,\epsilon}, v_{x_0,\epsilon}, x) \le f$. We extend φ such that $\varphi \in C^0(\overline{\Omega})$, choose the constant C_ϵ such that $\varphi(x_0) + \epsilon + C_\epsilon w(|x - z_0|) \ge \varphi(x)$ for $x \in \overline{\Omega}$ and set

$$v(x) := \Big(\inf_{x_0 \in \partial\Omega,\, \epsilon > 0} v_{x_0,\epsilon}(x) \Big)_*$$

for $x \in \overline{\Omega}$, i.e., v is the lower semicontinuous envelope of the function in brackets. From [4, Lemma 4.2] we infer that v is a supersolution. Moreover we get $v(x) - \varphi(x_0) \le \epsilon + C_\epsilon w(|x - z_0|)$, where $x_0 \in \partial\Omega$, $x \in \overline{\Omega}$, $\epsilon > 0$ and hence $v^*(x_0) = \varphi(x_0)$ on $\partial\Omega$. The construction of subsolutions is similar. Since F is Lipschitz continuous in x the hypotheses of the comparison result given by [4, Theorem 3.3] are satisfied. Therefore Perron's method [4, Theorem 4.1] is applicable. □

1.2. Maximum principle, Harnack inequality and Hölder regularity.

We state the Alexandroff-Bakelman-Pucci maximum principle for C^2-viscosity solutions; see [3, Appendix A] for a proof of the following Theorem.

Theorem 1.7 (Alexandroff-Bakelman-Pucci). *Assume that* $\Omega \subset\subset \mathbb{R}^n$ *is open and* $\operatorname{diam}(\Omega) \le d$, $u \in C^0(\overline{\Omega})$ *and* $f \in L^n(\Omega) \cap C^0(\Omega)$. *Then there exists a constant* $C(n, \lambda, \Lambda, b, d)$ *such that for all* C^2*-viscosity subsolutions* $u \in \underline{S}(b, f)$

$$\sup_\Omega u \le \sup_{\partial\Omega} u^+ + C(n, \lambda, \Lambda, b, d)\|f\|_{L^n(\Gamma_u^+ \cap \{u>0\})}$$

and for all C^2*-viscosity supersolutions* $u \in \overline{S}(b, f)$

$$\sup_\Omega u^- \le \sup_{\partial\Omega} u^- + C(n, \lambda, \Lambda, b, d)\|f\|_{L^n(\Gamma_u^- \cap \{u<0\})}$$

holds. The set $\Gamma_u^+ = \Gamma^+(u, \Omega)$ *is the upper contact set of* u, *defined as*

$$\Gamma^+(u, \Omega) := \{x \in \Omega; \exists\, p \in \mathbb{R}^n : u(y) \le u(x) + p(y - x) \text{ for all } y \in \Omega\}$$

and $\Gamma_u^- = \Gamma^-(u, \Omega)$ *is the lower contact set of* u,

$$\Gamma^-(u, \Omega) := \{x \in \Omega; \exists\, p \in \mathbb{R}^n : u(y) \ge u(x) + p(x - y) \text{ for all } y \in \Omega\}.$$

The Harnack inequality for C^2-viscosity solutions in $S^*(\lambda, \Lambda, 0, f)$ was proven by L. Caffarelli in [1], see also [2, Theorems 4.3 and 4.8]. In [7, Chapter 5] P. Fok proved the Harnack inequality and interior Hölder continuity for $W^{2,n-\epsilon_0}$-viscosity solutions in $S^*(\lambda, \Lambda, b, f)$, see [7, Theorem 5.20, Theorem 5.21].

The weak Harnack inequality, however, is proven only for the case $p > n$ and $W^{2,p}$-viscosity solutions in $\overline{S}(\lambda, \Lambda, b, f)$. Therefore we show how to prove the weak Harnack inequality for $p > n - \epsilon_0$.

Proposition 1.8 (Weak Harnack inequality). *Let $p > n - \epsilon_0\big(n, \frac{\Lambda}{\lambda}, b\big)$. Suppose $u \in \overline{S}(\lambda, \Lambda, b, f)$ in $B_\rho = B_\rho(0)$ in the $W^{2,p}$-viscosity sense satisfies $u \geq 0$ in B_ρ where $f \in L^p(B_\rho)$. Then*

$$\rho^{-\frac{n}{p_0}} \|u\|_{L^{p_0}(B_{\rho/2})} \leq C\Big(\inf_{B_{\rho/2}} u \ + \rho \|f\|_{L^p(B_\rho)}\Big),$$

where $p_0 > 0$, C depend only on n, λ, Λ, and b.

Proof. We prove the claim for $\rho = 1$. According to [3, Proposition 3.1], there exists a strong solution $v \in W^{2,p}_{\text{loc}}(B_1)$ of $-\mathcal{M}^+(D^2 v) - b|Dv| \geq |f|$ in B_1 such that $v = 0$ on ∂B_ρ, $v \geq 0$ in $B_1(0)$, and $\|v\|_{L^\infty(B_1)} \leq C\|f\|_{L^p(B_1)}$ where $C = C(n, \lambda, \Lambda, p, b)$. Consequently $u + v \in \bar{S}(\lambda, \Lambda, b, 0)$ and [7, Corollary 5.9] yields

$$\|u + v\|_{L^{p_0}(B_{1/2}(0))} \leq C(n, \lambda, \Lambda, b) \inf_{B_{1/2}(0)} (u + v).$$

Combining these estimates we derive the assertion for $\rho = 1$. A scaling argument completes the proof. □

Proposition 1.9 (Weak Harnack inequality at the boundary). *Assume $p > n - \epsilon_0\big(n, \frac{\Lambda}{\lambda}, b\big)$. Let $u \in \overline{S}(\lambda, \Lambda, b, f)$ in Ω in the $W^{2,p}$-viscosity sense satisfy $u \in C^0(\overline{\Omega})$, and $u \geq 0$. Suppose that $f \in L^p(\Omega)$ and define*

$$m := \inf_{\partial\Omega \cap Q_\rho(0)} u \qquad \text{and} \qquad u_-^m := \begin{cases} min(u, m) & \text{in } Q_\rho(0) \cap \Omega \\ m & \text{in } Q_\rho(0) \setminus \Omega\,. \end{cases}$$

Then

$$\rho^{-\frac{n}{p_0}} \|u_-^m\|_{L^{p_0}(Q_{\rho/4}(0))} \leq C\Big(\inf_{Q_{\rho/2}(0)} u_-^m \ + \rho \|f\|_{L^n(Q_\rho(0) \cap \Omega)}\Big),$$

where $p_0 > 0$, C are universal constants.

Proof. The assertion follows from Proposition 1.8, provided that we prove $u_-^m \in \overline{S}(\lambda, \Lambda, b, f^+)$ in $Q_\rho(0)$, where $f^+ := \max(f, 0)$. First of all we observe that $u_-^m \in C^0(Q_\rho(0))$. We extend f by 0 outside Ω and continue to denote the extension by f. Since $f^+ \geq f$ we get $u \in \overline{S}(\lambda, \Lambda, b, f^+)$ in Ω. Moreover, $f^+ \geq 0$

yields $v \in \overline{S}(\lambda, \Lambda, b, f^+)$ in Q_ρ for any constant function v. Consider $x_0 \in Q_\rho(0)$ and $\varphi \in W^{2,p}(B_r(x_0))$ such that

$$\mathcal{M}^-(D^2\varphi(x)) - b|D\varphi(x)| \geq f^+(x) + \epsilon$$

holds for a.e. $x \in B_r(x_0)$. If $u_-^m(x_0) = u(x_0) < m$, we have $x_0 \in \Omega$ and $u_-^m = u$ near x_0. Consequently, $u_-^m - \varphi$ cannot attain a local minimum at x_0.

If $u_-^m(x_0) = m$ we assume that $u_-^m - \varphi$ attains a local minimum at x_0 and deduce $m - \varphi(x_0) = u_-^m(x_0) - \varphi(x_0) \leq u_-^m - \varphi \leq m - \varphi$. This is a contradiction as we have already seen that constant functions are supersolutions. □

Combining the weak Harnack inequality at the boundary with the interior Hölder estimate, [7, Theorem 5.21], we obtain global Hölder continuity in the standard way.

Theorem 1.10. *Assume $p > n - \epsilon_0$ and that $u \in S^*(\lambda, \Lambda, b, f)$ in Ω in the $W^{2,p}$-viscosity sense satisfies $u = \varphi$ on an open boundary portion $T \subset \partial\Omega$ where $f \in L^p(\Omega)$, and $\varphi \in C^{0,\beta}(T)$. Assume that T satisfies a uniform exterior cone condition, i.e., for all $x_0 \in T$ there exists a cone V_{x_0} congruent to some fixed cone V, such that $\overline{V}_{x_0} \cap \overline{\Omega} = \{x_0\}$. Then $u \in C^{0,\alpha}(\Omega')$ for any $\Omega' \subset\subset \Omega \cup T$ where $\alpha = \alpha(n, \lambda, \Lambda, b, p, V, \beta)$ and*

$$\|u\|_{C^{0,\alpha}(\Omega')} \leq C \left(\|u\|_{L^\infty(\Omega)} + \|\varphi\|_{C^{0,\beta}(T)} + \|f\|_{L^p(\Omega)}\right)$$

for some constant $C = C(n, \lambda, \Lambda, b, p, V, d)$ where $d := \operatorname{dist}(\Omega', \partial\Omega \setminus T)$. If $\Omega' = \Omega$, d is to be replaced by $\operatorname{diam}(\Omega)$.

We complete this subsection with a further existence result for C^2-viscosity solutions which is needed in the following sections.

Proposition 1.11. *Let $\Omega \subset\subset \mathbb{R}^n$ and $\partial\Omega$ satisfy a uniform exterior sphere condition. Assume that $F(M, p, r, x)$ is continuous on $S(n) \times \mathbb{R} \times \mathbb{R} \times \Omega$, non-increasing in r, and satisfies structure condition* (5) *and $F(0, 0, 0, x) \equiv 0$. Then for $f \in C^0(\Omega)$, bounded, and $\varphi \in C^0(\partial\Omega)$ there exists at least one C^2-viscosity solution u of Dirichlet problem* (1).

Proof. We set $F_\delta(M, p, r, x) := F(M, p, r, x) - \delta r$ for $\delta > 0$ and observe that F_δ satisfies (7). We have $|F_\delta(M, p, r, x)| \leq \Lambda\|M\| + b\ |p| + 2\ c\ |r|$ if δ is chosen sufficiently small and consider, for $\epsilon > 0$, the sup-convolutions

$$F_\delta^\epsilon(M, p, r, x) := \sup_{y \in \Omega} \left(F_\delta(M, p, r, y) - \frac{1}{2\epsilon}|x - y|^2\right).$$

Sub-convolutions were introduced by R. Jensen in [9], see also Chapter 5 in [2]. Similar to the proof of [2, Lemma 5.2] we obtain that F_δ^ϵ is Lipschitz continuous in x with Lipschitz constant $\frac{3}{\epsilon}\mathrm{diam}(\Omega)$. It is easy to check that F_δ^ϵ satisfies (5), (7), and that $F_\delta^\epsilon(0,0,0,x) \equiv 0$ holds. The assumptions of Proposition 1.6 are satisfied, and hence there exists a C^2-viscosity solution u_δ^ϵ of

$$\begin{cases} F_\delta^\epsilon(D^2u_\delta^\epsilon, Du_\delta^\epsilon, u_\delta^\epsilon, x) = f & \text{in } \Omega \\ \qquad\qquad\qquad\quad u_\delta^\epsilon = \varphi & \text{on } \partial\Omega. \end{cases}$$

Next we prove that u_δ^ϵ converges to a C^2-viscosity solution u of the original Dirichlet problem. Therefore we check that $F_\delta^\epsilon(M,p,r,x) \to F(M,p,r,x)$ uniformly on compact subsets of $S(n) \times \mathbb{R}^n \times \mathbb{R} \times \Omega$ and, $u_\delta^\epsilon \to u$ uniformly on compact subsets of Ω. From [2, Lemma 5.2] we infer $F_\delta(M,p,r,x_0) \leq F_\delta^\epsilon(M,p,r,x_0) \leq F_\delta(M,p,r,x_0^*)$ and

$$|x_0 - x_0^*|^2 = 2\epsilon\left(F_\delta(M,p,r,x_0^*) - F_\delta^\epsilon(M,p,r,x_0)\right) \leq \epsilon\, C\left(\|M\| + |p| + |r|\right),$$

where $C = C(\Lambda, b, c)$, $x_0 \in \Omega' \subset\subset \Omega$, $x_0^* \in \overline{\Omega'}$, $M \in S(n)$, $p \in \mathbb{R}^n$, and $r \in \mathbb{R}$. If M, p, r are bounded we conclude $x_0^* \to x_0$ as $\epsilon \to 0$. Note that F is uniformly continuous on compact subsets of $S(n)\times\mathbb{R}^n\times\mathbb{R}\times\Omega$ and hence $F_\delta^\epsilon(M,p,r,x_0) \to F_\delta(M,p,r,x_0)$ uniformly on compact subsets of $S(n) \times \mathbb{R}^n \times \mathbb{R} \times \Omega$. Clearly, we also have $F_\delta(M,p,r,x_0) \to F(M,p,r,x_0)$ uniformly on compact subsets of $S(n) \times \mathbb{R}^n \times \mathbb{R} \times \Omega$.

Since F_δ^ϵ is non-increasing in r we may apply the Alexandroff maximum principle, Theorem 1.7 and obtain

$$\|u_\delta^\epsilon\|_{L^\infty(\Omega)} \leq \|\varphi\|_{L^\infty(\partial\Omega)} + C(n,\lambda,\Lambda,b,\mathrm{diam}(\Omega))\|f\|_{L^n(\Omega)} =: K.$$

We deduce $u_\delta^\epsilon \in S^*(\lambda,\Lambda,b,f+2cK)$ and thus Theorem 1.10 yields that u_δ^ϵ is uniformly bounded in $C^{0,\alpha}(\Omega')$ for $\Omega' \subset\subset \Omega$.

It remains to show that u_δ^ϵ achieves the boundary value in an equicontinuous manner. Fix $\varrho > 0$ and let $x_0 \in \partial\Omega$ be arbitrary. For $\bar{u} := v_{x_0,\varrho}$ from the proof of Proposition 1.6 we have that $\mathcal{M}^+(D^2\bar{u}) + b|D\bar{u}| + \bar{u}^- \leq f$. Moreover we may assume $\varphi \leq \bar{u}$ on $\partial\Omega$. Consequently, $v := u_\delta^\epsilon - \bar{u}$ is a C^2-viscosity subsolution of $F_\delta^\epsilon(D^2v, Dv, v, x) = 0$ in Ω. From the maximum principle we infer $\sup_\Omega v \leq \sup_{\partial\Omega}(\varphi - \bar{u}) \leq 0$ and hence $u_\delta^\epsilon \leq \bar{u}$. Similarly we obtain $u_\delta^\epsilon \geq \underline{u} := \varphi(x_0) - \varrho - C_\varrho w(|\cdot - z_0|)$ where w, z_0, C_ϱ are the same as in the proof of Proposition 1.6. Therefore we have $|u_\delta^\epsilon(x) - \varphi(x_0)| \leq \varrho + C_\varrho w(|x - z_0|) \leq 2\varrho$, provided that $|x - x_0|$ is sufficiently small.

Finally, by Arzela–Ascoli's theorem we obtain the existence of $u \in C^0(\overline{\Omega})$ and a subsequence such that $u_\delta^\epsilon \to u$ in $C^0(\overline{\Omega})$. □

2. $W^{2,p}$-estimates at the boundary

The aim of this section is to prove $W^{2,p}$-boundary estimates for C^2-viscosity solutions. Before we state the theorem, we introduce the function β in order to measure the oscillation of F in x:

Definition 2.1. Let $F : S(n) \times \Omega \to \mathbb{R}^n$ be continuous in x. We define

$$\beta_F(x,y) = \beta(x,y) = \sup_{M \in S(n)\setminus\{0\}} \frac{|F(M,x) - F(M,y)|}{\|M\|}.$$

An important hypothesis on F will be

Assumption A**:** We assume that the function F satisfies interior and boundary $C^{1,1}$-estimates, i.e. for $x_0 \in B_1$ and $w_0 \in C^0(\partial B_1)$ there exists a solution $w \in C^2(B_1) \cap C^0(\overline{B}_1)$ of

$$\begin{cases} F(D^2 w, x_0) = 0 & \text{in } B_1 \\ w = w_0 & \text{on } \partial B_1 \end{cases}$$

such that $\|w\|_{C^{1,1}(\overline{B}_{1/2})} \leq c_e \|w_0\|_{L^\infty(\partial B_1)}$. Additionally, we assume that for $x_0 \in B_1 \cap \{x_n = 0\}$ and $w_0 \in C^0(\partial B_1^+)$, $w_0 = 0$ on $B_1 \cap \{x_n = 0\}$, there exists a solution $w \in C^2(B_1^+) \cap C^0(\overline{B}_1^+)$ of

$$\begin{cases} F(D^2 w, x_0) = 0 & \text{in } B_1^+ \\ w = w_0 & \text{on } \partial B_1^+ \end{cases}$$

such that $\|w\|_{C^{1,1}(\overline{B^+}_{1/2})} \leq c_e \|w_0\|_{L^\infty(\partial B_1^+)}$. Taking $w_0 = 0$ we observe that Assumption A implies $F(0,\cdot) = 0$. Now, the main result of this section is

Theorem 2.2. *Let u be a bounded C^2-viscosity solution of*

$$\begin{cases} F(D^2 u, x) = f & \text{in } B_1^+ \\ u = 0 & \text{on } B_1 \cap \{x_n = 0\}. \end{cases}$$

Assume that F is uniformly elliptic with ellipticity constants λ, Λ, continuous in x, and that Assumption A is satisfied. Let $f \in L^p(B_1^+) \cap C^0(B_1^+)$ for $n < p < \infty$. Then there exist constants β_0 and C depending on n, λ, Λ, c_e, p such that

$$\left(\frac{1}{|B_r(x_0) \cap B_1^+|} \int_{B_r(x_0)\cap B_1^+} \beta(x_0,x)^n dx \right)^{\frac{1}{n}} \leq \beta_0$$

for all $x_0 \in B_1^+$ and all $r > 0$ implies $u \in W^{2,p}(B_{1/2}^+)$ and

$$\|u\|_{W^{2,p}(B_{1/2}^+)} \leq C\big(\|u\|_{L^\infty(B_1^+)} + \|f\|_{L^p(B_1^+)} \big).$$

Remark 2.3. One can show that the oscillation condition in Theorem 2.2 implies that the oscillation measured in the L^∞-norm is also small. In order to show this we consider $x, y \in B_1^+$, $z := \frac{x+y}{2}$ and $r := |\frac{x-y}{2}|$ and compute

$$\begin{aligned}\beta(x,y)^n &= \frac{1}{|B_r(z)\cap B_1^+|}\int_{B_r(z)\cap B_1^+}\beta(x,y)^n dw\\ &\le \frac{2^{n-1}}{|B_r(z)\cap B_1^+|}\left(\int_{B_r(z)\cap B_1^+}\beta(x,w)^n dw + \int_{B_r(z)\cap B_1^+}\beta(w,y)^n dw\right)\\ &\le \frac{2^{n-1}}{|B_{2r}(x)\cap B_1^+|}\int_{B_{2r}(x)\cap B_1^+}\beta(x,w)^n dw\\ &\quad + \frac{2^{n-1}}{|B_{2r}(y)\cap B_1^+|}\int_{B_{2r}(y)\cap B_1^+}\beta(w,y)^n dw\\ &\le 4^n\,\beta_0^n.\end{aligned}$$

Particularly with regard to the linear case we observe that the assumption on a small oscillation, measured in the L^n-norm, is not weaker than the corresponding assumption in the Calderon–Zygmund estimates.
However, in the present paper we continue using the L^n-condition.

Before we start to prove Theorem 2.2 we introduce some terminology. A function $P(x) = p_0 + p_1 x \pm \frac{M}{2}|x|^2$ is called a paraboloid with opening M. The paraboloid is convex in the case $+$ and concave in the case $-$. For $u \in C^0(\Omega)$, $\Omega' \subset \Omega$ and $M > 0$ we define

$$\underline{G}_M(u,\Omega') := \left\{x_0 \in \Omega';\ \begin{array}{l}\text{there is a concave paraboloid } P \text{ of opening } M,\\ \text{such that } P(x_0) = u(x_0), P(x) \le u(x)\ \forall\, x \in \Omega'\end{array}\right\}$$

and $\underline{A}_M(u,\Omega') := \Omega' \setminus \underline{G}_M(u,\Omega')$. Using convex paraboloids we similarly define $\overline{G}_M(u,\Omega')$ and $\overline{A}_M(u,\Omega')$ and set

$$\begin{aligned}G_M(u,\Omega') &:= \underline{G}_M(u,\Omega') \cap \overline{G}_M(u,\Omega')\\ A_M(u,\Omega') &:= \underline{A}_M(u,\Omega') \cap \overline{A}_M(u,\Omega').\end{aligned}$$

Moreover, we define $\overline{\Theta}(u,\Omega',x) := \inf\{M > 0; x \in \overline{G}_M(u,\Omega')\}$ and again we similarly define $\underline{\Theta}(u,\Omega',x)$ and $\Theta(u,\Omega',x)$.

We will need the following technical proposition whose proof is based on [8, Lemma 9.7], the details are left to the reader.

Proposition 2.4. *Let $f \ge 0$ be a measurable function, $\mu_f(t) := |\{f \ge t\}|$ be the distribution function and $\eta > 0, M > 1$ constants. Then we have*

$$f \in L^p(\Omega) \iff \sum_{j\in\mathbb{N}} M^{pj}\mu_f(\eta M^j) =: S_f < \infty$$

for every $p \in (0,\infty)$. In particular there exists a constant $C = C(n,\eta,M)$, such that

$$C^{-1}S_f \le \|f\|^p_{L^p(\Omega)} \le C\left(|\Omega| + S_f\right).$$

Note that, once we have estimates for paraboloids at the boundary, the arguments are similar to those of Caffarelli's proof of the interior estimates. We recall Cafarelli's idea: Consider the distribution function of Θ

$$\mu_{\Theta,\Omega'}(t) := |\{x \in \Omega';\ \Theta(x) > t\}|.$$

It is clear that $\mu_{\Theta,\Omega'}(t) = |A_t(u,\Omega')|$ and an application of Proposition 2.4 yields

$$\Theta(u,\Omega',\cdot) \in L^p(\Omega') \Longleftrightarrow \left(\sum_{j\geq 1} M^{pj} |A_{\eta M^j}(u,\Omega')| \right) =: S_\Theta < \infty.$$

From Proposition 1.1 in [2] we infer $\|D^2 u\|_{L^p(\Omega')} \leq C(\eta, M, p)\,(|\Omega| + S_\Theta)$. Therefore it suffices to prove estimates for $\sum_{j\geq 1} M^{pj}|A_{\eta M^j}(u,\Omega')|$ in order to derive $W^{2,p}$-estimates in Ω'.

2.1. Estimates for paraboloids at the boundary. The first step towards $W^{2,p}$-estimates at the boundary are estimates for paraboloids at the boundary. The goal of this subsection is the proof of a power decay at the boundary for $|A_t(u,\Omega)|$. We restrict ourselves to a flat boundary, more precisely we look at B_r^+. In this chapter we only consider equations without dependence on Du and u. Therefore we set $\underline{S}, \overline{S}, S(\lambda,\Lambda,0,f) = \underline{S}, \overline{S}, S(\lambda,\Lambda,f)$ as an abbreviation. Let $Q_r^d(x_0) := (x_0 - \frac{r}{2}, x_0 + \frac{r}{2})^d$ be the cube of dimension d, side-length r and center x_0. In case $x_0 = 0$ we write Q_r^d instead of $Q_r^d(0)$ and if $d = n$ we write $Q_r(x_0)$ instead of $Q_r^n(x_0)$. Throughout the paper, a constant is called *universal* if it depends only on the dimension n and the ellipticity constants λ, Λ. In the course of the proof we will need the Maximal function and the Calderon–Zygmund cube decomposition:

Proposition 2.5. *For $f \in L^1_{loc}(\mathbb{R}^n)$ the Maximal function $M(f)$ of f is defined by*

$$M(f)(x) = \sup_{\rho>0} \frac{1}{|B_\rho(x)|} \int_{B_\rho(x)} |f|dx.$$

The Maximal operator M is of weak type $(1,1)$ and of strong type (p,p) for $1 < p \leq \infty$. More precisely we have

$$|\{(M(f) > t)\}| \leq C(n)t^{-1}\|f\|_{L^1(\mathbb{R}^n)} \quad \text{for } f \in L^1(\mathbb{R}^n), t > 0$$

and

$$\|M(f)\|_{L^p(\mathbb{R}^n)} \leq C(n,p)\|f\|_{L^p(\mathbb{R}^n)} \quad \text{for } f \in L^p(\mathbb{R}^n), 1 < p \leq \infty.$$

Moreover, if $f \in L^p(\mathbb{R})$, $1 \leq p \leq \infty$ then $M(f)$ is finite almost everywhere.

See [10, Theorem 1] for a proof of Proposition 2.5. We turn to the Calderon–Zygmund cube decomposition. By (k-times) repeated bisection of the edges we split the unit cube Q_1 into 2^{kn} sub cubes of side-length 2^{-k}. The cubes obtained in this way are called dyadic cubes. By $\tilde{Q}$ we denote the unique predecessor of a dyadic cube Q; see [2, Lemma 4.2] for the proof of the following Lemma.

Lemma 2.6 (Calderon–Zygmund decomposition)**.** *Suppose $A \subset B \subset Q_1$, $|A| \leq \delta < 1$ and $|A \cap Q| > \delta|Q| \Longrightarrow \tilde{Q} \subset B$ hold whenever Q is a dyadic qube. Then $|A| \leq \delta|B|$.*

Now, all necessary preparations are completed and we start with a rescaled version of the interior power decay result for $|A_t(u,\Omega)|$ from [2]:

Lemma 2.7. *Let $B_{6r\sqrt{n}}(x_0) \subset \Omega$ and $u \in C^0(\Omega)$ satisfy $\|u\|_{L^\infty(\Omega)} \leq 1$ and $u \in \overline{S}(f)$ in $B_{6r\sqrt{n}}(x_0)$. Then there are universal constants $\mu, \delta_0 > 0$ and C, such that $\|f\|_{L^n(B_{6r\sqrt{n}}(x_0))} \leq r^{-1}\delta_0$ implies*

$$|\underline{A}_t(u,\Omega) \cap Q_r(x_0)| \leq Ct^{-\mu}r^{-2\mu}|Q_r(x_0)|. \tag{8}$$

Moreover, if $u \in S^*(f)$ in $B_{6r\sqrt{n}}(x_0)$, then (8) holds for $A_t(u,\Omega)$.

Lemma 2.8. *Let $u \in \overline{S}(f)$ in Ω satisfy $u \in C^0(\Omega)$ and $\|u\|_{L^\infty(\Omega)} \leq 1$. Then for any $\Omega' \subset\subset \Omega$ we have*

$$|\underline{A}_t(u,\Omega) \cap \Omega'| \leq C(n,\lambda,\Lambda,\Omega,\operatorname{dist}(\Omega',\partial\Omega))\left(1 + \|f\|_{L^n(\Omega)}\right)^\mu t^{-\mu}.$$

Proof. Fix $\epsilon > 0$ such that $6\epsilon\sqrt{n} < \operatorname{dist}(\Omega',\partial\Omega)$ and choose a finite cover of Ω' with axially parallel cubes having side-length ϵ and disjoint interior. The choice of ϵ implies $B_{6\epsilon\sqrt{n}}(x_i) \subset\subset \Omega$ for all centers x_i of cubes $Q_\epsilon(x_i)$. Without loss of generality we may assume $\epsilon < 1$. By $N = N(\Omega',\epsilon)$ we denote the total number of cubes. Set $\tilde{u} := \frac{\delta_0 u}{\delta_0 + \|f\|_{L^n(\Omega)}}$ and $\tilde{f} := \frac{\delta_0 f}{\delta_0 + \|f\|_{L^n(\Omega)}}$, where δ_0 is as in Lemma 2.7. Then $\tilde{u}$ and $\tilde{f}$ satisfy the hyotheses of Lemma 2.7 in $B_{6\epsilon\sqrt{n}}(x_i)$ and we get $|\underline{A}_t(\tilde{u},\Omega) \cap Q_\epsilon(x_i)| \leq C\epsilon^n t^{-\mu}\epsilon^{-2\mu}$. Now

$$|\underline{A}_t(\tilde{u},\Omega) \cap \Omega'| \leq \sum_{i=1}^{N} |\underline{A}_t(\tilde{u},\Omega) \cap Q_\epsilon(x_i)| \leq Ct^{-\mu}\epsilon^{-2\mu},$$

where we have used $N\epsilon^n \leq C(n)|\Omega|$ for the last estimate. Finally, by definition $\underline{A}_t(Ku,\Omega) = \underline{A}_{\frac{t}{K}}(u,\Omega)$ for any constant K. □

Our first estimate for $|G_t(u,\Omega)|$ at the boundary is a direct consequence of the preceeding Lemma.

Lemma 2.9. *Assume that $u \in \overline{S}(f)$ in $B^+_{12\sqrt{n}} \subset \Omega$, $u \in C^0(\Omega)$ and that $\|u\|_{L^\infty(\Omega)} \leq 1$. Then there exist universal constants $M > 1$ and $0 < \sigma < 1$, such that $\| f \|_{L^n(B^+_{12\sqrt{n}})} \leq 1$ implies*

$$\left|\underline{G}_M(u,\Omega) \cap \left(\left(Q_1^{n-1} \times (0,1)\right) + x_0\right)\right| \geq 1 - \sigma$$

for any $x_0 \in B_{9\sqrt{n}} \cap \{x_n \geq 0\}$.

Proof. Fix $0 < \sigma < 1$ and set $x_0 = (x_0', x_{0,n})$. In case $x_{0,n} \geq \frac{\sigma}{2}$ the assertion follows from Lemma 2.8. Otherwise we apply Lemma 2.8 in the following way:

$$\begin{aligned}&\left|\underline{A}_t(u,\Omega) \cap \left(\left(Q_1^{n-1} \times (0,1)\right) + x_0\right)\right| \\ &\leq \left|\underline{A}_t(u,\Omega) \cap \left(\left(Q_1^{n-1} \times \left(\frac{\sigma}{2}, x_{0,n}+1\right)\right) + (x_0', 0)\right)\right| + \left|\left(Q_1^{n-1} \times \left(x_{0,n}, \frac{\sigma}{2}\right)\right) + (x_0', 0)\right| \\ &\leq C(n,\lambda,\Lambda,\sigma)t^{-\mu} + \frac{\sigma}{2}.\end{aligned}$$

This estimate holds for $t > 1$. Therefore the lemma is proven if we choose $t = t(n,\lambda,\Lambda,\sigma)$ large enough. □

Lemma 2.10. *Assume that $u \in S^*(f)$ in $B^+_{12\sqrt{n}} \subset \Omega \subset \mathbb{R}^n_+$, $u \in C^0(\Omega)$ and let $G_1(u,\Omega) \cap \left(\left(Q_2^{n-1} \times (0,2)\right) + \tilde{x}_0\right) \neq \emptyset$ for some $\tilde{x}_0 \in B_{9\sqrt{n}} \cap \{x_n \geq 0\}$. Then there exist universal constants $M > 1$ and $0 < \sigma < 1$, such that $\|f\|_{L^n(B^+_{12\sqrt{n}})} \leq 1$ implies*

$$\left|G_M(u,\Omega) \cap \left(\left(Q_1^{n-1} \times (0,1)\right) + x_0\right)\right| \geq 1 - \sigma$$

for any $x_0 \in B_{9\sqrt{n}} \cap \{x_n \geq 0\}$.

Proof. Let $x_1 \in G_1(u,\Omega) \cap \left(\left(Q_2^{n-1} \times (0,2)\right) + \tilde{x}_0\right)$. Then there are paraboloids with opening 1 touching u in x_1 from above and below, i.e., we have $L(x) - \frac{1}{2}|x - x_1|^2 \leq u(x) \leq L(x) + \frac{1}{2}|x - x_1|^2$ for $x \in \Omega$ and an affine function L. Set $v := \frac{u-L}{C(n)}$, where the constant $C(n)$ is chosen large enough, such that $\|v\|_{L^\infty(B^+_{12\sqrt{n}})} \leq 1$, and $-|x|^2 \leq v(x) \leq |x|^2$ in $\Omega \setminus B^+_{12\sqrt{n}}$. Applying Lemma 2.9 to $v \in S(\frac{f}{C(n)})$ in $B^+_{12\sqrt{n}}$ we obtain $|G_M(v, B^+_{12\sqrt{n}}) \cap \left(\left(Q_1^{n-1} \times (0,1)\right) + x_0\right)| \geq 1 - \sigma$. From the preceeding estimates we infer

$$\left|G_N(v,\Omega) \cap \left(\left(Q_1^{n-1} \times (0,1)\right) + x_0\right)\right| \geq 1 - \sigma$$

for $N \geq M$ large enough. Finally, we have $G_M(v,\Omega) = G_{MC(n)}(u,\Omega)$ which finishes the proof. □

We proceed with an iteration Lemma to improve the estimate given by Lemma 2.9.

Lemma 2.11. *Let $u \in S^*(f)$ in $B^+_{12\sqrt{n}} \subset \Omega \subset \mathbb{R}^n_+$, $u \in C^0(\Omega)$, $\|u\|_{L^\infty(\Omega)} \leq 1$ and $\|f\|_{L^n(B^+_{12\sqrt{n}})} \leq 1$. Extend f by zero outside $B^+_{12\sqrt{n}}$ and define*

$$\begin{aligned}A &:= A_{M^{k+1}}(u,\Omega) \cap \left(Q_1^{n-1} \times (0,1)\right) \\ B &:= \left(A_{M^k}(u,\Omega) \cap \left(Q_1^{n-1} \times (0,1)\right)\right) \cup \{x \in Q_1^{n-1} \times (0,1); M(f^n) \geq (C_0 M^k)^n\}\end{aligned}$$

for $k \in \mathbb{N}_0$. Then $|A| \leq \sigma|B|$, where $C_0 = C_0(n)$ and $0 < \sigma < 1$, $M > 1$ are universal constants.

Proof. The proof is based on the Calderon–Zygmund cube decomposition, given by Lemma 2.6. We have $A \subset B \subset \left(Q_1^{n-1} \times (0,1)\right)$ by $A_{M^{k+1}}(u,\Omega) \subset A_{M^k}(u,\Omega)$ and by Lemma 2.9 we get $|A| \leq \sigma < 1$. It remains to show that for any dyadic cube Q with $|A \cap Q| > \sigma|Q|$ we obtain $\tilde{Q} \subset B$. Assume that for some $i \geq 1$, $Q = \left(Q^{n-1}_{1/2^i} \times (0, \frac{1}{2^i})\right) + x_0$ is a dyadic cube with predecessor $\tilde{Q} = \left(Q^{n-1}_{1/2^{i-1}} \times (0, \frac{1}{2^{i-1}})\right) + \tilde{x}_0$. Assume further that Q satisfies

$$|A \cap Q| = |A_{M^{k+1}}(u,\Omega) \cap Q| > \sigma|Q| \tag{9}$$

but $\tilde{Q} \not\subset B$. Then there exists $x_1 \in \tilde{Q} \setminus B$, i.e.,

$$x_1 \in \tilde{Q} \cap G_{M^k}(u,\Omega), \quad M(f^n)(x_1) < (C_0 M^k)^n. \tag{10}$$

Consider the transformation $T(y) := \tilde{x}_0 + \frac{y}{2^i}$ and set $\tilde{u}(y) := \frac{2^{2i}}{M^k} u(T(y))$, $\tilde{f}(y) := \frac{1}{M^k} f(T(y))$ and $\tilde{\Omega} = T^{-1}(\Omega)$. Since $i \geq 1$ and $\tilde{Q} \subset \left(Q_1^{n-1} \times (0,1)\right)$, we obtain $B^+_{12\sqrt{n}/2^i}(\tilde{x}_0) \subset B^+_{12\sqrt{n}}$ which implies $\tilde{u} \in S^*(\tilde{f})$ in $B^+_{12\sqrt{n}}$. Note that $|\tilde{x}_0 - x_1|_\infty < \frac{1}{2^{i-1}}$, implies $B^+_{12\sqrt{n}/2^i}(\tilde{x}_0) \subset Q_{28\sqrt{n}/2^i}(x_1)$. Therefore, we obtain from (10)

$$\|\tilde{f}\|_{L^n(B^+_{12\sqrt{n}})} \leq \frac{2^i}{M^k} \left(\int_{Q_{28\sqrt{n}/2^i}(x_1)} f(x)^n dx \right)^{\frac{1}{n}} \leq C(n) C_0 \leq 1$$

provided C_0 is sufficiently small. Moreover, from (10) we infer $G_1(\tilde{u}, \tilde{\Omega}) \cap \left(Q_2^{n-1} \times (0,2)\right) \neq \emptyset$ and hence we have shown that the hypotheses of Lemma 2.10 are satisfied in $\tilde{\Omega}$. Since $x_{0,n} \geq \tilde{x}_{0,n}$ and $|x_0 - \tilde{x}_0| \leq \frac{1}{2^i}\sqrt{n}$ we have $2^i(x_0 - \tilde{x}_0) \in B_{9\sqrt{n}} \cap \{x_n \geq 0\}$. Applying Lemma 2.10 we obtain

$$\left| G_M(\tilde{u}, \tilde{\Omega}) \cap \left(\left(Q_1^{n-1} \times (0,1)\right) + 2^i(x_0 - \tilde{x}_0) \right) \right| \geq 1 - \sigma$$

and hence $|G_{M^{k+1}}(u,\Omega) \cap Q| \geq (1-\sigma)\,|Q|$ which is a contradiction to (9). □

From Lemma 2.11 we derive the power decay for $|A_t(u,\Omega)|$ at the boundary.

Proposition 2.12. *Let $u \in S(f)$ in $B^+_{12\sqrt{n}} \subset \Omega \subset \mathbb{R}^n_+$, $u \in C^0(\Omega)$ and let $\|u\|_{L^\infty(\Omega)} \leq 1$. Then there exist universal constants C, μ such that $\|f\|_{L^n(B^+_{12\sqrt{n}})} \leq 1$ implies*

$$|A_t(u,\Omega) \cap \left(\left(Q_1^{n-1} \times (0,1)\right) + x_0 \right)| \leq C t^{-\mu}$$

for any $x_0 \in B_{9\sqrt{n}} \cap \{x_n \geq 0\}$ and $t > 1$.

Proof. If $x_0 = 0$, we define

$$\begin{aligned}
\alpha_k &:= |A_{M^k}(u,\Omega) \cap \left(Q_1^{n-1} \times (0,1)\right)| \\
\beta_k &:= |\{x \in Q_1^{n-1} \times (0,1);\ M(f^n)(x) \geq (C_0 M^k)^n\}|,
\end{aligned}$$

and apply Lemma 2.11 to obtain $\alpha_{k+1} \le \sigma(\alpha_k + \beta_k)$. Hence $\alpha_k \le \sigma^k + \sum_{j=0}^{k-1} \sigma^{k-j}\beta_j$. In order to estimate the second part we use the properties of the maximal function M, see Proposition 2.5:

$$\beta_j \le C(n)(c_0 M^j)^{-n} \|f^n\|_{L^1(B^+_{12\sqrt{n}})} \le C M^{-nj}.$$

Hence $\alpha_k \le \sigma^k + C\sum_{j=0}^{k-1} \sigma^{k-j} M^{-nj} \le (1+Ck)\max(\sigma, M^{-n})^k \le CM^{-\mu k}$. Now, choose μ suffiently small to finish the first part of the proof.

If $x_0 \ne 0$ choose a finite covering of $\left(Q_1^{n-1} \times (0,1)\right) + x_0$ with suitable cubes. From the first part of the proof and Lemma 2.7 we deduce the claim. □

2.2. Proof of Theorem 2.2. We proceed in a way similar to [2] und prove an approximation lemma at the boundary first. Using this result we iterate the estimates of the previous subsection to obatin $W^{2,p}$-estimates. Recall that Assumption A implies $F(0,\cdot) = 0$.

Proposition 2.13 (Approximation lemma)**.** *Let $0 < \epsilon < 1$, and let u be a C^2-viscosity solution of*

$$\begin{cases} F(D^2u, x) = f & \text{in } B^+_{14\sqrt{n}} \\ u = 0 & \text{on } B_{14\sqrt{n}} \cap \{x_n = 0\} \end{cases}$$

such that $\|u\|_{L^\infty(B^+_{14\sqrt{n}})} \le 1$. Assume $\|\beta(.,0)\|_{L^n(B^+_{13\sqrt{n}})} \le \epsilon$ and that Assumption A is satisfied. Then there exists a function $h \in C^2(\overline{B^+}_{12\sqrt{n}})$ such that for $\varphi = f - F(D^2h, .) \in C^0(B^+_{12\sqrt{n}})$ we have $u - h \in S(\varphi)$ and

$$\|h\|_{C^{1,1}(\overline{B^+}_{12\sqrt{n}})} \le C(n, \lambda, \Lambda, c_e)$$
$$\|u-h\|_{L^\infty(B^+_{12\sqrt{n}})} + \|\varphi\|_{L^n(B^+_{12\sqrt{n}})} \le C\left(\epsilon^\gamma + \|f\|_{L^n(B^+_{14\sqrt{n}})}\right),$$

where $\gamma \in (0,1)$ is universal and $C = C(n, \lambda, \Lambda, c_e)$.

Proof. Let $h \in C^2(B^+_{13\sqrt{n}})$ be the solution of

$$\begin{cases} F(D^2h, 0) = 0 & \text{in } B^+_{13\sqrt{n}} \\ h = u & \text{on } \partial B^+_{13\sqrt{n}}. \end{cases}$$

Assumption A implies $\|h\|_{C^{1,1}(\overline{B^+}_{12\sqrt{n}})} \le C\|u\|_{L^\infty(B^+_{13\sqrt{n}})} \le C(n, c_e)$. Moreover, from the Hölder estimate, Theorem 1.10, we infer $u \in C^{0,\beta}(B^+_{13\sqrt{n}})$ where $0 < \beta < 1$ is universal. Using Theorem 1.10 we get

$$\|h\|_{C^{0,\alpha}(\overline{B^+}_{13\sqrt{n}})} \le C\Big(1 + \|u\|_{C^{0,\beta}(\overline{B}^+_{13\sqrt{n}})}\Big) \le C\Big(1 + \|f\|_{L^n(B^+_{14\sqrt{n}})}\Big),$$

where $0 < \alpha < \beta$ is a universal constant. Since $u - h = 0$ on $\partial B^+_{13\sqrt{n}}$ we have $(u-h)(x) \le \delta^\alpha \operatorname{höl}_{\alpha, B^+_{13\sqrt{n}}}(u-h)$ where $0 < \delta < 1$ and $x \in \partial B^+_{13\sqrt{n}-\delta}$. Hence

$$\|u - h\|_{L^\infty(\partial B^+_{13\sqrt{n}-\delta})} \le C\delta^\alpha \big(1 + \|f\|_{L^n(B^+_{14\sqrt{n}})}\big). \tag{11}$$

Let $x_0 \in B^+_{13\sqrt{n}-\delta}$. If $B_{\delta/2}(x_0) \subset B^+_{13\sqrt{n}}$ we apply rescaled interior $C^{1,1}$-estimates in $B_{\delta/2}(x_0)$ to $h - h(x_0)$ and get

$$\frac{\delta^2}{16}\|D^2 h(x_0)\| \le C\|h - h(x_0)\|_{L^\infty(\partial B_{\delta/2}(x_0))} \le C\delta^\alpha\Big(1 + \|f\|_{L^n(B^+_{14\sqrt{n}})}\Big).$$

If not, there exists $z_0 \in B_{13\sqrt{n}-\delta} \cap \{x_n = 0\}$ such that $x_0 \in B^+_{\delta/2}(z_0)$. We apply rescaled $C^{1,1}$-estimates in $B^+_\delta(z_0)$ to $h - h(x_0)$ and obtain $\frac{\delta^2}{16}\|D^2h(x_0)\| \le C\delta^\alpha(1 + \|f\|_{L^n(B^+_{14\sqrt{n}})})$. From the definition of β we infer

$$|F(D^2h(x_0), x_0)| \le C\delta^{\alpha-2}\beta(x_0, 0)\Big(1 + \|f\|_{L^n(B^+_{14\sqrt{n}})}\Big), \tag{12}$$

where $x_0 \in B^+_{13\sqrt{n}-\delta}$. Proposition 1.4 yields $u - h \in S\big(\frac{\lambda}{n}, \Lambda, f(x) - F(D^2h(x), x)\big)$ in $B^+_{13\sqrt{n}}$. By the maximum principle, (11), (12) we obtain

$$\begin{aligned}
&\|u - h\|_{L^\infty(B^+_{13\sqrt{n}-\delta})} + \|f(x) - F(D^2h(x), x)\|_{L^n(B^+_{13\sqrt{n}-\delta})} \\
&\le C\Big(\delta^\alpha + \delta^{\alpha-2}\|\beta\|_{L^n(B^+_{13\sqrt{n}})}\Big)\Big(1 + \|f\|_{L^n(B^+_{14\sqrt{n}})}\Big) + C\|f\|_{L^n(B^+_{14\sqrt{n}})}.
\end{aligned}$$

Choose $\delta = \epsilon^{\frac{1}{2}}$ to finish the proof. □

In the next step we use the approximation lemma to improve Lemma 2.10, more precisely we prove $|G_M(u, \Omega) \cap Q| \ge 1 - \epsilon_0$ for arbitrary $\epsilon_0 > 0$.

Lemma 2.14. *Let $\epsilon_0 \in (0,1)$, $B^+_{14\sqrt{n}} \subset \Omega \subset \mathbb{R}^n_+$ and $u \in C^0(\Omega)$ be a C^2-viscosity solution of*

$$\begin{cases} F(D^2u, x) = f & \text{in } B^+_{14\sqrt{n}} \\ \qquad\quad u = 0 & \text{on } B_{14\sqrt{n}} \cap \{x_n = 0\}. \end{cases}$$

Assume that Assumption A holds and that $F(0, \cdot) \equiv 0$,

$$\|f\|_{L^n(B^+_{14\sqrt{n}})},\ \|\beta(\cdot, 0)\|_{L^n(B^+_{13\sqrt{n}})} \le \epsilon,$$

where $\epsilon = \epsilon(n, \lambda, \Lambda, \epsilon_0, c_e)$. Then $G_1(u, \Omega) \cap \big((Q_2^{n-1} \times (0,2)) + \tilde{x}_0\big) \neq \emptyset$ for some $\tilde{x}_0 \in B_{9\sqrt{n}} \cap \{x_n \ge 0\}$ implies

$$\Big|G_M(u, \Omega) \cap \Big(\big(Q^{n-1} \times (0,1)\big) + x_0\Big)\Big| \ge 1 - \epsilon_0,$$

where $x_0 \in B_{9\sqrt{n}} \cap \{x_n \ge 0\}$ and $M = M(n, c_e)$.

Proof. Similar to the proof of Lemma 2.10 we set $\tilde{u} := \frac{u-L}{C}$ where L is affine and $C = C(n)$ is chosen sufficiently large such that $\tilde{u}$ has the same properties as v in Lemma 2.10. We know that $\tilde{u}$ is a solution of

$$\tilde{F}(D^2\tilde{u}, x) := \frac{1}{C}F(CD^2\tilde{u}, x) = \frac{1}{C}f(x) =: \tilde{f}(x)$$

and that the ellipticity constants of F and $\tilde{F}$ agree. As in Proposition 2.13, let $h \in C^2(B^+_{13\sqrt{n}}) \cap C^0(\overline{B^+}_{13\sqrt{n}})$ be the solution of

$$\begin{cases} \tilde{F}(D^2h, 0) = 0 & \text{in } B^+_{13\sqrt{n}} \\ h = \tilde{u} & \text{on } \partial B^+_{13\sqrt{n}}. \end{cases}$$

From the maximum principle we infer $\|h\|_{L^\infty(B^+_{13\sqrt{n}})} \leq \|\tilde{u}\|_{L^\infty(\partial B^+_{13\sqrt{n}})} \leq 1$. Also $\|h\|_{C^{1,1}(\overline{B^+}_{12\sqrt{n}})} \leq C(n, c_e)$ implies $A_N(h, B^+_{12\sqrt{n}}) \cap ((Q_1^{n-1} \times (0,1)) + x_0) = \emptyset$ for some $N = N(n, c_e) > 1$. We extend $h|_{B^+_{12\sqrt{n}}}$ continuously outside $B^+_{12\sqrt{n}}$ such that $h = \tilde{u}$ outside $B^+_{13\sqrt{n}}$ and $\|\tilde{u} - h\|_{L^\infty(\Omega)} = \|\tilde{u} - h\|_{L^\infty(B^+_{12\sqrt{n}})}$. Hence $\|\tilde{u} - h\|_{L^\infty(\Omega)} \leq 2$ and $-2 - |x|^2 \leq h(x) \leq 2 + |x|^2$ in $\Omega \setminus B^+_{12\sqrt{n}}$. These estimates imply

$$A_{M_0}(h, \Omega) \cap \big((Q_1^{n-1} \times (0,1)) + x_0\big) = \emptyset \tag{13}$$

for some $M_0 = M_0(n, c_e) \geq N$. For $w := \tilde{u} - h$ Proposition 2.13 yields

$$\|w\|_{L^\infty(B^+_{12\sqrt{n}})} + \|\tilde{f} - \tilde{F}(D^2h, \cdot)\|_{L^n(B_{12\sqrt{n}})} \leq C\Big(\epsilon^\gamma + \|\tilde{f}\|_{L^n(B^+_{14\sqrt{n}})}\Big) \leq C\epsilon^\gamma$$

and $w \in S\big(\frac{\lambda}{n}, \Lambda, \tilde{f} - \tilde{F}(D^2h, \cdot)\big)$ in $B^+_{12\sqrt{n}}$. Hence $\|w\|_{L^\infty(\Omega)} = \|w\|_{L^\infty(B^+_{12\sqrt{n}})} \leq C\epsilon^\gamma$. Therefore $\tilde{w} = (C\epsilon^\gamma)^{-1}w$ satisfies the hypotheses of Proposition 2.12 and we obtain for $t > 1$:

$$|A_t(\tilde{w}, \Omega) \cap \big((Q_1^{n-1} \times (0,1)) + x_0\big)| \leq C(n, \lambda, \Lambda, c_e)t^{-\mu}.$$

Using $A_{2M_0}(\tilde{u}) \subset A_{M_0}(w) \cup A_{M_0}(h)$ and (13) we conclude

$$|A_{2M_0}(\tilde{u}, \Omega) \cap \big((Q_1^{n-1} \times (0,1)) + x_0\big)| \leq C(n, \lambda, \Lambda, c_e)\epsilon^{\gamma\mu}M_0^\mu.$$

Finally, since $A_{2M_0}(\tilde{u}, \Omega) = A_{2CM_0}(u, \Omega)$ we set $M = 2CM_0$ and choose $\epsilon = \epsilon(\lambda, \Lambda, n, c_e, \epsilon_0)$ sufficiently small to finish the proof. □

Lemma 2.15. *Let $0 < \epsilon_0 < 1$ and u be a C^2-viscosity solution of*

$$\begin{cases} F(D^2u, x) = f & \textit{in } B^+_{14\sqrt{n}} \\ u = 0 & \textit{on } B_{14\sqrt{n}} \cap \{x_n = 0\}. \end{cases}$$

Assume that Assumption A holds and that $\|u\|_{L^\infty(B^+_{14\sqrt{n}})} \leq 1, \|f\|_{L^n(B^+_{14\sqrt{n}})} \leq \epsilon$.

Extend f by zero outside $B^+_{14\sqrt{n}}$. Let

$$\left(\frac{1}{|B_r(x_0)\cap B^+_{14\sqrt{n}}|}\int_{B_r(x_0)\cap B^+_{14\sqrt{n}}}\beta(x_0,x)^n dx\right)^{\frac{1}{n}}\leq \epsilon.$$

for $x_0 \in B^+_{14\sqrt{n}}$ and $r>0$. For $k\in\mathbb{N}_0$ we set

$$A := A_{M^{k+1}}(u, B^+_{14\sqrt{n}})\cap Q_1^{n-1}\times(0,1)$$
$$B := \big(A_{M^k}(u, B^+_{14\sqrt{n}})\cap Q_1^{n-1}\times(0,1)\big)\cup\{x\in Q_1^{n-1}\times(0,1); M(f^n)\geq (C_0M^k)^n\}.$$

Then $|A|\leq \epsilon_0|B|$, where $M=M(n,c_e)>1$, $\epsilon=\epsilon(n,\lambda,\Lambda,c_e,\epsilon_0)$.

Proof. Like the proof of Lemma 2.11, this proof is also based upon the Calderon–Zygmund decomposition. We have $A\subset B\subset(Q_1^{n-1}\times(0,1))$ and from Lemma 2.14 we infer $|A|\leq\delta<1$ for $\delta=\epsilon_0$. Therefore it remains to show that for dyadic cubes Q with $|A\cap Q|>\epsilon_0|Q|$ we have $\tilde{Q}\subset B$. Let Q, $\tilde{Q}$ be the same as in Lemma 2.11. We assume that Q satisfies

$$|A\cap Q| = |A_{M^{k+1}}(u, B^+_{14\sqrt{n}})\cap Q| > \epsilon_0|Q| \tag{14}$$

but $\tilde{Q}\not\subset B$. Therefore there exists $x_1\in\tilde{Q}\setminus B$, i.e.,

$$x_1\in\tilde{Q}\cap G_{M^k}(u, B^+_{14\sqrt{n}}),\quad M(f^n)(x_1)<(C_0M^k)^n. \tag{15}$$

In case $|x_0-(x_0',0)|<\frac{8}{2^i}\sqrt{n}$ we consider $T(y):=(x_0',0)+2^{-i}y$ and define $\tilde{u}(y):=\frac{2^{2i}}{M^k}u(T(y))$, $\tilde{F}(X,y):=\frac{1}{M^k}F(M^kX,T(y))$ and $\tilde{f}(y):=\frac{1}{M^k}f(T(y))$. Now $Q\subset(Q_1^{n-1}\times(0,1))$ implies $B^+_{14\sqrt{n}/2^i}(x_0',0)\subset B^+_{14\sqrt{n}}$ and we have that $\tilde{u}$ is a C^2-viscosity solution of

$$\begin{cases}\tilde{F}(D^2\tilde{u},y)=\tilde{f} & \text{in } B^+_{14\sqrt{n}}\\ \tilde{u}=0 & \text{on } B_{14\sqrt{n}}\cap\{x_n=0\}.\end{cases}$$

The ellipticity constants of $\tilde{F}$ and F agree and $\tilde{F}$ satisfies the $C^{1,1}$-estimates with the same constant as F. Moreover, $\beta_{\tilde{F}}(y,0)=\beta_F(x,(x_0',0))$ and hence $\|\beta_{\tilde{F}}\|_{L^n(B^+_{13\sqrt{n}})}\leq C(n)\epsilon$. Similar to the proof of Lemma 2.11 we get $\|\tilde{f}\|_{L^n(B^+_{14\sqrt{n}})}\leq\epsilon$ from (15) for C_0 sufficiently small. Again by (15) we obtain

$$G_1\big(\tilde{u},T^{-1}\big(B^+_{14\sqrt{n}}\big)\big)\cap\big(\big(Q_2^{n-1}\times(0,2)\big)+2^i(\tilde{x}_0-(x_0',0))\big)\neq\emptyset.$$

From $|x_0-\tilde{x}_0|\leq\frac{1}{2^i}\sqrt{n}$ we get $|2^i(\tilde{x}_0-(x_0',0))|<9\sqrt{n}$ such that the hypotheses of Lemma 2.14 are satisfied. It follows

$$\big|G_M(\tilde{u},T^{-1}(B^+_{14\sqrt{n}}))\cap\big(\big(Q_1^{n-1}\times(0,1)\big)+2^i(x_0-(x_0',0)\big)\big|\geq 1-\epsilon_0$$

and hence $|G_{M^{k+1}}(u, B^+_{14\sqrt{n}}) \cap Q| \geq (1-\epsilon_0)|Q|$ which is a contradiction to (14).

If $|x_0 - (x_0', 0)| \geq \frac{8}{2^i}\sqrt{n}$, we conclude $B_{8\sqrt{n}/2^i}(x_0 + \frac{1}{2^{i+1}} e_n) \subset B^+_{8\sqrt{n}}$, where e_n is the n-th unit vector. Using the transformation $T(y) := (x_0 + \frac{1}{2^{i+1}}\, e_n) + \frac{1}{2^i} y$ we proceed in a way similar to the first part of the proof. Now we apply [2, Lemma 7.11] instead of Lemma 2.14 in order to obtain a contradiction to (14). □

Proof of Theorem 2.2. Fix $x_0 \in B_{1/2} \cap \{x_n = 0\}$, $0 < r < \frac{1-|x_0|}{14\sqrt{n}}$ and define

$$K := \frac{\epsilon r}{\epsilon r^{-1} \|u\|_{L^\infty(B^+_{14r\sqrt{n}}(x_0))} + \|f\|_{L^n(B^+_{14r\sqrt{n}}(x_0))}},$$

where $\epsilon = \epsilon(n, \lambda, \Lambda, p, c_e, \epsilon_0)$ is the same as in Lemma 2.14 and $0 < \epsilon_0 < 1$ will be chosen later in the proof. We set $\tilde{u}(y) := K r^{-2} u(ry + x_0)$, $\tilde{f} := K f(ry + x_0)$ and $\tilde{F}(M, y) := K F(K^{-1} M, ry + x_0)$. Then $\tilde{u}$ is a C^2-viscosity solution of

$$\begin{cases} \tilde{F}(D^2 \tilde{u}, x) = \tilde{f} & \text{in } B^+_{14\sqrt{n}} \\ \tilde{u} = 0 & \text{on } B_{14\sqrt{n}} \cap \{x_n = 0\}\,. \end{cases}$$

The ellipticity constants of $\tilde{F}$ and F agree. We have $\beta_{\tilde{F}}(y, 0) = \beta_F(ry + x_0, x_0)$ and $\|\beta_{\tilde{F}}\|_{L^n(B^+_{14\sqrt{n}})} \leq C(n)\beta_0 \leq \epsilon$, provided β_0 is chosen sufficiently small. Moreover $\|\tilde{u}\|_{L^\infty(B^+_{14\sqrt{n}})} \leq 1$ and

$$\|\tilde{f}\|_{L^n(B^+_{14\sqrt{n}})} = \frac{K}{r} \|f\|_{L^n(B^+_{14r\sqrt{n}}(x_0))} \leq \epsilon, \tag{16}$$

such that the hypotheses of Lemma 2.15 are satisfied. Let $M = M(n, c_e)$ and $C_0 = C_0(n, \lambda, \Lambda, p, c_e, \epsilon_0)$ be the same as in Lemma 2.15 and choose $\epsilon_0 = \frac{1}{2M^p}$. We define

$$\begin{aligned} \alpha_k &:= \left| A_{M^k}\big(\tilde{u}, B^+_{14\sqrt{n}}\big) \cap \big(Q_1^{n-1} \times (0,1)\big) \right| \\ \beta_k &:= \left| \{ x \in \big(Q_1^{n-1} \times (0,1)\big)\, ; M(\tilde{f}^n)(x) \geq (C_0 M^k)^n \} \right|. \end{aligned}$$

and apply Lemma 2.15 to obtain $\alpha_{k+1} \leq \epsilon_0 (\alpha_k + \beta_k)$ and hence

$$\alpha_k \leq \epsilon_0^k + \sum_{i=0}^{k-1} \epsilon_0^{k-i} \beta_i. \tag{17}$$

From Propositon 2.5 and (16) we infer

$$\|M(\tilde{f}^n)\|_{L^{\frac{p}{n}}} \leq C(n,p) \|\tilde{f}^n\|_{L^{\frac{p}{n}}} = C(n,p) \|\tilde{f}\|^n_{L^p} \leq C(n,p).$$

Since β_k is the distribution function of $M(\tilde{f}^n)$ we infer from Proposition 2.4

$$\sum_{k \in \mathbb{N}} M^{pk} \beta_k \leq C(n,p). \tag{18}$$

By the choice of ϵ_0, (17) and (18)

$$\sum_{k\in\mathbb{N}} M^{pk}\alpha_k \leq \sum_{k\in\mathbb{N}} 2^{-k} + \Big(\sum_{k\geq 0} M^{pk}\beta_k\Big)\Big(\sum_{k\in\mathbb{N}} M^{pk}\epsilon_0^k\Big) \leq C(n,p).$$

Consequently $\|D^2\tilde{u}\|_{L^p(B^+_{1/2})} \leq C(n,p,M)$ and hence

$$\|D^2u\|_{L^p(B^+_{r/2}(x_0))} \leq C(n,\lambda,\Lambda,p,c_e,r)\big(\|u\|_{L^\infty(B_1^+)} + \|f\|_{L^p(B_1^+)}\big). \tag{19}$$

Finally choose a suitable covering of $B^+_{1/2}$ with $B^+_r(x_0)$ for $x_0 \in B_{1/2} \cap \{x_n = 0\}$ and $B_r(x_0)$ for $x_0 \in B^+_{1/2}$ respectively where r is chosen to be suffiently small. The desired assertion is a consequence of (19) and [2, Theorem 7.1]. □

Caffarelli's interior $W^{2,p}$-estimates were generalised by L. Escauriaza to the range of $n-\epsilon_0 < p < \infty$ where $\epsilon_0 = \epsilon_0(\frac{\Lambda}{\lambda}, n)$. The boundary estimate, Theorem 2.2, can be generalised similarly. Using results from [5] we obtain the weak Harnack inequality (at the boundary) and global Hölder continuity for $W^{2,n-\epsilon_0}$-viscosity solutions. In the related estimates $\|f\|_{L^n}$ is replaced by $\|f\|_{L^{n-\epsilon_0}}$. Therefore, by repeating the arguments of Subsections 2.1 and 2.2 we obtain

Theorem 2.16. *Let u be a bounded C^2-viscosity solution of*

$$\begin{cases} F(D^2u,x) = f & \text{in } B_1^+ \\ \qquad\quad u = 0 & \text{on } B_1 \cap \{x_n = 0\}. \end{cases}$$

Assume that F is uniformly elliptic with ellipticity constants λ, Λ, continuous in x, $F(0,.) \equiv 0$ and that Assumption A holds. Then there exist constants $\epsilon_0 = \epsilon_0(\frac{\Lambda}{\lambda}, n)$, $C = C(n,\lambda,\Lambda,c_e,p)$ and $\beta_0 = \beta_0(n,\lambda,\Lambda,c_e,p)$, where $n-\epsilon_0 < p < \infty$ such that the following holds: If

$$\left(\frac{1}{|B_r(x_0)\cap B_1^+|}\int_{B_r(x_0)\cap B_1^+} \beta(x_0,x)^n dx\right)^{\frac{1}{n}} \leq \beta_0,$$

for $x_0 \in B_1^+$, $r > 0$ and $f \in L^p(B_1^+)$, then $u \in W^{2,p}(B^+_{1/2})$ and

$$\|u\|_{W^{2,p}(B^+_{1/2})} \leq C\big(\|u\|_{L^\infty(B_1^+)} + \|f\|_{L^p(B_1^+)}\big).$$

3. $W^{1,p}$-estimates at the boundary

The objective of this section is the proof of $W^{1,p}$-estimates at the boundary for equations with dependence on Du, u. Interior estimates of this type were proven by A. Świech in [11], see also [1] and [2]. Our proof is similar to the proof in [11]. Before we state the main theorem of this section we need some preparations.

Henceforth we assume that $F(M,p,r,\cdot)$ is measurable in x. Similar to Section 2 we define the function β to measure the oscillation of F in x:

$$\beta(x,x_0) := \sup_{M\in S(n)\setminus\{0\}} \frac{|F(M,0,0,x)-F(M,0,0,x_0)|}{\|M\|}.$$

Instead of Assumption A we will make use of

Assumption A*: We assume that F satisfies $C^{1,\bar{\alpha}}$ interior and boundary estimates, i.e., there exist constants $0<\bar{\alpha}<1$ and c_e such that for any $w_0\in C^0(\partial B_1)$ there exists a C^2-viscosity solution $w\in C^{1,\alpha}_{loc}(B_1)\cap C^0(\overline{B_1})$ of

$$\begin{cases} F(D^2w,0,0,0)=0 & \text{in } B_1\\ \qquad\qquad w=w_0 & \text{on } \partial B_1\end{cases}$$

such that

$$\|w\|_{C^{1,\bar{\alpha}}(\bar{B}_{1/2})}\le c_e\|w\|_{L^\infty(B_1)}.$$

Additionally, we assume that for any $w_0\in C^0(\partial B_1^+)\cap C^{1,\gamma}(B_1\cap\{x_n=0\})$ there exist a constant $0<\bar{\alpha}<1$, depending on γ, and a C^2-viscosity solution $w\in C^{1,\alpha}_{loc}(B_1^+\cup\{x_n=0\})\cap C^0(\overline{B_1^+})$ of

$$\begin{cases} F(D^2w,0,0,0)=0 & \text{in} B_1^+\\ \qquad\qquad w=w_0 & \text{on} B_1\cap\{x_n=0\}\end{cases}$$

such that

$$\|w\|_{C^{1,\bar{\alpha}}(\bar{B}^+_{1/2})}\le c_e\big(\|w\|_{L^\infty(B_1^+)}+\|w_0\|_{C^{1,\gamma}(B_1\cap\{x_n=0\})}\big).$$

The main result of this section is

Theorem 3.1. *Let $p>n-\epsilon_0(\frac{\Lambda}{\lambda},n,b)$ and u be a $W^{2,p}$-viscosity solution of*

$$\begin{cases} F(D^2u,Du,u,x)=f & \text{in } B_1^+\\ \qquad\qquad u=\varphi & \text{on } \partial B_1\cap\{x_n=0\},\end{cases}$$

where $f\in L^p(B_1^+)$, $\varphi\in C^{1,\gamma}(B_1\cap\{x_n=0\})$. Assume that F satisfies Assumption A^*, $F(0,0,0,x)=0$, and structure condition (5).

If $p>n$ let $\alpha<min(1-\frac{n}{p},\bar{\alpha}(1-\gamma))$. There exists $\beta_0=\beta_0(n,\lambda,\Lambda,p,\alpha,\bar{\alpha})$ such that the existence of $r_0>0$ with

$$\left(\frac{1}{|B_r(x_0)\cap B_1^+|}\int_{B_r(x_0)\cap B_1^+}\beta(x,x_0)^p dx\right)^{\frac{1}{p}}\le\beta_0$$

for all $x_0 \in B_1^+$ *and* $r \le r_0$ *implies:* $u \in C^{1,\alpha}(B_{1/2} \cap \{x_n \ge 0\})$ *and*

$$\|u\|_{C^{1,\alpha}(B_{1/2}\cap\{x_n\ge 0\})} \le C\big(\|u\|_{L^\infty(B_1^+)} + \|\varphi\|_{C^{1,\gamma}(B_1\cap\{x_n=0\})} + \|f\|_{L^p(B_1^+)}\big),$$

where $C = C(n, \lambda, \Lambda, b, c, p, q, r_0)$.

If $p \le n$ *there exists* $\beta_0 = \beta_0(n, \lambda, \Lambda, p)$ *such that the existence of* $r_0 > 0$ *with*

$$\left(\frac{1}{|B_r(x_0) \cap B_1^+|}\int_{B_r(x_0)\cap B_1^+} \beta(x, x_0)^p dx\right)^{\frac{1}{p}} \le \beta_0$$

for all $x_0 \in B_1^+$ *and* $r \le r_0$ *implies:* $u \in W^{1,q}\big(B_{1/2}^+\big)$ *for every* $q < p^* := \frac{np}{n-p}$ *and*

$$\|u\|_{W^{1,q}(B_{1/2}^+)} \le C\big(\|u\|_{L^\infty(B_1^+)} + \|\varphi\|_{C^{1,\gamma}(B_1\cap\{x_n=0\})} + \|f\|_{L^p(B_1^+)}\big),$$

where $C = C(n, \lambda, \Lambda, b, c, p, q, r_0)$.

For later application we remark that in case $\varphi = 0$ Theorem 3.1 requires the second part of Assumption A^* to hold for $w_0 = 0$ only.

3.1. Proof of Theorem 3.1. In the proof of Theorem 3.1, we will make use of the sets $B_r^\nu(x_0) := B_r(x_0) \cap \{x_n > -\nu\}$ for $\nu > 0$. We commence with an approximation result.

Proposition 3.2. *Let* $p > n - \epsilon_0(\frac{\Lambda}{\lambda}, n, b)$. *Assume that* F *satisfies* (5) *and* $F(0, 0, 0, x) = 0$ *in* $B_1^\nu(0)$ *for some* $0 \le \nu \le 1$. *Let* $\varphi \in C^{0,\gamma}(\partial B_1^\nu)$ *satisfy* $\|\varphi\|_{C^{0,\gamma}(\partial B_1^\nu)} \le C_0$. *Then for all* $\varrho > 0$ *there exists* $\delta = \delta(\varrho, n, \lambda, \Lambda, p, \gamma, C_0) < 1$ *such that*

$$\|\beta(0,\cdot)\|_{L^p(B_1^\nu)}, \|f\|_{L^p(B_1^\nu)}, b, c \le \delta$$

implies the following: Any two $W^{2,p}$*-viscosity solutions* u *and* v *of*

$$\begin{cases} F(D^2u, Du, u, x) = f & \text{in } B_1^\nu \\ u = \varphi & \text{on } \partial B_1^\nu \end{cases} \quad \text{and} \quad \begin{cases} F(D^2v, 0, 0, 0) = 0 & \text{in } B_1^\nu \\ v = \varphi & \text{on } \partial B_1^\nu \end{cases}$$

satisfy $\|u - v\|_{L^\infty(B_1^\nu)} \le \varrho$.

Proof. We argue by contradiction and assume that the claim is not satisfied. Then there exist $\varrho_0 > 0$, a sequence $0 \le \nu_k \le 1$, and sequences of functions F_k satisfying (5) with b replaced by b_{F_k} and c replaced by c_{F_k}, $\varphi_k \in C^{0,\gamma}(\partial B_1^{\nu_k})$ with $\|\varphi_k\|_{C^{0,\gamma}(\partial B_1^{\nu_k})} \le C_0$ and f_k, for which there exist viscosity solutions u_k, v_k of

$$\begin{cases} F_k(D^2u_k, Du_k, u_k, x) = f_k & \text{in } B_1^{\nu_k} \\ u_k = \varphi_k & \text{on } \partial B_1^{\nu_k} \end{cases}$$

and

$$\begin{cases} F_k(D^2 v_k, 0, 0, 0) = 0 & \text{in } B_1^{\nu_k} \\ \qquad\qquad v_k = \varphi_k & \text{on } \partial B_1^{\nu_k} \end{cases}$$

such that $\|\beta_{F_k}(0,\cdot)\|_{L^p(B_1^{\nu_k})}, \|f_k\|_{L^p(B_1^{\nu_k})}, b_{F_k}, c_{F_k} \leq \delta_k \to 0$ as $k \to \infty$, and

$$\|u_k - v_k\|_{L^\infty(B_1^{\nu_k})} > \varrho_0. \tag{20}$$

Since the functions F_k are Lipschitz continuous in M, p, r we infer from (5) and Arzela–Ascoli's theorem that there exists a function F_∞ and a subsequence such that $F_k(\cdot,\cdot,\cdot,0) \to F_\infty(\cdot)$ uniformly on compact subsets of $S(n) \times \mathbb{R}^n \times \mathbb{R}$. The maximum principle yields $\|u_k\|_{L^\infty(B_1^{\nu_k})} \leq C_0 + C(n,\lambda,\Lambda)(\delta_k + c_{F_k}\|u_k\|_{L^\infty(B_1^{\nu_k})})$ and consequently $\|u_k\|_{L^\infty(B_1^{\nu_k})}, \|v_k\|_{L^\infty(B_1^{\nu_k})} \leq C(C_0)$ if k is sufficiently large. Moreover, from Theorem 1.10 we get

$$\|u_k\|_{C^{0,\alpha}(\overline{B_1^{\nu_k}})}, \|v_k\|_{C^{0,\alpha}(\overline{B_1^{\nu_k}})} \leq C(n,\lambda,\Lambda,b,C_0). \tag{21}$$

We may assume that there exist $0 \leq \nu_\infty \leq 1$ and a subsequence such that $\nu_k \to \nu_\infty$ as $k \to \infty$. Choosing another subsequence, if necessary, we may also assume that ν_k is monotonous. Thus we have either $B_1^{\nu_\infty} \subset B_1^{\nu_k}$ or $B_1^{\nu_k} \subset B_1^{\nu_{k+1}}$. In the first case we apply Arzela–Ascoli's theorem in $B_1^{\nu_\infty}$ directly. In the second case, there is an elementary extension of φ_k to $B_1 \cap \{-\nu_\infty \leq x_n \leq -\nu_k\}$ such that $\| \varphi_k \|_{C^{0,\gamma}(B_1 \cap \{\nu_\infty \leq x_k \leq -\nu_k\})} \leq C_0$ and hence we may suppose that (21) holds in $\overline{B_1^{\nu_\infty}}$ for the extended u_k, v_k.

Therefore, in both cases, we apply Arzela–Ascoli's theorem in $B_1^{\nu_\infty}$ and obtain the existence of functions $u_\infty, v_\infty \in C^0(\overline{B_1^{\nu_\infty}})$, $\varphi_\infty \in C^0(\partial B_1^{\nu_\infty})$ and subsequences such that $u_k \to u_\infty$, $v_k \to v_\infty$ uniformly on $B_1^{\nu_\infty}$ and $u_\infty = v_\infty = \varphi_\infty$ on $\partial B_1^{\nu_\infty}$. Clearly, v_∞ is a C^2-viscosity solution of

$$\begin{cases} F_\infty(D^2 v_\infty, 0, 0, 0) = 0 & \text{in } B_1^{\nu_\infty} \\ \qquad\qquad v_\infty = \varphi_\infty & \text{on } \partial B_1^{\nu_\infty}\,. \end{cases} \tag{22}$$

Finally, we use Proposition 1.5 to prove that u_∞ is also a viscosity solution of (22). In order to check the hypothesis of that lemma we take $\phi \in C^2(\Omega)$ and apply (5) to obtain

$$\begin{aligned} &|F_k(D^2\phi, D\phi, u_k, x) - f_k(x) - F_\infty(D^2\phi, 0, 0, 0)| \\ &\leq c_{F_k} C(C_0) + b_{F_k}|D\phi| + \beta_k(0,x)|D^2\phi| + |f_k|. \end{aligned}$$

The L^p-Norm of this term goes to 0 as $k \to \infty$. Hence, Proposition 1.5 is applicable. Since (22) is uniquely solvable we get $u_\infty = v_\infty$ which contradicts (20). □

Remark 3.3. Consider a C^2-viscosity solution w of

$$\begin{cases} F(D^2w, 0, 0, 0) = 0 & \text{in } B_1^\nu \\ \qquad\qquad w = w_0 & \text{on } \partial B_1^\nu, \end{cases}$$

where $w_0 \in C^0(\partial B_1^\nu) \cap C^{1,\gamma}(B_1 \cap \{x_n = -\nu\})$. By rescaling Assumption A* and using a covering argument if necessary we obtain

$$\|w\|_{C^{1,\bar{\alpha}}(\overline{B_{1/2}^\nu})} \leq K_2 \left(\|w\|_{L^\infty(B_1^\nu)} + \|w_0\|_{C^{1,\gamma}(B_1 \cap \{x_n = -\nu\})}\right)$$

for a constant $K_2 = K_2(n, c_e)$.

Proof of Theorem 3.1. Let $p > p' > n - \epsilon_0$. Fix $y = (y', y_n) \in B_{1/2} \cap \{\, x_n \geq 0 \,\}$ and set $d := \min(\frac{1}{2}, r_0)$. Initially we rescale the equation such that the assumptions of Proposition 3.2 are satisfied. Therefore we choose a constant σ such that

$$\sigma \leq \frac{d}{2}, \quad \sigma b \leq \frac{\delta}{32M}, \quad \sigma^2 c \leq \frac{\delta}{32(M+1)}, \tag{23}$$

where δ is the constant from Proposition 3.2 and M will be chosen later. If $y_n < \frac{\sigma}{2}$ we define

$$\begin{aligned} K = K(y) &:= \|u\|_{L^\infty(B_d(y) \cap \{x_n \geq 0\})} + \|\varphi\|_{C^{1,\gamma}(B_d(y) \cap \{x_n = 0\})} \\ &\quad + \frac{1}{\beta_0} \sup_{r \leq d} \left(r^{1-\alpha} \left(r^{-n} \int_{B_r(y) \cap \{x_n \geq 0\}} |f(x)|^{p'} dx \right)^{\frac{1}{p'}} \right). \end{aligned}$$

Since $K(y) \leq \|u\|_{L^\infty(B_1^+)} + \|\varphi\|_{C^{1,\gamma}(B_1 \cap \{x_n = 0\})} + C(n, \beta_0) \left(M(f^p)(y)\right)^{\frac{1}{p}}$ we observe that $K(y)$ is finite almost everywhere. We proceed under the assumption that $K(y) < \infty$ and consider $\tilde{u}(x) := \frac{1}{K} u(\sigma x + y)$. Set $\tilde{f}(x) := \frac{\sigma^2}{K} f(\sigma x + y)$, $\tilde{F}(M, p, r, x) := \frac{\sigma^2}{K} F\left(\frac{K}{\sigma^2} M, \frac{K}{\sigma} p, Kr, \sigma x + y\right)$, and $\tilde{\varphi}(x) := \frac{1}{K} \varphi(\sigma x + y)$. It is easy to check, using (23), that $\tilde{u}$ is a $W^{2,p}$-viscosity solution of

$$\begin{cases} \tilde{F}(D^2\tilde{u}, D\tilde{u}, \tilde{u}, x) = \tilde{f} & \text{in } B_2^\nu \\ \qquad\qquad \tilde{u} = \tilde{\varphi} & \text{on } B_2 \cap \{x_n = -\nu\}, \end{cases}$$

where $\nu := \frac{y_n}{\sigma}$. We have that $\tilde{F}$ satisfies (5) with b replaced by $b_{\tilde{F}} := \sigma b$ and c replaced by $c_{\tilde{F}} := \sigma^2 c$. Moreover, we obtain for all $0 < r < 2$

$$r^{1-\alpha} \left(\frac{1}{r^n} \int_{B_r^\nu(0)} |\tilde{f}(x)|^{p'} dx \right)^{\frac{1}{p'}} \leq \sigma^{1+\alpha} \beta_0.$$

Since $\beta_{\tilde{F}}(0, x) = \beta(y, \sigma x + y)$ we estimate

$$\|\beta_{\tilde{F}}(0, \cdot)\|_{L^{p'}(B_1^\nu(0))} = \left(\frac{1}{\sigma^n} \int_{B_\sigma(y) \cap \{x_n > 0\}} |\beta(y, x)|^{p'} dx \right)^{\frac{1}{p'}} \leq C(n) \beta_0 \leq \delta$$

provided β_0 is chosen sufficiently small.

We proceed similar to [11], [1] and [2] and show that there exist positive constants μ, K_1, K_2, $C(K_2)$, $0 < \alpha, \beta < 1$, and a sequence of affine functions $l_k(x) := a_k + b_k x$ for $k \in \mathbb{N} \cup \{-1, 0\}$ such that

(i) $\|\tilde{u} - l_k\|_{L^\infty(B_{\mu^k} \cap \{x_n \geq -\nu\})} \leq \mu^{k(1+\alpha)}$

(ii) $|a_{k-1} - a_k| + \mu^{k-1}|b_{k-1} - b_k| \leq 2K_2 \mu^{(k-1)(1+\alpha)}$

(iii) $|(\tilde{u} - l_k)(\mu^k x) - (\tilde{u} - l_k)(\mu^k z)| \leq C(K_2) K_1 \mu^{k(1+\alpha)} |x - z|^\beta$

for all $x, z \in B_1 \cap \{x_n \geq -\nu\}$, and $k \geq 0$. Set $l_{-1} = 0$ and $l_0 = 0$.

To prove the claim let $K_1 := C(n, \lambda, \Lambda, p)$, $\beta := \alpha(n, \lambda, \Lambda, p)$ where C, α are the constants from Theorem 1.10 when it is applied to a function $\tilde{u} \in S^*(\lambda, \Lambda, 1, \tilde{f})$ in B_2^ν. Furthermore, let K_2 and $\bar{\alpha}$ be the constants from Remark 3.3. We take $\alpha < \bar{\alpha}(1 - \gamma)$, choose $\mu \leq \frac{1}{4}$ such that

$$2\, K_2 (2\, \mu)^{1+\bar{\alpha}} \leq \mu^{1+\alpha}, \tag{24}$$

and set

$$M = 4K_2 \sum_{i=0}^{\infty} \left(\frac{1}{4}\right)^{i\alpha} \geq 4K_2 \sum_{i=0}^{\infty} \mu^{i\alpha}. \tag{25}$$

By definition, (i) and (ii) are satisfied for $k = 0$. Since $\tilde{u} \in S^*(\lambda, \Lambda, 1, \tilde{f} + \frac{\delta}{32})$ we can apply Theorem 1.10 and obtain $\|\tilde{u}\|_{C^{0,\beta}(B_1^\nu)} \leq 4K_1$ which is (iii) for $k = 0$. Assume now, that (i)–(iii) hold for some $k \geq 0$. We will prove that they hold for $k + 1$. Define

$$v(x) := \frac{\tilde{u}(\mu^k x) - l_k(\mu^k x)}{\mu^{k(1+\alpha)}}.$$

We have that v is a viscosity solution of

$$\begin{cases} F_k(D^2 v, Dv, v, x) = f_k + g_k & \text{in } B_2^{\frac{\nu}{\mu^k}} \\ v = \varphi_k & \text{on } B_2 \cap \left\{x_n = -\dfrac{\nu}{\mu^k}\right\}, \end{cases}$$

where

$$\begin{aligned} F_k(M, p, r, x) &:= \mu^{k(1-\alpha)} \tilde{F}(\mu^{k(\alpha-1)} M, \mu^{k\alpha} p, \mu^{k(\alpha+1)} r, \mu^k x) \\ g_k(x) &:= F_k(D^2 v, Dv, v, x) - F_k(D^2 v, Dv + \mu^{-k\alpha} b_k, v + \mu^{-k(1+\alpha)} l_k(\mu^k x), x) \\ f_k(x) &:= \mu^{k(1-\alpha)} \tilde{f}(\mu^k x) \\ \varphi_k &:= \mu^{-k(1+\alpha)} \left(\tilde{\varphi}(\mu^k x) - l_k(\mu^k x)\right). \end{aligned}$$

Using (i) and $\|\tilde{\varphi}\|_{C^{1,\gamma}(B_1 \cap \{x_n = -\nu\})} \leq 1$ one can check that $\|\varphi_k\|_{C^{1,\gamma}(B_1 \cap \{x_n = -\nu\mu^{-k}\})} \leq 4$. We have that $\beta_{F_k}(0, x) = \beta_{\tilde{F}}(0, \mu^k x)$ and also F_k satisfies (5) with $b_{F_k} = \mu^k b_{\tilde{F}}$ and $c_{F_k} = \mu^{2k} c_{\tilde{F}}$. For $x \in B_1^{\nu/\mu^k}$ we infer from (5)

$$|g_k(x)| \leq b_{F_k} \mu^{-k\alpha} |b_k| + c_{F_k} \mu^{-k(1+\alpha)} |l_k(\mu^k x)|.$$

From (ii) and (25) we derive $\|l_k\|_{L^\infty(B_1^\nu)}, |b_k| \le M$ and hence $|g_k(x)| \le \frac{\delta}{16}\mu^{k(1-\alpha)}$ where we made use of (23). Therefore

$$\|f_k\|_{L^{p'}(B_1^{\nu/\mu^k})} + \|g_k\|_{L^{p'}(B_1^{\nu/\mu^k})} \le \frac{\delta}{2} + \mu^{k(1-\alpha)}\frac{\delta}{2} < \delta.$$

We get $\|v\|_{C^{0,\alpha}(B_1\cap\{x_n\ge \frac{\nu}{\mu^k}\})} \le 1 + C(K_2)K_1 =: C_0$ from (i) and (iii). Let $h \in C^0(B_{1/2}^{\nu/\mu^k})$ be a C^2-viscosity solution of

$$\begin{cases} F_k(D^2h,0,0,0) = 0 & \text{in } B_1^{\nu/\mu^k} \\ h = v & \text{on } \partial B_1^{\nu/\mu^k}. \end{cases}$$

The maximum principle yields $\|h\|_{L^\infty(B_1^{\nu/\mu^k})} \le 1$ and we can apply Proposition 3.2 with $\varrho = K_2(2\mu)^{1+\bar\alpha}$ to obtain

$$\|v-h\|_{L^\infty(B_1^{\nu/\mu^k})} \le K_2(2\mu)^{1+\bar\alpha} \tag{26}$$

provided δ is chosen sufficiently small. From Assumption A^* (see Remark 3.3) we derive

$$\|h\|_{C^{1,\bar\alpha}(B_{1/2}^{\nu/\mu^k})} \le K_2. \tag{27}$$

Setting $\bar l(x) = h(0) + Dh(0)x$ we derive from (24), (26) and (27)

$$\|v-\bar l\|_{L^\infty(B_{2\mu}^{\nu/\mu^k})} \le \mu^{1+\alpha}.$$

Set $l_{k+1}(x) := l_k(x) + \mu^{k(1+\alpha)}\bar l(\mu^{-k}x)$. Since $(\tilde u - l_{k+1})(\mu^{1+k}x) = \mu^{k(1+\alpha)}(v-\bar l)(\mu x)$ we have that (i) ist satisfied for $k+1$. By definition we have $a_{k+1} = a_k + \mu^{k(1+\alpha)}h(0)$ and $b_{k+1} = b_k + \mu^{k\alpha}Dh(0)$. Therefore (ii) is an immediate consequence of (27). It remains to check (iii) for $k+1$. Therefore we utilise $(v-\bar l) \in S^*(\lambda,\Lambda,b_k,f_k+g_k+\frac{\delta}{8})$ in B_2^{ν/μ^k}. Theorem 1.10, properly scaled, yields $v-\bar l \in C^{0,\beta}(B_\mu \cap \{x_n \ge -\frac{\nu}{\mu^k}\})$ and

$$\|v-\bar l\|_{C^{0,\beta}(B_\mu^{\nu/\mu^k})} \le K_1\mu^{-\beta}\Big(\mu^{1+\alpha} + \mu^\gamma\|\varphi_k - \bar l\|_{C^{0,\gamma}(B_\mu\cap\{x_n=-\frac{\nu}{\mu^k}\})} + 2\mu^{2-\frac{n}{p'}}\delta\Big).$$

By choosing δ smaller, if necessary, we have $2\,\delta \le \mu^{\alpha+\frac{n}{p'}-1}$. In order to get an appropriate estimate for the $C^{0,\gamma}$ norm of $\varphi_k - \bar l$ on the flat part of the boundary we recall that $h = v = \varphi_k$ on $B_1 \cap \{x_n = -\frac{\nu}{\mu^k}\}$. Hence

$$\Big|\varphi_k\Big(x',\frac{\nu}{\mu^k}\Big) - \bar l\Big(x',\frac{\nu}{\mu^k}\Big)\Big| \le K_2\Big|\Big(x',-\frac{\nu}{\mu^k}\Big)\Big|^{1+\bar\alpha}.$$

Clearly, $B_\mu \cap \{x_n = -\frac{\nu}{\mu^k}\}$ is empty if $\frac{\nu}{\mu^k} > \mu$. Therefore, we have

$$\|\ \varphi_k\ -\ \bar l\ \|_{L^\infty(B_\mu\cap\{x_n=-\frac{\nu}{\mu^k}\})} \le 2\ K_2\mu^{1+\bar\alpha}.$$

Moreover, we compute

$$\begin{aligned}
&|(\varphi_k - \bar{l})(x) - (\varphi_k - \bar{l})(z)| \\
&\quad = |(\varphi_k - \bar{l})(x) - (\varphi_k - \bar{l})(z)|^\gamma |(\varphi_k - \bar{l})(x) - (\varphi_k - \bar{l})(z)|^{1-\gamma} \\
&\quad \le (4 + K_2)^\gamma (4\, K_2)^{1-\gamma} |x - z|^\gamma \mu^{(1+\bar{\alpha})(1-\gamma)}
\end{aligned}$$

for $x, z \in B_\mu \cap \{x_n = -\frac{\nu}{\mu^i}\}$. From the last estimate we derive (iii).

From (i)–(iii) we obtain the existence of an affine function l such that

$$\begin{aligned}
|l(0)|, |Dl(0)| &\le CK(y), \\
\|u - l\|_{L^\infty(B_r(y)\cap\{x_n \ge 0\})} &\le Cr^{1+\alpha} K(y).
\end{aligned} \tag{28}$$

If $y_n \ge \frac{\sigma}{2}$ we obtain (28) from the proof of [11, Theorem 2.1]. Thus, (28) holds for every $y \in B^+_{1/2}$ with $K(y) < \infty$. Choosing $p' = n$ we may deduce the first assertion of the theorem from (28), provided $K(\cdot)$ is finite. Applying Hölder's inequality to K we get

$$K(y) \le \|u\|_{L^\infty(B_1^+)} + \|\varphi\|_{C^{1,\gamma}(B_1\cap\{x_n=0\})} + \frac{1}{\beta_0} \sup_{r \le d} \left(r^{1-\alpha-\frac{n}{p}} \|f\|_{L^p(B_1^+)} \right).$$

Thus, in this case $K(y) < \infty$ since α is assumed to satisfy $\alpha \le 1 - \frac{n}{p}$.

In order to prove the second assertion, we remark that (28) implies

$$\frac{|u(y+x) - u(y)|}{|x|} \le CK(y)$$

for a.e. $y \in B^+_{1/2}$. Therefore

$$\left(\int_{B^+_{1/2}} \frac{|u(y+x) - u(y)|^q}{|x|^q} dy \right)^{\frac{1}{q}} \le C\big(\|u\|_{L^\infty(B_1^+)} + \|\varphi\|_{C^{1,\gamma}(B_1\cap\{x_n=0\})} + I \big),$$

where

$$\begin{aligned}
I^q &:= \int_{B^+_{1/2}} \sup_{r\le d} \left(r^{q(1-\alpha)} \left(r^{-n} \int_{B_r(y)\cap\{x_n\ge 0\}} |f(x)|^{p'} dx \right)^{\frac{q}{p'}} \right) dy \\
&= \int_{B^+_{1/2}} \sup_{r\le d} \left(r^{q(1-\alpha)-\frac{qn}{p'}+n} \left(r^{-n} \int_{B_r(y)\cap\{x_n\ge 0\}} |f(x)|^{p'} dx \right) \|f\|^{q-p'}_{L^{p'}(B_1^+)} \right) dy \\
&\le C \sup_{r\le d} \left(r^{q(1-\alpha)-\frac{qn}{p'}+n} \right) \left(\int_{B_1^+} M(f^{p'})(y) dy \right) \|f\|^{q-p'}_{L^{p'}(B_1^+)} \\
&\le C \sup_{r\le d} \left(r^{(1-\alpha)-\frac{n}{p'}+\frac{n}{q}} \right)^q \|f\|^{p'}_{L^p(B_1^+)} \|f\|^{q-p'}_{L^p(B_1^+)} \\
&\le C\|f\|^q_{L^p(B_1^+)}
\end{aligned}$$

provided $p' \le q \le \frac{np'}{n-p'(1-\alpha)}$. By choosing α, $p-p'$ sufficiently small we observe that the last estimate holds for every $p' \le q < p^* = \frac{np}{n-p}$. We have shown that

$$\sup_{|x|<d}\left(\int_{B^+_{1/2}} \frac{|u(y+x)-u(y)|^q}{|x|^q}dy\right)^{\frac{1}{q}}$$
$$\le C\big(\|u\|_{L^\infty(B_1^+)} + \|\varphi\|_{C^{1,\gamma}(B_1\cap\{x_n=0\})} + \|f\|_{L^p(B_1^+)}\big)$$

from which we deduce the second assertion of the theorem. □

4. $W^{2,p}$-estimates in the measurable ingredients context

In this last section we will relax the continuity assumptions on f and show that the $W^{2,p}$-estimates of Section 2 still hold for equations with a merely measurable right hand side f. Moreover we extend the results to equations with dependence on Du and u. These results lead to global $W^{2,p}$-estimates and an existence result for $W^{2,p}$-viscosity solutions of the Dirichlet problem

$$\begin{cases} F(D^2u, Du, u, x) = f & \text{in } \Omega \\ u = \varphi & \text{on } \partial\Omega. \end{cases}$$

The standard reference for $W^{2,p}$-viscosity solutions is [3] and we frequently refer to this work. Note that our definition of uniformly elliptic functions differs in the sign from that in [3]. Since we will frequently apply [3, Lemma 2.6] we want to emphasise that this result is also valid under structure condition (5), which can be observed after an examination of the proof.

We intend to apply Theorem 3.1 in the next subsection to obatin $W^{2,p}$-estimates for equations with dependence on Du and u and boundary data $\varphi = 0$. Therefore we have to ensure that Assumption A^* is satisfied. For the interior $C^{1,\bar\alpha}$ estimates we refer to [2, Corollary 5.7] or [12, Theorem 2.1] whereas for the estimates at the boundary we have the following proposition.

Proposition 4.1. *Let $u \in C^0(\overline{B_1^+})$ be a bounded C^2-viscosity solution of*

$$\begin{cases} F(D^2u) = 0 & \text{in } B_1^+ \\ u = 0 & \text{on } B_1 \cap \{x_n = 0\}. \end{cases}$$

Assume that F is uniformly elliptic with ellipticity constants λ, Λ. Then there exists $\alpha = \alpha(n, \lambda, \Lambda)$ such that $u \in C^{1,\alpha}(B^+_{1/2})$ and

$$\|u\|_{C^{1,\alpha}(B^+_{1/2})} \le C\big(\|u\|_{L^\infty(B_1^+)} + |F(0)|\big)$$

for some positive constant $C = C(n, \lambda, \Lambda)$.

Proof. Like in the proof of [2, Corollary 5.7] we use an iteration argument. Consider $v_{k,\gamma}(x) := \frac{1}{h^\gamma}(u(x+he_k)-u(x))$ for $0<\gamma\le 1$, $0<h<\frac{1}{8}$ and $1\le k\le n$, where e_k is the k-th unit vector. Note that [2, Proposition 5.5] yields $v_{k,\gamma}\in S(0)$ in $B^+_{7/8}$. From [12, Lemma 2.2] we derive $v_{n,\gamma}\in C^{0,\beta}(B_r\cap\{x_n=0\})$ and

$$\|v_{n,\gamma}\|_{C^{0,\beta}(B_r\cap\{x_n=0\})}\le h^{1-\gamma}\,C(n,\lambda,\Lambda,r)\big(\|u\|_{L^\infty(B_1^+)}+|F(0)|\big),$$

where $0<r<1$ and $\beta=\beta(n,\lambda,\Lambda)>0$. For $1\le k\le n-1$ we have $v_{k,\gamma}=0$ on $B_1\cap\{x_n=0\}$. From Theorem 1.10 we obtain

$$\|u\|_{C^{0,\alpha}(B_r^+)}\le C(n,\lambda,\Lambda,r)\big(\|u\|_{L^\infty(B_1^+)}+|F(0)|\big)=:C(n,\lambda,\Lambda,r)K\,,$$

where $\alpha=\alpha(n,\lambda,\Lambda)$ and $0<r<1$. By making α smaller if necessary, we may assume that there exists $N=N(n,\lambda,\Lambda)\in\mathbb{N}$ such that $N\alpha<1<(N+1)\alpha$. We fix constants $\frac{3}{4}<r_{N+1}<r_N<\cdots<r_1=\frac{7}{8}$ and choose h sufficiently small such that $\frac{1}{2}(r_{i+1}+r_i)<\frac{1}{2}(r_{i+1}+r_i)+h<r_i$ for $1\le i\le N$.

We start with the iteration process. Theorem 1.10 yields

$$\begin{aligned}\|v_{n,\gamma}\|_{C^{0,\alpha}(B^+_{r_2})}&\le C\Big(\|v_{n,\gamma}\|_{L^\infty(B^+_{1/2(r_2+r_1)})}+\|u\|_{L^\infty(B_1^+)}+|F(0)|\Big)\\&\le C\Big(\|u\|_{C^{0,\gamma}(B^+_{r_1})}+\|u\|_{L^\infty(B_1^+)}+|F(0)|\Big),\end{aligned}$$

where $C=C(n,\lambda,\Lambda,r_1,r_2)$. Choosing $\gamma=\alpha$ we obtain $\|v_{n,\alpha}\|_{C^{0,\alpha}(B^+_{r_2})}\le CK$. After repeating the preceeding argument for $v_{1,\gamma},\ldots,v_{n-1,\gamma}$ we infer from [2, Lemma 5.6] $\|u\|_{C^{0,2\alpha}(B^+_{r_1})}\le CK$. In the next step we choose $\gamma=2\alpha$ in order to get $u\in C^{0,3\alpha}(B^+_{r_3})$ and the corresponding estimate.

We repeat this process until we can choose $\gamma=N\alpha$. In this case [2, Lemma 5.6] yields $u\in C^{0,1}(B^+_{3/4})$ and $\|u\|_{C^{0,1}(B_{3/4})}\le C(n,\lambda,\Lambda)K$. Finally, we carry out the iteration argument for $\gamma=1$ to derive the claim. □

4.1. $W^{2,p}$-estimates for viscosity solutions.

Theorem 4.2. *Let u be a bounded $W^{2,p}$-viscosity solution of*

$$F(D^2u,Du,u,x)=f(x)\ \textit{in}\ B_1.$$

Assume that $f\in L^p(B_1)$ and that F is convex in M, and satisfies $F(0,0,0,\cdot)\equiv 0$ and structure condition (5) for a.e. x in B_1. Then there exist constants $\epsilon_0=\epsilon_0(\frac{\Lambda}{\lambda},n,b)$, $\beta_0=\beta_0(n,\lambda,\Lambda,p)$ and $C=C(n,\lambda,\Lambda,b,c,p,r_0)$ for $n-\epsilon_0<p<\infty$ such that the following holds: If

$$\left(\frac{1}{|B_r(x_0)|}\int_{B_r(x_0)}\beta(x,x_0)^n dx\right)^{\frac{1}{n}}\le\beta_0$$

for all $x_0\in B_1$ and $0<r<r_0$, then $u\in W^{2,p}(B_{1/2})$ and

$$\|u\|_{W^{2,p}(B_{1/2})}\le C\left(\|u\|_{L^\infty(B_1)}+\|f\|_{L^p(B_1)}\right).$$

Theorem 4.3. *Let u be a bounded $W^{2,p}$-viscosity solution of*

$$\begin{cases} F(D^2u, Du, u, x) = f & \text{in } B_1^+ \\ u = 0 & \text{on } B_1 \cap \{x_n = 0\}. \end{cases} \tag{29}$$

Assume that $f \in L^p(B_1^+)$ and that F is convex in M, and satisfies $F(0,0,0,\cdot) \equiv 0$ and structure condition (5) for a.e. x in B_1^+. Then there exist constants $\epsilon_0 = \epsilon_0(\frac{\Lambda}{\lambda}, n, b)$, $\beta_0 = \beta_0(n, \lambda, \Lambda, p)$ and $C = C(n, \lambda, \Lambda, b, c, p, r_0)$ where $n - \epsilon_0 < p < \infty$, such that the following holds: If

$$\left(\frac{1}{|B_r(x_0) \cap B_1^+|} \int_{B_r(x_0)\cap B_1^+} \beta(x_0, x)^n dx \right)^{\frac{1}{n}} \le \beta_0$$

for any $x_0 \in B_1^+$ and $0 < r < r_0$, then $u \in W^{2,p}(B_{1/2}^+)$ and

$$\|u\|_{W^{2,p}(B_{1/2}^+)} \le C\big(\|u\|_{L^\infty(B_1^+)} + \|f\|_{L^p(B_1^+)}\big). \tag{30}$$

Remark 4.4. Unlike [5, Theorem 1] and Theorem 2.16 we have replaced the hypotheses on $C^{1,1}$-estimates by convexity of F in M. This is because we prove Theorems 4.2 and 4.3 by approximating F by functions F_j that satisfy the assumptions of [5, Theorem 1] and Theorem 2.16. Therefore it is required that the F_j satisfy the hypothesis on $C^{1,1}$-estimates uniformly in j. Convexity in M guarantees this.

Since the proofs of both theorems are very similar we restrict ourselves to the proof of Theorem 4.3.

Proof of Theorem 4.3. Initially we show that it suffices to prove the assertion for equations without dependence on Du and u. We infer from [3, Theorem 3.6] that u is pointwise twice differentiable a.e. and satisfies $F(D^2u, Du, u, x) = f$ pointwise a.e. Let $\tilde{f}(x) := F(D^2u, 0, 0, x)$. From (5) we derive $|\tilde{f}(x)| \le b|Du(x)| + c|u(x)| + |f(x)|$ for a.e. $x \in B_1^+$, and hence by Theorem 3.1 $\tilde{f} \in L^p(B_1^+)$ provided β_0 is chosen small enough. Thus [11, Corollary 1.6] yields that $F(D^2u, 0, 0, x) = \tilde{f}$ in B_1^+ in the $W^{2,p}$-viscosity sense. Assuming that the theorem is already proven for equations without dependence on Du and u we get

$$\|u\|_{W^{2,p}(B_{1/2}^+)} \le C\big(\|u\|_{L^\infty(B_1^+)} + \|\tilde{f}\|_{L^p(B_1^+)}\big)$$

from which we derive (30). Henceforth we assume u to be a $W^{2,p}$-viscosity solution of

$$\begin{cases} F(D^2u, x) = f & \text{in } B_1^+ \\ u = 0 & \text{on } B_1 \cap \{x_n = 0\}, \end{cases}$$

where $F(M, x) := F(M, 0, 0, x)$.

Let $\phi \in C_0^\infty(\mathbb{R}^n)$ with supp $\phi \subset \mathbb{R}^n_-$, $\phi \geq 0$, $\int \phi = 1$ and set $\phi_j := j^n \phi(jx)$ for $j \in \mathbb{N}$. For $x \in B_1^+$ we consider the convolution with F

$$F_j(M, x) := \int \phi_j(x - y) F(M, y) dy,$$

where F is extended by 0 outside B_1^+. Note that F_j is convex in M, Lipschitz continuous, $F_j(0, \cdot) = 0$, and uniformly elliptic with the same ellipticity constants as F. We approximate f in L^p by functions $f_j \in C^\infty(\overline{B_1^+}) \cap L^p(B_1^+)$. From Proposition 1.11 we derive the existence of C^2-viscosity solutions u_j of

$$\begin{cases} F_j(D^2 u_j, x) = f_j & \text{in } B_1^+ \\ u_j = 0 & \text{on } \partial B_1^+, \end{cases}$$

where $\partial B_1^+ = (B_1 \cap \{x_n = 0\}) \cup (\partial B_1 \cap \{x_n > 0\})$. Convexity of F in M implies the hypothesis on $C^{1,1}$-estimates in Theorem 2.16. So it remains to check that β_{F_j} satisfies the assumptions of Theorem 2.16. For $x_0 \in B_{1-\delta}^+$ and $0 < \delta < 1$ we obtain $\beta_{F_j}(x, x_0)^n \leq \int_{B_{1/j}(0)} \phi_j(y) \beta_F(x - y, x_0 - y)^n dy$. And hence for $0 < r < \delta$

$$\begin{aligned} &\left(\frac{1}{|B_r(x_0) \cap B_1^+|} \int_{B_r(x_0) \cap B_1^+} \beta_{F_j}(x, x_0)^n dx \right)^{\frac{1}{n}} \\ &\leq \left(\int_{B_{1/j}(0)} \phi_j(y) \frac{1}{|B_r(x_0) \cap B_1^+|} \int_{B_r(x_0 - y) \cap B_1^+} \beta_F(x, x_0 - y)^n dx dy \right)^{\frac{1}{n}} \\ &\leq \beta_0. \end{aligned}$$

Therefore the hypotheses of Theorem 2.16 are satisfied and hence

$$\|u_j\|_{W^{2,p}\left(B_{1/2}^+\right)} \leq C\big(\|u_j\|_{L^\infty(B_1^+)} + \|f_j\|_{L^p(B_1^+)}\big).$$

A standard covering argument yields $u_j \in W^{2,p}_{loc}(B_1^+)$. From the generalised maximum principle, [6, Theorem 1.2], we infer that $\{u_j\}_{j \in \mathbb{N}}$ is uniformly bounded in $W^{2,p}(B_\rho^+)$ for $0 < \rho < 1$. Since $\mathcal{M}^-(D^2 u_j - D^2 u_k) \leq f_j - f_k \leq \mathcal{M}^+(D^2 u_j - D^2 u_k)$ we apply [6, Theorem 1.2] again and obtain

$$\|u_j - u_k\|_{L^\infty(B_1^+)} \leq C(n, \lambda, \Lambda, b) \|f_j - f_k\|_{L^p(B_1^+)}.$$

We have shown that $\{u_j\}$ is a Cauchy-sequence in $C^0(\overline{B_1^+})$ which implies $u_j \longrightarrow v$ in $C^0(\overline{B_1^+})$ for some $v \in C^0(\overline{B_1^+})$. Since u_j is bounded in $W^{2,p}\big(B_{1/2}^+\big)$ we have $u_j \longrightarrow v$ weakly in $W^{2,p}\big(B_{1/2}^+\big)$ and hence by the lower semicontinuity of the norm

$$\|v\|_{W^{2,p}(B_{1/2}^+)} \leq C\big(\|v\|_{L^\infty(B_1^+)} + \|f\|_{L^p(B_1^+)}\big).$$

Next, we prove that v is a $W^{2,p}$-viscosity solution of the original Dirichlet-problem. We consider test functions $\psi \in W^{2,p}(B_r(x_0))$ for $B_r(x_0) \subset B_1^+$ and get $F_j(M,x) \to F(M,x)$ whenever x is a Lebesgue point of $F(M,.)$. Since F is uniformly elliptic and $F(0,x) = 0$ we get

$$|F(M,x)| \leq \Lambda \|M\| \tag{31}$$

which implies that, for fixed M, a.e. $x \in B_1^+$ is a Lebesgue point of F. Let $\tilde{S}(n)$ be a countable, dense subset of $S(n)$, and $L(M)$ be the set of Lebesgue points of $F(M,\cdot)$. Then $|\bigcap_{M\in\tilde{S}(n)} L(M)| = |B_1^+|$, as a countable union of Nullsets is a Nullset. Since F is uniformly continuous in M, almost every $x \in \Omega$ is a Lebesgue point of $F(D^2\psi,.)$ and hence

$$F_j(D^2\psi(x),x) \to F(D^2\psi(x),x)$$

for a.e. $x \in B_1^+$. From (31) it follows that $F_j(D^2\psi,\cdot)$ is dominated in L^p. Lebesgue's Theorem implies that the hypotheses of Proposition 1.5 are satisfied and hence v is a $W^{2,p}$-viscosity solution of (29).

It remains to show that we have $u = v$. Since $v \in W^{2,p}_{loc}(B_1^+) \cap C^0(\overline{B_1^+})$ we get that $w := u - v$ is a $W^{2,p}$-viscosity solution in $S(0)$. The maximum principle and $w = 0$ on ∂B_1^+ yield $\|w\|_{L^\infty(B_1^+)} \leq 0$. □

Theorem 4.5. *Let $n - \epsilon_0 < p < \infty$, $\Omega \subset\subset \mathbb{R}^n$, $\partial\Omega \in C^{1,1}$ and u be a $W^{2,p}$-visocsity solution of*

$$\begin{cases} F(D^2u, Du, u, x) = f & \text{in } \Omega \\ u = \varphi & \text{on } \partial\Omega\,, \end{cases}$$

where $f \in L^p(\Omega)$, $\varphi \in W^{2,p}(\Omega)$. Assume that F satisfies (5) *for a.e. x, $F(0,0,0,\cdot) \equiv 0$ in Ω and that F is convex in M. Then there exists a constant β_0 such that the following holds: If*

$$\left(\frac{1}{|B_r(x_0)\cap\Omega|}\int_{B_r(x_0)\cap\Omega} \beta(x,x_0)^n dx\right)^{\frac{1}{n}} \leq \beta_0$$

for $x_0 \in \overline{\Omega}$, $0 < r < r_0$ and $\beta_0 = \beta_0(n,\lambda,\Lambda,p,r_0)$, then $u \in W^{2,p}(\Omega)$ and

$$\|u\|_{W^{2,p}(\Omega)} \leq C\left(\|u\|_{L^\infty(\Omega)} + \|\varphi\|_{W^{2,p}(\Omega)} + \|f\|_{L^p(\Omega)}\right), \tag{32}$$

where $C = C(n,\lambda,\Lambda,b,c,p,r_0,\Omega)$.

Proof. At first we show that it suffices to prove the claim for $\varphi = 0$. For $u = u - \varphi + \varphi =: w + \varphi$ we have that w is a $W^{2,p}$-viscosity solution of

$$\begin{cases} G(D^2w, Dw, w, x) = g & \text{in } \Omega \\ w = 0 & \text{on } \partial\Omega\,, \end{cases}$$

where $G(M,p,r,x) := F(M + D^2\varphi, p + D\varphi, r + \varphi, x) - F(D^2\varphi, D\varphi, \varphi, x)$ and $g(x) := f(x) - F(D^2\varphi, D\varphi, \varphi, x)$. We infer from (5)

$$|g(\cdot)| \le |f(\cdot)| + C(\lambda, \Lambda, b, c)\left(\|D^2\varphi(\cdot)\| + |D\varphi(\cdot)| + |\varphi(\cdot)|\right)$$

and hence $g \in L^p(\Omega)$. Assuming that the theorem is already proven for $\varphi = 0$ we obtain $\|w\|_{W^{2,p}(\Omega)} \le C(\|w\|_{L^\infty(\Omega)} + \|g\|_{L^p(\Omega)})$ which implies (32). From now on we assume $\varphi = 0$.

Now the claim follows from a standard covering argument and Theorems 4.2 and 4.3. In order to apply Theorem 4.3 we have to flatten the boundary first. Since $\partial\Omega \in C^{1,1}$, for any $x_0 \in \partial\Omega$ there exists a neighborhood $U(x_0)$ and a $C^{1,1}$-diffeomorphism

$$\Psi : U(x_0) \xrightarrow{\cong} B_1(0)$$

such that $\Psi(x_0) = 0$, $\Psi(U(x_0) \cap \Omega) = B_1^+$. For $\tilde\varphi \in W^{2,p}(B_1^+)$ we set $\varphi = \tilde\varphi \circ \Psi \in W^{2,p}(U(x_0))$ and obtain $D\varphi = (D\tilde\varphi \circ \Psi)\, D\Psi$, and

$$D^2\varphi = D\Psi^T\left(D^2\tilde\varphi \circ \Psi\right) D\Psi + \left((D\tilde\varphi \circ \Psi)\, \partial_{i,j}\Psi\right)_{1 \le i,j \le n}.$$

Therefore we have for $\tilde u = u \circ \Psi^{-1} \in C^0(B_1^+)$

$$\begin{aligned} F(D^2\varphi, D\varphi, u, x) \circ \Psi^{-1} = F(&D\Psi^T \circ \Psi^{-1} D^2\tilde\varphi\; D\Psi \circ \Psi^{-1} \\ &+ \left(D\tilde\varphi\; \partial_{i,j}\Psi \circ \Psi^{-1}\right)_{1 \le i,j \le n}, \\ D\tilde\varphi\; D\Psi \circ \Psi^{-1}, \tilde u, \Psi^{-1}(x)) &=: \tilde F(D^2\tilde\varphi, D\tilde\varphi, \tilde u, x). \end{aligned}$$

Consequently $\tilde u$ is a $W^{2,p}$-viscosity solution of $\tilde F(D\tilde u, D\tilde u, \tilde u, x) = \tilde f(x)$ in B_1^+ where $\tilde f := f \circ \Psi^{-1}$. Note that the function $\tilde F$ ist convex in M, $\tilde F(0,0,0,x) = 0$, and $\tilde F(M,0,0,x) = F(D\Psi^T \circ \Psi^{-1} M\; D\Psi \circ \Psi^{-1}, 0, 0, \Psi^{-1}(x))$ from which we conclude $\beta_{\tilde F}(x, x_0) \le C(\Psi)\beta_F(\Psi^{-1}(x), \Psi^{-1}(x_0))$. Moreover, $\tilde F$ is uniformly elliptic with ellipticity constants $\lambda C(\Psi), \Lambda C(\Psi)$. Therefore $\tilde F$ satisfies the assumptions of Theorem 4.3. □

Finally, we use the previous estimates to derive an existence result for $W^{2,p}$-viscosity solutions.

Theorem 4.6. *Assume that the hypotheses of Theorem* 4.5 *hold. Additionally we assume that F is non-increasing in r. Then there exists a unique $W^{2,p}$-viscosity solution u of*

$$\begin{cases} F(D^2u, Du, u, x) = f & \text{in } \Omega \\ u = \varphi & \text{on } \partial\Omega. \end{cases} \tag{33}$$

Moreover $u \in W^{2,p}(\Omega)$ and

$$\|u\|_{W^{2,p}(\Omega)} \le C\left(\|u\|_{L^\infty(\Omega)} + \|\varphi\|_{W^{2,p}(\Omega)} + \|f\|_{L^p(\Omega)}\right).$$

Proof. We proceed similar to the proof of Theorem 4.3 and consider a standard mollifier $\phi \in C_0^\infty(\mathbb{R}^n)$ with supp $\phi \subset \mathbb{R}^n_-$, $\phi \geq 0$, $\int \phi = 1$ and set $\phi_j := j^n \phi(jx)$ for $j \in \mathbb{N}$. For $x \in B_1^+$ we consider the convolution with F

$$F_j(M,p,r,x) := \int \phi_j(x-y) F(M,p,r,y)dy,$$

where F is extended by 0 outside Ω. Note that F_j is convex in M, non-increasing in r, and satisfies (5), and $F_j(0,0,0,x) = 0$. We approximate f in L^p by functions $f_j \in C^\infty(\overline{B_1^+}) \cap L^p(B_1^+)$. From Proposition 1.11 we derive the existence of C^2-viscosity solutions u_j of

$$\begin{cases} F_j(D^2u_j, Du_j, u_j, x) = f_j & \text{in } \Omega \\ u_j = \varphi & \text{on } \partial\Omega. \end{cases}$$

Similar to the proof of Theorem 4.3 we obtain that $\beta_{\tilde{F}_j}$ satisfies the hypotheses of Theorem 4.5 and hence

$$\|u_j\|_{W^{2,p}(\Omega)} \leq C\left(\|u_j\|_{L^\infty(\Omega)} + \|\varphi\|_{W^{2,p}(\Omega)} + \|f_j\|_{L^p(\Omega)}\right). \tag{34}$$

From the last inequality and the generalised maximum principle we infer that u_j is uniformly bounded in $W^{2,p}(\Omega)$. Since $W^{2,p}(\Omega)$ is reflexive there exists $u \in W^{2,p}(\Omega)$ and a subsequence such that $u_j \to u$ weakly in $W^{2,p}(\Omega)$. We have $p > \frac{n}{2}$ and hence there exists another subsequence such that $u_j \to u$ in $C^0(\overline{\Omega})$. From the weak convergence we infer that (34) holds for u, and similarly to the proof of Theorem 4.3 we obtain

$$\|F_j(D^2\phi, D\phi, u_j, \cdot) - F(D^2\phi, D\phi, u, \cdot)\|_{L^p(B_r(x_0))} \to 0\,,$$

where $\phi \in W^{2,p}(B_r(x_0))$, $B_r(x_0) \subset \Omega$. By Proposition 1.5, u is a viscosity solution of (33) and hence it is also a strong solution. Uniqueness follows from [3, Theorem 2.10]. □

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