

Low degree bounded cohomology and L^2 -invariants for negatively curved groups

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Abstract. We study the subgroup structure of discrete groups that share cohomological properties which resemble non-negative curvature. Examples include all Gromov hyperbolic groups.

We provide strong restrictions on the possible s -normal subgroups of a ‘negatively curved’ group. Another result says that the image of a group, which is boundedly generated by a finite set of amenable subgroups, in a group, which admits a proper quasi-1-cocycle into the regular representation, has to be amenable. These results extend to a certain class of randomorphisms in the sense of Monod.

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1. Introduction

This note is a companion to a paper by J. Peterson and the author [PT07], to which we refer for background and notation. Cohomology and bounded cohomology of an infinite group with coefficients in the left regular representation have proved to be useful tools to understand properties of the group. For the study of ℓ^2 -homology and cohomology we refer to the book by W. Lück [Lüc02], for information about bounded cohomology the standard reference is the book by N. Monod [Mon01].

Non-vanishing of the second bounded cohomology with coefficients in the left regular representation is the key condition in the work of Burger–Monod [BM02], Monod–Shalom [MS04], [MS06] and Mineyev–Monod–Shalom [MMS04] on rigidity theory. In [PT07], Peterson and the author studied non-vanishing of the first cohomology with coefficients in the left regular representation and derived results about the subgroup structure. Now we link first ℓ^2 -cohomology and second bounded cohomology via an exact sequence of LG -modules, where LG denotes the group von Neumann algebra. Moreover, we extend the methods of [PT07] to apply to a wider class of groups including all non-elementary Gromov hyperbolic groups. The key

notion here is the notion of quasi-1-cocycle, which appeared already in several places including Monod's foundational work [Mon01].

The organization of the article is as follows. Section 1 is the introduction. After studying quasi-1-cocycles and the first quasi-cohomology group in connection with Lück's dimension theory in Section 1 and 2, we introduce a class of groups \mathcal{D}_{reg} closely related to the class \mathcal{C}_{reg} , which was studied in [MS04], [MS06]. Examples of groups in the class \mathcal{D}_{reg} include non-elementary Gromov hyperbolic groups and all groups with a positive first ℓ^2 -Betti number. In the sequel we prove two main results. In Section 3 we prove that an s-normal subgroup of a group in \mathcal{D}_{reg} is again in \mathcal{D}_{reg} using methods from [PT07]. In Section 4 we show that all group homomorphisms from a group, which is boundedly generated by a finite set of amenable subgroups, to a group, which admits a proper quasi-1-cocycle into an infinite sum of the left regular representation, have amenable image. If one adds property (T) to the assumptions on the source group, the image has to be finite. Since the combination of bounded generation and property (T) applies to many lattices in higher rank Lie groups, this gives a new viewpoint towards some well-known results in the field.

In the last section, we study the class of groups which admit such proper quasi-1-cocycles and show that it is closed under formation of free products and a notion of ℓ^2 -orbit equivalence. The methods imply that the results of Section 4 extend to certain randommorphisms in the sense of Monod; see [Mon06].

Throughout the article, G will be a discrete countable group, $\ell^2 G$ denotes the Hilbert space with basis G endowed with the left regular representation.

Definition 1.1. Let $\pi: G \rightarrow U(H_\pi)$ be a unitary representation of G . A map $c: G \rightarrow H_\pi$ is called a *quasi-1-cocycle* if the map

$$G \times G \ni (g, h) \rightarrow \pi(g)c(h) - c(gh) + c(g) \in H_\pi$$

is uniformly bounded on $G \times G$. We denote the vector space of quasi-1-cocycles with values in H_π by $QZ^1(G, H_\pi)$. The subspace of uniformly bounded quasi-1-cocycles is denoted by $QB^1(G, H_\pi)$.

In analogy to the definition of $H^1(G, H_\pi)$, we define the following:

Definition 1.2. The first quasi-cohomology of G with coefficients in the unitary G -representation H_π is defined by

$$QH^1(G, H_\pi) = QZ^1(G, H_\pi)/QB^1(G, H_\pi).$$

The relevance of $QH^1(G, H_\pi)$ becomes obvious in the next theorem, which links low degree cohomology and second bounded cohomology with coefficients in H_π .

Theorem 1.3. *Let G be a discrete countable group and H_π be a unitary G -representation. There exists an exact sequence as follows:*

$$0 \rightarrow H^1(G, H_\pi) \rightarrow QH^1(G, H_\pi) \xrightarrow{\delta} H_b^2(G, H_\pi) \rightarrow H^2(G, H_\pi).$$

Here δ denotes the Hochschild coboundary, which maps a 1-chain to a 2-cocycle.

Proof. The proof is contained in [Mon01]. However, since this sequence is relevant for our work, we give a short argument. Clearly, the first non-trivial map is injective, since the trivial 1-cocycles are precisely the bounded ones, by the Bruhat–Tits fixed point lemma. The kernel of δ is given precisely by those 1-chains which are cycles and define elements in $H^1(G, H_\pi)$. It remains to prove exactness at $H_b^2(G, H_\pi)$. This is obvious since the elements which are mapped to zero are precisely those which are coboundaries of 1-chains. Thus, these are precisely those which are in the image of the coboundary map δ . \square

It is clear that homological algebra can fit every comparison map between additive functors, which is defined on a suitable *chain level*, into a long exact sequence. Hence the above exact sequence can be extended to the right; see [Mon01]. Since we will not need this extension we do not dwell on this.

2. Second bounded cohomology and dimension theory

The following theorem was first observed for finitely generated groups by Burger–Monod in [BM02] as a consequence of the new approach to bounded cohomology developed by N. Monod in [Mon01]. Later the result was extended to arbitrary countable groups by V. Kaimanovich [Kai03]. Note that we do not state the most general form of the result but rather a version which we can readily apply to the problems we study.

Theorem 2.1. *Let G be a discrete countable group and H_π be a separable dual Banach G -module. There exists a standard probability space S with a Borel G -action, leaving the probability measure quasi-invariant, such that there exists a natural isometric isomorphism*

$$H_b^2(G, H_\pi) \cong ZL_{\text{alt}}^\infty(S^3, H_\pi)^G.$$

Here $ZL_{\text{alt}}^\infty(S^3, H_\pi)$ denotes the space of alternating 3-cocycles with values in H_π .

The naturality of the isomorphism immediately implies that, in case H_π carries an additional module structure over a ring R (which commutes with the G -action), the isomorphism is an isomorphism of R -modules. One highly non-trivial consequence

of the preceding theorem (from our point of view) is that $H_b^2(G, H_\pi)$ can be viewed as a subspace of $L^\infty(S^3, H_\pi)$ for some *standard* probability space S . The following corollary makes use of this fact.

In the sequel, we are freely using Lück’s dimension function for LG -modules, which is defined for *all* modules over the group von Neumann algebra LG . For details about its definition see [Lüc02]. Note that ℓ^2G carries a commuting right LG -module structure, which induces LG -module structures on all its (quasi-)cohomological invariants.

Definition 2.2. An LG -module M is called *rank separated* if for every non-zero element $\xi \in M$

$$[\xi] := 1 - \sup\{\tau(p) \mid p^2 = p^* = p \in LG, \xi p = 0\} > 0.$$

For more information on the notion of rank, we refer to [Tho07]. The only consequence we need is the following lemma.

Lemma 2.3. *An LG -module is rank separated if and only if every non-zero LG -submodule has a positive dimension.*

Corollary 2.4. *Let G be a discrete countable group and $K \subset G$ a subgroup. Then $H_b^2(K, \ell^2G)$ is a rank separated LG -module. In particular, every non-zero element generates a sub-module of positive dimension.*

Proof. If we apply Theorem 2.1 to K , it follows that $H_b^2(K, \ell^2G) \subset L^\infty(S^3, \ell^2G)$. Let $\xi \in H_b^2(K, \ell^2G)$ and assume that there exists a sequence of projections $p \in LG$ such that $p_n \uparrow 1$ and $\xi p_n = 0$ for all $n \in \mathbb{N}$. If $\xi p_n = 0$, then for a co-null set $X_n \subset S^3$, we have $\xi(x)p_n = 0$ for all $x \in X_n$. Clearly, the intersection $\bigcap_{n \in \mathbb{N}} X_n$ is still co-null and hence $\xi(x) = 0$ for almost all $x \in S^3$. This implies that $\xi = 0$. \square

Corollary 2.5. *Let G be a discrete countable group and $K \subset G$ a subgroup. The LG -module*

$$QH^1(K, \ell^2G)$$

is rank separated if and only if K is non-amenable.

Proof. In view of the exact sequence

$$0 \rightarrow H^1(K, \ell^2G) \rightarrow QH^1(K, \ell^2G) \rightarrow H_b^2(K, \ell^2G),$$

the LG -module $QH^1(K, \ell^2G)$ is rank separated if both $H^1(K, \ell^2G)$ and $H_b^2(K, \ell^2G)$ are rank separated. Hence, the proof is finished by Corollary 2.4 and Hulanicki’s Theorem; see also the proof of Corollary 2.4 in [PT07]. \square

Following [MS06], we denote by \mathcal{C}_{reg} the class of groups for which

$$H_b^2(G, \ell^2 G) \neq 0.$$

This class was studied extensively in [MS06], and strong results about rigidity and superrigidity were obtained. An a priori slightly different class is of importance in the results we obtain.

Definition 2.6. We denote by \mathcal{D}_{reg} the class of groups with

$$\dim_{LG} QH^1(G, \ell^2 G) \neq 0.$$

Remark 2.7. Neither of the possible inclusions between \mathcal{C}_{reg} and \mathcal{D}_{reg} is known to hold. Any positive result in this direction would be very interesting. Both of the inclusions seem to be likely. Indeed, it is very likely that $\dim_{LG} QH^1(G, \ell^2 G)$ and $\dim_{LG} H_b^2(G, \ell^2 G)$ can only take the values 0 or ∞ . At least if G is of type FP_2 , this would imply $G \in \mathcal{C}_{\text{reg}} \iff G \in \mathcal{D}_{\text{reg}}$. Moreover, it would give the implication

$$\beta_1^{(2)}(G) \in (0, \infty) \implies G \in \mathcal{C}_{\text{reg}},$$

which seems natural. However, we did not succeed in proving the required restriction on the values of the dimension.

Both \mathcal{C}_{reg} and \mathcal{D}_{reg} consist of groups which all remember some features of negatively curved metric spaces. It is therefore permissible to call these groups ‘negatively curved’. We will see this more directly in the examples below.

Lemma 2.8. *Let G be a countable discrete group. If the group G is in \mathcal{D}_{reg} , then either the first ℓ^2 -Betti number of G or the second bounded cohomology of G with coefficients in $\ell^2 G$ does not vanish. The converse holds if $\beta_2^{(2)}(G) = 0$.*

Proof. This is immediate from Theorem 1.3 and Corollary 2.4. □

A large class of groups in \mathcal{D}_{reg} is provided by a result from [MMS04], Theorem 3.

Theorem 2.9. *All non-elementary hyperbolic groups are in \mathcal{D}_{reg} .*

Proof. In [MMS04], it was shown that all hyperbolic groups have non-vanishing $QH^1(G, \ell^2 G)$. In fact, it was shown that there exists some element in $QH^1(G, \ell^2 G)$ which even maps non-trivially to $H_b^2(G, \ell^2 G)$. It follows from Corollary 2.5 that all non-elementary (i.e. non-amenable) hyperbolic groups are in \mathcal{D}_{reg} . □

In [MS04], the class \mathcal{C}_{reg} is studied more extensively. One result we want to mention is the following:

Theorem 2.10 (Corollary 7.6 in [MS04]). *Let G be a discrete group acting non-elementarily and properly by isometries on some proper CAT(−1) space. Then*

$$H_b^2(G, \ell^2 G) \neq 0.$$

Note that in view of Lemma 2.8, the preceding theorem provides examples of groups in \mathcal{D}_{reg} as soon as the second ℓ^2 -Betti number of the corresponding group vanishes.

3. Non-existence of infinite s-normal subgroups

The following notion of normality was studied by Peterson and the author [PT07] in connection with a non-vanishing first ℓ^2 -Betti number. The definition of s-normality goes back to the seminal work of S. Popa, who studied similar definitions in [Pop06].

Definition 3.1. Let G be a discrete countable group. An infinite subgroup $K \subset G$ is said to be *s-normal* if $gKg^{-1} \cap K$ is infinite for all $g \in G$.

Example 3.2. The inclusions

$$\text{GL}_n(\mathbb{Z}) \subset \text{GL}_n(\mathbb{Q}) \quad \text{and} \quad \mathbb{Z} = \langle a \rangle \subset \langle a, b \mid ba^p b^{-1} = a^q \rangle = \text{BS}_{p,q}$$

are inclusions of s-normal subgroups.

Given Banach space valued functions $f, g: X \rightarrow B$, defined on a set X , we write $f \asymp g$ if the function

$$X \ni x \mapsto \|f(x) - g(x)\| \in \mathbb{R}$$

is uniformly bounded on X .

A unitary G -representation is said to be *strongly mixing* if $\langle g\xi, \eta \rangle \rightarrow 0$ for $g \rightarrow \infty$. The following lemma is the key observation which leads to our first main results.

Lemma 3.3. *Let G be a discrete countable group and let $K \subset G$ be an infinite s-normal subgroup. Let H_π be a strongly mixing unitary representation of the group G . The restriction map*

$$\text{res}_K^G: \mathcal{QH}^1(G, H_\pi) \rightarrow \mathcal{QH}^1(K, H_\pi)$$

is injective.

Proof. Let $c : G \rightarrow H_\pi$ be a quasi-1-cocycle, which is bounded on K . Therefore $c(k) \asymp 0$ as a function of $k \in K$. We compute

$$c(gkg^{-1}) \asymp (1 - gkg^{-1})c(g) + gc(k) \asymp (1 - gkg^{-1})c(g),$$

as a function of $(g, k) \in G \times K$. For $k \in g^{-1}Kg \cap K$, we obtain that

$$\|(1 - gkg^{-1})c(g)\| \leq \|c(gkg^{-1})\| + C' \leq C$$

for some constants $C', C \geq 0$. Since $K \subset G$ is s -normal, the subgroup $g^{-1}Kg \cap K$ is infinite. Now, since H_π is strongly mixing, we conclude that

$$2\|c(g)\|^2 = \lim_{k \rightarrow \infty} \|(1 - gkg^{-1})c(g)\|^2 \leq C^2.$$

Hence $g \rightarrow \|c(g)\|$ is uniformly bounded by $2^{-1/2}C$. This proves the claim. \square

The following theorem is a non-trivial consequence about the subgroup structure for groups in the class \mathcal{D}_{reg} . It is our first main result.

Theorem 3.4. *Let G be a discrete countable group in \mathcal{D}_{reg} and let $K \subset G$ be an infinite s -normal subgroup. The group K satisfies at least one of the following properties:*

- (i) *the first ℓ^2 -Betti number of K does not vanish, or*
- (ii) *the second bounded cohomology of K with coefficients in $\ell^2 K$ does not vanish.*

In particular, K can neither be amenable nor a product of infinite groups.

Proof. By Lemma 3.3, the restriction map

$$QH^1(G, \ell^2 G) \rightarrow QH^1(K, \ell^2 G)$$

is injective. One easily sees that $QH^1(K, \ell^2 K) \subset QH^1(K, \ell^2 G)$ generates a rank-dense LG -submodule in $QH^1(K, \ell^2 G)$. Hence $QH^1(K, \ell^2 K)$ cannot be zero-dimensional and we see that K is in \mathcal{D}_{reg} , and the claim follows from Lemma 2.8. We conclude that K can neither be amenable or a product of infinite groups, since both classes of groups have vanishing first ℓ^2 -Betti number (see [Lüc02]) and vanishing second bounded cohomology with coefficients in the left regular representation; see [MS06]. \square

Remark 3.5. The result easily extends to ws -normal subgroups; see [PT07].

4. Bounded generation and finiteness theorems

It has been observed by many people that boundedly generated groups and non-elementary hyperbolic groups are opposite extremes in geometric group theory. In

this section we support this view by showing that there are essentially no group homomorphisms from a boundedly generated with property (T) to a Gromov hyperbolic group. Later, in Section 5.2, we can even extend this result to a suitable class of randomorphisms in the sense of Monod; see [Mon06].

A group G is said to be *boundedly generated* by a subset X if there exists $k \in \mathbb{N}$ such that each element of G is a product of less than k elements from $X \cup X^{-1}$. We say that G is boundedly generated by a finite set of subgroups $\{G_i, i \in I\}$ if G is boundedly generated by the set $\bigcup_{i \in I} G_i$.

Lemma 4.1. *Let G be a non-amenable group which is boundedly generated by a finite set of amenable subgroups. Then the group $QH^1(G, \ell^2 G^{\oplus \infty})$ is zero.*

Proof. We view $\ell^2 G^{\oplus \infty} \cong \ell^2(G \times \mathbb{Z})$ and consider it as an $L(G \times \mathbb{Z})$ -module. In view of Corollary 2.5, we can assume that $QH^1(G, \ell^2 G^{\oplus \infty})$ is rank separated. Hence, given an arbitrary element

$$c \in QH^1(G, \ell^2(G \times \mathbb{Z})),$$

in order to show that it is zero, we have to provide a sequence of projections $p_n \in L(G \times \mathbb{Z})$ such that $p_n \uparrow 1$ and $cp_n = 0$ for all $n \in \mathbb{N}$.

Let G be boundedly generated by amenable subgroups G_1, \dots, G_n . The restriction of a quasi-1-cocycle onto G_i is almost bounded, i.e., there exists a projection $q_i \in L(G \times \mathbb{Z})$ of trace $\tau(q_i) \geq 1 - \varepsilon/n$ such that cq_i is bounded on G_i . Setting $p = \inf_{1 \leq i \leq n} q_i$, we obtain a projection p with trace $\tau(p) \geq 1 - \varepsilon$ such that cp is bounded on G_i for all $1 \leq i \leq n$. The cocycle identity and bounded generation imply that cp is bounded on the whole of G . Hence $cp = 0 \in QH^1(G, \ell^2(G \times \mathbb{Z}))$. The sequence p_n is constructed by choosing $\varepsilon < 1/n$. This proves the claim. \square

As we have seen, non-elementary Gromov hyperbolic groups are in \mathcal{D}_{reg} , but more is true:

Lemma 4.2. *Let G be a Gromov hyperbolic group. There exists a proper quasi-1-cocycle on G with values in $\ell^2 G^{\oplus \infty}$.*

Proof. This is an immediate consequence of the proof of Theorem 7.13 in [MS04], which builds on Mineyev’s work on equivariant bicomings; see Theorem 10 in [Min01]. \square

The following theorem is the second main result of this article. A cocycle version of it will be presented in the last section as Corollary 5.12.

Theorem 4.3. *Let G be a group which admits a proper quasi-1-cocycle into $\ell^2 G^{\oplus \infty}$, and let H be a group which is boundedly generated by a finite set of amenable subgroups. Then every group homomorphism $\phi: H \rightarrow G$ has amenable image.*

Proof. We may assume that ϕ is injective, since any quotient of a group which is boundedly generated by amenable groups is of the same kind. If the quotient is non-amenable, then the restriction of the proper quasi-1-cocycle coming from Lemma 4.2 has to be bounded on H by Lemma 4.1. Indeed, $\ell^2 H^{\oplus\infty} \cong \ell^2 G^{\oplus\infty}$ as unitary H -representations using a coset decomposition and hence Lemma 4.1 applies. However, the quasi-1-cocycle is unbounded on any infinite subset. This is a contradiction since H follows to be finite and hence amenable. \square

Remark 4.4. The result applies in particular to the case when G is Gromov hyperbolic. In this case we can even conclude that the image is finite or virtually cyclic, since all amenable subgroups of a Gromov hyperbolic group are finite or virtually cyclic.

From the above theorem we can derive the following corollary.

Corollary 4.5. *Let G be a group which admits a proper quasi-1-cocycle into $\ell^2 G^{\oplus\infty}$, and let H be a group which*

- (i) *is boundedly generated by a finite set of amenable subgroups, and*
- (ii) *has property (T) of Kazhdan–Margulis.*

Then every group homomorphism $\phi: H \rightarrow G$ has finite image.

Proof. Any quotient of a property (T) group has also property (T). However, the image of ϕ is amenable by Theorem 4.3 and the only amenable groups with property (T) are finite. This finishes the proof. \square

Remark 4.6. Note that results like the preceding corollary are well known if one assumes the target to be a-T-menable, whereas here: many Gromov hyperbolic groups have property (T). Examples of groups H which satisfy assumptions (i) and (ii) of Corollary 4.5 include $SL_n(\mathbb{Z})$ for $n \geq 3$ and many other lattices in higher rank semi-simple Lie groups; see [Tav90]. Conjecturally, all irreducible, non-cocompact lattices in higher rank Lie groups share these properties. In [BM02], it was shown that higher rank lattices in certain algebraic groups over local fields have property (TT) of Monod; see Theorem 13.4.1 in [Mon01] and the definitions therein. A similar proof can be carried out in this situation.

Note that the groups which satisfy the conditions (i) and (ii) do not always satisfy property (TT) of Monod. Indeed, in an appendix of [Man06] Monod–Rémy construct boundedly generated groups (in fact lattices in higher rank semi-simple Lie groups) with property (T) which fail to have property (QFA) of Manning (see [Man06]) and property (TT) of Monod. Hence, the plain quasification of property (T), which would yield some version of property (TT) of Monod (and should of course also imply property (QFA) by an extension of Watatani’s proof, see [Wat82]) is too strong to hold for all lattices in higher rank semi-simple Lie group. Hence, the combination

of conditions (i) and (ii) is perhaps the appropriate set of conditions that encodes the way in which higher rank lattices satisfy a *strong form* of property (T).

Remark 4.7. The mechanism of properness vs. boundedness works in the context of ordinary first cohomology with coefficients in ℓ^p -spaces as well. In fact, G. Yu [Yu05] provided proper ℓ^p -cocycles for hyperbolic groups, whereas Bader–Furman–Gelander–Monod [BFGM07] studied the necessary strengthening of property (T) for higher rank lattices. A combination of Theorem B in [BFGM07] and Yu’s result allows to conclude Corollary 4.5 for lattices in certain algebraic groups (see the assumption of Theorem B in [BFGM07]). However, even for this special case, our approach seems more elementary, just using the notion of quasi-1-cocycle.

5. Groups with proper quasi-1-cocycles

5.1. Subgroups and free products. Every quasi-1-cocycle is close to one for which

$$c(g^{-1}) = -g^{-1}c(g)$$

holds on the nose. Indeed, $\tilde{c}(g) = \frac{1}{2}(c(g) - gc(g^{-1}))$ is anti-symmetric and only bounded distance away from c . Note also that \tilde{c} is proper if and only c is proper. We call the quasi-1-cocycles which satisfy this additional property *anti-symmetric*.

Lemma 5.1. *Let G, H be discrete countable groups and let H_π be a unitary representation of $G * H$. Moreover, let $c_1: G \rightarrow H_\pi$ and $c_2: H \rightarrow H_\pi$ be anti-symmetric quasi-1-cocycles. Then there is a natural anti-symmetric quasi-1-cocycle*

$$c = (c_1 * c_2): G * H \rightarrow H_\pi$$

which extends c_1 and c_2 .

Proof. Let $w = g_1h_1g_2h_2 \dots g_nh_n$ be a reduced element in $G * H$ (i.e., only g_1 or h_n might be trivial). We define

$$c(w) = c_1(g_1) + g_1c_2(h_2) + g_1h_1c_1(g_2) + \dots + g_1h_1 \dots g_nc_2(h_n).$$

Clearly, c is anti-symmetric, i.e., $c(w^{-1}) = -w^{-1}c(w)$ just by construction and using that c_1 and c_2 were anti-symmetric. Let us now check that it is indeed a quasi-1-cocycle. Let w_1 and w_2 be elements of $G * H$ and assume that $w_1 = w'_1r$ and $w_2 = r^{-1}w'_2$, such that the products $w'_1w'_2$, w'_1r and $r^{-1}w'_2$ are reduced in the sense that the block length drops at most by one.

Then the identities

$$\begin{aligned} c(w'_1w'_2) &= w'_1c(w'_2) + c(w'_1), \\ c(w'_1r) &= w'_1c(r) + c(w'_1), \\ c(r^{-1}w'_2) &= r^{-1}c(w'_2) + c(r^{-1}) \end{aligned}$$

hold up to a uniformly bounded error. Hence, using the three equations above, we can compute

$$\begin{aligned} c(w_1 w_2) &= c(w'_1 w'_2) \\ &= w'_1 c(w'_2) + c(w'_1) \\ &= w'_1 r c(r^{-1} w'_2) - w'_1 r c(r^{-1}) + c(w'_1) \\ &= w'_1 r c(r^{-1} w'_2) - w'_1 r c(r^{-1}) + c(w'_1 r') - w'_1 c(r) \\ &= w_1 c(w_2) + c(w_2), \end{aligned}$$

again up to uniformly bounded error. In the last step we used that c is anti-symmetric. In fact, the construction seems to fail at this point if one does not assume c_1 and c_2 to be anti-symmetric, since we cannot assure that $c(r^{-1}) + r^{-1}c(r)$ is uniformly bounded. This finishes the proof. \square

Remark 5.2. Note that obviously the class of c in the first quasi-cohomology does depend heavily on c_1 and c_2 and not only on their classes in the first quasi-cohomology.

Theorem 5.3. *The class of groups G which admit proper quasi-1-cocycles $c : G \rightarrow \ell^2 G^{\oplus \infty}$ is closed under subgroups and free products.*

Proof. The assertion concerning subgroups is obvious since we can just restrict the cocycles and decompose the regular representation according to the cosets.

Let us now turn to the question about free products. Given proper quasi-1-cocycles $c_1 : G \rightarrow \ell^2 G^{\oplus \infty}$ and $c_2 : H \rightarrow \ell^2 H^{\oplus \infty}$ we can regard both as taking values in $\ell^2(G * H)^{\oplus \infty}$ and can assume that they are anti-symmetric. Moreover, since the set of values of c_1 and c_2 in a bounded region is finite, we can add a bounded 1-cocycle and assume that the minimum of $g \mapsto \|c_1(g)\|_2$ and $h \mapsto \|c_2(h)\|$ is non-zero on $G \setminus \{e\}$ resp. $H \setminus \{e\}$.

We claim that the quasi-1-cocycle $(c_1 * 0) \oplus (0 * c_2) : G * H \rightarrow \ell^2(G * H)^{\oplus \infty}$ is proper. Here we are using the notation of Lemma 5.1. Let $w = g_1 h_1 g_2 h_2 \dots g_n h_n$ be a reduced element in $G * H$. Clearly,

$$\|(c_1 * 0)(w)\|^2 = \sum_{i=1}^n \|c(g_i)\|^2.$$

Hence, using the properness of c_1 , the set of g_i 's that can appear with a given bound on $\|(c_1 * 0)(w)\|$ is a finite subset of G . Moreover, we find an upper bound on i since the minimum of $g \rightarrow \|c(g)\|$ is assumed to be non-zero. The same is true for the set of h_i 's. Hence, the set of elements $w \in G * H$ for which $\|(c_1 * 0) \oplus (0 * c_2)(w)\|$ is less than a constant is finite. This finishes the proof. \square

5.2. Orbit equivalence. In this section we study the stability of the class of groups which admit proper quasi-1-cocycles in a multiple of the regular representation under orbit equivalence. For the notion of *orbit equivalence*, which goes back to work of Dye (see [Dye59], [Dye63]), we refer to Gaboriau’s nice survey [Gab05] and the references therein; see also [Gro93]. Although it remains open whether the class is closed under this relation, we are able to prove that it is closed under a slightly more restricted relation, which we call ℓ^2 -orbit equivalence. (To our knowledge, the idea of ℓ^2 -orbit equivalence goes back to unpublished work of R. Sauer.) Unfortunately, we cannot say much more about the class of groups which are ℓ^2 -orbit equivalent to Gromov hyperbolic groups. This is subject of future work. Literally everything extends to the suitable notions of measure equivalence, but for sake of simplicity we restrict to orbit equivalence.

Let G be a discrete countable group. Let (X, μ) be a standard probability space and $G \curvearrowright (X, \mu)$ be a measure preserving (m.p.) action by Borel automorphisms. We denote by $X \rtimes G$ the inverse semigroup of partial isomorphisms which are implemented by the action (not just the equivalence relation). Two partial isomorphisms ϕ, ψ which are induced by the action are said to be *orthogonal* if $\text{dom}(\phi) \cap \text{dom}(\psi) = \emptyset$ and $\text{ran}(\phi) \cap \text{ran}(\psi) = \emptyset$. They are said to be *disjoint* if they are disjoint as subsets of the set of morphisms of the associated discrete measured groupoid. Clearly, orthogonal partial isomorphisms are disjoint. All equalities which concern subsets of a probability space or partial maps between probability spaces are supposed to hold almost everywhere, i.e., up to a set of measure zero, as usual.

Every partial isomorphism can be written as an infinite orthogonal sum

$$\phi = \bigoplus_{i=1}^{\infty} \phi_{A_i} g_i$$

for some Borel subsets A_i and $g_i \in G$. The sub-inverse-semigroup of those for which there is a finite sum as above is denoted by $X \rtimes_{\text{fin}} G$. If G is finitely generated and $l : G \rightarrow \mathbb{N}$ is a word length function on G , there is yet another sub-inverse-semigroup, which is formed by those infinite sums for which

$$\sum_{i=1}^{\infty} \mu(A_i) l(g_i)^2 < \infty.$$

We denote it by $X \rtimes_2 G$. Since $l(gh)^2 \leq (l(g) + l(h))^2 \leq 2(l(g)^2 + l(h)^2)$, it is obvious that $X \rtimes_2 G$ is closed under composition. Note also that the summability does not depend on the set of generators we choose to define the length function.

Definition 5.4. Let (X, μ) be a standard probability space and $G, H \curvearrowright (X, \mu)$ essentially free m.p. actions by Borel automorphisms. The data is said to induce an orbit-equivalence if the orbits of the two actions agree up to measure zero. In this case injective natural homomorphisms of inverse semigroups,

$$\phi_1 : G \rightarrow X \rtimes H \quad \text{and} \quad \phi_2 : H \rightarrow X \rtimes G,$$

are defined. We say that an orbit equivalence is an ℓ^2 -orbit-equivalence if the image of ϕ_1 (resp. ϕ_2) is contained in $X \rtimes_2 H$ (resp. $X \rtimes_2 G$).

Remark 5.5. If the images are contained even in $X \rtimes_{\text{fin}} G$ (resp. $X \rtimes_{\text{fin}} H$), one usually speaks about a uniform orbit equivalence. Using Gromov’s dynamical criterion, this also implies that G is quasi-isometric to H . Thus ℓ^2 -orbit equivalence is somehow halfway between quasi-isometry and usual orbit equivalence.

Definition 5.6. Let H_π be a unitary G -representation which carries a compatible normal action of $L^\infty(X)$. A 1-cocycle of $X \rtimes_2 G$ with values in H_π is defined to be a map $c : X \rtimes_2 G \rightarrow H_\pi$ such that

- (1) $c(\phi) \in \chi_{\text{ran}(\phi)} H_\pi$,
- (2) c is compatible with infinite orthogonal decompositions of the domain,
- (3) $c(\psi\phi) = \psi c(\phi) + c(\psi)$ if $\text{dom}(\psi) = \text{ran}(\phi)$.

A 1-cocycle is said to be inner if $c(\phi) = (\phi - \chi_{\text{ran}(\phi)})\xi$ for some vector $\xi \in H_\pi$. In analogy to the group case we call a map $c : X \rtimes_2 G \rightarrow H_\pi$ satisfying (1) and (2) from above a quasi-1-cocycle if $\|\psi c(\phi) - c(\psi\phi) + c(\psi)\|$ is uniformly bounded for ψ, ϕ with $\text{dom}(\psi) = \text{ran}(\phi)$.

Definition 5.7. A quasi-1-cocycle $c : X \rtimes_2 G \rightarrow H_\pi$ is said to be *proper* if for every sequence of disjoint partial isomorphisms $\phi_i \in X \rtimes G$ with $\liminf_{i \rightarrow \infty} \mu(\text{dom}(\phi_i)) > 0$, we have that $\lim_{i \rightarrow \infty} \|c(\phi_i)\| = \infty$.

Lemma 5.8. *Let G be a discrete countable group. Let (X, μ) be a standard probability space and $G \curvearrowright (X, \mu)$ be an m.p. action by Borel automorphisms. Let $c : G \rightarrow H_\pi$ be a quasi-1-cocycle. There is a natural extension of the (quasi-)1-cocycle c to a (quasi-)1-cocycle $\tilde{c} : X \rtimes_2 G \rightarrow L^2(X, \mu) \otimes_2 H_\pi$ where G acts diagonally.*

Proof. We define

$$\tilde{c}\left(\bigoplus_{i=1}^\infty \chi_{A_i} g_i\right) = \sum_{i=1}^\infty \chi_{A_i} \otimes c(g_i).$$

Since $\|c(g)\| \leq C \cdot l(g)$ for some constant $C > 0$, the right-hand side is well defined in $L^2(X, \mu) \otimes_2 H_\pi$. It can be easily checked that all relations are satisfied. \square

Lemma 5.9. *If the quasi-1-cocycle $c : G \rightarrow H_\pi$ is proper, then so is the quasi-1-cocycle $\tilde{c} : X \rtimes_2 G \rightarrow L^2(X, \mu) \otimes_2 H_\pi$, which we obtain from the construction in Lemma 5.8.*

Proof. Let ϕ_i be a sequence of disjoint partial isomorphisms with $\liminf_{i \rightarrow \infty} \mu(\phi_i) \geq \varepsilon > 0$. In order to derive a contradiction, we can assume that $\|\tilde{c}(\phi_i)\| < C$ for some constant C and all $i \in \mathbb{N}$.

Hence, for every $i \in \mathbb{N}$, at least half of ϕ_i is supported at group elements $g \in G$ with $\|c(g)\| \leq 2C/\varepsilon$. Indeed, if not, then $\|\tilde{c}(\phi_i)\| \geq 2^{-1/2}\varepsilon \cdot 2C/\varepsilon > C$. Since this holds for all $i \in \mathbb{N}$ and the support of $g \in G$ has measure 1, the set of $g \in G$ with $\|c(g)\| \leq 2C/\varepsilon$ has to be infinite. This is a contradiction since we assume c to be proper. \square

Theorem 5.10. *The class of groups which admit a proper quasi-1-cocycle with values in an infinite sum of the regular representation is closed under ℓ^2 -orbit equivalence.*

Proof. Let G and H be ℓ^2 -orbit equivalent groups. We show that if G admits a proper quasi-1-cocycle with values in $\ell^2 G^{\oplus\infty}$, then H admits a proper quasi-1-cocycle with values in $\ell^2 H^{\oplus\infty}$.

Let $c: G \rightarrow \ell^2 G^{\oplus\infty}$ be a proper quasi-1-cocycle. Lemma 5.8 says that we can extend c to a quasi-1-cocycle which is defined on $X \rtimes_2 G$ and takes values in $L^2(X, \mu) \otimes_2 \ell^2 G^{\oplus\infty}$. Note that the homomorphism $\phi: H \rightarrow X \rtimes_2 G$ is compatible in the sense that the obvious actions are intertwined with the natural isomorphism

$$L^2(X, \mu) \otimes_2 \ell^2 G^{\oplus\infty} \cong L^2(X, \mu) \otimes_2 \ell^2 H^{\oplus\infty}.$$

We conclude by noting that the restriction $\tilde{c}|_H: H \rightarrow L^2(X, \mu) \otimes_2 \ell^2 H^{\oplus\infty}$ is proper by Lemma 5.9, and that $L^2(X, \mu) \otimes_2 \ell^2 H^{\oplus\infty} \cong \ell^2 H^{\oplus\infty}$ as unitary H -representations. \square

Remark 5.11. A similar proof applies to the case where H merely embeds into $X \rtimes_2 G$ (i.e., is not necessarily part of an orbit equivalence). This type of sub-object is called *random subgroup* in [Mon06]. However, note that an arbitrary random subgroup does not necessarily satisfy the ℓ^2 -condition we impose. We could speak of ℓ^2 -random subgroups in case it satisfies the ℓ^2 -condition.

The following strengthening of Theorem 4.3 is also an immediate consequence of Theorem 5.10.

Corollary 5.12. *Let G be a group which is boundedly generated by a finite set of amenable subgroups and let H be a Gromov hyperbolic group. Let $H \curvearrowright (X, \mu)$ be an m.p. action by Borel automorphisms on a standard probability space. Any homomorphism of inverse semi-groups $\phi: G \rightarrow X \rtimes_2 H$ has amenable image. Moreover, if G has property (T) of Kazhdan, then the image of ϕ has finite measure.*

Remark 5.13. In terms of randommorphisms, the last corollary says that any ℓ^2 -randommorphism from G to H has finite image; in the sense that the relevant G -invariant measure μ on the polish space $[G, H]_\bullet$. (see [Mon06] for details) is a.e. supported on maps with finite image such that

$$\int_{[G, H]_\bullet} \#\phi(G) d\mu(\phi) < \infty.$$

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References

- [BFGM07] U. Bader, A. Furman, T. Gelander, and N. Monod, Property (T) and rigidity for actions on Banach spaces. *Acta Math.* **198** (2007), 57–105. [Zbl 05166604](#) [MR 2316269](#)
- [BM02] M. Burger and N. Monod, Continuous bounded cohomology and applications to rigidity theory. *Geom. Funct. Anal.* **12** (2002), 219–280. [Zbl 1006.22010](#) [MR 1911660](#)
- [Dye59] H. A. Dye, On groups of measure preserving transformation. I. *Amer. J. Math.* **81** (1959), 119–159. [Zbl 0087.11501](#) [MR 0131516](#)
- [Dye63] H. A. Dye, On groups of measure preserving transformations. II. *Amer. J. Math.* **85** (1963), 551–576. [Zbl 0191.42803](#) [MR 0158048](#)
- [Gab05] D. Gaboriau, Examples of groups that are measure equivalent to the free group. *Ergodic Theory Dynam. Systems* **25** (2005), 1809–1827. [Zbl 1130.37311](#) [MR 2183295](#)
- [Gro93] M. Gromov, *Geometric group theory* (Sussex, 1991), vol. 2: Asymptotic invariants of infinite groups. London Math. Soc. Lecture Note Ser. 182, Cambridge University Press, Cambridge 1993. [Zbl 0841.20039](#) [MR 1253544](#)
- [Kai03] V. A. Kaimanovich, Double ergodicity of the Poisson boundary and applications to bounded cohomology. *Geom. Funct. Anal.* **13** (2003), 852–861. [Zbl 1027.60038](#) [MR 2006560](#)
- [KN04] A. Karlsson and G. A. Noskov, Some groups having only elementary actions on metric spaces with hyperbolic boundaries. *Geom. Dedicata* **104** (2004), 119–137. [Zbl 1062.20047](#) [MR 2043957](#)
- [Lüc02] *L^2 -invariants: theory and applications to geometry and K -theory*. *Ergeb. Math. Grenzgeb.* (3) 44, Springer-Verlag, Berlin 2002. [Zbl 1009.55001](#) [MR 1926649](#)
- [Man06] J. F. Manning, Quasi-actions on trees and property (QFA). *J. London Math. Soc.* (2) **73** (2006), 84–108. [Zbl 05014368](#) [MR 2197372](#)
- [Min01] I. Mineyev, Straightening and bounded cohomology of hyperbolic groups. *Geom. Funct. Anal.* **11** (2001), 807–839. [Zbl 1013.20034](#) [MR 1866802](#)
- [MMS04] I. Mineyev, N. Monod, and Y. Shalom, Ideal bicomings for hyperbolic groups and applications. *Topology* **43** (2004), 1319–1344. [Zbl 1137.20033](#) [MR 2081428](#)
- [Mon01] N. Monod, *Continuous bounded cohomology of locally compact groups*. Lecture Notes in Math. 1758, Springer-Verlag, Berlin 2001. [Zbl 0967.22006](#) [MR 1840942](#)

- [Mon06] N. Monod, An invitation to bounded cohomology. In *Proc. Internat. Congr. Mathematicians*, Vol. II, EMS Publ. House, Zürich 2006, 1183–1211. [Zbl 1127.55002](#) [MR 2275641](#)
- [MS04] N. Monod and Y. Shalom, Cocycle superrigidity and bounded cohomology for negatively curved spaces. *J. Differential Geom.* **67** (2004), 395–455. [Zbl 1127.53035](#) [MR 2153026](#)
- [MS06] N. Monod and Y. Shalom, Orbit equivalence rigidity and bounded cohomology. *Ann. of Math. (2)* **164** (2006), 825–878. [Zbl 1129.37003](#) [MR 2259246](#)
- [PT07] J. Peterson and A. Thom, Group cocycles and the ring of affiliated operators. Preprint 2007. [arXiv:0708.4327](#)
- [Pop06] S. Popa, Some computations of 1-cohomology groups and construction of non-orbit-equivalent actions. *J. Inst. Math. Jussieu* **5** (2006), 309–332. [Zbl 1092.37003](#) [MR 2225044](#)
- [Tav90] O. I. Tavgen', Bounded generation of Chevalley groups over rings of S -integer algebraic numbers. *Izv. Akad. Nauk SSSR Ser. Mat.* **54** (1990), 97–122; English transl. *Math. USSR-Izv.* **36** (1991), 101–128. [Zbl 0697.20032](#) [MR 1044049](#)
- [Tho07] A. Thom, L^2 -invariants and rank metric. In *C^* -algebras and elliptic theory II*, Trends Math., Birkhäuser, Basel 2008, 267–280. [Zbl 05296155](#) [MR 2408147](#)
- [Wat82] Y. Watatani, Property T of Kazhdan implies property FA of Serre. *Math. Japon.* **27** (1982), 97–103. [Zbl 0489.20022](#) [MR 649023](#)
- [Yu05] G. Yu, Hyperbolic groups admit proper affine isometric actions on l^p -spaces. *Geom. Funct. Anal.* **15** (2005), 1144–1151. [Zbl 1112.46054](#) [MR 2221161](#)

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