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Strongly self-absorbing C*-algebras are $\mathcal{Z}$-stable

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Abstract. We prove the title. This characterizes the Jiang–Su algebra $\mathcal{Z}$ as the uniquely determined initial object in the category of strongly self-absorbing C*-algebras.

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Introduction

A separable unital C*-algebra $\mathcal{D} \neq \mathbb{C}$ is called strongly self-absorbing if there is an isomorphism $\mathcal{D} \to \mathcal{D} \otimes \mathcal{D}$ that is approximately unitarily equivalent to the first factor embedding, cf. [12]. The interest in such algebras largely arises from Elliott's program to classify nuclear C*-algebras by K-theoretic invariants. In fact, examples suggest that classification will only be possible up to $\mathcal{D}$-stability (i.e., up to tensoring with $\mathcal{D}$) for a strongly self-absorbing $\mathcal{D}$, cf. [11], [4], [15]. While the known strongly self-absorbing examples are quite well understood, and are entirely classified, it remains an open problem whether these are the only ones. From a more general perspective, the question is in how far abstract properties allow for comparison with concrete examples. For nuclear C*-algebras, this question prominently manifests itself as the UCT problem (i.e., is every nuclear C*-algebra KK-equivalent to a commutative one); a positive answer even in the special setting of strongly self-absorbing C*-algebras would be highly satisfactory, and likely shed light on the general case.

In this note we shall be concerned with a closely related interpretation of the aforementioned question: we will show that any strongly self-absorbing C*-algebra $\mathcal{D}$ admits a unital embedding of a specific example, the Jiang–Su algebra $\mathcal{Z}$ (we refer to [8] and to [10] for an introduction and various characterizations of $\mathcal{Z}$). It then follows immediately that $\mathcal{D}$ is in fact $\mathcal{Z}$-stable. The result answers some problems left open in [12] and in [1]; in particular it implies that strongly self-absorbing C*-algebras are always K_1-injective. Moreover, it shows that the Jiang–Su algebra is an initial object in the category of strongly self-absorbing C*-algebras (with unital

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$*$-homomorphisms); there can only be one such initial object, whence $\mathcal{Z}$ is characterized this way. It is interesting to note that the Cuntz algebra $\mathcal{O}_2$ is the uniquely determined final object in this category, and that $\mathcal{O}_\infty$ can be characterized as the initial object in the category of infinite strongly self-absorbing C*-algebras.

The proof of our main result builds on ideas from [10] and from [1], where the problem was settled in the case where $\mathcal{D}$ contains a nontrivial projection.

1. Small elementary tensors

In this section, we generalize a technical result from [1] to a setting that does not require the existence of projections, see Lemma 1.4 below. We refer to [9] for a brief account of the Cuntz semigroup.

Proposition 1.1. *Let A be a unital C*-algebra, $0 \le g \le \mathbf{1}_A$.*

Then, for any $0 \neq n \in \mathbb{N}$, we have

$$\begin{aligned} \mathbf{1}_{A^{\otimes n}} - g^{\otimes n} \ge\ & (\mathbf{1}_A - g) \otimes g \otimes \cdots \otimes g \\ & + g \otimes (\mathbf{1}_A - g) \otimes g \otimes \cdots \otimes g \\ & \vdots \\ & + g \otimes \cdots \otimes g \otimes (\mathbf{1}_A - g). \end{aligned}$$

Proof. The statement is trivial for $n = 1$. Suppose now we have shown the assertion for some $0 \neq n \in \mathbb{N}$. We obtain

$$\begin{aligned} \mathbf{1}_{A^{\otimes(n+1)}} - g^{\otimes(n+1)} &= \mathbf{1}_{A^{\otimes n}} \otimes g - g^{\otimes n} \otimes g + \mathbf{1}_{A^{\otimes n}} \otimes (\mathbf{1}_A - g) \\ &= (\mathbf{1}_{A^{\otimes n}} - g^{\otimes n}) \otimes g + \mathbf{1}_{A^{\otimes n}} \otimes (\mathbf{1}_A - g) \\ &\ge ((\mathbf{1}_A - g) \otimes g \otimes \cdots \otimes g) \otimes g \\ &\quad + (g \otimes (\mathbf{1}_A - g) \otimes g \otimes \cdots \otimes g) \otimes g \\ &\quad \vdots \\ &\quad + (g \otimes \cdots \otimes g \otimes (\mathbf{1}_A - g)) \otimes g \\ &\quad + g^{\otimes n} \otimes (\mathbf{1}_A - g), \end{aligned}$$

where for the inequality we have used our induction hypothesis as well as the fact that $\mathbf{1}_{A^{\otimes n}} \otimes (\mathbf{1}_A - g) \ge g^{\otimes n} \otimes (\mathbf{1}_A - g)$. Therefore, the statement also holds for $n + 1$. □

Proposition 1.2. *Let $\mathcal{D}$ be strongly self-absorbing, $0 \le d \le \mathbf{1}_{\mathcal{D}}$.*

Then, for any $0 \neq k \in \mathbb{N}$,

$$[\mathbf{1}_{\mathcal{D}^{\otimes k}} - d^{\otimes k}] \le k \cdot [(\mathbf{1}_{\mathcal{D}} - d) \otimes \mathbf{1}_{\mathcal{D}^{\otimes(k-1)}}] \ \ in \ \ W(\mathcal{D}^{\otimes k}).$$

Proof. The assertion holds trivially for $k = 1$. Suppose now it has been verified for some $k \in \mathbb{N}$. Then

$$\begin{aligned}[\mathbf{1}_{\mathcal{D}^{\otimes(k+1)}} - d^{\otimes(k+1)}] &= [\mathbf{1}_{\mathcal{D}^{\otimes k}} \otimes (\mathbf{1}_{\mathcal{D}} - d) + \mathbf{1}_{\mathcal{D}^{\otimes k}} \otimes d - d^{\otimes k} \otimes d] \\ &\leq [\mathbf{1}_{\mathcal{D}^{\otimes k}} \otimes (\mathbf{1}_{\mathcal{D}} - d)] + [(\mathbf{1}_{\mathcal{D}^{\otimes k}} - d^{\otimes k}) \otimes \mathbf{1}_{\mathcal{D}}] \\ &\leq [(\mathbf{1}_{\mathcal{D}} - d) \otimes \mathbf{1}_{\mathcal{D}^{\otimes k}}] + k \cdot [(\mathbf{1}_{\mathcal{D}} - d) \otimes \mathbf{1}_{\mathcal{D}^{\otimes(k-1)}} \otimes \mathbf{1}_{\mathcal{D}}] \\ &= (k+1) \cdot [(\mathbf{1}_{\mathcal{D}} - d) \otimes \mathbf{1}_{\mathcal{D}^{\otimes k}}]\end{aligned}$$

(using that $\mathcal{D}$ is strongly self-absorbing as well as our induction hypothesis for the second inequality), so the assertion also holds for $k + 1$. □

The following is only a mild generalization of [1], Lemma 1.3.

Lemma 1.3. *Let $\mathcal{D}$ be strongly self-absorbing and let $0 \leq f \leq g \leq \mathbf{1}_{\mathcal{D}}$ be positive elements of $\mathcal{D}$ satisfying $\mathbf{1}_{\mathcal{D}} - g \neq 0$ and $fg = f$.*

Then there is $0 \neq n \in \mathbb{N}$ such that

$$[f^{\otimes n}] \leq [\mathbf{1}_{\mathcal{D}^{\otimes n}} - g^{\otimes n}] \ \textit{in} \ W(\mathcal{D}^{\otimes n}).$$

Proof. Since $\mathcal{D}$ is simple and $\mathbf{1}_{\mathcal{D}} - g \neq 0$, there is $n \in \mathbb{N}$ such that

$$[f] \leq n \cdot [\mathbf{1}_{\mathcal{D}} - g].$$

Then

$$\begin{aligned}[f^{\otimes n}] &\leq n \cdot [(\mathbf{1}_{\mathcal{D}} - g) \otimes f \otimes \cdots \otimes f] \\ &= [(\mathbf{1}_{\mathcal{D}} - g) \otimes f \otimes \cdots \otimes f] + \cdots + [f \otimes \cdots \otimes f \otimes (\mathbf{1}_{\mathcal{D}} - g)] \\ &= [(\mathbf{1}_{\mathcal{D}} - g) \otimes f \otimes \cdots \otimes f + \cdots + f \otimes \cdots \otimes f \otimes (\mathbf{1}_{\mathcal{D}} - g)] \\ &\leq [(\mathbf{1}_{\mathcal{D}} - g) \otimes g \otimes \cdots \otimes g + \cdots + g \otimes \cdots \otimes g \otimes (\mathbf{1}_{\mathcal{D}} - g)] \\ &\leq [\mathbf{1}_{\mathcal{D}^{\otimes n}} - g^{\otimes n}],\end{aligned}$$

where for the first equality we have used that $\mathcal{D}$ is strongly self-absorbing, for the second equality we have used that the terms are pairwise orthogonal by our assumptions on f and g, and the last inequality follows from Proposition 1.1. □

The following is a version of [1], Lemma 2.4, for positive elements rather than projections.

Lemma 1.4. *Let $\mathcal{D}$ be strongly self-absorbing and let $0 \leq f \leq g \leq \mathbf{1}_{\mathcal{D}}$ be positive elements satisfying $\mathbf{1}_{\mathcal{D}} - g \neq 0$ and $fg = f$; let $0 \neq d \in \mathcal{D}_+$.*

Then there is $0 \neq m \in \mathbb{N}$ such that

$$[f^{\otimes m}] \leq [d \otimes \mathbf{1}_{\mathcal{D}^{\otimes(m-1)}}] \ \textit{in} \ W(\mathcal{D}^{\otimes m}).$$

Proof. By Lemma 1.3, there is $0 \neq n \in \mathbb{N}$ such that

$$[f^{\otimes n}] \leq [\mathbf{1}_{\mathcal{D}^{\otimes n}} - g^{\otimes n}];$$

since $f^{\otimes n} \perp \mathbf{1}_{\mathcal{D}^{\otimes n}} - g^{\otimes n}$, this implies that

$$2 \cdot [f^{\otimes n}] \leq [\mathbf{1}_{\mathcal{D}^{\otimes n}}].$$

By an easy induction argument we then have

$$2^k \cdot [f^{\otimes nk}] \leq [\mathbf{1}_{\mathcal{D}^{\otimes nk}}]$$

for any $k \in \mathbb{N}$.

By simplicity of $\mathcal{D}$ and since d is nonzero, there is $\bar{k} \in \mathbb{N}$ such that

$$[f] \leq 2^{\bar{k}} \cdot [d].$$

Set

$$m := n\bar{k} + 1.$$

Then

$$\begin{aligned} [f^{\otimes m}] &\leq 2^{\bar{k}} \cdot [d \otimes f^{\otimes (m-1)}] \\ &= 2^{\bar{k}} \cdot [d \otimes f^{\otimes n\bar{k}}] \\ &\leq [d \otimes \mathbf{1}_{\mathcal{D}^{\otimes n\bar{k}}}] \\ &= [d \otimes \mathbf{1}_{\mathcal{D}^{\otimes (m-1)}}]. \end{aligned}$$

□

2. Large order zero maps

Below we establish the existence of nontrivial order zero maps from matrix algebras into strongly self-absorbing C*-algebras, and we show certain systems of such maps give rise to order zero maps with small complements. We refer to [16] and [17] for an introduction to order zero maps.

Proposition 2.1. *Let $\mathcal{D}$ be strongly self-absorbing and $0 \neq d \in \mathcal{D}_+$.*

Then, for any $0 \neq k \in \mathbb{N}$, there is a nonzero completely positive contractive (*henceforth abbreviated as c.p.c.*) *order zero map*

$$\psi : M_k \to \overline{d\mathcal{D}d}.$$

Proof. Let us first prove the assertion in the case where $d = \mathbf{1}_{\mathcal{D}}$ and $k = 2$. Since $\mathcal{D}$ is infinite dimensional, there are orthogonal positive normalized elements $e, f \in \mathcal{D}$.

Since $\mathcal{D} \cong \mathcal{D} \otimes \mathcal{D}$ is strongly self-absorbing, there is a sequence of unitaries $(u_n)_{n\in\mathbb{N}} \subset \mathcal{D} \otimes \mathcal{D}$ such that

$$u_n(e \otimes f)u_n^* \xrightarrow{n\to\infty} f \otimes e;$$

since $e \otimes f \perp f \otimes e$, this implies that there is a c.p.c. order zero map

$$\bar{\sigma}\colon M_2 \to \prod_{\mathbb{N}} \mathcal{D} \otimes \mathcal{D} / \bigoplus_{\mathbb{N}} \mathcal{D} \otimes \mathcal{D}$$

given by

$$\bar{\sigma}(e_{11}) = e \otimes f, \quad \bar{\sigma}(e_{22}) = f \otimes e, \quad \bar{\sigma}(e_{21}) = \pi((u_n(e \otimes f))_{n\in\mathbb{N}})$$

(cf. [16]), where $\pi\colon \prod_{\mathbb{N}} \mathcal{D} \otimes \mathcal{D} \to \prod_{\mathbb{N}} \mathcal{D} \otimes \mathcal{D} / \bigoplus_{\mathbb{N}} \mathcal{D} \otimes \mathcal{D}$ denotes the quotient map.

Since order zero maps with finite dimensional domains are semiprojective (cf. [16]), $\bar{\sigma}$ has a c.p.c. order zero lift $M_2 \to \prod_{\mathbb{N}} \mathcal{D} \otimes \mathcal{D}$, which in turn implies that there is a nonzero completely positive contractive c.p.c. order zero map

$$\tilde{\sigma}\colon M_2 \to \mathcal{D} \otimes \mathcal{D} \cong \mathcal{D}.$$

Next, if $k = 2^r$ for some $r \in \mathbb{N}$, then

$$M_{2^r} \cong (M_2)^{\otimes r} \xrightarrow{\tilde{\sigma}^{\otimes r}} \mathcal{D}^{\otimes r} \cong \mathcal{D}$$

is a nonzero c.p.c. order zero map; for an arbitrary $k \in \mathbb{N}$, we may take r large enough and restrict $\tilde{\sigma}^{\otimes r}$ to $M_k \subset M_{2^r}$ to obtain a nonzero c.p.c. order zero map

$$\sigma\colon M_k \to \mathcal{D}.$$

This settles the proposition for arbitrary k and for $d = \mathbf{1}_{\mathcal{D}}$. Now if d is an arbitrary nonzero positive element (which we may clearly assume to be normalized), we can define a c.p.c. map

$$\bar{\psi}\colon M_k \to \prod_{\mathbb{N}} \overline{d\mathcal{D}d} / \bigoplus_{\mathbb{N}} \overline{d\mathcal{D}d} \subset \prod_{\mathbb{N}} \mathcal{D} / \bigoplus_{\mathbb{N}} \mathcal{D}$$

by setting

$$\bar{\psi}(x) := \pi((d\sigma_n(x)d)_{n\in\mathbb{N}}) \text{ for } x \in M_k,$$

where again $\pi\colon \prod_{\mathbb{N}} \overline{d\mathcal{D}d} \to \prod_{\mathbb{N}} \overline{d\mathcal{D}d} / \bigoplus_{\mathbb{N}} \overline{d\mathcal{D}d}$ denotes the quotient map and $\sigma_n\colon M_k \to \mathcal{D}$ is a sequence of c.p.c. maps lifting the c.p.c. order zero map

$$\mu\sigma\colon M_k \to \left(\prod_{\mathbb{N}} \mathcal{D} / \bigoplus_{\mathbb{N}} \mathcal{D}\right) \cap \mathcal{D}',$$

with

$$\mu\colon \mathcal{D} \to \left(\prod_{\mathbb{N}} \mathcal{D} / \bigoplus_{\mathbb{N}} \mathcal{D}\right) \cap \mathcal{D}'$$

being a unital $*$-homomorphism as in [12], Theorem 2.2. It is straightforward to check that $\bar{\psi}$ is nonzero and has order zero. Again by semiprojectivity of order zero maps, this implies the existence of a nonzero c.p.c. order zero map

$$\psi\colon M_k \to \overline{d\mathcal{D}d}. \qquad \square$$

Proposition 2.2. *Let B be a unital C*-algebra and $\varrho\colon M_2 \to B$ a unital $*$-homomorphism. Define*

$$E := \{f \in \mathcal{C}([0,1], B \otimes M_2) \mid f(0) \in B \otimes \mathbf{1}_{M_2},\ f(1) \in \mathbf{1}_B \otimes M_2\}.$$

Then there is a unital $*$-homomorphism

$$\tilde{\varrho}\colon M_2 \to E.$$

Proof. This follows from simply connecting the two embeddings $\varrho \otimes \mathbf{1}_{M_2}$ and $\mathbf{1}_{M_2} \otimes \mathrm{id}_{M_2}$ of M_2 into $\varrho(M_2) \otimes M_2 \cong M_2 \otimes M_2$ along the unit interval. □

Lemma 2.3. *Let $m \in \mathbb{N}$ and A a unital C*-algebra. Let*

$$\varphi_1, \ldots, \varphi_m\colon M_2 \to A$$

be c.p.c. order zero maps such that

$$\sum_{i=1}^{m} \varphi_i(\mathbf{1}_{M_2}) \le \mathbf{1}_A$$

and

$$[\varphi_i(M_2), \varphi_j(M_2)] = 0 \quad \text{if } i \neq j.$$

Then, there is a c.p.c. order zero map

$$\bar{\varphi}\colon M_2 \to C^*(\varphi_i(M_2) \mid i = 1, \ldots, m) \subset A$$

such that

$$\bar{\varphi}(\mathbf{1}_{M_2}) = \sum_{i=1}^{m} \varphi_i(\mathbf{1}_{M_2}).$$

Moreover, if $d \in A_+$ satisfies $\varphi_m(e_{11})d = d$, we may assume that $\bar{\varphi}(e_{11})d = d$.

Proof. In the following, we write C_i, $i = 1, \ldots, m$, for various copies of the C*-algebra $\mathcal{C}_0((0,1], M_2)$; these come equipped with c.p.c. order zero maps $\varrho_i\colon M_2 \to C_i$ given by

$$\varrho_i(x)(t) = t \cdot x \quad \text{for } t \in (0,1] \text{ and } x \in M_2.$$

By [14], Proposition 3.2 (a), the c.p.c. order zero maps $\varphi_i\colon M_2 \to A$ induce unique $*$-homomorphisms $C_i \to A$ via $\varrho_i(x) \mapsto \varphi_i(x)$ for $x \in M_2$.

We now define a universal C*-algebra

$$B := C^*\big(C_i, \mathbf{1} \mid \sum_{l=1}^{m} \varrho_l(\mathbf{1}_{M_2}) \le \mathbf{1},\ [C_i, C_j] = 0 \text{ if } i \neq j \in \{1, \ldots, m\}\big).$$

Then, B is generated by the $\varrho_i(x)$, $i \in \{1, \ldots, m\}$ and $x \in M_2$; the assignment

$$\varrho_i(x) \mapsto \varphi_i(x) \quad \text{for } i \in \{1, \ldots, m\} \text{ and } x \in M_2$$

induces a unital $*$-homomorphism

$$\pi\colon B \to C^*(\varphi_i(M_2), \mathbf{1}_A \mid i \in \{1, \dots, m\}) \subset A$$

satisfying

$$\sum_{l=1}^{m} \pi \varrho_l(\mathbf{1}_{M_2}) = \sum_{l=1}^{m} \varphi_l(\mathbf{1}_{M_2}).$$

Now if we find a c.p.c. order zero map

$$\bar{\varrho}\colon M_2 \to B$$

satisfying

$$\bar{\varrho}(\mathbf{1}_{M_2}) = \sum_{l=1}^{m} \varrho_l(\mathbf{1}_{M_2}).$$

Then

$$\bar{\varphi} := \pi \bar{\varrho}$$

will have the desired properties, proving the first assertion of the lemma. We proceed to construct $\bar{\varrho}$.

For $k = 1, \dots, m$, let

$$J_k := \mathcal{J}(\mathbf{1} - \textstyle\sum_{l=k}^{m} \varrho_l(\mathbf{1}_{M_2})) \lhd B$$

denote the ideal generated by $\mathbf{1} - \sum_{l=k}^{m} \varrho_l(\mathbf{1}_{M_2})$ in B; let

$$B_k := B/J_k$$

denote the quotient. We clearly have

$$J_1 \subset J_2 \subset \cdots \subset J_m$$

and surjections

$$B \xrightarrow{\pi_1} B_1 \xrightarrow{\pi_2} \to \cdots \xrightarrow{\pi_m} B_m.$$

Observe that

$$\pi_m \circ \cdots \circ \pi_1 \circ \varrho_m\colon M_2 \to B_m$$

is a unital surjective c.p. order zero map, hence a $*$-homomorphism by [14], Proposition 3.2 (b); therefore,

$$B_m \cong M_2.$$

For $k = 1, \dots, m-1$, set

$$E_k := \{f \in \mathcal{C}([0,1], B_{k+1} \otimes M_2) \mid f(0) \in B_{k+1} \otimes \mathbf{1}_{M_2}, f(1) \in \mathbf{1}_{B_{k+1}} \otimes M_2\}. \tag{1}$$

One easily checks that the maps

$$\sigma_k\colon B_k \to E_k$$

induced by

$$\pi_k \dots \pi_1 \varrho_i(x) \mapsto \begin{cases} (t \mapsto (1-t)\cdot \pi_{k+1}\dots\pi_1\varrho_i(x) \otimes \mathbf{1}_{M_2}) \text{ for } i = k+1,\dots,m \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{and } x \in M_2, \\ (t \mapsto t \cdot \mathbf{1}_{B_{k+1}} \otimes x) \text{ for } i = k \text{ and } x \in M_2 \end{cases}$$

are well defined $*$-isomorphisms. Similarly, the map

$$\sigma_0 \colon B \to E_0 := \{ f \in \mathcal{C}([0,1], B_1) \mid f(1) \in \mathbb{C} \cdot \mathbf{1}_{B_1} \}$$

induced by

$$\begin{aligned} \varrho_i(x) &\mapsto (t \mapsto (1-t)\cdot \pi_1\varrho_i(x)) \quad \text{for } i = 1,\dots,m \text{ and } x \in M_2, \\ \mathbf{1}_B &\mapsto \mathbf{1}_{E_0}, \end{aligned}$$

is a well-defined $*$-isomorphism; note that

$$\sigma_0\big(\textstyle\sum_{l=1}^m \varrho_l(\mathbf{1}_{M_2})\big) = (t \mapsto (1-t)\cdot \mathbf{1}_{B_1}).$$

By (1) together with Proposition 2.2 and an easy induction argument, the unital $*$-homomorphism

$$\pi_m \dots \pi_1 \varrho_m \colon M_2 \to B_m$$

pulls back to a unital $*$-homomorphism

$$\tilde{\varrho} \colon M_2 \to B_1.$$

This in turn induces a c.p.c. order zero map

$$\check{\varrho} \colon M_2 \to E_0$$

by

$$\check{\varrho}(x) := (t \mapsto (1-t)\cdot \tilde{\varrho}(x)).$$

Note that this map satisfies

$$\check{\varrho}(\mathbf{1}_{M_2}) = (t \mapsto (1-t)\cdot \mathbf{1}_{B_1}).$$

We now define a c.p.c. order zero map

$$\bar{\varrho} := \sigma_0^{-1} \circ \check{\varrho} \colon M_2 \to B.$$

Note that $\bar{\varrho}(\mathbf{1}_{M_2}) = \sum_{l=1}^m \varrho_l(\mathbf{1}_{M_2})$, whence $\bar{\varrho}$ is as desired.

For the second assertion of the lemma, note that $\bar{\varrho}$ and ϱ_m agree modulo J_m. Therefore, $\bar{\varphi} = \pi\bar{\varrho}$ and $\varphi_m = \pi\varrho_m$ agree up to $\pi(J_m)$. However, one checks that $\pi(J_m) \perp d$, whence $(\bar{\varphi}(x) - \varphi_m(x))d = 0$ for all $x \in M_2$. This implies that $\bar{\varphi}(e_{11})d = \varphi_m(e_{11})d = d$. □

Proposition 2.4. *Let $\mathcal{D}$ be strongly self-absorbing, $0 \neq m \in \mathbb{N}$ and*

$$\varphi_0 \colon M_2 \to \mathcal{D}$$

a c.p.c. order zero map.

Then there are c.p.c. order zero maps

$$\varphi_1, \ldots, \varphi_m \colon M_2 \to \mathcal{D}^{\otimes m}$$

such that

(i) $\varphi_1 = \varphi_0 \otimes \mathbf{1}_{\mathcal{D}^{\otimes(m-1)}}$,

(ii) $[\varphi_i(M_2), \varphi_j(M_2)] = 0$ *if* $i \neq j$,

(iii) $\mathbf{1}_{\mathcal{D}^{\otimes m}} - \sum_{i=1}^{m} \varphi_i(\mathbf{1}_{M_2}) = (\mathbf{1}_{\mathcal{D}} - \varphi_0(\mathbf{1}_{M_2}))^{\otimes m}$.

Proof. For $k \in \{1, \ldots, m\}$, define

$$\varphi_k := (\mathbf{1}_{\mathcal{D}} - \varphi_0(\mathbf{1}_{M_2}))^{\otimes(k-1)} \otimes \varphi_0 \otimes \mathbf{1}_{\mathcal{D}^{\otimes(m-k)}}.$$

Then the φ_k obviously satisfy 2.4 (i) and (ii).

A simple induction argument shows that, for $k = 1, \ldots, m$,

$$\mathbf{1}_{\mathcal{D}^{\otimes m}} - \sum_{i=1}^{k} \varphi_i(\mathbf{1}_{M_2}) = (\mathbf{1}_{\mathcal{D}} - \varphi_0(\mathbf{1}_{M_2}))^{\otimes k} \otimes \mathbf{1}_{\mathcal{D}^{\otimes(m-k)}},$$

which is 2.4 (iii) when we take $k = m$. □

3. $\mathcal{Z}$-stability

We now assemble the techniques of the preceding sections and a result from [10] to prove our main result; we also derive some consequences.

Theorem 3.1. *Any strongly self-absorbing C*-algebra $\mathcal{D}$ absorbs the Jiang–Su algebra $\mathcal{Z}$ tensorially.*

Proof. Let $k \in \mathbb{N}$. By Proposition 2.1, there is a nonzero c.p.c. order zero map $\varphi \colon M_2 \to \mathcal{D}$. Using functional calculus for order zero maps (cf. [17]), we may assume that there is

$$0 \leq d \leq \varphi(e_{11})$$

such that

$$d \neq 0 \quad \text{and} \quad \varphi(e_{11})d = d.$$

Note that

$$\varphi(e_{22})d = 0 \quad \text{and} \quad (\mathbf{1}_{\mathcal{D}} - \varphi(\mathbf{1}_{M_2}))(\mathbf{1}_{\mathcal{D}} - d) = \mathbf{1}_{\mathcal{D}} - \varphi(\mathbf{1}_{M_2}).$$

By Proposition 2.1, there is a nonzero c.p.c. order zero map

$$\psi\colon M_k \to \overline{d\mathcal{D}d};$$

observe that

$$\varphi(e_{11})\psi(x) = \psi(x) \quad \text{for } x \in M_k.$$

Apply Lemma 1.4 (with $\mathcal{D}^{\otimes k}$, $\psi(e_{11})^{\otimes k}$, $(\mathbf{1}_{\mathcal{D}} - \varphi(\mathbf{1}_{M_2})) \otimes \mathbf{1}_{\mathcal{D}^{\otimes(k-1)}}$ and $(\mathbf{1}_{\mathcal{D}} - d) \otimes \mathbf{1}_{\mathcal{D}^{\otimes(k-1)}}$ in place of $\mathcal{D}$, d, f and g, respectively) to obtain $0 \neq m \in \mathbb{N}$ such that

$$[((\mathbf{1}_{\mathcal{D}} - \varphi(\mathbf{1}_{M_2})) \otimes \mathbf{1}_{\mathcal{D}^{\otimes(k-1)}})^{\otimes m}] \leq [\psi(e_{11})^{\otimes k} \otimes \mathbf{1}_{(\mathcal{D}^{\otimes k})^{\otimes(m-1)}}] \tag{2}$$

in $W((\mathcal{D}^{\otimes k})^{\otimes m})$. From Proposition 2.4 (with $\mathcal{D}^{\otimes k}$ in place of $\mathcal{D}$ and $\varphi_0 := \varphi \otimes \mathbf{1}_{\mathcal{D}^{\otimes(k-1)}}$) we obtain c.p.c. order zero maps

$$\varphi_1, \ldots, \varphi_m\colon M_2 \to (\mathcal{D}^{\otimes k})^{\otimes m}$$

satisfying 2.4 (i), (ii) and (iii). By relabeling the φ_i we may assume that actually $\varphi_m = \varphi_0 \otimes \mathbf{1}_{(\mathcal{D}^{\otimes k})^{\otimes(m-1)}}$ in 2.4 (i).

From Lemma 2.3, we obtain a c.p.c. order zero map

$$\bar{\varphi}\colon M_2 \to C^*(\varphi_i(M_2) \mid i = 1, \ldots, m) \subset (\mathcal{D}^{\otimes k})^{\otimes m}$$

such that

$$\bar{\varphi}(\mathbf{1}_{M_2}) = \sum_{i=1}^{m} \varphi_i(\mathbf{1}_{M_2}).$$

By the second assertion of Lemma 2.3 and since

$$\begin{aligned}\varphi_m(e_{11})(\psi(\mathbf{1}_{M_k}) \otimes \mathbf{1}_{\mathcal{D}^{\otimes(km-1)}}) &= (\varphi(e_{11}) \otimes \mathbf{1}_{\mathcal{D}^{\otimes(km-1)}})(\psi(\mathbf{1}_{M_k}) \otimes \mathbf{1}_{\mathcal{D}^{\otimes(km-1)}}) \\ &= \psi(\mathbf{1}_{M_k}) \otimes \mathbf{1}_{\mathcal{D}^{\otimes(km-1)}},\end{aligned}$$

we may furthermore assume that

$$\bar{\varphi}(e_{11})(\psi(\mathbf{1}_{M_k}) \otimes \mathbf{1}_{\mathcal{D}^{\otimes(km-1)}}) = \psi(\mathbf{1}_{M_k}) \otimes \mathbf{1}_{\mathcal{D}^{\otimes(km-1)}},$$

which in turn yields

$$\psi(\mathbf{1}_{M_k}) \otimes \mathbf{1}_{\mathcal{D}^{\otimes(km-1)}} \leq \bar{\varphi}(e_{11}) \tag{3}$$

since ψ is contractive. Note that we have

$$\begin{aligned}[\mathbf{1}_{(\mathcal{D}^{\otimes k})^{\otimes m}} - \bar{\varphi}(\mathbf{1}_{M_2})] &\overset{2.4\,\text{(iii)}}{=} [(\mathbf{1}_{\mathcal{D}^{\otimes k}} - \varphi_0(\mathbf{1}_{M_2}))^{\otimes m}] \\ &= [((\mathbf{1}_{\mathcal{D}} - \varphi(\mathbf{1}_{M_2})) \otimes \mathbf{1}_{\mathcal{D}^{\otimes(k-1)}})^{\otimes m}] \\ &\overset{(2)}{\leq} [\psi(e_{11})^{\otimes k} \otimes \mathbf{1}_{(\mathcal{D}^{\otimes k})^{\otimes(m-1)}}]\end{aligned} \tag{4}$$

in $W((\mathcal{D}^{\otimes k})^{\otimes m})$. Define a c.p.c. order zero map

$$\Phi\colon M_{2^k} \cong (M_2)^{\otimes k} \to ((\mathcal{D}^{\otimes k})^{\otimes m})^{\otimes k} \cong \mathcal{D}^{\otimes kmk}$$

by

$$\Phi := \bar{\varphi}^{\otimes k}.$$

We have

$$\begin{aligned}
[\mathbf{1}_{((\mathcal{D}^{\otimes k})^{\otimes m})^{\otimes k}} - \Phi(\mathbf{1}_{(M_2)^{\otimes k}})] &\overset{1.2}{\leq} k \cdot [(\mathbf{1}_{(\mathcal{D}^{\otimes k})^{\otimes m}} - \bar{\varphi}(\mathbf{1}_{M_2})) \otimes \mathbf{1}_{((\mathcal{D}^{\otimes k})^{\otimes m})^{\otimes (k-1)}}] \\
&\overset{(4)}{\leq} k \cdot [\psi(e_{11})^{\otimes k} \otimes \mathbf{1}_{(\mathcal{D}^{\otimes k})^{\otimes (m-1)}} \\
&\qquad\qquad \otimes \mathbf{1}_{((\mathcal{D}^{\otimes k})^{\otimes m})^{\otimes (k-1)}}] \\
&\leq [\psi(\mathbf{1}_{M_k})^{\otimes k} \otimes \mathbf{1}_{(\mathcal{D}^{\otimes (km-1)})^{\otimes k}}] \\
&\overset{(3)}{\leq} [\bar{\varphi}(e_{11})^{\otimes k}] \\
&= [\Phi(e_{11})]
\end{aligned}$$

in $W(((\mathcal{D}^{\otimes k})^{\otimes m})^{\otimes k})$. From [10], Proposition 5.1, we now see that there is a unital $*$-homomorphism

$$\varrho\colon Z_{2^k,2^k+1} \to \mathcal{D}^{\otimes kmk} \cong \mathcal{D}.$$

Since k was arbitrary, by [13], Proposition 2.2, this implies that $\mathcal{D}$ is $\mathcal{Z}$-stable. □

Corollary 3.2. *The Jiang–Su algebra is the uniquely determined* (*up to isomorphism*) *initial object in the category of strongly self-absorbing C*-algebras (with unital $*$-homomorphisms).*

Proof. By Theorem 3.1, the Jiang–Su algebra does embed unitally into any strongly self-absorbing C*-algebra, so it is an initial object. If $\mathcal{D}$ is another initial object, then $\mathcal{Z}$ and $\mathcal{D}$ embed unitally into one another, whence they are isomorphic by [12], Proposition 5.12.

Sometimes an object in a category is called initial only if there is a *unique* morphism to any other object; this remains true in our setting if one takes approximate unitary equivalence classes of unital $*$-homomorphisms as morphisms. □

Remark 3.3. By [9], $\mathcal{Z}$-stable C*-algebras are K_1-injective, whence K_1-injectivity is unnecessary in the hypotheses of the main results of [12], [3], [2], [5], [6] and [7].

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