

Square-free words as products of commutators

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Abstract. Elements of the commutator subgroup of a free group \mathbb{F} can be presented as values of canonical forms, called Wicks forms. We show that, starting from sufficiently high genus g , there is a sequence of words w_g which can be presented by $f(g)$ distinct Wicks forms, where $f(g) > g!$. Moreover we may choose these words w_g to be square-free.

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Introduction

Let \mathbb{F} be a free group and let $[\mathbb{F}, \mathbb{F}]$ be its commutator subgroup. We define the *genus* of a word $w \in [\mathbb{F}, \mathbb{F}]$ to be the least positive integer g such that w is a product of g commutators in \mathbb{F} . Every element of genus g in $[\mathbb{F}, \mathbb{F}]$ can be presented by non-cancelling substitution in a Wicks form of genus g (cf. [4], [3]) as described below. In genus one there is only one Wicks form but for $g \geq 2$ there are finitely many Wicks forms and their number grows quite fast (factorially) with g . Thus a natural question (posed by E. Rips at the geometric group theory conference in Anogia in 1996) is whether it is possible to find a word of genus g which can be presented by non-cancelling substitution in “many” Wicks forms. We show that there is a sequence of words w_2, w_3, \dots such that w_g has genus g and the number of Wicks forms from which it can be obtained by non-cancelling substitution is bounded below by $g!$, when g is sufficiently large.

Wicks forms are not affected by Dehn twists. Since, by a classical result of M. Dehn ([5]), the modular group is generated by Dehn twists, Wicks forms are invariant under the action of $\text{Mod}(S_g)$. In [2] Bestvina and Feighn consider genus g representations (defined below) of words of genus g . They describe an equivalence relation on representations in terms of partial Dehn twists, which are a generalisation of Dehn twists. They show that for certain types of word w_g the number of distinct equivalence classes of representations of w_g grows exponentially with g . It would be interesting to know what happens to our Wicks form representations under fractional Dehn twists.

1. Wicks forms

An *alphabet* consists of a countably enumerable set S equipped with a fixed-point free involution τ and a fixed set of representatives S^+ of the orbits of τ . We shall consider words in a fixed alphabet $A = \{a_1^{\pm 1}, a_2^{\pm 1}, \dots\}$ of letters a_1, a_2, \dots and their inverses $a_1^{-1}, a_2^{-1}, \dots$, where $\tau(a_i) = a_i^{-1}$ and $\tau(a_i^{-1}) = a_i$ and $A^+ = \{a_1, a_2, \dots\}$. Alphabets B, A' are defined analogously, replacing a_i by b_i or a'_i , in the obvious way. A *word (over A)* is an element of the free monoid A^* on A . A *cyclic word (over A)* is the set of cyclic permutations $[w]$ of a word w . Words w and w' are said to *define the same cyclic word* if $[w] = [w']$. A word u is said to be a *factor* of the cyclic word $[w]$ if u is a subword of some element of $[w]$. A word is said to be *reduced* if it has no factor of the form aa^{-1} or $a^{-1}a$, where $a \in A^+$. A word is said to be *cyclically reduced* if every element of $[w]$ is reduced. An element a^ϵ , where $a \in A^+$ and $\epsilon = \pm 1$, is said to *occur n -times* in the word w if $w = u_1 a^\epsilon u_2 \dots u_n a^\epsilon u_{n+1}$, for some elements u_1, \dots, u_{n+1} of $(A \setminus \{a^\epsilon\})^*$: we count the number of occurrences of a^{+1} and a^{-1} separately.

Definition 1. An *orientable Wicks form* is a cyclic word $[w]$ over A such that

- (i) if $a^\epsilon \in A$ occurs in w (for $a \in A^+$ and $\epsilon \in \{\pm 1\}$) then $a^{-\epsilon}$ occurs exactly once in w ;
- (ii) w is cyclically reduced, and
- (iii) if $a_i^\epsilon a_j^\delta$ is a factor of $[w]$ then $a_j^{-\delta} a_i^{-\epsilon}$ is not a factor of $[w]$.

We shall abuse notation by referring to a Wicks form $[w]$ as w when convenient. An orientable Wicks form $[w] = [w_1 w_2 \dots]$, over A , is *isomorphic* to $[w'] = [w'_1 w'_2 \dots]$, over the alphabet A' , if there exists a bijection $\varphi: A \rightarrow A'$ with $\varphi(a^{-1}) = \varphi(a)^{-1}$ such that $[\varphi(w)] = [w']$ (where $\varphi(w) = \varphi(w_1)\varphi(w_2) \dots$). The relation “is isomorphic to” is an equivalence relation on the orientable Wicks forms over A .

If $[w]$ is an orientable Wicks form then w , when considered as an element of the free group \mathbb{F} generated by a_1, a_2, \dots , is an element of the commutator subgroup. We define the *algebraic genus* $g_a(w)$ of $[w]$ to be the least positive integer g_a such that w is a product of g_a commutators in \mathbb{F} .

The *topological genus* $g_t(w)$ of an orientable Wicks form $w = w_1 \dots w_{2e-1} w_{2e}$ is defined to be the topological genus of the orientable compact connected surface obtained by labelling and orienting the edges of a $2e$ -gon (which we consider as a subset of the oriented plane) according to w and then identifying edges with the same labels (respecting orientation).

Proposition (cf. [4], [3]). *The algebraic genus and the topological genus of an orientable Wicks form coincide.*

We define the *genus* $g(w)$ of an orientable Wicks form $[w]$ to be $g(w) = g_a(w) = g_t(w)$.

Consider the orientable compact surface S associated to an orientable Wicks form $w = w_1 \dots w_{2e}$. This surface carries an embedded graph $\Gamma \subset S$ such that $S \setminus \Gamma$ is an open polygon with $2e$ sides (and hence connected and simply connected). Moreover, conditions (ii) and (iii) on the Wicks form imply that Γ contains no vertices of degree 1 or 2 (or equivalently that the dual graph of $\Gamma \subset S$ contains no faces which are 1-gons or 2-gons). This construction also works in the opposite direction: given a graph $\Gamma \subset S$ with e edges and no vertices of degree 1 or 2 on an orientable compact connected surface S of genus g such that $S \setminus \Gamma$ is connected and simply connected, we get an orientable Wicks form of genus g and length $2e$ by labelling and orienting the edges of Γ and by cutting S open along the graph Γ . The associated orientable Wicks form is defined as the word which appears in this way on the boundary of the resulting polygon with $2e$ sides. Henceforth we identify orientable Wicks forms with their associated embedded graphs $\Gamma \subset S$, speaking of vertices and edges of orientable Wicks forms.

The formula for the Euler characteristic

$$\chi(S) = 2 - 2g = v - e + 1$$

(where v denotes the number of vertices and e the number of edges in $\Gamma \subset S$) shows that an orientable Wicks form of genus g has length at least $4g$ (the associated graph has then a unique vertex of degree $4g$ and $2g$ edges) and length at most $6(2g - 1)$ (the associated graph has then $2(2g - 1)$ vertices of degree three and $3(2g - 1)$ edges).

We call an orientable Wicks form of genus g *maximal* if it has length $6(2g - 1)$. It is convenient to interpret a genus g Wicks form as an *oriented circuit* in the graph Γ ; that is a circuit which traverses every edge of the graph exactly twice, once in each direction, but contains no instance of an edge followed by its reverse (for more details concerning oriented circuits see [9]). The dual graph of a Wicks form is an ideal triangulation, as defined by L. Mosher [6]. It is shown in [6] that ideal triangulations are invariant under Dehn twists.

Let g be a positive integer, $A = \{a_1^{\pm 1}, \dots, a_{6g-3}^{\pm 1}\}$ and $B = \{b_1^{\pm 1}, \dots, b_n^{\pm 1}\}$ be alphabets and let A^* and B^* be the free monoids on A and B , respectively. Define $(a_i^{-1})^{-1} = a_i$ and for a word $w = x_1 \dots x_n \in A^*$ with $x_i \in A$ define $w^{-1} = x_n^{-1} \dots x_1^{-1}$. Let Φ be a map from A to B^* , such that $\Phi(a_i)$ is freely reduced and $\Phi(a_i^{-1}) = \Phi(a_i)^{-1}$, for all i . Then Φ induces a map, also denoted Φ , from A^* to B^* . Let $u \in A^*$ and w be the word obtained from $\Phi(u)$ by free reduction. Then we say that (Φ, u) is a *representation* of w . If $\Phi(u)$ is cyclically reduced then (Φ, u) is said to be a *non-cancelling representation* of w .

Given a Wicks form $[u]$ of genus g we say that (Φ, u) is a *genus g Wicks representation* of $w \in B^*$ if there is $u' \in [u]$ such that (Φ, u') is a representation of w . If u and v are isomorphic genus g Wicks forms then there exists a genus g Wicks representation (Φ, u) of w if and only if there exists a genus g Wicks representation (Ψ, v) of w . If there exists a genus g Wicks representation of w then there also exists a non-cancelling genus g representation of w (see [3]).

Definition 2. For $w \in B^*$ we define $M(g, w)$ to be the number of isomorphism classes I of maximal Wicks forms over A , of genus g , such that there exists a non-cancelling Wicks representation (Φ, u) of w , for some $u \in I$.

We shall say a word $w \in B^*$ has *genus* g if w has a genus g Wicks representation but no Wicks representation of genus less than g . If w does not have a Wicks representation, of any genus, we define its genus to be ∞ . A cyclically reduced word w has genus $g < \infty$ if and only if it represents an element of the commutator subgroup of the free group on $\{b_1, \dots, b_n\}$ (see [3]).

2. Words with many representations

In the sequel, if $u \in A^*$ and $w \in B^*$, for alphabets A and B , and there exists a non-cancelling representation (Φ, u) of w then we shall say that w is *obtained from* u by *non-cancelling substitution*.

Theorem. *There is a sequence w_2, w_3, w_4, \dots of words over an alphabet B of size 24 such that*

- (i) w_g has genus g and
- (ii) $M(g, w_g) > g!$, for $g > 10^{10}$.

Proof. Consider a maximal Wicks form w of genus g and length $12g - 6$. We colour vertices of the graph Γ of w so that no edge is incident to two vertices of the same colour. A straightforward induction on the number of vertices shows that, for a simple graph whose vertices have degree at most d , vertices may be coloured in this way using at most $d + 1$ colours. In our case Γ has no vertex of degree more than 3 (in fact is regular of degree 3) so four colours 1, 2, 3, 4 are sufficient.

We construct a new labelled oriented graph Γ' on the surface S associated to w , as follows. The underlying graph of Γ' is the barycentric subdivision of Γ . Thus Γ' has two types of vertex; the original vertices of Γ which are of degree 3, and new vertices of degree 2 corresponding to edges of G . All edges of G' join a vertex of degree 3 to a vertex of degree 2 and we orient each edge from the degree 3 vertex to the degree 2 vertex. The vertices of Γ' of degree 3 inherit a colouring from the corresponding vertices of Γ . Now let $B = \{\alpha_j^{\pm 1}, \beta_j^{\pm 1}, \gamma_j^{\pm 1} \mid j = 1, 2, 3, 4\}$ be an alphabet disjoint from A . Let u be a vertex of Γ' of degree 3 and colour j . Choose one of the 3 edges emanating from u and call it e_1 . Choose one of the other edges and call it e_2 and call the third e_3 . Label the oriented edges e_1, e_2 and e_3 coming out of u with α_j, β_j by γ_j , respectively, as shown in Figure 1. Repeat this for all degree 3 vertices of Γ' . Now reading the oriented circuit C with this labelling we obtain a word $v \in B^*$. Moreover, the condition on the colouring of vertices of Γ means that v is obtained from w by non-cancelling substitution.

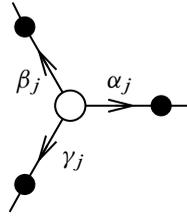


Figure 1. Labelling of edges of Γ' .

To simplify the estimation of the lower bound below we now make a further adjustment to the current labelling of Γ' . Let u be a vertex of degree 3 with outgoing edges labelled $\alpha_j, \beta_j, \gamma_j \in B$. Then either $\alpha_j^{-1}\beta_j$ or $\beta_j^{-1}\alpha_j$ is a factor of w'' . Moreover, if $\alpha_j^{-1}\beta_j$ is a factor of w'' then so are $\beta_j^{-1}\gamma_j$ and $\gamma_j^{-1}\alpha_j$, whereas if $\beta_j^{-1}\alpha_j$ is a factor then so are $\alpha_j^{-1}\gamma_j$ and $\gamma_j^{-1}\beta_j$. In the latter case we alter the labelling on the edges incident to u by interchanging labels γ_j and β_j : as shown in Figure 2, where the curved arcs indicate the direction in which the boundary of $S \setminus \Gamma'$ is read. The

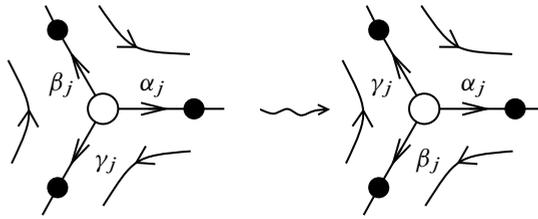


Figure 2. Adjustment of labels on Γ' .

result is that $\alpha_j^{-1}\beta_j, \beta_j^{-1}\gamma_j$ and $\gamma_j^{-1}\alpha_j$ are subwords of the (altered) word w'' . We perform this alteration, if necessary, on every vertex, of degree 3, of Γ' . By reading the circuit C with these altered labels we obtain a new word v ; also obtained from w by non-cancelling substitution. Furthermore this word $v \in B^*$ satisfies the following conditions.

- (i) Every letter of v from B^+ of colour j is followed by a letter of colour $k \neq j$ from $B \setminus B^+$.
- (ii) Every letter of v from $B \setminus B^+$ is followed by a letter of the same colour from B^+ .
- (iii) None of $\beta_j^{-1}\alpha_j, \alpha_j^{-1}\gamma_j$ or $\gamma_j^{-1}\beta_j$ occur as subwords of v .
- (iv) The length of v is $24g - 12$.

An example of this construction is given after the proof.

Let us show that v cannot be obtained by a non-cancelling substitution from a Wicks form of genus k , where $k < g$. Suppose that (Φ, u) is a genus k non-

cancelling representation of v . Since a Wicks form of genus k has length at most $12k - 6 \leq 12(g - 1) - 6$ and v has length $24g - 12$ there must be some letter a occurring in u such that $\Phi(a) > 2$. Let a be such a letter. Then $\Phi(a)$ contains a subword xyz , where $x, y, z \in B$. As a^{-1} also occurs in u both xyz and $z^{-1}y^{-1}x^{-1}$ must be subwords of v . However, from (i) and (ii) above, positive and negative powers of letters of v alternate. Moreover, if j is fixed and $x, y \in \{\alpha_j, \beta_j, \gamma_j\}$ then v may have subword $x^{-1}y$ but does not have subword xy^{-1} . This means that a subword of v of length three always has form $x_i^{-1}y_i z_j^{-1}$ or $x_i y_j^{-1} z_j$, where $x_i^{\pm 1}, y_i^{\pm 1}, z_j^{\pm 1}$ and $z_j^{\pm 1}$ are elements of B . Therefore v cannot contain both subwords xyz and $z^{-1}y^{-1}x^{-1}$, and the result follows. Hence the genus of v is g .

Now we wish to count the number of words v that may arise in this way. First note that we may read the word v starting from any vertex of Γ' and so we may choose v so that it begins with α_j , for some $j \in \{1, 2, 3, 4\}$. Let $V(g)$ denote the set of all freely reduced words of B^* which satisfy (i) to (iv) above and which begin with α_j , for some j ; so $v \in V(g)$. If u belongs to $V(g)$ and $x \in \{\alpha_s, \beta_s, \gamma_s\}$ occurs in u then the letter following x belongs to $\bigcup_{i \neq s} \{\alpha_i^{-1}, \beta_i^{-1}, \gamma_i^{-1}\}$; so there are nine possibilities for this letter. If y is a letter of $B \setminus B^+$ occurring in u then the letter following y is completely determined by conditions (ii) and (iii) above. Given that there are 4 choices for the first letter α_j of u and that u has length $24g - 12$ this means that $|V(g)| = 4 \cdot 9^{12g-7}$.

Let $M(g) = \max\{M(g, v) \mid v \text{ is a reduced word of genus } g \text{ and length } 24g - 12\}$ and let $W(g)$ be the number of isomorphism classes of maximal oriented Wicks forms of genus g . Then $|V(g)|M(g) \geq W(g)$. However it follows from [1] that $W(g)$ grows faster than $g!$; in fact setting

$$m(g) = \left(\frac{1}{12}\right)^g \frac{(6g - 4)!}{g!(3g - 2)}$$

we have $W(g) \geq m(g)$. (In [1] $m(g) = m_1^g$.) Using upper and lower bounds for $n!$, from [8], straightforward calculations give

$$M(g) \geq W(g)/M(g) \geq m(g)/(4 \cdot 9^{12g-7}) > g!$$

for $g > 10^{10}$, and the statement of the theorem follows. □

Example. The quadratic word

$$w = a_1 a_2 a_3 a_4 a_5 a_1^{-1} a_6 a_2^{-1} a_5^{-1} a_7 a_8 a_3^{-1} a_6^{-1} a_9 a_7^{-1} a_4^{-1} a_8^{-1} a_9^{-1}$$

gives rise to the labelled graph of Figure 3, embedded on an orientable surface of genus 2; so this word is a maximal Wicks form of genus 2. (The small curved arrows indicate the direction to be taken at each vertex in reading the circuit C .) A colouring of the graph with colours 1, 2 and 3 is also shown in the figure.

The barycentric subdivision of this graph with its new orientation and a choice of new labelling is shown in Figure 4.

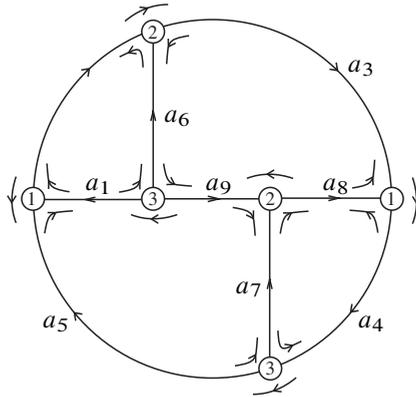


Figure 3. A maximal Wicks form.

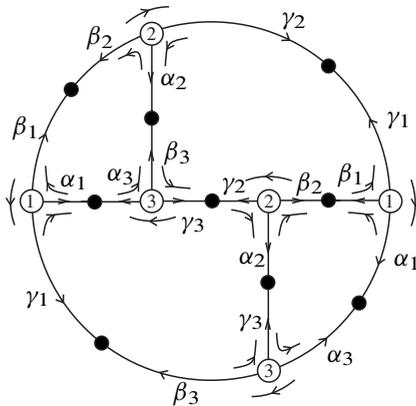


Figure 4. Barycentric subdivision and new labelling.

Starting by reading the edge labelled α_1 coming out of the rightmost vertex we obtain the word

$$v = \alpha_1 \alpha_3^{-1} \beta_3 \gamma_1^{-1} \alpha_1 \alpha_3^{-1} \beta_3 \alpha_2^{-1} \beta_2 \beta_1^{-1} \gamma_1 \beta_3^{-1} \gamma_3 \alpha_2^{-1} \beta_2 \beta_1^{-1} \gamma_1 \gamma_2^{-1} \\ \alpha_2 \beta_3^{-1} \gamma_3 \gamma_2^{-1} \alpha_2 \gamma_3^{-1} \alpha_3 \alpha_1^{-1} \beta_1 \beta_2^{-1} \gamma_2 \gamma_3^{-1} \alpha_3 \alpha_1^{-1} \beta_1 \beta_2^{-1} \gamma_2 \gamma_1^{-1}.$$

No adjustment of the labels is required as v contains none of the forbidden subwords.

As mentioned in the introduction the objects counted in [2] are equivalence classes of representations of particular words w_g ; whereas here we count (isomorphism classes of) representations themselves. It seems plausible that by choosing the words w_g carefully the size of equivalence classes could be made small, but we have no

proof of any such claim. We can however offer some evidence that it is possible, by making a moderate increase in the size of the alphabet, to choose the words w_g from a much more limited set than $V(g)$ above. A word in B^* is said to be *square-free* if it contains no subword of the form uu , where u is a non-trivial element of B^* . Such square-free words exist (see for example [7], Chapter 4).

Corollary. *In the theorem above, replacing B with an alphabet of size 80, the words w_g may be chosen to be square-free and the conclusion of the theorem then holds for $g > 10^{17}$.*

Proof. Let w be a maximal Wicks form of genus g and length $12g - 6$ and, as in the proof of the theorem, let v be the label of the oriented circuit C on the surface corresponding to w with the labelling constructed in the proof. Now for each letter $x_j \in \{\alpha_j, \beta_j, \gamma_j\} \subseteq B$, where $j = 1, 2, 3$ or 4 , define a set $E^+(x_j) = \{x_{j,1}, x_{j,2}, x_{j,3}\}$ and define $E(x_j) = E^+(x_j) \cup \{x^{-1} | x \in E^+(x_j)\}$ and $E = \bigcup_{j=1}^4 (E(\alpha_j) \cup E(\beta_j) \cup E(\gamma_j))$. Further let $E_S = \{x_{1,0}^{\pm 1}, x_{2,0}^{\pm 1}, x_{3,0}^{\pm 1}, x_{4,0}^{\pm 1}\}$ and let $\hat{E} = E \cup E_S$.

Fix some element $x_i^{+1} \in B$ which occurs in v . If x_i occurs s times in v , then v can be written as $v = u_1 x_i u_2 x_i u_3 \dots x_i u_s x_i u_{s+1}$ where the words $u_1, u_2, \dots, u_s, u_{s+1}$ do not contain x_i (they contain x_i^{-1} , of course, but we are interested in positive powers of x_i only). Now v is transformed as follows. Choose a square-free word $t_1 \dots t_{s-1}$ of length $s - 1$ in $E(x_i)^*$. Replace the first occurrence of x_i with $x_{i,0}$ and replace the j th occurrence of x_i with t_{j-1} , for $j = 2, \dots, s$. Reading the oriented circuit C each occurrence of x_i is paired to an occurrence of x_i^{-1} in v ; as each edge is read in both directions. The occurrence x_i^{-1} paired to the j th occurrence of x_i is now replaced by $x_{i,0}^{-1}$ if $j = 1$, and by t_{j-1}^{-1} otherwise. (The special letter $x_{i,0}$ is introduced to ensure that all cyclic permutations of the resulting word are square-free, as we are dealing with cyclic words.) Repeat this transformation with respect to every letter of positive exponent occurring in v and denote the resulting word by z . Then $z \in \hat{E}^*$ is obtained by non-cancelling substitution from w .

It is easy to see that z is square-free as follows. Suppose not: then z has a subword uu . Let $u = r_1 r_2 \dots r_p$, where $r_i \in E$. Set $j_1 = 1$ if r_1 is an element of E of exponent $+1$ and set $j_1 = 2$ otherwise. Since z consists of alternating positive and negative powers then r_{j_1} will be a positive power. Then $r_{j_1} = e_i \in E(x_i)$, for some i . Suppose that there are p occurrences of e_i in u , namely r_{j_1}, \dots, r_{j_p} , with $j_1 < \dots < j_p$. Since uu is a subword of z the word $r_{j_1} \dots r_{j_p} r_{j_1} \dots r_{j_p}$ is a subword of the word $t = t_1 \dots t_{s-1}$ used to construct z , above. This contradicts the choice of t as a square-free word. Hence z is square-free as claimed.

The argument of the proof of the theorem shows that z has genus g . As v begins with α_j the first letter of z is $x_{j,0}$ and so there are 4 possibilities for this letter. Each other letter of z of positive exponent is obtained by substitution of one of the three elements of $E(x_j)$ for x_j , for some $x_j \in B^+$; or is the first letter of its kind in z in

which case it is a uniquely determined element of E_S . It follows that there are at most $4 \cdot 27^{12g-7}$ possibilities for z . The result now follows. \square

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