

C*-algebras associated with integral domains and crossed products by actions on adèle spaces

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Abstract. We compute the K-theory for C*-algebras naturally associated with rings of integers in number fields.

The main ingredient is a duality theorem for arbitrary global fields. It allows us to identify the crossed product arising from affine transformations on the finite adèles with the analogous crossed product algebra over the infinite adèle space.

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1. Introduction

Let R be a countable ring. The elements of R act by addition and multiplication on $\ell^2(R)$. Denote by $\mathfrak{A}[R]$ the C*-algebra generated by all the corresponding operators in $\mathcal{L}(\ell^2(R))$. In [Cun] the first named author had studied (using a different notation) this ring C*-algebra for $R = \mathbb{Z}$ and had shown that it has an intriguing structure. In particular, it is purely infinite simple (thus a Kirchberg algebra) and can be described as a universal C*-algebra given by generators and relations. It is also Morita equivalent to a crossed product of the algebra of functions on the finite adèle space for \mathbb{Q} by the action of the $ax + b$ -group over \mathbb{Q} . These results were generalized in [CuLi] to the case where R is an integral domain with finite quotients and in [Li] to general rings.

It is an obvious problem to determine the K-theory of $\mathfrak{A}[R]$. In [Cun] the case of $R = \mathbb{Z}$ was discussed and it was stated that $K_*(\mathfrak{A}[\mathbb{Z}])$ is given as an exterior \mathbb{Z} -algebra with one generator for each prime number in \mathbb{Z} . A proof for this was sketched. This proof however was not complete. Moreover, in [Cun] a duality theorem was stated claiming that $\mathfrak{A}[\mathbb{Z}]$ can be described also as a crossed product of $C_0(\mathbb{R})$ by the natural action of the $ax + b$ -group over \mathbb{Q} . Again a proof was sketched which was not complete.

In the present paper we give complete proofs for these two results generalizing them at the same time to a substantially more general setting. It turns out that the

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two results are related. We first prove a general duality theorem that holds for any global field K and states that the crossed products $C_0(\mathbb{A}_f) \rtimes P_K$ and $C_0(\mathbb{A}_\infty) \rtimes P_K$ are isomorphic. Here \mathbb{A}_f and \mathbb{A}_∞ denote the spaces of finite and infinite adeles, respectively, and P_K denotes the $ax + b$ -group over K . Both crossed products describe $\mathfrak{A}[\mathfrak{o}]$ up to Morita equivalence, where \mathfrak{o} is the ring of integers in K . We note at this point that we also determine the crossed product $C_0(\mathbb{A}) \rtimes P_K$ for the full adèle space $\mathbb{A} = \mathbb{A}_f \times \mathbb{A}_\infty$ and show that it is Morita equivalent to the group C*-algebra $C^*(P_K)$. Moreover, we point out that in the case of number fields, the Bost–Connes system and its generalizations considered in [BoCo], [CMR], [HaPa] and [LLN] are carried by a natural subalgebra of $\mathfrak{A}[\mathfrak{o}]$. This is explained in [CuLi] in more detail.

In a second step, we use our duality theorem to determine the K-theory for $\mathfrak{A}[\mathfrak{o}]$ in the case where \mathfrak{o} is the ring of integers in a number field K which contains only $1, -1$ as roots of unity. The point is that the description of $\mathfrak{A}[\mathfrak{o}]$ as $C_0(\mathbb{A}_\infty) \rtimes P_K$ is much better suited for this computation since it allows for certain homotopy arguments which do not apply in the totally disconnected space \mathbb{A}_f . We find that the K-theory depends on the number of real embeddings of K : Roughly speaking, we get the exterior \mathbb{Z} -algebra over the torsion-free part of (K^\times, \cdot) . But if the number of real embeddings of K is even and at least 2, we will get an additional copy of this exterior algebra with coefficients in $\mathbb{Z}/2\mathbb{Z}$. These results indicate that the K-theory of $\mathfrak{A}[\mathfrak{o}]$ as such does not contain information on the class number of K . Therefore, one is forced to investigate finer structures in $\mathfrak{A}[\mathfrak{o}]$ to find out more about the class number (compare [Li], 6.5).

For an arbitrary number field K , we also determine the K-theory of the subalgebra of $\mathfrak{A}[\mathfrak{o}]$ which is generated by the addition operators for elements in \mathfrak{o} and the multiplication operators coming from the torsion-free part of K^\times . We think of this subalgebra as $\mathfrak{A}[\mathfrak{o}]$ “without roots of unity”. We find that its K-theory can be described as the exterior \mathbb{Z} -algebra over the torsion-free part of K^\times , with coefficients in \mathbb{Z} or $\mathbb{Z}/2\mathbb{Z}$ depending on the real embeddings of K .

The paper is structured as follows:

In Section 2, we give an overview of certain aspects of algebraic number theory which we will need. We also briefly recall the notion of ring C*-algebras.

Then we compute the K-theory of $\mathfrak{A}[\mathbb{Z}]$. There are several reasons why we choose to treat this special case first. On the one hand, it was this case from which all our investigations started. It serves as a guide through our computations in the general setting and thereby helps to understand the general arguments. On the other hand, at various points we can considerably shorten the calculations using special features of the case $R = \mathbb{Z}$. Moreover, it is possible to work out several steps explicitly in the concrete situation of \mathbb{Z} . This leads to a self-contained exposition, in the sense that we do not need to refer to results from algebraic number theory in this special case.

Finally, we consider the general situation. As a first step, we establish a duality theorem for arbitrary global fields (Section 4). Using this duality, we carry out the K-theoretic computations, first for the subalgebras “without roots of unity” of arbitrary number fields (Section 5) and then for the whole ring C*-algebras, but under the

assumption that the number fields only contain the roots of unity $1, -1$ (Section 6).

We are indebted to W. Lück for useful discussions and for bringing Lemma 6.1 to our attention.

2. Preliminaries

2.1. Algebraic number theory. Let us very briefly mention aspects from algebraic number theory which will be of interest for us. First of all, the classical objects of study in algebraic number theory are the so called number fields, which are finite (separable) extensions of \mathbb{Q} , and the corresponding rings of integers, which are the integral closures of \mathbb{Z} in those fields. Moreover, it turns out that the theory of the so called function fields, which are finite separable extensions of $\mathbb{F}_p(T)$, can be – at least to some extent – developed parallelly. For this reason, our constructions and some of the results (our duality theorem) will apply to both situations.

However, the final step in our K-theoretic computation is only carried out for number fields. For this, it is useful to note the following:

Lemma 2.1. *Let K be a number field and μ be the set of roots of unity in K . There exists an infinitely generated free abelian group $\Gamma \subseteq K^\times$ with $K^\times = \mu \times \Gamma$.*

Proof. Consider the exact sequence after Corollary (3.9) in [Neu], Chapter I, §4. \square

Now let K be a global field, which means a number field or a function field, and let \mathfrak{o} be the integral closure of \mathbb{Z} or $\mathbb{F}_p[T]$ in K . We will be concerned with the following objects:

$$\begin{aligned} \text{the infinite adele ring } \mathbb{A}_\infty &= \prod_{v|\infty} K_v, \\ \text{the finite adele ring } \mathbb{A}_f &= \prod'_{v \nmid \infty} K_v, \end{aligned}$$

where the restricted product is taken with respect to the maximal compact subrings $\mathfrak{o}_v \subseteq K_v$, and

$$\text{the full adele ring } \mathbb{A} = \mathbb{A}_\infty \times \mathbb{A}_f.$$

These products are taken over equivalence classes of valuations of K ; infinite ones for \mathbb{A}_∞ , finite ones for \mathbb{A}_f . At this point, we should note that for function fields, we choose the valuations satisfying $|T|_v > 1$ to be the infinite ones (compare [Weil], Chapter III, §1, Theorem 2). Instead of equivalence classes of valuations, one can equivalently take equivalence classes of embeddings of K into locally compact, nondiscrete fields, these are called places in [Weil]. Thus, we can always embed K diagonally into \mathbb{A}_∞ , \mathbb{A}_f or \mathbb{A} as each valuation v (we choose one representative for each class) gives rise to an embedding $K \hookrightarrow K_v$. We will not distinguish between K and its image under these embeddings. This will be our convention in general as it will become clear from the context into which object we embed.

Remark 2.2. Adeles and their multiplicative analogues, the ideles, play an important role in class field theory. The reader is referred to [Neu] or [Weil] for more information.

Starting with \mathfrak{o} , we can form the profinite completion $\varprojlim\{\mathfrak{o}/I\}$ over the set of nontrivial ideals in \mathfrak{o} ordered by inclusion. It turns out that this completion coincides with the maximal compact subring $\prod_{v \nmid \infty} \mathfrak{o}_v$ of \mathbb{A}_f :

$$\varprojlim_{(0) \neq I} \{\mathfrak{o}/I\} \cong \prod \mathfrak{o}_v. \quad (1)$$

Moreover, we have K as well as $\prod \mathfrak{o}_v$ sitting inside \mathbb{A}_f . For their intersection, we get the following

Lemma 2.3. $\mathfrak{o} = K \cap (\prod_{v \nmid \infty} \mathfrak{o}_v)$.

Proof. Compare [Weil], Theorem 1 of Chapter V, §2, for number fields. The proof for function fields is analogous using [Weil], Chapter VI. \square

For an infinite place v of a number field, we either have $K_v \cong \mathbb{R}$ or $K_v \cong \mathbb{C}$. In the first case v is called real, and it is called a complex place otherwise. We will write $v_{\mathbb{R}}$ for a real place and $v_{\mathbb{C}}$ for a complex one. Thus, we get

$$\mathbb{A}_{\infty} \cong \mathbb{R}^{\#\{v_{\mathbb{R}}\}} \times \mathbb{C}^{\#\{v_{\mathbb{C}}\}}$$

as topological rings. Note that we consider equivalence classes of embeddings, which means that two complex embeddings which are conjugate give rise to the same place. As additive topological groups, we have $\mathbb{A}_{\infty} \cong \mathbb{R}^n$ where n is the degree of K over \mathbb{Q} .

The last point we would like to talk about is duality. Let K be a global field.

Theorem 2.4. *There exists a nontrivial character χ of \mathbb{A} which is trivial on K . Any such character yields a pairing*

$$\mathbb{A} \times \mathbb{A} \rightarrow \mathbb{T}, \quad (x, y) \mapsto \langle x, y \rangle := \chi(xy).$$

This pairing induces an isomorphism of topological groups: $\mathbb{A} \cong \widehat{\mathbb{A}}$, $x \mapsto [y \mapsto \langle y, x \rangle]$. Thus, we also obtain

$$\mathbb{A}/K \cong \widehat{K} \quad \text{via} \quad \pi(x) \mapsto [a \mapsto \langle a, x \rangle], \quad (2)$$

where π is the projection $\mathbb{A} \rightarrow \mathbb{A}/K$. Moreover, we can choose χ so that $\langle \cdot, \cdot \rangle$ restricted to \mathbb{A}_{∞} yields an isomorphism

$$\mathbb{A}_{\infty} \cong \widehat{\mathbb{A}_{\infty}}, \quad t \mapsto [s \mapsto \langle t, s \rangle]. \quad (3)$$

Proof. For the first two pairings, compare [Weil], Chapter IV, §2, Theorem 3, or [Lang], Chapter XIV, §6, Theorem 10. The second result, (3), is proven in [Lang], Chapter XIV, §1, Theorem 1, for number fields, and follows from [Weil], Chapter II, §5, Theorem 3, in the general case. For the case of number fields, the reader may also consult [Tate]. \square

2.2. Review of our constructions. Let us recall the concept of ring C*-algebras. We will only consider the integral closures of \mathbb{Z} or $\mathbb{F}_p[T]$ in a global field. This is a nice situation as far as the construction of ring C*-algebras is concerned because these rings are integral domains with finite quotients. We mention that it is possible to extend the construction to arbitrary rings (see [Li]).

Now let \mathfrak{o} be the integral closure of \mathbb{Z} or $\mathbb{F}_p[T]$ in a global field K . Consider the following operators on the Hilbert space $\ell^2(\mathfrak{o})$:

$$U^a \xi_r = \xi_{a+r} \quad \text{for } a \in \mathfrak{o}, \quad S_b \xi_r = \xi_{br} \quad \text{for } b \in \mathfrak{o}^\times.$$

Here $\{\xi_r\}$ is the canonical orthonormal basis of $\ell^2(\mathfrak{o})$, and \mathfrak{o}^\times is the set of nontrivial elements in \mathfrak{o} .

In analogy to the group case, we define the reduced ring C*-algebra as

$$\mathfrak{A}_r[\mathfrak{o}] := C^*(\{U^a, S_b \mid a \in \mathfrak{o}, b \in \mathfrak{o}^\times\}) \subseteq \mathcal{L}(\ell^2(\mathfrak{o})).$$

The full ring C*-algebra $\mathfrak{A}[\mathfrak{o}]$ is defined as the universal C*-algebra generated by unitaries $\{u^a \mid a \in \mathfrak{o}\}$ and isometries $\{s_b \mid b \in \mathfrak{o}^\times\}$ satisfying

$$(I) \quad u^a s_b u^c s_d = u^{a+bc} s_{bd},$$

$$(II) \quad \sum u^a s_b s_b^* u^{-a} = 1,$$

where we sum over $\mathfrak{o}/(b) = \{a + (b) \mid a \in \mathfrak{o}\}$ in (II).

We use the notation $(b) := b \cdot \mathfrak{o}$ for principal ideals.

There is a canonical homomorphism $\pi : \mathfrak{A}[\mathfrak{o}] \rightarrow \mathfrak{A}_r[\mathfrak{o}]$ which is called the regular representation (as for groups). Moreover, it turns out that $\mathfrak{A}[\mathfrak{o}]$ is purely infinite and simple (see [CuLi], Theorem 1), so that π is an isomorphism. This allows us to identify $\mathfrak{A}[\mathfrak{o}]$ with its image under π on $\ell^2(\mathfrak{o})$.

These ring C*-algebras are closely related to the number-theoretic objects we introduced before. Namely, it turns out that there is a canonical commutative C*-subalgebra $\mathfrak{D}[\mathfrak{o}] := C^*(\{u^a e_b u^{-a} \mid a \in \mathfrak{o}, b \in \mathfrak{o}^\times\})$, where e_b is the range projection $s_b s_b^*$ of s_b . And the bridge to number theory is built by the observation $\text{Spec } \mathfrak{D}[\mathfrak{o}] \cong \prod \mathfrak{o}_v$ (see [CuLi], Observation 1; the argument essentially uses (I)). Furthermore, $\mathfrak{A}[\mathfrak{o}]$ can be described as a semigroup crossed product (see [CuLi], Remark 3 and [La]):

$$\mathfrak{A}[\mathfrak{o}] \cong \mathfrak{D}[\mathfrak{o}] \rtimes^{\mathfrak{o}} \mathfrak{o} \rtimes \mathfrak{o}^\times \sim_M C_0(\mathbb{A}_f) \rtimes K \rtimes K^\times = C_0(\mathbb{A}_f) \rtimes P_K, \quad (4)$$

where $\rtimes^{\mathfrak{o}}$ denotes the crossed product by endomorphisms (following the notation in [Li]). Recall that, by definition, the $ax + b$ -group P_K is $K \rtimes K^\times$.

From now on, we will omit the argument \mathfrak{o} and simply write \mathfrak{A} , \mathfrak{A}_r or \mathfrak{D} respectively as it will become clear from the context which ring we mean.

3. Computations for \mathbb{Q}

As announced, we study a special case first: the integers \mathbb{Z} in \mathbb{Q} . This leads to a program which serves as a guide through more general computations. Roughly speaking, the idea is to compute the K-groups explicitly for the C*-subalgebra $C^*(u^1, s_{-1}, \{e_b \mid b \in \mathbb{Z}^\times\})$ by choosing a suitable filtration and then to apply the Pimsner–Voiculescu sequence iteratively to get the K-theory of the whole ring C*-algebra. Actually, this idea is already contained in [Cun]. However, to work out the second step rigorously we will need as a new ingredient the comparison between finite and infinite places.

We obtain, as announced in [Cun], $K_*(\mathfrak{A}) \cong \Lambda^*(\mathbb{Q}_{>0})$ as $\mathbb{Z}/2\mathbb{Z}$ -graded abelian groups. Here and in the sequel, Λ^* denotes the $\mathbb{Z}/2\mathbb{Z}$ -graded exterior \mathbb{Z} -algebra and K_* is the direct sum of K_0 and K_1 with the canonical grading.

3.1. K-theoretic computations I

Lemma 3.1. $K_0(C^*(u^1, s_{-1}, \{e_b\})) \cong \mathbb{Q} \oplus \mathbb{Z}$ via

$$[e_b]_0 \mapsto \left(\frac{1}{b}, 0\right) \text{ for all } b \in \mathbb{Z}_{>0}, \quad \left[\frac{1}{2}(1 + s_{-1})\right]_0 - \left[\frac{1}{2}(1 + u^1 s_{-1})\right]_0 \mapsto (0, 1),$$

whereas $K_1(C^*(u^1, s_{-1}, \{e_b\}))$ is trivial.

Moreover, we have the following relation in $K_0(C^*(u^1, s_{-1}, \{e_b\}))$:

$$[1]_0 = 2 \cdot \left[\frac{1}{2}(1 + u^1 s_{-1})\right]_0. \quad (5)$$

We write $[\cdot]_0$ or $[\cdot]_1$ for the classes in K_0 or K_1 , respectively. Furthermore, we use curly brackets $\{\cdot\}$ to indicate that we consider a whole family of generators of a certain type. For instance, $\{e_b\}$ means $\{e_b \mid b \in \mathfrak{o}^\times\}$.

Proof. By the universal relation (II), e_b lies in $C^*(u^1, e_{bd})$. Therefore, we can form $\varinjlim \{C^*(u^1, s_{-1}, e_{b_i})\}$ (over $\mathbb{Z}_{>0}$ ordered by divisibility), and we get in the inductive limit $C^*(u^1, s_{-1}, \{e_b\})$. Therefore, to determine $K_*(C^*(u^1, s_{-1}, \{e_b\}))$, we have to compute $K_*(C^*(u^1, s_{-1}, e_b))$ for single b and how the inclusion $\iota_{b,bd}: C^*(u^1, s_{-1}, e_b) \hookrightarrow C^*(u^1, s_{-1}, e_{bd})$ acts on K-theory.

First of all, it is well known that $C^*(u^1, s_{-1}) \cong C^*(\mathbb{Z} \rtimes (\mathbb{Z}/2\mathbb{Z})) \cong (\mathbb{C} * \mathbb{C})^\sim$. This follows by comparing the universal properties of these C*-algebras. Mutually inverse isomorphisms $C^*(u^1, s_{-1}) \cong (\mathbb{C} * \mathbb{C})^\sim$ are given by

$$s_{-1} \mapsto 2p - 1, \quad u^1 s_{-1} \mapsto 2q - 1 \quad \text{and} \quad \frac{1}{2}(1 + s_{-1}) \leftarrow p, \quad \frac{1}{2}(1 + u^1 s_{-1}) \leftarrow q,$$

where p and q are the canonical generators of $\mathbb{C} * \mathbb{C}$.

The K-theory of $(\mathbb{C} * \mathbb{C})^\sim$ is known, it is given by

$$K_j((\mathbb{C} * \mathbb{C})^\sim) \cong \begin{cases} \mathbb{Z}[1]_0 \oplus \mathbb{Z}[p]_0 \oplus \mathbb{Z}[q]_0 & \text{if } j = 0, \\ \{0\} & \text{for } j = 1. \end{cases}$$

This determines the K-groups of $C^*(u^1, s_{-1})$. Let us fix the identification

$$\begin{aligned} \mathbb{Z}^3 &\cong K_0(C^*(u^1, s_{-1})); \\ e_1 &\mapsto [1]_0, \quad e_2 \mapsto [\tfrac{1}{2}(1 + u^1 s_{-1})]_0, \quad e_3 \mapsto [\tfrac{1}{2}(1 + s_{-1})]_0. \end{aligned}$$

This also allows us to compute $K_*(C^*(u^1, s_{-1}, e_b))$ for any $b \in \mathbb{Z}_{>0}$ since

$$C^*(u^1, s_{-1}, e_b) \cong M_b(C^*(u^1, s_{-1})). \quad (6)$$

The idea is that the projections $e_b, u^1 e_b u^{-1}, \dots, u^{b-1} e_b u^{-(b-1)}$ decompose $\ell^2(\mathbb{Z})$ into b mutually isomorphic subspaces $\ell^2(b\mathbb{Z}), \ell^2(1 + b\mathbb{Z}), \dots, \ell^2((b-1) + b\mathbb{Z})$ (see Lemma 5.2 for more details). Thus,

$$K_j(C^*(u^1, s_{-1}, e_b)) \cong \begin{cases} \mathbb{Z}^3 & \text{if } j = 0, \\ \{0\} & \text{for } j = 1. \end{cases} \quad (7)$$

From these calculations, it already follows that $K_1(C^*(u^1, s_{-1}, \{e_b\}))$ is trivial.

It remains to compute $K_0(\iota_{b,bd})$. However, it turns out that taking (7) into account, we get $K_0(\iota_{b,bd}) = K_0(\iota_{b',b'd})$ for any $b, b' \in \mathbb{Z}_{>0}$ (see the proof of Lemma 5.1). Thus, it suffices to consider $\iota_d := \iota_{1,d}$. Under the identification (6), we get the following:

For $d = 2$, we have $\iota_2(u^1) = \begin{pmatrix} 0 & u^1 \\ 1 & 0 \end{pmatrix}$ and $\iota_2(s_{-1}) = \begin{pmatrix} s_{-1} & 0 \\ 0 & u^{-1}s_{-1} \end{pmatrix}$, which implies on K_0 :

$$K_0(\iota_2)([\tfrac{1}{2}(1 + s_{-1})]_0) = [\tfrac{1}{2}(1 + s_{-1})]_0 + [\tfrac{1}{2}(1 + u^1 s_{-1})]_0$$

and

$$K_0(\iota_2)([\tfrac{1}{2}(1 + u^1 s_{-1})]_0) = [1]_0.$$

Therefore, we get $K_0(\iota_2) = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$.

$$\text{For } d \text{ odd we have } \iota_d(u^1) = \begin{pmatrix} 0 & \dots & 0 & u^1 \\ 1 & & & \\ & \ddots & & \vdots \\ & & 1 & 0 \end{pmatrix}, \quad \iota_d(s_{-1}) = \begin{pmatrix} s_{-1} & 0 & \dots & 0 \\ 0 & & & u^{-1}s_{-1} \\ \vdots & & \ddots & \\ 0 & u^{-1}s_{-1} & & \end{pmatrix},$$

which implies on K_0 :

$$K_0(\iota_d)([\tfrac{1}{2}(1 + s_{-1})]_0) = [\tfrac{1}{2}(1 + s_{-1})]_0 + \frac{d-1}{2}[1]_0$$

and

$$K_0(\iota_d)([\tfrac{1}{2}(1 + u^1 s_{-1})]_0) = [\tfrac{1}{2}(1 + u^1 s_{-1})]_0 + \frac{d-1}{2}[1]_0.$$

Thus we conclude that $K_0(\iota_d) = \begin{pmatrix} d & \frac{d-1}{2} & \frac{d-1}{2} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

Putting these facts together, we get by choosing a cofinal sequence b_i in $\mathbb{Z}_{>0}$ with $b_{i+1} = 2d_i b_i$:

$$K_0(C^*(u^1, s_{-1}, \{e_b\})) \cong \varinjlim \left\{ \mathbb{Z}^3; \begin{pmatrix} 2d_i & d_i & d_i^{-1} \\ 0 & 0 & 1 \end{pmatrix} \right\} \cong \mathbb{Q} \oplus \mathbb{Z}. \quad (8)$$

The map of the i -th K_0 -group \mathbb{Z}^3 into $\mathbb{Q} \oplus \mathbb{Z}$ is given by

$$\mathbb{Z}^3 \rightarrow \mathbb{Q} \oplus \mathbb{Z}, \quad (x, y, z) \mapsto \left(\frac{1}{b_i} \left(x + \frac{1}{2}y + \frac{1}{2}z \right), y \right).$$

This immediately implies $[1]_0 = 2 \cdot [\frac{1}{2}(1 + u^1 s_{-1})]_0$. Moreover, K_0 is generated by $[e_b]_0$ corresponding to $(\frac{1}{b}, 0)$ and $[\frac{1}{2}(1 + s_{-1})]_0 - [\frac{1}{2}(1 + u^1 s_{-1})]_0$ which corresponds to $(0, 1)$ under the identification in (8). \square

The next step is to adjoin the isometries s_b . We consider to this end

$$\mathfrak{A}^{(m)} := C^*(u^1, s_{-1}, \{e_b\}, s_{p_1}, \dots, s_{p_m}).$$

Here, $p_1 < p_2 < \dots$ are the prime numbers in $\mathbb{Z}_{>0}$. By construction, we have $\mathfrak{A} \cong \varinjlim \{\mathfrak{A}^{(0)} \hookrightarrow \mathfrak{A}^{(1)} \hookrightarrow \dots\}$. Therefore, it suffices to determine $K_*(\mathfrak{A}^{(m)})$. Similarly to (4), $\mathfrak{A}^{(m)}$ can be described as a semigroup crossed product. This yields

$$\begin{aligned} \mathfrak{A}^{(m)} &\sim_M C_0(\Gamma_m \cdot (\prod \mathbb{Z}_p)) \rtimes (\Gamma_m \cdot \mathbb{Z}) \rtimes (\mu \times \Gamma_m) \\ &\cong \varinjlim \{ \mathfrak{A}^{(m-1)}; \text{Ad}(s_{p_m}) \} \rtimes_{\text{Ad}(s_{p_m})} \mathbb{Z} \end{aligned} \quad (9)$$

where $\Gamma_m = \langle p_1, \dots, p_m \rangle \subseteq \mathbb{Q}^\times$. We have taken the inductive limit of

$$\{ \dots \xrightarrow{\text{Ad}(s_{p_m})} \mathfrak{A}^{(m-1)} \xrightarrow{\text{Ad}(s_{p_m})} \mathfrak{A}^{(m-1)} \xrightarrow{\text{Ad}(s_{p_m})} \dots \}$$

to formally invert $\text{Ad}(s_{p_m})$.

Just a remark on notation: When we write a product like $\Gamma_m \cdot (\prod \mathbb{Z}_p)$ (or $\Gamma_m \cdot \mathbb{Z}$), it means that we embed the factors into an object carrying a multiplicative structure, for instance \mathbb{A}_f (or \mathbb{Q}), and take the product there. It will be clear from the context which object we mean.

(9) is the reason why we can apply the Pimsner–Voiculescu sequence. First, we compute:

Lemma 3.2. $K_j(\mathfrak{A}^{(1)}) \cong \mathbb{Z}$ for $j = 0, 1$.

Proof. First of all, it follows from Lemma 3.1 that $\text{Ad}(s_2)$ induces $\frac{1}{2}\text{id}_{\mathbb{Q}}$ on the summand \mathbb{Q} of $K_0(\mathfrak{A}^{(0)})$.

To calculate $K_0(\text{Ad}(s_2))([\frac{1}{2}(1 + s_{-1})]_0 - [\frac{1}{2}(1 + u^1 s_{-1})]_0)$, let us consider the identification $\mathfrak{A}^{(0)} \cong M_2(\mathfrak{A}^{(0)})$ analogous to (6) under which

$$\frac{1}{2}(1 + s_{-1}) \text{ corresponds to } \begin{pmatrix} \frac{1}{2}(1+s_{-1}) & 0 \\ 0 & \frac{1}{2}(1+u^1 s_{-1}) \end{pmatrix},$$

$$\begin{aligned} \frac{1}{2}(1 + u^1 s_{-1}) \text{ corresponds to } \frac{1}{2} \begin{pmatrix} 1 & s_{-1} \\ s_{-1} & 1 \end{pmatrix} &\sim \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \\ \text{Ad}(s_2) \left(\frac{1}{2}(1 + s_{-1}) \right) \text{ corresponds to } &\begin{pmatrix} \frac{1}{2}(1+s_{-1}) & 0 \\ 0 & 0 \end{pmatrix}, \\ \text{Ad}(s_2) \left(\frac{1}{2}(1 + u^1 s_{-1}) \right) \text{ corresponds to } &\begin{pmatrix} \frac{1}{2}(1+u^1 s_{-1}) & 0 \\ 0 & 0 \end{pmatrix}. \end{aligned}$$

Thus, on K-theory, this isomorphism maps both

$$\left[\frac{1}{2}(1 + s_{-1}) \right]_0 - \left[\frac{1}{2}(1 + u^1 s_{-1}) \right]_0$$

and

$$K_0(\text{Ad}(s_2)) \left(\left[\frac{1}{2}(1 + s_{-1}) \right]_0 - \left[\frac{1}{2}(1 + u^1 s_{-1}) \right]_0 \right)$$

to $\left[\frac{1}{2}(1 + s_{-1}) \right]_0 - \left[\frac{1}{2}(1 + u^1 s_{-1}) \right]_0$, where we used (5).

This shows that $K_0(\text{Ad}(s_2))$ is given by $\begin{pmatrix} \frac{1}{2} \text{id}_{\mathbb{Q}} & 0 \\ 0 & \text{id}_{\mathbb{Z}} \end{pmatrix}$ on $K_0(\mathfrak{A}^{(0)}) \cong \mathbb{Q} \oplus \mathbb{Z}$.

Hence, the Pimsner–Voiculescu sequence applied to (9), together with Lemma 3.1, gives

$$\begin{array}{ccccc} \mathbb{Q} \oplus \mathbb{Z} & \xrightarrow{-\text{id}_{\mathbb{Q}} \oplus 0} & \mathbb{Q} \oplus \mathbb{Z} & \longrightarrow & K_0(\mathfrak{A}^{(1)}) \\ \uparrow & & \downarrow & & \downarrow \\ K_1(\mathfrak{A}^{(1)}) & \longleftarrow & 0 & \longleftarrow & 0, \end{array}$$

which implies $K_j(\mathfrak{A}^{(1)}) \cong \mathbb{Z}$ for $j = 0, 1$. □

Actually, we can go one step further and show $(\text{Ad}(s_3))_* = \text{id}_{K_*(\mathfrak{A}^{(1)})}$, but at this point we cannot show directly $(\text{Ad}(s_{p_{m+1}}))_* = \text{id}_{K_*(\mathfrak{A}^{(m)})}$ in general.

3.2. Infinite and finite places over \mathbb{Q} . To solve our problem given in the last section, we compare the infinite place of \mathbb{Q} with the finite ones. To be more precise, our goal is to prove that the crossed products arising from the $ax + b$ -group $P_{\mathbb{Q}}$ acting on the finite adeles $\mathbb{A}_f = \mathbb{Q}^{\times} \cdot (\prod \mathbb{Z}_p)$ and on the infinite place \mathbb{R} of \mathbb{Q} respectively are Morita equivalent. This can be written in a slightly more complicated way as

$$C_0(\mathbb{R}) \rtimes \mathbb{Q} \rtimes \mathbb{Q}^{\times} \sim_M C_0(\mathbb{Q}^{\times} \cdot (\prod \mathbb{Z}_p)) \rtimes (\mathbb{Q}^{\times} \cdot \mathbb{Z}) \rtimes \mathbb{Q}^{\times}.$$

The point is that we actually need this result not only for \mathbb{Q}^{\times} but – more generally – for any subgroup of \mathbb{Q}^{\times} in place of the full group \mathbb{Q}^{\times} . This will be proven along the way as well.

The central idea of the proof is that the infinite place and the finite ones are connected via duality (see Lemma 3.5). That is why we think of our result as a duality theorem.

3.2.1. Fourier transform for \mathbb{R} . Let us consider some very basic constructions (mainly to set up the notation):

We have an action of \mathbb{Q} on $C_0(\mathbb{R})$ given by translation:

$$\hat{\tau}: \mathbb{Q} \rightarrow \text{Aut}(C_0(\mathbb{R})), \quad \hat{\tau}_a(g)(t) = g(t - a), \quad \text{for all } g \in C_0(\mathbb{R}), a \in \mathbb{Q}, t \in \mathbb{R}.$$

Moreover, the Fourier transform on $C_c(\mathbb{R})$ is given by

$$F_{\mathbb{R}}: C_c(\mathbb{R}) \rightarrow C_0(\mathbb{R}), \quad f \mapsto \hat{f} = \left[t \mapsto \int_{\mathbb{R}} e(ts) f(s) ds \right],$$

where we set

$$e(t) := \exp(2\pi i t)$$

and identify \mathbb{R} with $\hat{\mathbb{R}}$ by $t \mapsto [s \mapsto e(ts)]$. $F_{\mathbb{R}}$ extends to an isomorphism $F_{\mathbb{R}}: C^*(\mathbb{R}) \rightarrow C_0(\mathbb{R})$.

Now we can consider the action $\tau: \mathbb{Q} \rightarrow \text{Aut}(C^*(\mathbb{R}))$ given by conjugating $\hat{\tau}$ by $F_{\mathbb{R}}$. By construction, $F_{\mathbb{R}}$ is a covariant isomorphism with respect to τ and $\hat{\tau}$, and it thus extends to an isomorphism $F_{\mathbb{R}}: C^*(\mathbb{R}) \rtimes_{\tau} \mathbb{Q} \rightarrow C_0(\mathbb{R}) \rtimes_{\hat{\tau}} \mathbb{Q}$. To simplify the notation, we will not distinguish between covariant homomorphisms and their extensions to crossed product algebras. τ is explicitly given by $\tau_a(f)(t) = e(-at)f(t)$ for all $f \in C_c(\mathbb{R}) \subseteq C^*(\mathbb{R})$.

Furthermore, consider the action $\hat{\beta}: \mathbb{Q}^{\times} \rightarrow \text{Aut}(C_0(\mathbb{R}) \rtimes_{\hat{\tau}} \mathbb{Q})$ given by

$$\hat{\beta}_b(gu^a) = g(b^{-1}\sqcup)u^{ab} \quad \text{for all } g \in C_0(\mathbb{R}), a \in \mathbb{Q}.$$

Again, conjugating $\hat{\beta}$ by $F_{\mathbb{R}}$ gives an action $\beta: \mathbb{Q}^{\times} \rightarrow \text{Aut}(C^*(\mathbb{R}) \rtimes_{\tau} \mathbb{Q})$ such that $F_{\mathbb{R}}$ induces an isomorphism

$$F_{\mathbb{R}}: C^*(\mathbb{R}) \rtimes_{\tau} \mathbb{Q} \rtimes_{\beta} \mathbb{Q}^{\times} \rightarrow C_0(\mathbb{R}) \rtimes_{\hat{\tau}} \mathbb{Q} \rtimes_{\hat{\beta}} \mathbb{Q}^{\times}.$$

This action β is given by $\beta_b(fu^a) = |b|f(b\sqcup)u^{ab}$.

3.2.2. Identification of crossed products. From this point of departure, we will now move towards the finite adeles, and the bridge between the infinite place and the finite ones is given by the additive group of our global field \mathbb{Q} , in the following sense: Start with the action $\lambda: \mathbb{R} \rightarrow \text{Aut}(C^*(\mathbb{Q}))$ given by

$$\lambda_t(u^a) = e(at)u^a \quad \text{for all } t \in \mathbb{R}, a \in \mathbb{Q},$$

where $C^*(\mathbb{Q})$ denotes the group C^* -algebra of $(\mathbb{Q}, +)$. We will show that the crossed product C^* -algebras $C^*(\mathbb{R}) \rtimes_{\tau} \mathbb{Q}$ and $C^*(\mathbb{Q}) \rtimes_{\lambda} \mathbb{R}$ are isomorphic.

To this end, define a linear map

$$\varphi: C_c(\mathbb{Q}, C_c(\mathbb{R})) \rightarrow C^*(\mathbb{Q}) \rtimes_{\lambda} \mathbb{R}, \quad \sum_a f_a u^a \mapsto [t \mapsto \sum_a e(at) f_a(t) u^a].$$

Lemma 3.3. φ identifies $C_c(\mathbb{Q}, C_c(\mathbb{R}))$ – viewed as a *-subalgebra of $C^*(\mathbb{R}) \rtimes_{\tau} \mathbb{Q}$ – with the *-subalgebra $C_c(\mathbb{Q} \times \mathbb{R})$ of $C^*(\mathbb{Q}) \rtimes_{\lambda} \mathbb{R}$.

Proof. This follows by computations as in the proof of Lemma 4.3. \square

Lemma 3.4. φ extends to an isomorphism $\varphi: C^*(\mathbb{R}) \rtimes_{\tau} \mathbb{Q} \cong C^*(\mathbb{Q}) \rtimes_{\lambda} \mathbb{R}$.

Proof. φ extends to an isometric isomorphism $\ell^1(\mathbb{Q}, L^1(\mathbb{R})) \cong L^1(\mathbb{R}, \ell^1(\mathbb{Q}))$, where we view $\ell^1(\mathbb{Q}, L^1(\mathbb{R}))$ and $L^1(\mathbb{R}, \ell^1(\mathbb{Q}))$ as *-subalgebras of $C^*(\mathbb{R}) \rtimes_{\tau} \mathbb{Q}$ and $C^*(\mathbb{Q}) \rtimes_{\lambda} \mathbb{R}$, respectively. Moreover, $C^*(\mathbb{R}) \rtimes_{\tau} \mathbb{Q}$ is the enveloping C*-algebra of $\ell^1(\mathbb{Q}, L^1(\mathbb{R}))$ and $C^*(\mathbb{Q}) \rtimes_{\lambda} \mathbb{R}$ is the enveloping C*-algebra of $L^1(\mathbb{R}, \ell^1(\mathbb{Q}))$. Thus, we indeed get an isomorphism $\varphi: C^*(\mathbb{R}) \rtimes_{\tau} \mathbb{Q} \cong C^*(\mathbb{Q}) \rtimes_{\lambda} \mathbb{R}$ (compare the proof of Lemma 4.3 for the details). \square

Once again, the \mathbb{Q}^{\times} -action on $C^*(\mathbb{R}) \rtimes_{\tau} \mathbb{Q}$, conjugated by φ , yields an action $\alpha: \mathbb{Q}^{\times} \rightarrow \text{Aut}(C^*(\mathbb{Q}) \rtimes_{\lambda} \mathbb{R})$. α is given by the formula

$$\alpha_b([t \mapsto \sum_a f_a(t)u^a]) = [t \mapsto \sum_a |b| f_a(bt)u^{ab}]$$

for all $[t \mapsto \sum_a f_a(t)u^a] \in C_c(\mathbb{R}, \ell^1(\mathbb{Q}))$.

By construction, φ induces an isomorphism

$$(C^*(\mathbb{R}) \rtimes_{\tau} \mathbb{Q}) \rtimes_{\beta} \mathbb{Q}^{\times} \xrightarrow{\cong} (C^*(\mathbb{Q}) \rtimes_{\lambda} \mathbb{R}) \rtimes_{\alpha} \mathbb{Q}^{\times}.$$

3.2.3. Fourier transform for \mathbb{Q} . At this point, the following well-known result brings the finite adèle ring or rather its maximal compact subring into the game:

Lemma 3.5. *The dual group of \mathbb{Q} can be identified with*

$$Y := \mathbb{R} \times_{\mathbb{Z}} (\prod \mathbb{Z}_p) = (\mathbb{R} \times \widehat{\mathbb{Z}}) / (r, z) \sim (r+1, z+1).$$

Proof. We use the well-known result that $\prod \mathbb{Z}_p$ can be identified with $(\widehat{\mathbb{Q}/\mathbb{Z}})$ via

$$\prod \mathbb{Z}_p \ni z \mapsto ([\frac{m}{n}] \mapsto e(z(n) \cdot \frac{m}{n})) \in (\widehat{\mathbb{Q}/\mathbb{Z}}), \quad (10)$$

where we view the maximal compact subring $\prod \mathbb{Z}_p$ of \mathbb{A}_f as the projective limit of quotients of \mathbb{Z} which is realized as a subspace of $\prod_{n>0} \mathbb{Z}/n\mathbb{Z}$.

Now define $Y \xrightarrow{\gamma} \widehat{\mathbb{Q}}$, $[r, z] \mapsto [\frac{m}{n} \mapsto e((r-z(n)) \cdot \frac{m}{n})]$. The map γ is well defined and continuous. Since both spaces are compact, we just have to show bijectivity to prove that γ is a homeomorphism.

To prove surjectivity, take any $\chi \in \widehat{\mathbb{Q}}$. Restricting χ to \mathbb{Z} yields a character of \mathbb{Z} which is of the form $e(r\sqcup)$ for some $r \in \mathbb{R}$. Therefore, $\chi \cdot e(-r\sqcup)$ has constant value 1 on \mathbb{Z} , hence it induces a character of \mathbb{Q}/\mathbb{Z} . In other words, there exists $z \in \prod \mathbb{Z}_p$

such that $\chi(\frac{m}{n})e(-r \cdot \frac{m}{n}) = e(-z(n) \cdot \frac{m}{n})$ for all $\frac{m}{n} \in \mathbb{Q}$ because of (10). This means $\chi = \gamma([r, z])$.

The map γ is injective as well: As one immediately checks, γ is actually a group homomorphism (where addition on Y is defined componentwise). Thus, we just have to show that γ has trivial kernel. Given $[r, z] \in \ker(\gamma)$, we have $1 \equiv \gamma([r, z])|_{\mathbb{Z}} = e(r\sqcup)|_{\mathbb{Z}}$ which implies $r \in \mathbb{Z}$. Furthermore, this shows that $r - z$ is an element in $\prod \mathbb{Z}_p$ yielding the trivial character on \mathbb{Q}/\mathbb{Z} . Hence, by (10), it must be 0, which means $[r, z] \sim [0, 0]$. \square

This result can be viewed as a special case of Theorem 2.4, (2).

The map γ can be used to identify $C^*(\mathbb{Q})$ and $C(Y)$ via the Fourier transform given by

$$C_c(\mathbb{Q}) \xrightarrow{F_{\mathbb{Q}}} C(Y); F_{\mathbb{Q}}(u^{m/n})([r, z]) = \text{ev}_{m/n}(\gamma([r, z])) = e((r - z(n)) \cdot \frac{m}{n}).$$

Conjugating $\lambda: \mathbb{R} \rightarrow \text{Aut}(C^*(\mathbb{Q}))$ by $F_{\mathbb{Q}}$ yields an action $\hat{\lambda}$ on $C(Y)$ given by $\hat{\lambda}_t(f)([r, z]) = f \circ \hat{\lambda}_t^*([r, z])$ with $\hat{\lambda}_t^*([r, z]) = [r + t, z]$. This follows from

$$\begin{aligned} (F_{\mathbb{Q}} \circ \lambda_t(u^{m/n}))([r, z]) &= F_{\mathbb{Q}}(e(\frac{m}{n} \cdot t)u^{m/n})([r, z]) \\ &= e(((r + t) - z(n)) \cdot \frac{m}{n}) = F_{\mathbb{Q}}(u^{m/n})([r + t, z]). \end{aligned}$$

Again, we get an isomorphism $C^*(\mathbb{Q}) \rtimes_{\lambda} \mathbb{R} \xrightarrow{F_{\mathbb{Q}}} C(Y) \rtimes_{\hat{\lambda}} \mathbb{R}$.

As the last step, we describe the action $\hat{\alpha}$ of \mathbb{Q}^{\times} on $C(Y) \rtimes_{\hat{\lambda}} \mathbb{R}$ induced by α conjugated by $F_{\mathbb{Q}}$. For any $\mathbb{Q}^{\times} \ni b = \frac{m_b}{n_b}$ ($m_b \in \mathbb{Z}, n_b \in \mathbb{Z}_{>0}$), consider

$$Y \xrightarrow{\hat{\alpha}_b^*} Y, \quad [r, z] \mapsto [(r - z(n_b)) \cdot b, (z(\sqcup \cdot n_b) - z(n_b)) \cdot b].$$

Multiplication with $b = \frac{m_b}{n_b}$ makes sense since $z(\sqcup \cdot n_b) - z(n_b)$ is in $\prod \mathbb{Z}_p$ with $z(ln_b) - z(n_b) \in n_b \mathbb{Z}$ for all $l \in \mathbb{Z}_{>0}$ and because it is independent of the representation of b . Moreover, the expression defining $\hat{\alpha}_b^*$ is compatible with \sim so that $\hat{\alpha}_b^*$ is well defined. Furthermore, $\hat{\alpha}_b^*$ is continuous and thus a homeomorphism since $\hat{\alpha}_b^* \circ \hat{\alpha}_{1/b}^* = \text{id}_Y$.

Now we claim that $\hat{\alpha}: \mathbb{Q}^{\times} \rightarrow \text{Aut}(C(Y) \rtimes_{\hat{\lambda}} \mathbb{R})$ given by $\hat{\alpha}_b = F_{\mathbb{Q}} \circ \alpha_b \circ F_{\mathbb{Q}}^{-1}$ is of the form $\hat{\alpha}_b(f \cdot g) = ([r, z] \mapsto (f \circ \hat{\alpha}_b^*) \cdot (|b|g(b\sqcup)))$ for all $f \in C(Y), g \in C_c(\mathbb{R})$. This follows from

$$\begin{aligned} (F_{\mathbb{Q}} \circ \alpha_b([s \mapsto g(s)u^a])(t))[r, z] &= |b|g(bt) \cdot e((r - z(n_a n_b))ab) \\ &= |b|g(bt)e(((r - z(n_b)) \cdot b - ((z(\sqcup \cdot n_b) - z(n_b)) \cdot b)(n_a)) \cdot a) \\ &= |b|g(bt)F_{\mathbb{Q}}(u^a)([(r - z(n_b)) \cdot b, (z(\sqcup \cdot n_b) - z(n_b)) \cdot b]) \\ &= (|b|g(bt)F_{\mathbb{Q}}(u^a) \circ \hat{\alpha}_b^*)[r, z]. \end{aligned}$$

Remark 3.6. It is useful to consider the action $\kappa: \mathbb{R} \rtimes \mathbb{Q}^\times \rightarrow \text{Aut}(C(Y))$ given by $\kappa(t, b)(f) = f \circ \hat{\alpha}_b^* \circ \hat{\lambda}_t^*$, where the semidirect product is taken with respect to the action $\mathbb{Q}^\times \rightarrow \text{Aut}(\mathbb{R}); b \mapsto [t \mapsto t/b]$. The action κ is a group homomorphism since $\hat{\lambda}_t^* \circ \hat{\alpha}_b^* = \hat{\alpha}_b^* \circ \hat{\lambda}_{t/b}^*$. Using a general result on crossed products by semidirect products (compare [Wil], Proposition 3.11), one immediately deduces

$$(C(Y) \rtimes_{\hat{\lambda}} \mathbb{R}) \rtimes_{\hat{\alpha}} \mathbb{Q}^\times \cong C(Y) \rtimes_{\kappa} (\mathbb{R} \rtimes \mathbb{Q}^\times).$$

Remark 3.7. Up to now, we could just as well consider a subgroup of \mathbb{Q}^\times instead of the whole group. So, to sum up, we have shown that for any subgroup Γ of \mathbb{Q}^\times , we have an isomorphism

$$(C_0(\mathbb{R}) \rtimes_{\hat{\iota}} \mathbb{Q}) \rtimes_{\hat{\beta}} \Gamma \cong C(Y) \rtimes_{\kappa} (\mathbb{R} \rtimes \Gamma). \quad (11)$$

3.2.4. Morita equivalent crossed product C*-algebras

Proposition 3.8. *The transformation groupoids associated to the action of*

$$\mathbb{R} \rtimes \mathbb{Q}^\times \text{ on } Y \text{ via } [r, z] \cdot (t, b) = \hat{\alpha}_b^* \circ \hat{\lambda}_t^*([r, z]),$$

denoted by \mathcal{G} , and of

$$\mathbb{Q} \rtimes \mathbb{Q}^\times \text{ on } \mathbb{A}_f \text{ by } z \cdot (a, b) = b^{-1}(z - a),$$

denoted by $\tilde{\mathcal{G}}$, are equivalent in the sense of [MRW].

Proof. We will show that both groupoids are equivalent to certain subgroupoids which we can identify.

First, consider the closed subset $\tilde{N} := \prod \mathbb{Z}_p \subseteq \mathbb{A}_f = \tilde{\mathcal{G}}^0$. As $\mathbb{Q}^\times \cdot (\prod \mathbb{Z}_p) = \mathbb{A}_f$, \tilde{N} meets every orbit in $\tilde{\mathcal{G}}^0$. Moreover, the restricted range and source maps are open (details can be found in Lemma 4.4). Thus, by [MRW], Example 2.7, $\tilde{\mathcal{G}}$ and $\tilde{\mathcal{G}}_{\tilde{N}}^{\tilde{N}}$ are equivalent, where

$$\tilde{\mathcal{G}}_{\tilde{N}}^{\tilde{N}} = \{(z, (a, b)) \in (\prod \mathbb{Z}_p) \times (\mathbb{Q} \rtimes \mathbb{Q}^\times) \mid b(z + a) \in \prod \mathbb{Z}_p\}.$$

As a second step, consider the closed subset $\pi(\{0\} \times (\prod \mathbb{Z}_p)) =: N$ of Y where π is the canonical projection $\mathbb{R} \times (\prod \mathbb{Z}_p) \xrightarrow{\pi} (\mathbb{R} \times (\prod \mathbb{Z}_p))/\mathbb{Z} = Y$. N meets every orbit in $Y = \mathcal{G}^0$ because $\bigcup_{t \in \mathbb{R}} \hat{\lambda}_t^*(N) = Y$. Again, the restricted range and source maps are open (compare Lemma 4.4 for the details). Thus, \mathcal{G} and \mathcal{G}_N^N are equivalent by Example 2.7 of [MRW].

We have $\mathcal{G}_N^N = \{([0, z], (t, b)) \in N \times (\mathbb{R} \rtimes \mathbb{Q}^\times) \mid \hat{\alpha}_b^*([t, z]) \in N\}$. Now,

$$\begin{aligned} \hat{\alpha}_b^*([t, z]) &= [(t - z(n_b)) \cdot b, (z(\sqcup \cdot n_b) - z(n_b)) \cdot b] \in N \\ &\iff [(t - z) \cdot m_b](n_b) = (t - z(n_b)) \cdot m_b \in n_b \mathbb{Z} \\ &\iff (t - z) \cdot b \in \prod \mathbb{Z}_p. \end{aligned}$$

In particular, this implies $t \in \mathbb{Q}$. Therefore, $\tilde{\mathcal{G}}_N^{\tilde{N}}$ and \mathcal{G}_N^N can be identified (as in Lemma 4.4) via $\tilde{\mathcal{G}}_N^{\tilde{N}} \ni (z, (a, b)) \mapsto ([0, z], (a, b^{-1})) \in \mathcal{G}_N^N$. \square

If we replace \mathbb{Q}^\times by an arbitrary subgroup Γ of \mathbb{Q}^\times , we have to consider the action of $(\Gamma \cdot \mathbb{Z}) \rtimes \Gamma$ on $\Gamma \cdot (\prod \mathbb{Z}_p)$ and the action of $\mathbb{R} \rtimes \Gamma$ on Y . With these modifications, everything works out as above.

Corollary 3.9. $C_0(\Gamma \cdot (\prod \mathbb{Z}_p)) \rtimes (\Gamma \cdot \mathbb{Z}) \rtimes \Gamma \sim_M C(Y) \rtimes_{\kappa} (\mathbb{R} \rtimes \Gamma)$ for any subgroup Γ of \mathbb{Q}^\times .

Proof. This follows from Proposition 3.8 (applied to Γ instead of \mathbb{Q}^\times) together with [MRW], Theorem 2.8, and the well-known fact that for a transformation groupoid the (full) groupoid C^* -algebra and the corresponding (full) crossed product are isomorphic. \square

Corollary 3.10. For any subgroup Γ of \mathbb{Q}^\times , $C_0(\Gamma \cdot (\prod \mathbb{Z}_p)) \rtimes (\Gamma \cdot \mathbb{Z}) \rtimes \Gamma$ and $(C_0(\mathbb{R}) \rtimes_{\hat{\tau}} \mathbb{Q}) \rtimes_{\hat{\beta}} \Gamma$ are Morita equivalent.

Proof. This result follows by combining the last corollary with (11). \square

3.3. K-theoretic computations II. Corollary 3.10 enables us to continue with our computations of Section 3.1. The crucial point is that on \mathbb{R} , we can work with homotopies to compute the multiplicative action of \mathbb{Q}^\times on K-theory.

By (4), $\mathfrak{A} \sim_M C_0(\mathbb{A}_f) \rtimes \mathbb{Q} \rtimes \mathbb{Q}^\times$. Thus, by Corollary 3.10, we have to determine the K-theory of $C_0(\mathbb{R}) \rtimes \mathbb{Q} \rtimes \mathbb{Q}^\times$.

As a first step, the K-theory of $C_0(\mathbb{R}) \rtimes_{\hat{\beta}_{-1}} \mu$ can be computed with the help of the split exact sequence $C_0(\mathbb{R}) \rtimes \mu \hookrightarrow C(\mathbb{T}) \rtimes \mu \twoheadrightarrow C^*(\mu)$ (recall that $\mu = \{\pm 1\}$ in this case). We get

$$K_j(C_0(\mathbb{R}) \rtimes_{\hat{\beta}_{-1}} \mu) \cong \begin{cases} \mathbb{Z} & \text{if } j = 0, \\ 0 & \text{for } j = 1. \end{cases} \quad (12)$$

As a next step, we have

$$K_j(C_0(\mathbb{R}) \rtimes \mathbb{Q} \rtimes_{\hat{\beta}_{-1}} \mu) \cong \begin{cases} \mathbb{Q} \oplus \mathbb{Z} & \text{for } j = 0, \\ 0 & \text{if } j = 1, \end{cases}$$

because of Lemma 3.1, (9) for $m = 0$ and Corollary 3.10 for $\Gamma = \mu$.

Similarly, Lemma 3.2 implies $K_j(C_0(\mathbb{R}) \rtimes \mathbb{Q} \rtimes (\mu \times \Gamma_1)) \cong \mathbb{Z}$ ($j = 0, 1$) because of (9) for $m = 1$ and Corollary 3.10 for $\Gamma = \mu \times \Gamma_1$. Recall that Γ_m is $\langle p_1, \dots, p_m \rangle$, the subgroup of \mathbb{Q}^\times generated by the first m primes.

The inclusion $i: C_0(\mathbb{R}) \hookrightarrow C_0(\mathbb{R}) \rtimes \mathbb{Q}$ is covariant with respect to $\hat{\beta}$ and thus induces homomorphisms between the corresponding crossed products.

Lemma 3.11. $i : C_0(\mathbb{R}) \rtimes (\mu \times \Gamma_1) \rightarrow C_0(\mathbb{R}) \rtimes \mathbb{Q} \rtimes (\mu \times \Gamma_1)$ induces $C \cdot \text{id}_{\mathbb{Z}}$ for some $0 \neq C \in \mathbb{Z}$ on both K_0 and K_1 .

Proof. First of all, we claim that $i : C_0(\mathbb{R}) \rtimes_{\hat{\beta}_{-1}} \mu \rightarrow C_0(\mathbb{R}) \rtimes_{\hat{\tau}} \mathbb{Q} \rtimes_{\hat{\beta}_{-1}} \mu$ induces $\mathbb{Z} \xrightarrow{0 \oplus (C \cdot \text{id})} \mathbb{Q} \oplus \mathbb{Z}$ on K_0 for some $0 \neq C \in \mathbb{Z}$.

To show this, we consider the $\hat{\beta}_{-1}$ -invariant inclusion $C_0(\mathbb{R}) \hookrightarrow C_0(\mathbb{R}) \rtimes_{\hat{\tau}_1} \mathbb{Z}$. It yields, using the Pimsner–Voiculescu sequence and its naturality, the following commutative diagram with exact rows:

$$\begin{array}{ccccccc} \cdots & \longrightarrow & \mathbb{Z} & \xrightarrow{2\text{id}} & \mathbb{Z} & \longrightarrow & K_1(C_0(\mathbb{R}) \rtimes_{\hat{\beta}_{-1}} \mathbb{Z}) \longrightarrow 0 \\ & & \downarrow \cong & & \downarrow \cong & & \downarrow K_1(i) \\ \cdots & \longrightarrow & \mathbb{Z} & \xrightarrow{2\text{id}} & \mathbb{Z} & \longrightarrow & K_1((C_0(\mathbb{R}) \rtimes_{\hat{\tau}_1} \mathbb{Z}) \rtimes_{\hat{\beta}_{-1}} \mathbb{Z}) \longrightarrow \cdots \end{array}$$

Therefore, $i : C_0(\mathbb{R}) \rtimes_{\hat{\beta}_{-1}} \mathbb{Z} \rightarrow (C_0(\mathbb{R}) \rtimes_{\hat{\tau}_1} \mathbb{Z}) \rtimes_{\hat{\beta}_{-1}} \mathbb{Z}$ does not induce the trivial map on K_1 .

Now, by [Bla], Theorem 10.7.1 (the sequence described therein is natural with respect to covariant homomorphisms), we get the following commutative diagram with exact rows:

$$\begin{array}{ccccccc} K_0(C_0(\mathbb{R}) \rtimes_{\hat{\beta}_{-1}} \mu) & \longrightarrow & K_1(C_0(\mathbb{R}) \rtimes_{\hat{\beta}_{-1}} \mathbb{Z}) & \longrightarrow & 0 \\ & & \downarrow K_0(i) & & \downarrow K_1(i) \\ K_0((C_0(\mathbb{R}) \rtimes_{\hat{\tau}_1} \mathbb{Z}) \rtimes_{\hat{\beta}_{-1}} \mu) & \longrightarrow & K_1((C_0(\mathbb{R}) \rtimes_{\hat{\tau}_1} \mathbb{Z}) \rtimes_{\hat{\beta}_{-1}} \mathbb{Z}) & \longrightarrow & \cdots \end{array}$$

In the commutative square, going right and then down does not yield the trivial map, and hence $K_0(i)$ is not trivial.

As $K_0(C_0(\mathbb{R}) \rtimes_{\hat{\beta}_{-1}} \mu) \cong \mathbb{Z}$ by (12) and $\hat{\beta}_b \sim_h \text{id}$ on $C_0(\mathbb{R}) \rtimes_{\hat{\beta}_{-1}} \mu$, the nontrivial image of $K_0(i)$ is fixed by $K_0(\hat{\beta}_b)$ for all $b \in \mathbb{Z}_{>0}$. Hence it follows that

$$C_0(\mathbb{R}) \rtimes_{\hat{\beta}_{-1}} \mu \xrightarrow{i} C_0(\mathbb{R}) \rtimes_{\hat{\tau}} \mathbb{Q} \rtimes_{\hat{\beta}_{-1}} \mu \cong \lim_{b \in \mathbb{Z}_{>0}} \{(C_0(\mathbb{R}) \rtimes_{\hat{\tau}_1} \mathbb{Z}) \rtimes \mu; \hat{\beta}_b\}$$

does not yield the trivial homomorphism on K_0 , either.

Now,

$$K_j(C_0(\mathbb{R}) \rtimes \mathbb{Q} \rtimes \mu) \cong \begin{cases} \mathbb{Q} \oplus \mathbb{Z} & \text{if } j = 0, \\ \{0\} & \text{for } j = 1, \end{cases}$$

and $K_j(C_0(\mathbb{R}) \rtimes \mathbb{Q} \rtimes (\mu \times \Gamma_1)) \cong \mathbb{Z}$ for $j = 0, 1$ as we already know. Therefore, studying the Pimsner–Voiculescu sequence and going through the possibilities yield that $K_0(\hat{\beta}_2)$ must be of the form $\mathbb{Q} \oplus \mathbb{Z} \xrightarrow{? \oplus \text{id}_{\mathbb{Z}}} \mathbb{Q} \oplus \mathbb{Z}$ with $? \neq \text{id}_{\mathbb{Q}}$ on $K_0(C_0(\mathbb{R}) \rtimes$

$\mathbb{Q} \rtimes_{\hat{\beta}_{-1}} \mu$). But we have just seen that $K_0(i)(1)$ is fixed by $K_0(\hat{\beta}_2)$, where 1 is the generator of $\mathbb{Z} \cong K_0(C_0(\mathbb{R}) \rtimes_{\hat{\beta}_{-1}} \mu)$. Thus, $K_0(i)(1) = (0, C)$ for some $0 \neq C \in \mathbb{Z}$ (C is nontrivial as $K_0(i) \neq 0$). This proves our claim.

Secondly, the Pimsner–Voiculescu sequence, together with its naturality, implies that the assertion of the lemma is true. \square

Theorem 3.12. *We have $K_j(\mathfrak{A}^{(m)}) \cong \mathbb{Z}^{2^{m-1}}$ for all $m \in \mathbb{Z}_{>0}$ ($j = 0, 1$).*

Proof. We prove by induction on m that $K_j(C_0(\mathbb{R}) \rtimes \mathbb{Q} \rtimes (\mu \times \Gamma_m)) \cong \mathbb{Z}^{2^{m-1}}$ for $j = 0, 1$ and that $C_0(\mathbb{R}) \rtimes (\mu \times \Gamma_m) \xrightarrow{i} C_0(\mathbb{R}) \rtimes \mathbb{Q} \rtimes (\mu \times \Gamma_m)$ induces $\begin{pmatrix} C & * \\ \vdots & \vdots \\ 0 & C \end{pmatrix}$ on K-theory.

The case $m = 1$ has just been shown in the last lemma.

Now assume that we have proven our assertion for m . We have (for $j = 0, 1$) the following commutative diagram

$$\begin{array}{ccc} K_j(C_0(\mathbb{R}) \rtimes (\mu \times \Gamma_m)) & \xrightarrow{0} & K_j(C_0(\mathbb{R}) \rtimes (\mu \times \Gamma_m)) \\ \downarrow i_* & & \downarrow i_* \\ K_j(C_0(\mathbb{R}) \rtimes \mathbb{Q} \rtimes (\mu \times \Gamma_m)) & \xrightarrow{\text{id} - (\hat{\beta}_{p_{m+1}})_*^{-1}} & K_j(C_0(\mathbb{R}) \rtimes \mathbb{Q} \rtimes (\mu \times \Gamma_m)). \end{array}$$

As we know by induction hypothesis that $K_j(C_0(\mathbb{R}) \rtimes \mathbb{Q} \rtimes (\mu \times \Gamma_m))$ ($j = 0, 1$) is torsion-free and that $i_* = \begin{pmatrix} C & * \\ \vdots & \vdots \\ 0 & C \end{pmatrix}$, it follows that $\text{id} - (\hat{\beta}_{p_{m+1}})_*^{-1}$ must be trivial.

Therefore,

$$\begin{aligned} & K_j(C_0(\mathbb{R}) \rtimes \mathbb{Q} \rtimes (\mu \times \Gamma_{m+1})) \\ & \cong K_j(C_0(\mathbb{R}) \rtimes \mathbb{Q} \rtimes (\mu \times \Gamma_m)) \oplus K_{j+1}(C_0(\mathbb{R}) \rtimes \mathbb{Q} \rtimes (\mu \times \Gamma_m)) \\ & \cong \mathbb{Z}^{2^m} \end{aligned}$$

for $j = 0, 1$, and the inclusion i induces

$$\left(\begin{array}{ccc|ccc} C & & * & & & \\ & \ddots & & & * & \\ 0 & & C & & & \\ \hline & & & C & & * \\ & 0 & & 0 & \ddots & C \end{array} \right)$$

on K-theory under this decomposition of $K_j(C_0(\mathbb{R}) \rtimes \mathbb{Q} \rtimes (\mu \times \Gamma_{m+1}))$ ($j = 0, 1$), as we wanted to prove.

Now the theorem follows from (9) and Corollary 3.10. \square

We can instantly derive the following consequences:

Corollary 3.13. $\text{Ad}(s_{p_{m+1}})$ induces the identity on $K_*(\mathfrak{A}^{(m)})$.

Corollary 3.14. $K_*(\mathfrak{A}) \cong \Lambda^*(\mathbb{Q}_{>0})$ where K_0 corresponds to products of even and K_1 corresponds to products of odd numbers of pairwise distinct primes.

Remark 3.15. Using analogous arguments, we can determine the K-theory of $C^*(u^1, \{s_b \mid b \in \mathbb{Z}_{>0}\})$. This case has already been investigated in [Cun], where $C^*(u^1, \{s_b \mid b \in \mathbb{Z}_{>0}\})$ is denoted by $\mathcal{Q}_{\mathbb{N}}$. Again, the main point is that Corollary 3.10 allows us to compute the multiplicative action of $\mathbb{Z}_{>0}$ or $\mathbb{Q}_{>0}$ on K-theory. As the final result, we get $K_*(C^*(u^1, \{s_b \mid b \in \mathbb{Z}_{>0}\})) \cong \Lambda^*(\mathbb{Q}_{>0})$ where K_0 corresponds to products of odd numbers, K_1 corresponds to products of even numbers of pairwise distinct primes.

Remark 3.16. Looking back at our explicit calculations for \mathbb{Q} , we see the following main steps:

1. Compute the K-theory of $\mathfrak{A}^{(0)} = C^*(\{u^a\}, s_\zeta, \{e_b\})$. Here ζ is a root of unity which generates μ .
2. Compare the finite adèle ring and the infinite one.
3. Show that it is enough to consider the multiplicative action of K^\times on the infinite adeles.
4. Apply the Pimsner–Voiculescu sequence iteratively, together with a homotopy argument showing that the multiplicative action of the torsion-free part of K^\times is trivial on K-theory.

4. A duality theorem

First of all, let us concentrate on the second step of our program. We can generalize Corollary 3.10 to arbitrary global fields (number fields or function fields). Our result can be viewed as a duality theorem based on the duality results of Theorem 2.4. So we prove the following result:

Theorem 4.1. *Let K be a global field and Γ be a subgroup of K^\times .*

The C-algebras $C_0(\mathbb{A}_\infty) \rtimes K \rtimes \Gamma$ and $C_0(\Gamma \cdot (\prod \mathfrak{o}_v)) \rtimes (\Gamma \cdot \mathfrak{o}) \rtimes \Gamma$ are Morita equivalent, where the groups act via inverse affine transformations.*

With $P_K = K \rtimes K^\times$ we get as a special case ($\Gamma = K^\times$):

Corollary 4.2. $C_0(\mathbb{A}_\infty) \rtimes P_K \sim_M C_0(\mathbb{A}_f) \rtimes P_K$.

As in the case of \mathbb{Q} , this result allows us to compute the action of K^\times by homotopies. But first of all, let us prove Theorem 4.1. We need two lemmas.

4.1. Crossed products by subgroups of the dual group

Lemma 4.3. *Assume that $(G, +)$ is a locally compact abelian group and H is a subgroup of the Pontryagin dual \widehat{G} . Equip H with a topology such that H becomes a locally compact group, and $\delta_h(f) = [g \mapsto h(g)f(g)]$ and $\varepsilon_g(\tilde{f}) = [h \mapsto h(-g)\tilde{f}(h)]$ extend to strongly continuous actions of H on $C^*(G)$ and of G on $C^*(H)$, respectively.*

Then $\varphi: C_c(G \times H) \rightarrow C_c(H \times G)$, $f \mapsto [(h, g) \mapsto h(-g)f(g, h)]$, extends to an isomorphism $C^(G) \rtimes_{\delta} H \cong C^*(H) \rtimes_{\varepsilon} G$.*

Before we come to the proof, just note that the discrete topology on H is always a possible choice. Actually, this is the case of interest for our applications.

Moreover, Lemma 4.3 generalizes our result in Section 3.2.2.

Proof. The strategy is to show that φ is an isomorphism of $*$ -algebras and that φ is isometric with respect to the norms $\|\cdot\|_{L^1(H, L^1(G))}$ and $\|\cdot\|_{L^1(G, L^1(H))}$. Then we just have to see that $C^*(G) \rtimes H$ and $C^*(H) \rtimes G$ are the enveloping C^* -algebras of $L^1(H, L^1(G))$ and $L^1(G, L^1(H))$.

The central idea is that infinitesimally, we have the relation

$$w_h v_g = h(g) v_g w_h \tag{13}$$

in both crossed products $C^*(G) \rtimes_{\delta} H$ and $C^*(H) \rtimes_{\varepsilon} G$. Here, v_g and w_h are the infinitesimal generators corresponding to G and H respectively.

So, as a first step, integrating (13) gives $\varphi(f_1 * f_2) = \varphi(f_1) * \varphi(f_2)$. Thus, φ is multiplicative. Moreover, a simple computation shows that φ is involutive as well.

Secondly, applying Fubini, we see that φ extends to an isometric isomorphism

$$L^1(H, L^1(G)) \cong L^1(G, L^1(H)).$$

Finally, $C^*(G) \rtimes H$ is defined as the norm closure of $L^1(H, C^*(G))$ with respect to the norm

$$\|f\| = \sup\{\|\pi(f)\| \mid \pi \text{ nondegenerate representation of } L^1(H, C^*(G))\}.$$

Now we claim that we can equally well take the norm closure of $L^1(H, L^1(G))$ in the norm

$$\|f\|' = \sup\{\|\pi(f)\| \mid \pi \text{ nondegenerate representation of } L^1(H, L^1(G))\}.$$

To see this, it suffices to prove $\|\cdot\| = \|\cdot\|'$ on $L^1(H, L^1(G))$, since this algebra is dense in $L^1(H, C^*(G))$. So it remains to show that any nondegenerate representation of $L^1(H, L^1(G))$ extends to a representation of $L^1(H, C^*(G))$ (which will automatically be nondegenerate, too). Now any nondegenerate representation of $L^1(H, L^1(G))$ is the integrated form of a covariant representation. Actually,

one can adapt the proof of the analogous statement for $L^1(H, C^*(G))$ (see for instance [Ped]). The only thing one has to use is that $L^1(G)$ has an approximate unit. But then the integrated form of the corresponding covariant representation defines a (nondegenerate) representation of $L^1(H, C^*(G))$ extending the original one. This shows that $C^*(G) \rtimes H$ is the enveloping C*-algebra of $L^1(H, L^1(G))$. Analogously, $C^*(H) \rtimes G$ is the enveloping C*-algebra of $L^1(G, L^1(H))$. But we already know that φ extends to an isometric isomorphism $L^1(H, L^1(G)) \cong L^1(G, L^1(H))$. Thus φ also extends to an isomorphism $C^*(G) \rtimes H \cong C^*(H) \rtimes G$. \square

4.2. Comparison of groupoids. As a second step, consider the following transformation groupoids which are closely related to the C*-algebras appearing in Theorem 4.1: Fix a subgroup Γ of K^\times . Each $b \in \Gamma$ acts on \mathbb{A}_∞ via multiplication by b^{-1} . This gives rise to an action of Γ on \mathbb{A}_∞ and thus to the semidirect product $\mathbb{A}_\infty \rtimes \Gamma$. Let \mathcal{G} be the groupoid associated to the right action of $\mathbb{A}_\infty \rtimes \Gamma$ on \mathbb{A}/K via affine transformations (given by $\pi(x) \cdot (t, b) = \pi(b((t, 0) + x))$), with the canonical projection $\pi : \mathbb{A} \rightarrow \mathbb{A}/K$.

Now let \mathfrak{o} be the integral closure of \mathbb{Z} in K if K is a number field, and the integral closure of $\mathbb{F}_p[T]$ in case K is a function field of characteristic p . Moreover, $\prod \mathfrak{o}_v$ is the maximal compact subring of \mathbb{A}_f , as above. Γ acts on $\Gamma \cdot \mathfrak{o}$ by multiplication as well (this time we do not take inverses) and we can form $(\Gamma \cdot \mathfrak{o}) \rtimes \Gamma$. Denote by $\tilde{\mathcal{G}}$ the groupoid associated to the right action of $\Gamma \cdot \mathfrak{o} \times \Gamma$ on $\Gamma \cdot (\prod \mathfrak{o}_v) \subseteq \mathbb{A}_f$ via inverse affine transformations ($z \cdot (a, b) = b^{-1}(z - a)$).

Lemma 4.4. *\mathcal{G} and $\tilde{\mathcal{G}}$ are equivalent as groupoids, in the sense of [MRW].*

This is the analogue of Proposition 3.8, but now in the general context.

Proof. We will use Example 2.7 of [MRW] to reduce our assertion to certain subgroupoids. The remaining groupoids will even be isomorphic.

First of all, it is shown in [MRW], Example 2.7, that a locally compact (Hausdorff) groupoid G is equivalent to G_N^N if N is a closed subset of G^0 such that

- (i) N meets every orbit in G^0 ,
- (ii) the restricted range and source maps $G_N \rightarrow G$ are open.

We would like to apply this result to \mathcal{G} and $\tilde{\mathcal{G}}$: Consider the first groupoid with $N = \pi(\{0\} \times (\prod \mathfrak{o}_v)) \subseteq \mathbb{A}/K = \mathcal{G}^0$; N is closed because $\{0\} \times (\prod \mathfrak{o}_v)$ is compact and π is continuous.

Moreover, N satisfies (i) since given $x = (x_\infty, x_f) \in \mathbb{A}$, we can find $a \in K$, $z \in \prod \mathfrak{o}_v$ such that $a + z = x_f$ (we have $\mathbb{A}_f = K + (\prod \mathfrak{o}_v)$, see [Weil], Chapter IV, Lemma 2.1, and [Weil], Chapter I, Corollary 4.2). Thus, $x = (x_\infty - a, z) + a$, which implies

$$\pi(x) = \pi(0, z) \cdot (x_\infty - a, 1) = r(\pi(0, z), (x_\infty - a, 1))$$

where r is the range map of \mathcal{G} . This shows that N meets every orbit in \mathcal{G}^0 .

To prove (ii), note that $\mathcal{G}_N = \{(\pi(x), (t, b)) \in \mathcal{G} \mid \pi(b((t, 0) + x)) \in N\}$ by definition. $s|_{\mathcal{G}_N} : \mathcal{G}_N \rightarrow N$, $(\pi(x), (t, b)) \mapsto \pi(b((t, 0) + x))$, is open because, for any open subset $U \subseteq \mathcal{G}$, $s(U \cap \mathcal{G}_N) = s(U \cap s^{-1}(N)) = s(U) \cap N$ is an open subset of N since the source map s is open. It remains to prove that $r|_{\mathcal{G}_N} : \mathcal{G}_N \rightarrow \mathbb{A}/K$, $(\pi(x), (t, b)) \mapsto \pi(x)$, is open. To see this, take open sets $U \subseteq \mathbb{A}/K$, $V \subseteq \mathbb{A}_\infty$ and $b \in \Gamma$. Consider the open subset $U \times (V \times \{b\})$ of \mathcal{G} . It suffices to look at open sets of this form since they form a basis for the topology of \mathcal{G} .

Now we have $r(U \times (V \times \{b\}) \cap \mathcal{G}_N) = U \cap \pi((-V) \times b^{-1} \cdot (\prod \phi_v))$ because of the following reason: For any $x \in \mathbb{A}$, $\pi(x) \in r(U \times (V \times \{b\}) \cap \mathcal{G}_N)$ means that $\pi(x)$ lies in U and that $(\pi(x), (t, b)) \in \mathcal{G}_N$ for some $t \in V$. The second statement is equivalent to: “There exists $t \in V$ with $\pi(x) \in \pi((-t, 0) + b^{-1} \cdot N)$ ”, which is again equivalent to “ $\pi(x) \in \pi((-V) \times b^{-1} \cdot (\prod \phi_v))$ ”. This proves our claim.

But since π is open and $b^{-1} \cdot (\prod \phi_v)$ is open in \mathbb{A}_f , $U \cap \pi((-V) \times b^{-1} \cdot (\prod \phi_v))$ is open in $\mathcal{G}^0 = \mathbb{A}/K$. Therefore, (i) and (ii) hold true and thus \mathcal{G} is equivalent to $\mathcal{G}_N^N = \{(\pi(x), (t, b)) \in \mathbb{A}/K \times (\mathbb{A}_\infty \rtimes \Gamma) \mid \pi(x) \in N \text{ and } \pi(b((t, 0) + x)) \in N\}$.

We study $\tilde{\mathcal{G}}$ in a similar way: Consider the closed subset $\tilde{N} := \prod \phi_v$ of $\Gamma \cdot (\prod \phi_v) = \tilde{\mathcal{G}}^0$. By construction, \tilde{N} meets every orbit in $\tilde{\mathcal{G}}^0$.

Moreover, $\tilde{\mathcal{G}}_{\tilde{N}}$ is given as $\{(z, (a, b)) \in \tilde{\mathcal{G}} \mid b^{-1}(z - a) \in \tilde{N}\}$. Let \tilde{r}, \tilde{s} be the range and source maps of $\tilde{\mathcal{G}}$.

Now $\tilde{s}|_{\tilde{\mathcal{G}}_{\tilde{N}}}$ is open as $\tilde{s}(U \cap \tilde{\mathcal{G}}_{\tilde{N}}) = \tilde{s}(U \cap \tilde{s}^{-1}(\tilde{N})) = \tilde{s}(U) \cap \tilde{N}$ is open in $\tilde{N} = \prod \phi_v$ for any open subset $U \subseteq \tilde{\mathcal{G}}$ because \tilde{s} is open. And $\tilde{r}|_{\tilde{\mathcal{G}}_{\tilde{N}}}$ is open since given any open subset $U \subseteq \Gamma \cdot (\prod \phi_v)$, $\tilde{r}(U \times \{(a, b)\} \cap \tilde{\mathcal{G}}_{\tilde{N}}) = U \cap (a + b(\prod \phi_v))$ is again open in $\Gamma \cdot (\prod \phi_v) = \tilde{\mathcal{G}}^0$. As above, it is sufficient to consider open subsets of this type as they form a basis for the topology of $\tilde{\mathcal{G}}$.

Thus, we have seen that (i) and (ii) hold. This implies that $\tilde{\mathcal{G}}$ is equivalent to $\tilde{\mathcal{G}}_{\tilde{N}}^{\tilde{N}} = \{(z, (a, b)) \in \Gamma \cdot (\prod \phi_v) \times (\Gamma \cdot \phi \rtimes \Gamma) \mid z \in \prod \phi_v \text{ and } b^{-1}(z - a) \in \prod \phi_v\}$.

Finally, we want to show that $\Phi : \tilde{\mathcal{G}}_{\tilde{N}}^{\tilde{N}} \rightarrow \mathcal{G}_N^N$, $(z, (a, b)) \mapsto (\pi(0, z), (a, b^{-1}))$, defines an isomorphism of topological groupoids.

First of all, Φ is well defined as $(z, (a, b)) \in \tilde{\mathcal{G}}_{\tilde{N}}^{\tilde{N}}$ means $b^{-1}(z - a) \in \tilde{N} = \prod \phi_v$ and therefore $\pi(b^{-1}((a, 0) + (0, z))) = \pi(b^{-1}a, b^{-1}z) = \pi(0, b^{-1}z - b^{-1}a) \in N$. Furthermore, Φ is injective since $(\pi(0, z), (a, b^{-1})) = (\pi(0, z'), (c, d^{-1}))$ implies $a = c, b = d$ and $(0, z - z') \in K \iff z = z'$. Φ is surjective: Given $(\pi(x), (t, b))$ in \mathcal{G}_N^N , $\pi(x) \in N$ means that there exists $z \in \prod \phi_v$ with $\pi(x) = \pi(0, z)$. Moreover, we know $\pi(bt, bz) = \pi(b((t, 0) + (0, z))) \in N$. This implies that there exists $z' \in \prod \phi_v$ with $(bt, bz) - (0, z') = (bt, bz - z') \in K$. Thus, $bt \in K$ and $bt = bz - z'$, which yield $t \in K \cap (\Gamma \cdot (\prod \phi_v)) = \Gamma \cdot (K \cap (\prod \phi_v)) = \Gamma \cdot \phi$ by Lemma 2.3. Therefore, we have found $(z, (t, b^{-1})) \in \tilde{\mathcal{G}}_{\tilde{N}}^{\tilde{N}}$ with $\Phi(z, (t, b^{-1})) = (\pi(x), (t, b))$.

Moreover, it is easy to check that Φ respects the groupoid structures as well.

As the last point, we have to check that Φ is a homeomorphism. It is immediate that Φ is continuous. To prove that Φ^{-1} is continuous, choose a sequence $(z_n, (a_n, b_n)) \in$

$\tilde{\mathcal{G}}_{\tilde{N}}^{\tilde{N}}$ with $\lim_{n \rightarrow \infty} \Phi(z_n, (a_n, b_n)) = \Phi(z, (a, b)) \in \mathcal{G}_N^N$ for $(z, (a, b)) \in \tilde{\mathcal{G}}_{\tilde{N}}^{\tilde{N}}$. This means $\lim_{n \rightarrow \infty} (\pi(0, z_n), (a_n, b_n^{-1})) = (\pi(0, z), (a, b^{-1}))$. We have to show $\lim_{n \rightarrow \infty} (z_n, (a_n, b_n)) = (z, (a, b))$ in $\tilde{\mathcal{G}}_{\tilde{N}}^{\tilde{N}}$. As $\lim_{n \rightarrow \infty} b_n^{-1} = b^{-1}$ in the discrete group Γ , we conclude that $b_n = b$ for almost all n . Thus, we can assume without loss of generality that $b_n = b$ for all n . Moreover, we have for all n that a_n lies in K and $b(z_n - a) \in \tilde{N} = \prod \mathfrak{o}_v$. With $b = \frac{l}{m}$ for some l, m in $\mathfrak{o}, m \neq 0$, we conclude that $a_n \in ((\prod \mathfrak{o}_v) + b^{-1} \cdot (\prod \mathfrak{o}_v)) \cap K \subseteq (\frac{1}{l} \cdot (\prod \mathfrak{o}_v)) \cap K = \frac{1}{l} \cdot \mathfrak{o}$ by Lemma 2.3. Since $\frac{1}{l} \cdot \mathfrak{o}$ sits discretely in \mathbb{A}_∞ , it follows that $a_n = a$ for almost all n . Finally, to see that $\lim_{n \rightarrow \infty} z_n = z$ in $\tilde{N} = \prod \mathfrak{o}_v$, observe that $\tilde{N} = \prod \mathfrak{o}_v \rightarrow N, z \mapsto \pi(0, z)$, is a homeomorphism as it is a bijective, continuous map between compact (Hausdorff) spaces.

This shows that $\tilde{\mathcal{G}}_{\tilde{N}}^{\tilde{N}} \cong \mathcal{G}_N^N$ as topological groupoids. Hence, we have shown

$$\tilde{\mathcal{G}} \sim_M \tilde{\mathcal{G}}_{\tilde{N}}^{\tilde{N}} \cong \mathcal{G}_N^N \sim_M \mathcal{G}. \quad \square$$

4.3. End of proof. With these two lemmas, we are ready for the

Proof of Theorem 4.1. Start with the additive action of K on $C_0(\mathbb{A}_\infty)$ given by $\hat{\tau}_a(g) = g(\sqcup - a)$ for all $g \in C_0(\mathbb{A}_\infty)$. Since $\mathbb{A}_\infty \cong \widehat{\mathbb{A}_\infty}$ by Theorem 2.4, (3), Fourier transform yields

$$C^*(\mathbb{A}_\infty) \cong C_0(\mathbb{A}_\infty), \quad C_c(\mathbb{A}_\infty) \ni f \mapsto \left[t \mapsto \int_{\mathbb{A}_\infty} \langle s, t \rangle f(s) ds \right].$$

Under this identification, $\hat{\tau}$ corresponds to the action τ on $C^*(\mathbb{A}_\infty)$ given by $\tau_a(f) = \langle -a, \sqcup \rangle \cdot f$. Thus, we are precisely in the situation of Lemma 4.3 with $G = \mathbb{A}_\infty$, $H = K$ (H viewed as a discrete group). Lemma 4.3 yields

$$C_0(\mathbb{A}_\infty) \rtimes_{\hat{\tau}} K \cong C^*(\mathbb{A}_\infty) \rtimes_{\tau} K \cong C^*(K) \rtimes_{\lambda} \mathbb{A}_\infty$$

with $\lambda_t(u^a) = \langle a, t \rangle u^a$ for all $a \in K, t \in \mathbb{A}_\infty$. Applying again Fourier transform, together with Theorem 2.4, (2), we end up with

$$C^*(K) \rtimes_{\lambda} \mathbb{A}_\infty \cong C(\hat{K}) \rtimes \mathbb{A}_\infty \cong C(\mathbb{A}/K) \rtimes_{\hat{\lambda}} \mathbb{A}_\infty$$

with $\hat{\lambda}_t(f) = f(\pi(t, 0) + \sqcup)$.

So, to sum up these observations, we have an isomorphism

$$C_0(\mathbb{A}_\infty) \rtimes_{\hat{\tau}} K \cong C(\mathbb{A}/K) \rtimes_{\hat{\lambda}} \mathbb{A}_\infty.$$

Now let Γ be a subgroup of K^\times . Under the last identification, the action of Γ on $C_0(\mathbb{A}_\infty) \rtimes_{\hat{\tau}} K$ given by $\hat{\beta}_b(g \cdot u^a) = g(b^{-1}\sqcup) \cdot u^{ab}$ for all $g \in C_0(\mathbb{A}_\infty), a \in K, b \in K^\times$, corresponds to the following action of Γ on $C(\mathbb{A}/K) \rtimes_{\hat{\lambda}} \mathbb{A}_\infty$: $\hat{\alpha}_b([t \mapsto f_t]) = [t \mapsto |N(b)| f_{bt}(b\sqcup)]$ where N denotes the norm on K^\times .

Using [Wil], Proposition 3.11, we deduce

$$(C_0(\mathbb{A}_\infty) \rtimes_{\hat{\tau}} K) \rtimes_{\hat{\beta}} \Gamma \cong (C(\mathbb{A}/K) \rtimes_{\hat{\lambda}} \mathbb{A}_\infty) \rtimes_{\hat{\alpha}} \Gamma \cong C(\mathbb{A}/K) \rtimes (\mathbb{A}_\infty \rtimes \Gamma).$$

The semidirect product $\mathbb{A}_\infty \rtimes \Gamma$ is taken with respect to the action of Γ on \mathbb{A}_∞ which we already had in Theorem 4.1, and the action of $\mathbb{A}_\infty \rtimes \Gamma$ on $C(\mathbb{A}/K)$ is given by $(t, b) \cdot f(x) = f(x \cdot (t, b)) = f(\pi(b((t, 0) + x)))$. Thus, Lemma 4.4, combined with [MRW], Theorem 2.8, yields

$$\begin{aligned} C_0(\mathbb{A}_\infty) \rtimes (K \rtimes \Gamma) &\cong (C_0(\mathbb{A}_\infty) \rtimes_{\hat{\tau}} K) \rtimes_{\hat{\beta}} \Gamma \\ &\cong C(\mathbb{A}/K) \rtimes (\mathbb{A}_\infty \rtimes \Gamma) \\ &\cong C^*(\mathcal{G}) \sim_M C^*(\tilde{\mathcal{G}}) \\ &\cong C_0(\Gamma \cdot (\prod \varphi_v)) \rtimes (\Gamma \cdot \varphi) \rtimes \Gamma. \end{aligned}$$

The first identification follows again from [Wil], Proposition 3.11. Moreover, the first and the last crossed products are given by the corresponding actions via inverse affine transformations. \square

5. Computations “without roots of unity”

Let us concentrate on number fields now. Fix such a field K and let \mathfrak{o} be the ring of integers in K . Before we turn to the whole ring C^* -algebra of \mathfrak{o} , let us study the C^* -algebra “without roots of unity” first. More precisely, we fix a subgroup Γ of K^\times with $K^\times = \mu \times \Gamma$ and consider the decomposition $\mathfrak{o}^\times = \mu \times (\Gamma \cap \mathfrak{o}^\times)$. Note that $\mu \subseteq \mathfrak{o}$ by definition. Then we consider $\mathfrak{B} = C^*(\{u^a, s_b \mid a \in \mathfrak{o}, b \in \Gamma \cap \mathfrak{o}^\times\})$. This C^* -subalgebra is generated by all unitaries given by addition but only those isometries which come from the torsion-free part Γ of K^\times . We point out that \mathfrak{B} depends on the choice of Γ .

The reason why we first compute the K -groups of \mathfrak{B} is twofold: On the one hand, it is possible to carry out the calculation in complete generality, in contrast to the computation of $K_*(\mathfrak{A})$ (at least up to now). On the other hand, some of the results which we prove along our way to determining $K_*(\mathfrak{B})$ will enter into the calculation of $K_*(\mathfrak{A})$ later on.

As far as our strategy is concerned, we more or less follow the program described in Remark 3.16. This means that we will first compute the K -theory of $\mathfrak{B}^{(0)} := C^*(\{u^a\}, \{e_b\})$ via a suitable filtration and then use Theorem 4.1 to adjoin the isometries by a homotopy argument.

The final result is

$$K_*(\mathfrak{B}) \cong \begin{cases} \Lambda^*(\Gamma) & \text{if } \#\{v_{\mathbb{R}}\} = 0, \\ \Lambda^*(\Gamma) & \text{if } \#\{v_{\mathbb{R}}\} \text{ odd and } \#\{v_{\mathbb{R}} \mid v_{\mathbb{R}}(b) < 0\} \text{ even } \forall b \in \Gamma, \\ (\mathbb{Z}/2\mathbb{Z}) \otimes_{\mathbb{Z}} \Lambda^*(\Gamma) & \text{if } \#\{v_{\mathbb{R}}\} \text{ odd and } \exists b \in \Gamma: \#\{v_{\mathbb{R}} \mid v_{\mathbb{R}}(b) < 0\} \text{ odd,} \\ (\mathbb{Z}/2\mathbb{Z}) \otimes_{\mathbb{Z}} \Lambda^*(\Gamma) & \text{if } \#\{v_{\mathbb{R}}\} \geq 2 \text{ even.} \end{cases}$$

Here, $\mathbb{Z}/2\mathbb{Z}$ is trivially graded and we consider graded tensor products.

5.1. K-theory of $\mathfrak{B}^{(0)}$. As a first step, the K-theory of $\mathfrak{B}^{(0)}$ can be computed with the help of a suitable filtration.

Lemma 5.1.

$$K_j(\mathfrak{B}^{(0)}) \cong \begin{cases} \mathbb{Q}^{2^{n-1}} & \text{if } j \equiv n+1 \pmod{2}, \\ \mathbb{Q}^{2^{n-1}-1} \oplus \mathbb{Z} & \text{if } j \equiv n \pmod{2}. \end{cases}$$

If $\omega_1, \dots, \omega_n$ is a \mathbb{Z} -basis for \mathfrak{o} and if we write $u(i) := u^{\omega_i}$, then the copy of \mathbb{Z} will be generated by $[u(1)]_1 \times \dots \times [u(n)]_1 \in K_n(\mathfrak{B}^{(0)})$. Here, we take the product on K-theory as it is described in [HiRo], 4.7.

Proof. By relation (II), p. 5, we have $e_b = \sum u^{ba} e_{bd} u^{-ba}$ where we sum over $\mathfrak{o}/(d) = \{a + (d)\}$. Thus, we obtain the inductive system $\{C^*(\{u^a\}, e_b); \iota_{b,bd}\}$ with the inclusions $\iota_{b,bd} : C^*(\{u^a\}, e_b) \hookrightarrow C^*(\{u^a\}, e_{bd})$. The associated inductive limit coincides with $C^*(\{u^a\}, \{e_b\}) = \mathfrak{B}^{(0)}$. Hence, we have to determine $K_*(C^*(\{u^a\}, e_b))$ for single b as well as $(\iota_{b,bd})_*$.

First of all, $C^*(\{u^a\}) \cong C^*(\mathfrak{o}) \cong C^*(\mathbb{Z}^n)$. Thus, $K_*(C^*(\{u^a\})) \cong \mathbb{Z}^{2^n}$ and $\{[u(i_1)]_1 \times \dots \times [u(i_k)]_1 \mid 1 \leq i_1 < \dots < i_k \leq n, 0 \leq k \leq n\}$ is a \mathbb{Z} -basis. For $k = 0$ we get $[1]_0$. Moreover, we have

Lemma 5.2. For any $b \in \mathbb{Z}_{>0}$, we have $C^*(\{u^a\}, e_b) \cong M_{b^n}(C^*(\{u^a\}))$.

Proof of Lemma 5.2. Choose a minimal system \mathcal{R}_b of representatives for $\mathfrak{o}/(b)$ in \mathfrak{o} , in the sense that for any c, c' in \mathcal{R}_b , $c - c' \in (b)$ implies $c = c'$. We always assume $0 \in \mathcal{R}_b$. From this data – using $\ell^2(\mathfrak{o}) \cong \ell^2(c + (b))$, $\xi_r \mapsto \xi_{c+br}$ – we construct a unitary $\ell^2(\mathfrak{o}) \cong \bigoplus_{c \in \mathcal{R}_b} \ell^2(c + (b)) \cong \bigoplus_{\mathcal{R}_b} \ell^2(\mathfrak{o})$. Conjugation with this unitary yields an isomorphism

$$\varphi_b : \mathcal{L}(\ell^2(\mathfrak{o})) \cong M_{b^n}(\mathcal{L}(\ell^2(\mathfrak{o}))), \quad T \mapsto (s_b^* u^{-c} T u^{c'} s_b)_{c,c'}.$$

We now show that $\varphi_b(C^*(\{u^a\}, e_b)) = M_{b^n}(C^*(\{u^a\}))$: The universal property of $M_{b^n}(C^*(\{u^a\})) \cong M_{b^n}(\mathbb{C}) \otimes C^*(\{u^a\})$ gives rise to a homomorphism $\phi_b : M_{b^n}(C^*(\{u^a\})) \rightarrow C^*(\{u^a\}, e_b)$ via $\phi_b(e_{c,c'} \otimes 1) = u^c e_b u^{-c'}$, $\phi_b(1 \otimes u^a) = u^{ba}$. ϕ_b is surjective, and we have $\varphi_b \circ \phi_b = \text{id}_{M_{b^n}(C^*(\{u^a\}, e_b))}$ by construction. This implies that $\varphi_b(C^*(\{u^a\}, e_b)) = M_{b^n}(C^*(\{u^a\}))$. \square

Thus, $C^*(\{u^a\}) \xrightarrow{e_{0,0} \otimes \text{id}} M_{b^n}(C^*(\{u^a\})) \xrightarrow{\phi_b} C^*(\{u^a\}, e_b)$ induces an isomor-

phism on K-theory. By continuity of K_* , we get

$$\begin{aligned} K_*(\mathfrak{B}^{(0)}) &\cong \lim_{\mathbb{Z}_{>0}} \{K_*(C^*(\{u^a\}, e_b)); (\iota_{b,bd})_*\} \\ &\cong \lim_{\mathbb{Z}_{>0}} \{K_*(C^*(\{u^a\})); (e_{0,0} \otimes \text{id})_*^{-1} \circ (\varphi_{bd})_* \\ &\quad \circ (\iota_{b,bd})_* \circ (\phi_b)_* \circ (e_{0,0} \otimes \text{id})_*\}, \end{aligned}$$

where we used $\phi_{bd}^{-1} = \varphi_{bd}$ (see Lemma 5.2). It suffices to take the inductive limit over $\mathbb{Z}_{>0}$ as $\mathbb{Z}_{>0}$ is cofinal in \mathcal{O}^\times .

To understand the structure maps, note that by a modified version of Lemma 5.2, $\phi_{bd} \circ (e_{0,0} \otimes \text{id})$ can be written as the composition

$$\begin{aligned} C^*(\{u^a\}) &\xrightarrow{e_{0,0} \otimes \text{id}} M_{d^n}(C^*(\{u^a\})) \xrightarrow{\phi_d} C^*(\{u^a\}, e_d) \\ &\xrightarrow{e_{0,0} \otimes \text{id}} M_{b^n}(C^*(\{u^a\}, e_d)) \xrightarrow{\phi_b} C^*(\{u^a\}, e_{bd}) \end{aligned}$$

and that the diagram

$$\begin{array}{ccc} C^*(\{u^a\}) & \xrightarrow{\phi_b \circ (e_{0,0} \otimes \text{id})} & C^*(\{u^a\}, e_b) \\ \iota_{1,d} \downarrow & & \downarrow \iota_{b,bd} \\ C^*(\{u^a\}, e_d) & \xrightarrow{\phi_b \circ (e_{0,0} \otimes \text{id})} & C^*(\{u^a\}, e_{bd}) \end{array}$$

commutes (these observations follow from $\phi_b \circ (e_{0,0} \otimes \text{id}) = \text{Ad}(s_b)$). Thus,

$$\begin{aligned} &(e_{0,0} \otimes \text{id})_*^{-1} \circ (\varphi_{bd})_* \circ (\iota_{b,bd})_* \circ (\phi_b)_* \circ (e_{0,0} \otimes \text{id})_* \\ &= (e_{0,0} \otimes \text{id})_*^{-1} \circ (\varphi_d)_* \circ (e_{0,0} \otimes \text{id})_*^{-1} \circ (\varphi_b)_* \circ (\phi_b)_* \circ (e_{0,0} \otimes \text{id})_* \circ (\iota_{1,d})_* \\ &= (e_{0,0} \otimes \text{id})_*^{-1} \circ (\varphi_d)_* \circ (\iota_{1,d})_*. \end{aligned}$$

Therefore, with $\iota_d := \iota_{1,d}$, it remains to determine $(e_{0,0} \otimes \text{id})_*^{-1} \circ (\varphi_d)_* \circ (\iota_d)_*$.

Now define $\nu_d: C^*(\{u^a\}) \rightarrow C^*(\{u^a\}); u^a \mapsto u^{da}$ for $d \in \mathbb{Z}_{>0}$. Functoriality of the K-theoretic product yields

$$(\nu_d)_*([u(i_1)]_1 \times \cdots \times [u(i_k)]_1) = d^k [u(i_1)]_1 \times \cdots \times [u(i_k)]_1.$$

The crucial observation for our purposes is that we have

$$\varphi_d \circ \iota_d \circ \nu_d = 1_{d^n} \otimes \text{id}. \quad (14)$$

This follows from the construction. We get

$$(e_{0,0} \otimes \text{id})_*^{-1} \circ (\varphi_d)_* \circ (\iota_d)_* \circ (\nu_d)_* = (e_{0,0} \otimes \text{id})_*^{-1} \circ (1_{d^n} \otimes \text{id})_* = d^n \cdot \text{id}_*$$

and thus $(e_{0,0} \otimes \text{id})_*^{-1} \circ (\varphi_d)_* \circ (\iota_d)_* = d^n \cdot (v_d)_*^{-1}$. We conclude that

$$\begin{aligned} & (e_{0,0} \otimes \text{id})_*^{-1} \circ (\varphi_d)_* \circ (\iota_d)_* ([u(i_1)]_1 \times \cdots \times [u(i_k)]_1) \\ &= d^{n-k} \cdot ([u(i_1)]_1 \times \cdots \times [u(i_k)]_1). \end{aligned} \quad (15)$$

This allows us to calculate the K-groups of $\mathfrak{B}^{(0)}$, and we get the desired results. \square

5.2. K-theory of \mathfrak{B} . We distinguish between the following cases:

1. $\#\{v_{\mathbb{R}}\} = 0$.

By [Neu], Chapter I, Proposition (8.4), there are infinitely many primes in $\mathbb{Z} \subseteq \mathfrak{o}$ which are unramified in \mathfrak{o} . Hence, since μ is finite, by taking an appropriate product of unramified primes, we find a number $p \in \mathbb{Z}_{>0}$ in Γ which we can extend to a \mathbb{Z} -basis for Γ of the form $\{p, p_1, p_2, \dots\}$.

Let Γ_m be the subgroup $\langle p, p_1, \dots, p_m \rangle$ of Γ and consider $\mathfrak{B}^{(1)} := C^*(\mathfrak{B}^{(0)}, s_p)$.

Since $\text{Ad}(s_p)_*([u(i_1)]_1 \times \cdots \times [u(i_k)]_1) = p^{k-n}[u(i_1)]_1 \times \cdots \times [u(i_k)]_1$ by (15) and

$$\begin{aligned} \mathfrak{B}^{(1)} &\cong \mathfrak{B}^{(0)} \rtimes_{\text{Ad}(s_p)}^e \mathbb{N} \sim_M \varinjlim \{\mathfrak{B}^{(0)}; \text{Ad}(s_p)\} \rtimes_{\text{Ad}(s_p)} \mathbb{Z} \\ &\sim_M C_0(\Gamma_0 \cdot (\prod \mathfrak{o}_v)) \rtimes \Gamma_0 \cdot \mathfrak{o} \rtimes \Gamma_0 \end{aligned}$$

(similar to (4)), we conclude that $K_j(\mathfrak{B}^{(1)}) \cong \mathbb{Z}$ for $j = 0, 1$. This result, Lemma 5.1 and Theorem 4.1 show that

$$\begin{aligned} K_j(C_0(\mathbb{A}_{\infty}) \rtimes K) &\cong K_j(C(\prod \mathfrak{o}_v) \rtimes \mathfrak{o}) \\ &\cong K_j(\mathfrak{B}^{(0)}) \cong \begin{cases} \mathbb{Q}^{2^{n-1}} & \text{for } j \equiv n+1 \pmod{2}, \\ \mathbb{Q}^{2^{n-1}-1} \oplus \mathbb{Z} & \text{for } j \equiv n \pmod{2}; \end{cases} \end{aligned} \quad (16)$$

$$\begin{aligned} K_j(C_0(\mathbb{A}_{\infty}) \rtimes K \rtimes \Gamma_0) &\cong K_j(C(\Gamma_0 \cdot (\prod \mathfrak{o}_v)) \rtimes (\Gamma_0 \cdot \mathfrak{o}) \rtimes \Gamma_0) \\ &\cong K_j(\mathfrak{B}^{(1)}) \cong \mathbb{Z} \text{ for } j = 0, 1. \end{aligned} \quad (17)$$

Moreover, as the multiplicative action of K^{\times} on \mathbb{A}_{∞} is homotopic to the trivial action, the Pimsner–Voiculescu sequence yields $K_j(C_0(\mathbb{A}_{\infty}) \rtimes \Gamma_0) \cong \mathbb{Z}$ ($j = 0, 1$). Now we claim that $K_j(C_0(\mathbb{A}_{\infty}) \rtimes K \rtimes \Gamma_m) \cong K_j(C_0(\mathbb{A}_{\infty}) \rtimes \Gamma_m) \cong \mathbb{Z}^{2^m}$ for $j = 0, 1$ and that $i_m: C_0(\mathbb{A}_{\infty}) \rtimes \Gamma_m \rightarrow C_0(\mathbb{A}_{\infty}) \rtimes K \rtimes \Gamma_m$, the map induced by the K^{\times} -covariant inclusion $C_0(\mathbb{A}_{\infty}) \hookrightarrow C_0(\mathbb{A}_{\infty}) \rtimes K$, yields an injective map on K-theory. Let us prove this assertion by induction on m .

$m = 0$: We already know $K_j(C_0(\mathbb{A}_{\infty}) \rtimes K \rtimes \Gamma_0) \cong K_j(C_0(\mathbb{A}_{\infty}) \rtimes \Gamma_0) \cong \mathbb{Z}$ for $j = 0, 1$. It remains to show the injectivity of $(i_0)_*$: By (16), (17) and the Pimsner–Voiculescu sequence, $C_0(\mathbb{A}_{\infty}) \rtimes K \hookrightarrow C_0(\mathbb{A}_{\infty}) \rtimes K \rtimes \Gamma_0$ must be nontrivial on the copy of \mathbb{Z} in K_j ($j \equiv n \pmod{2}$). Moreover, $C_0(\mathbb{A}_{\infty}) \hookrightarrow C_0(\mathbb{A}_{\infty}) \rtimes \mathfrak{o}$ is injective on K-theory by the Pimsner–Voiculescu sequence. And we have $C_0(\mathbb{A}_{\infty}) \rtimes K \cong \varinjlim \{C_0(\mathbb{A}_{\infty}) \rtimes \mathfrak{o}\}$ where the structure maps fix the image of $C_0(\mathbb{A}_{\infty})$ in $C_0(\mathbb{A}_{\infty}) \rtimes \mathfrak{o}$ on K-theory. It follows that $C_0(\mathbb{A}_{\infty}) \hookrightarrow C_0(\mathbb{A}_{\infty}) \rtimes K$ is injective on K-theory and

its image is fixed by Γ_0 in K-theory and thus must be contained in the copy of \mathbb{Z} ($j \equiv n \pmod{2}$). Thus, $(i_0)_*$ is injective.

Now assume that we have proven the claim for m . The diagram

$$\begin{array}{ccc} K_j(C_0(\mathbb{A}_\infty) \rtimes \Gamma_m) & \xrightarrow{0} & K_j(C_0(\mathbb{A}_\infty) \rtimes \Gamma_m) \\ (i_m)_* \downarrow & & \downarrow (i_m)_* \\ K_j(C_0(\mathbb{A}_\infty) \rtimes K \rtimes \Gamma_m) & \xrightarrow{\text{id} - (\hat{\beta}_{p_{m+1}})_*^{-1}} & K_j(C_0(\mathbb{A}_\infty) \rtimes K \rtimes \Gamma_m) \end{array}$$

commutes, where we used that the multiplicative action of K^\times on $C_0(\mathbb{A}_\infty)$ is equivariantly homotopic to the identity. As $K_j(C_0(\mathbb{A}_\infty) \rtimes K \rtimes \Gamma_m)$ ($j = 0, 1$) is torsion-free and $(i_m)_*$ is injective by hypothesis, it follows that $(\hat{\beta}_{p_{m+1}})_* = \text{id}_{K_*(C_0(\mathbb{A}_\infty) \rtimes K \rtimes \Gamma_m)}$. Thus, we get $K_j(C_0(\mathbb{A}_\infty) \rtimes K \rtimes \Gamma_{m+1}) \cong \mathbb{Z}^{2^{m+1}}$ for $j = 0, 1$ by the Pimsner–Voiculescu sequence. Injectivity of $(i_{m+1})_* = \begin{pmatrix} (i_m)_* & * \\ 0 & (i_m)_* \end{pmatrix}$ follows by induction hypothesis. This proves our claim.

Therefore,

$$K_*(C_0(\mathbb{A}_\infty) \rtimes K \rtimes \Gamma) \cong \lim_{\substack{\longrightarrow \\ m}} K_*(C_0(\mathbb{A}_\infty) \rtimes K \rtimes \Gamma_m) \cong \lim_{\substack{\longrightarrow \\ m}} \Lambda^*(\Gamma_m) \cong \Lambda^*(\Gamma).$$

Since $\mathfrak{B} \sim_M C_0(\mathbb{A}_f) \rtimes K \rtimes \Gamma \sim_M C_0(\mathbb{A}_\infty) \rtimes K \rtimes \Gamma$ by the analogue of (4) and Theorem 4.1, we conclude that $K_*(\mathfrak{B}) \cong \Lambda^*(\Gamma)$.

The remaining cases are very similar to the first one. The only difference lies in the fact that the multiplicative action does not need to be homotopic to the identity any more.

2. $\#\{v_{\mathbb{R}}\}$ odd and $\#\{v_{\mathbb{R}} \mid v_{\mathbb{R}}(b) < 0\}$ is even for all $b \in \Gamma$.

We still have that the action of Γ is equivariantly homotopic to the identity on $C_0(\mathbb{A}_\infty)$. Thus, we can adapt the computations of the first case. Again,

$$K_*(\mathfrak{B}) \cong \Lambda^*(\Gamma).$$

3. $\#\{v_{\mathbb{R}}\}$ odd and $\#\{v_{\mathbb{R}} \mid v_{\mathbb{R}}(b) < 0\}$ is odd for some $b \in \Gamma$.

Choose a basis $\{p, p_1, \dots\}$ of Γ as in the first case ($\#\{v_{\mathbb{R}}\} = 0$), but we additionally require $\#\{v_{\mathbb{R}} \mid v_{\mathbb{R}}(p_1) < 0\}$ to be odd and $\#\{v_{\mathbb{R}} \mid v_{\mathbb{R}}(p_i) < 0\}$ to be even for all $i > 1$. Now let $\Gamma_m = \langle p, p_1, \dots, p_m \rangle \subseteq \Gamma$ as above, and let Γ'_m be the subgroup generated by $\{p, p_2, \dots, p_m\}$ such that $\Gamma_m = \Gamma'_m \times \langle p_1 \rangle$.

We have $\hat{\beta}_{p_m} \sim_h \hat{\beta}_{t_m}$ where $t_m := (v(p_m)/|v(p_m)|)_{v|\infty}$. Here, we extend the multiplicative action of K^\times on $C_0(\mathbb{A}_\infty)$ to \mathbb{A}_∞^\times and still denote this action by $\hat{\beta}$. For $m \neq 1$, $\hat{\beta}_{t_m}$ is of period 2 and induces the identity on $K_*(C_0(\mathbb{A}_\infty))$. Thus, going through the Pimsner–Voiculescu sequences, we get that $\hat{\beta}_{t_m}$ and therefore $\hat{\beta}_{p_m}$ induces the identity on $K_*(C_0(\mathbb{A}_\infty) \rtimes \Gamma'_{m-1})$.

Hence, for the same reasons as above (in the first case), we get

$$K_*(C_0(\mathbb{A}_\infty) \rtimes K \rtimes \Gamma'_m) \cong \Lambda^*(\Gamma'_m).$$

But now we have to add p_1 . $\hat{\beta}_{p_1}$ induces $-\text{id}$ on $K_*(C_0(\mathbb{A}_\infty) \rtimes \Gamma'_m)$ as the map is of order 2 on K-theory but gives $-\text{id}$ on $K_*(C_0(\mathbb{A}_\infty))$. So, using the Pimsner–Voiculescu sequence again, we see that $\hat{\beta}_{p_1}$ induces $-\text{id}$ on $K_*(C_0(\mathbb{A}_\infty) \rtimes \Gamma'_m)$ and thus on $K_*(C_0(\mathbb{A}_\infty) \rtimes K \rtimes \Gamma'_m)$ as well. Hence we get

$$K_*(C_0(\mathbb{A}_\infty) \rtimes K \rtimes \Gamma_m) \cong (\mathbb{Z}/2\mathbb{Z}) \otimes_{\mathbb{Z}} \Lambda^*(\Gamma'_m)$$

and finally, in the inductive limit,

$$K_*(\mathfrak{B}) \cong K_*(C_0(\mathbb{A}_\infty) \rtimes K \rtimes \Gamma) \cong (\mathbb{Z}/2\mathbb{Z}) \otimes_{\mathbb{Z}} \Lambda^*(\Gamma).$$

4. $\#\{v_{\mathbb{R}}\} \geq 2$ even.

Since $\mathbb{R} \cdot K = \mathbb{A}_\infty$, there must be some $b \in \Gamma$ with $\#\{v_{\mathbb{R}} \mid v_{\mathbb{R}}(b) < 0\}$ odd. Then the same arguments as in the previous case show that

$$K_*(\mathfrak{B}) \cong K_*(C_0(\mathbb{A}_\infty) \rtimes K \rtimes \Gamma) \cong (\mathbb{Z}/2\mathbb{Z}) \otimes_{\mathbb{Z}} \Lambda^*(\Gamma).$$

6. General results for $\mu = \{\pm 1\}$

As in the previous section, let K be a number field of degree $n = [K : \mathbb{Q}]$ and let \mathfrak{o} be the ring of integers in K . We follow the program of Remark 3.16 and compute the K-groups of the ring C*-algebra associated to \mathfrak{o} under the assumption that the only roots of unity in K are ± 1 ($\mu = \{\pm 1\}$). We will explain below why we cannot treat the general case up to now.

The final result (with $K^\times = \mu \times \Gamma$) is as follows:

$$K_*(\mathfrak{A}) \cong \begin{cases} K_0(C^*(\mu)) \otimes_{\mathbb{Z}} \Lambda^*(\Gamma) & \text{if } \#\{v_{\mathbb{R}}\} = 0, \\ \Lambda^*(\Gamma) & \text{if } \#\{v_{\mathbb{R}}\} \text{ odd}, \\ \Lambda^*(\Gamma) \oplus ((\mathbb{Z}/2\mathbb{Z}) \otimes_{\mathbb{Z}} \Lambda^*(\Gamma)) & \text{if } \#\{v_{\mathbb{R}}\} \geq 2 \text{ even.} \end{cases}$$

Here we consider graded tensor products where $K_0(C^*(\mu))$ and $\mathbb{Z}/2\mathbb{Z}$ are trivially graded. We take the diagonal grading on the direct sum. Moreover, note that the condition “ $\#\{v_{\mathbb{R}}\}$ is odd” is equivalent to “ n is odd”.

6.1. The K-theory of $\mathfrak{A}^{(0)}$. The first step is to calculate the K-groups of $C^*(\{u^a, s_{-1}\})$, or – what amounts to the same – of the group C*-algebra of $\mathfrak{o} \rtimes \mu \cong \mathbb{Z}^n \rtimes \mathbb{Z}/2\mathbb{Z}$. Let $\omega_1, \dots, \omega_n$ be a \mathbb{Z} -basis of \mathfrak{o} and set $u(i) = u^{\omega_i}$ for $1 \leq i \leq n$. We thank W. Lück for pointing out the following result to us.

Lemma 6.1. *The K-theory of $C^*(\mathfrak{o} \rtimes \mu)$ is given by*

$$K_0(C^*(\mathfrak{o} \rtimes \mu)) = G_{\text{fin}} \oplus G_{\text{inf}},$$

where $G_{\text{fin}} \cong \mathbb{Z}^{2^n}$ is the part coming from finite subgroups and $G_{\text{inf}} \cong \mathbb{Z}^{2^n-1}$. $K_1(C^*(\vartheta \rtimes \mu))$ is trivial.

Moreover, if we identify $C^*(\vartheta \rtimes \mu)$ with $C^*(\{u^a\}, s_{-1})$, we obtain the following projections whose classes in $K_0(C^*(\{u^a\}, s_{-1}))$ form a \mathbb{Z} -basis for G_{fin} :

$$\frac{1}{2}(1 + u(i_1) \dots u(i_k)s_{-1}) \quad \text{where } 1 \leq i_1 < \dots < i_k \leq n, 0 \leq k \leq n.$$

Furthermore, the inclusion

$$i: C^*(\{u^a\}) \cong C^*(\vartheta) \hookrightarrow C^*(\vartheta \rtimes \mu) \cong C^*(\{u^a\}, s_{-1})$$

maps $K_0(C^*(\{u^a\}))$ into G_{inf} injectively. Thus, its cokernel is finite.

Proof. See [ELPW], Theorem 0.4. □

The next step is to compute the K-theory of $\mathfrak{A}^{(0)} := C^*(\{u^a\}, s_{-1}, \{e_b\})$.

Lemma 6.2.

$$K_0(\mathfrak{A}^{(0)}) \cong \begin{cases} \mathbb{Z} \oplus \mathbb{Q}^{2^n-1} & \text{if } n \text{ odd,} \\ \mathbb{Z} \oplus \mathbb{Q}^{2^n-1-1} \oplus \mathbb{Z} & \text{if } n \text{ even.} \end{cases}$$

$K_1(\mathfrak{A}^{(0)})$ is trivial.

Proof. Again, $\mathfrak{A}^{(0)} \cong \varinjlim \{C^*(\{u^a\}, s_{-1}, e_b)\}$ and with analogous arguments as in the proof of Lemma 5.1, we get

$$K_*(\mathfrak{A}^{(0)}) \cong \varinjlim_{d \in \mathbb{Z}_{>0}} \{K_*(C^*(\{u^a\}, s_{-1})); (e_{0,0} \otimes \text{id})_*^{-1}(\varphi_d)_*(\iota_d)_*\}.$$

It remains to determine

$$\kappa_d := (e_{0,0} \otimes \text{id})_*^{-1}(\varphi_d)_*(\iota_d)_*: K_*(C^*(\{u^a\}, s_{-1})) \rightarrow K_*(C^*(\{u^a\}, s_{-1}))$$

for $d \in \mathbb{Z}_{>0}$. In our computations of $K_*(\mathfrak{B}^{(0)})$ in Lemma 5.1, we have already seen in (15) that

$$\kappa_d([u(i_1)]_1 \times \dots \times [u(i_k)]_1) = d^{n-k} [u(i_1)]_1 \times \dots \times [u(i_k)]_1$$

for $1 \leq i_1 < \dots < i_k \leq n, 0 \leq k \leq n$. Thus, by Lemma 6.1, $\kappa_d|_{K_0(C^*(\{u^a\}))}$ is given by a diagonal matrix whose entries are powers of d . Among these diagonal entries, $1 = d^0$ appears if and only if n is even, this entry then corresponds to $([u(1)]_1 \times \dots \times [u(n)]_1) \in K_n(C^*(\{u^a\}, s_{-1}))$.

To determine $\kappa_d|_{G_{\text{fin}}}$, we distinguish between two cases:

1. $d = 2$: As in the proof of Lemma 5.2, we choose a set of representatives for $\mathfrak{o}/(2)$ in \mathfrak{o} , $\mathcal{R}_2 := \{0, \omega_1, \omega_2, \omega_1 + \omega_2, \omega_3, \dots, \omega_1 + \dots + \omega_n\}$. With this choice, and under analogous identifications as in Lemma 5.2,

$$\varphi_2 \circ \iota_2(s_{-1}) = \begin{pmatrix} u(1)^* s_{-1} & & & & \\ & u(2)^* s_{-1} & & & \\ & & u(1)^* u(2)^* s_{-1} & & \\ & & & \ddots & \\ & & & & u(1)^* \dots u(n)^* s_{-1} \end{pmatrix}.$$

This shows that $\kappa_2([\frac{1}{2}(1 + s_{-1})]_0) = \sum [\frac{1}{2}(1 + u(i_1) \dots u(i_k) s_{-1})]_0$ where the sum is taken over all $1 \leq i_1 < \dots < i_k \leq n, 0 \leq k \leq n$.

Now the remaining symmetries $u(i_1) \dots u(i_k) s_{-1}$ with $1 \leq i_1 < \dots < i_k \leq n, 1 \leq k \leq n$ map $\ell^2(c + (2))$ bijectively into $\ell^2(c' + (2))$ for $c \neq c'$ in \mathcal{R}_2 . Thus, a typical building block in $\varphi_2 \circ \iota_2(u(i_1) \dots u(i_k) s_{-1})$ is of the form

$$\begin{pmatrix} 0 & \dots & V^* \\ \vdots & 0 & \vdots \\ V & \dots & 0 \end{pmatrix}$$

for a unitary V . As

$$\frac{1}{2} \begin{pmatrix} 1 & 0 & \dots & 0 & V^* \\ 0 & & & & 0 \\ \vdots & & 0 & & \vdots \\ 0 & & & & 0 \\ V & 0 & \dots & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & & & \\ \vdots & & 0 & \\ 0 & & & \end{pmatrix},$$

we get $\kappa_2([\frac{1}{2}(1 + u(i_1) \dots u(i_k))]_0) = 2^{n-1}[1]_0$. Thus, κ_2 is given by

$$\left(\begin{array}{ccc|ccc} 1 & & & & & \\ \vdots & & & & & \\ \hline 0 & 2^{n-1} & \dots & 2^{n-1} & 2^n & \\ \vdots & & & & \ddots & \\ 0 & & & & & 2^? \end{array} \right),$$

where $?$ is 0 or 1 depending on the parity of n .

2. Let d be odd, say $d = 2d' + 1$:

This time let \mathcal{R}_d be $\{\sum_{m=1}^n l_m \omega_m \mid -d' \leq l_m \leq d'\}$, a set of representatives for $\mathfrak{o}/(d)$ in \mathfrak{o} as in the proof of Lemma 5.2.

For any $0 \leq k \leq n, 1 \leq i_1 < \dots < i_k \leq n, u(i_1) \dots u(i_k) s_{-1}$ maps $\ell^2(c + (d))$ bijectively into itself if and only if $c = -\sum_{m=1}^k d' \omega_{i_m}$, and a calculation shows

that for this $c \in \mathcal{R}_d$, $u(i_1) \dots u(i_k) s_{-1}$ acts on $\ell^2(c + (d))$ like $u(i_1)^* \dots u(i_k)^* s_{-1}$ under the identification $\ell^2(\varphi) \cong \ell^2(c + (d))$, $\xi_r \mapsto \xi_{c+dr}$.

Thus, $\kappa_d([\frac{1}{2}(1+u(i_1) \dots u(i_k) s_{-1})]_0) = [\frac{1}{2}(1+u(i_1) \dots u(i_k) s_{-1})]_0 + \frac{d^n-1}{2}[1]_0$. Then κ_d is given by

$$\left(\begin{array}{ccc|ccc} 1 & & & & & \\ & \ddots & & & 0 & \\ & & 1 & & & \\ \hline \frac{d^n-1}{2} & \dots & \frac{d^n-1}{2} & d^n & & \\ & & 0 & & \ddots & \\ & & & & & d^? \end{array} \right),$$

where ? is 0 or 1 depending on the parity of n .

Therefore, κ_{2d} (for d odd) is represented by the matrix

$$\left(\begin{array}{cccc|ccc} 1 & & & & & & \\ \vdots & & & & 0 & & 0 \\ 1 & & & & & & \\ \hline 2^{n-1}d^n - 2^{n-1} & (2d)^{n-1} & \dots & (2d)^{n-1} & (2d)^n & & \\ \vdots & & & 0 & & \ddots & \\ 0 & & & & & & (2d)^? \end{array} \right)$$

where ? is 0 or 1 depending on the parity of n .

The result on the K-theory of $\mathfrak{A}^{(0)}$ follows. \square

Our computations show that for any $d \in \mathbb{Z}_{>1}$, $\text{Ad}(s_d)$ induces id on the copies of \mathbb{Z} and multiplies the generators of \mathbb{Q} by some constant > 1 in K_0 . Thus,

$$K_j(C^*(\{u^a\}, s_{-1}, \{e_b\}, s_d)) \cong \begin{cases} \mathbb{Z} & \text{if } n \text{ odd,} \\ \mathbb{Z}^2 & \text{for } n \text{ even} \end{cases} \quad (18)$$

for $j = 0, 1$. Here we have used the Pimsner–Voiculescu sequence together with the description $C^*(\{u^a\}, s_{-1}, \{e_b\}, s_d) \sim_M \varinjlim \{\mathfrak{A}^{(0)}; \text{Ad}(s_d)\} \rtimes_{\text{Ad}(s_d)} \mathbb{Z}$.

6.2. Comparison between infinite and finite places. The main result (Theorem 4.1) has already been proven in Section 4. Applying Theorem 4.1 to our situation, we obtain, together with Lemma 6.2,

$$K_0(C_0(\mathbb{A}_\infty) \rtimes K \rtimes \mu) \cong \begin{cases} \mathbb{Z} \oplus \mathbb{Q}^{2^{n-1}} & \text{if } n \text{ odd,} \\ \mathbb{Z} \oplus \mathbb{Q}^{2^{n-1}-1} \oplus \mathbb{Z} & \text{if } n \text{ even,} \end{cases}$$

$$K_1(C_0(\mathbb{A}_\infty) \rtimes K \rtimes \mu) \cong \{0\}$$

and, using (18),

$$K_j(C_0(\mathbb{A}_\infty) \rtimes K \rtimes (\mu \times \langle d \rangle)) \cong \begin{cases} \mathbb{Z} & \text{if } n \text{ odd,} \\ \mathbb{Z}^2 & \text{if } n \text{ even} \end{cases}$$

for $j = 0, 1$. Furthermore, studying the Pimsner–Voiculescu sequence associated to

$$C_0(\mathbb{A}_\infty) \rtimes K \rtimes (\mu \times \langle d \rangle) \cong (C_0(\mathbb{A}_\infty) \rtimes K \rtimes \mu) \times \langle d \rangle$$

($\langle d \rangle \cong \mathbb{Z}$), we get that the only $\mathbb{Q}_{>0}$ -invariant part of $K_0(C_0(\mathbb{A}_\infty) \rtimes K \rtimes \mu)$ is given by the copies of \mathbb{Z} . We will need this observation later on.

6.3. The multiplicative action. By equivariant Bott periodicity, we obtain

$$K_0(C_0(\mathbb{A}_\infty) \rtimes \mu) \cong \begin{cases} \mathbb{Z} & \text{if } n \text{ odd,} \\ \mathbb{Z}^2 & \text{if } n \text{ even,} \end{cases}$$

and

$$K_1(C_0(\mathbb{A}_\infty) \rtimes \mu) \cong \{0\}$$

since $\mathbb{A}_\infty \cong \mathbb{R}^n$ (compare Section 2.1).

Now consider $i : C_0(\mathbb{A}_\infty) \rtimes \mu \rightarrow C_0(\mathbb{A}_\infty) \rtimes K \rtimes \mu$, the homomorphism induced by the inclusion $C_0(\mathbb{A}_\infty) \hookrightarrow C_0(\mathbb{A}_\infty) \rtimes K$. Our aim is to prove that i_* is injective on K-theory.

First of all, note that it suffices to prove that $i' : C_0(\mathbb{A}_\infty) \rtimes \mu \rightarrow C_0(\mathbb{A}_\infty) \rtimes \mathfrak{o} \rtimes \mu$ is injective on K_0 since $C_0(\mathbb{A}_\infty) \rtimes K \rtimes \mu$ can be written as an inductive limit (with $C_0(\mathbb{A}_\infty) \rtimes \mathfrak{o} \rtimes \mu$ as C*-algebra in each step) where the structure maps leave the image of $C_0(\mathbb{A}_\infty) \rtimes \mu$ fixed on K_0 . Thus, we have to prove the following

Lemma 6.3. *(i') $_*$ is injective on K_0 .*

Proof. The main ingredient is Theorem A1 of [Nat], which is some sort of Mayer–Vietoris sequence relating the K-theory of crossed products by a free product of two groups with the K-theory of the single crossed products. We apply this result to the group $\mathfrak{o} \rtimes \mu \cong \mathbb{Z}^n \rtimes \mathbb{Z}/2\mathbb{Z}$. Namely, the identification $\mathbb{Z} \rtimes \mathbb{Z}/2\mathbb{Z} \cong \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$, $z \mapsto t_2 t_1, t \mapsto t_1$ (z, t, t_1, t_2 are the canonical generators) is compatible with the action on \mathbb{Z}^{n-1} and on $C_0(\mathbb{A}_\infty)$. Thus, Theorem A1 of [Nat] yields the following sequence which is exact in the middle:

$$\begin{aligned} K_0(C_0(\mathbb{A}_\infty) \rtimes \mathbb{Z}^k) &\xrightarrow{\kappa_* - \kappa'_*} K_0(C_0(\mathbb{A}_\infty) \rtimes \mathbb{Z}^k \rtimes_{\hat{\beta}_{-1}} \mathbb{Z}/2\mathbb{Z}) \\ &\quad \oplus K_0(C_0(\mathbb{A}_\infty) \rtimes \mathbb{Z}^k \rtimes_{\hat{\alpha}_{\omega_{k+1}} \hat{\beta}_{-1}} \mathbb{Z}/2\mathbb{Z}) \\ &\xrightarrow{\varepsilon_* + \varepsilon'_*} K_0(C_0(\mathbb{A}_\infty) \rtimes \mathbb{Z}^{k+1} \rtimes_{\hat{\beta}_{-1}} \mathbb{Z}/2\mathbb{Z}) \end{aligned} \quad (19)$$

where $\kappa, \kappa', \varepsilon$ and ε' are the canonical maps and $0 \leq k \leq n-1$.

Now consider the translation-invariant isomorphism

$$C_0(\mathbb{A}_\infty) \rightarrow C_0(\mathbb{A}_\infty), \quad f \mapsto f(\sqcup - \frac{1}{2}\omega_{k+1}).$$

It yields an isomorphism

$$C_0(\mathbb{A}_\infty) \rtimes \mathbb{Z}^k \xrightarrow{\psi} C_0(\mathbb{A}_\infty) \rtimes \mathbb{Z}^k, \quad f \cdot u^a \mapsto f(\sqcup - \frac{1}{2}\omega_{k+1}) \cdot u^a.$$

We have

$$\begin{aligned} \psi \circ \hat{\beta}_{-1}(f \cdot u^a) &= f(-\sqcup + \frac{1}{2}\omega_{k+1}) \cdot u^{-a} \\ &= \hat{\alpha}_{\omega_{k+1}} \circ \hat{\beta}_{-1}(f(\sqcup - \frac{1}{2}\omega_{k+1}) \cdot u^a) \\ &= \hat{\alpha}_{\omega_{k+1}} \circ \hat{\beta}_{-1}(\psi(f \cdot u^a)). \end{aligned}$$

Thus, ψ induces an isomorphism

$$\psi : C_0(\mathbb{A}_\infty) \rtimes \mathbb{Z}^k \rtimes_{\hat{\beta}_{-1}} \mathbb{Z}/2\mathbb{Z} \cong C_0(\mathbb{A}_\infty) \rtimes \mathbb{Z}^k \rtimes_{\hat{\alpha}_{\omega_{k+1}} \hat{\beta}_{-1}} \mathbb{Z}/2\mathbb{Z}.$$

Now, the crucial point is that the following diagram commutes:

$$\begin{array}{ccc} C_0(\mathbb{A}_\infty) \rtimes \mathbb{Z}^k & \xrightarrow{\kappa} & C_0(\mathbb{A}_\infty) \rtimes \mathbb{Z}^k \rtimes_{\hat{\beta}_{-1}} \mathbb{Z}/2\mathbb{Z} \\ \psi \downarrow & & \downarrow \psi \\ C_0(\mathbb{A}_\infty) \rtimes \mathbb{Z}^k & \xrightarrow{\kappa'} & C_0(\mathbb{A}_\infty) \rtimes \mathbb{Z}^k \rtimes_{\hat{\alpha}_{\omega_{k+1}} \hat{\beta}_{-1}} \mathbb{Z}/2\mathbb{Z}. \end{array}$$

This fact, together with $\psi \sim_h \text{id}$ on $C_0(\mathbb{A}_\infty) \rtimes \mathbb{Z}^k$, implies that

$$\psi_* \kappa_* = \kappa'_* \tag{20}$$

on K-theory.

We would like to show that ε_* is injective. Assume that $x \in \ker(\varepsilon_*)$. Then $(x, 0) \in \ker(\varepsilon_* + \varepsilon'_*) = \text{im}(\kappa_* - \kappa'_*)$ because of (19). Thus, there exists y in $K_0(C_0(\mathbb{A}_\infty) \rtimes \mathbb{Z}^k)$ with $\kappa_*(y) = x$ and $\kappa'_*(y) = 0$. But by (20), we have $0 = \kappa'_*(y) = \psi_* \kappa_*(y) = \psi_*(x)$, and since ψ is an isomorphism, this implies $x = 0$. Thus, for any $0 \leq k \leq n-1$, ε_* is injective. This proves our claim, since $i' = \varepsilon^{(n-1)} \circ \dots \circ \varepsilon^{(0)}$ if $\varepsilon^{(k)}$ denotes the k -th map

$$C_0(\mathbb{A}_\infty) \rtimes \mathbb{Z}^k \rtimes \mu \rightarrow C_0(\mathbb{A}_\infty) \rtimes \mathbb{Z}^{k+1} \rtimes \mu. \quad \square$$

6.4. The general result. It remains to put everything together. We distinguish between three cases.

a) $\#\{v_{\mathbb{R}}\} = 0$:

Choose an unramified prime $p \in \mathbb{Z}_{>0}$ and a \mathbb{Z} -basis $\{p, p_1, p_2, \dots\}$ of Γ , where $K^\times = \mu \times \Gamma$. Let $\Gamma_m = \langle p, \dots, p_m \rangle$.

We have seen that $K_j(C_0(\mathbb{A}_\infty) \rtimes K \rtimes (\mu \times \Gamma_0)) \cong \mathbb{Z}^2$ for $j = 0, 1$ ($\#\{v_{\mathbb{R}}\} = 0$ implies that n is even). Moreover, it follows from our results of Section 6.3 and the

Pimsner–Voiculescu sequence that $i_* : K_*(C_0(\mathbb{A}_\infty) \rtimes (\mu \times \Gamma_0)) \rightarrow K_*(C_0(\mathbb{A}_\infty) \rtimes K \rtimes (\mu \times \Gamma_0))$ is injective. Now let us prove inductively that

$$K_j(C_0(\mathbb{A}_\infty) \rtimes (\mu \times \Gamma_m)) \cong K_j(C_0(\mathbb{A}_\infty) \rtimes K \rtimes (\mu \times \Gamma_m)) \cong \mathbb{Z}^{2^{m+1}}$$

for $j = 0, 1$ and that $i_* : K_*(C_0(\mathbb{A}_\infty) \rtimes (\mu \times \Gamma_m)) \rightarrow K_*(C_0(\mathbb{A}_\infty) \rtimes K \rtimes (\mu \times \Gamma_m))$ is injective. The case $m = 0$ has already been proven above.

If the claim is proven for m , the Pimsner–Voiculescu sequence, together with its naturality, will yield the result for $m + 1$ (analogously to the first case of Section 5.2, we have to use that $\hat{\beta}_b \sim_h \text{id}$ equivariantly).

Thus, we conclude $K_*(\mathfrak{A}) \cong K_*(C_0(\mathbb{A}_\infty) \rtimes K \rtimes K^\times) \cong K_0(C^*(\mu)) \otimes_{\mathbb{Z}} \Lambda^*(\Gamma)$.

b) $\#\{v_{\mathbb{R}}\}$ odd:

Again, choose $p \in \mathbb{Z}_{>0}$ prime and unramified and a \mathbb{Z} -basis $\{p, p_1, \dots\}$ of Γ with $K^\times = \mu \times \Gamma$. As $\#\{v_{\mathbb{R}}\}$ is odd, we can arrange by multiplying with -1 that $\#\{v_{\mathbb{R}} \mid v_{\mathbb{R}}(p_i) < 0\}$ is even for all i . As above, let $\Gamma_m = \langle p, \dots, p_m \rangle$. We can show for each m that

$$K_j(C_0(\mathbb{A}_\infty) \rtimes (\mu \times \Gamma_m)) \cong K_j(C_0(\mathbb{A}_\infty) \rtimes K \rtimes (\mu \times \Gamma_m)) \cong \mathbb{Z}^{2^m}$$

for $j = 0, 1$ and that $i_* : K_*(C_0(\mathbb{A}_\infty) \rtimes (\mu \times \Gamma_m)) \rightarrow K_*(C_0(\mathbb{A}_\infty) \rtimes K \rtimes (\mu \times \Gamma_m))$ is injective.

All we have to show is that $(\hat{\beta}_{p_{m+1}})_* = \text{id}$ on $K_*(C_0(\mathbb{A}_\infty) \rtimes (\mu \times \Gamma_m))$. This follows from $\hat{\beta}_{p_{m+1}} \sim_h \text{id}$ on $C_0(\mathbb{A}_\infty) \rtimes \mu$ and $\hat{\beta}_{p_{m+1}} \sim_h \hat{\beta}_{(v(p_{m+1})/|v(p_{m+1})|)v|_\infty}$. The second fact implies that $\hat{\beta}_{p_{m+1}}$ is of period 2, while the first one, together with the Pimsner–Voiculescu sequence, shows that $\hat{\beta}_{p_{m+1}}$ can be described by an upper triangular matrix where all the diagonal elements are 1. These two facts imply our claim, namely $(\hat{\beta}_{p_{m+1}})_* = \text{id}$ on $K_*(C_0(\mathbb{A}_\infty) \rtimes (\mu \times \Gamma_m))$.

Thus, $K_*(\mathfrak{A}) \cong K_*(C_0(\mathbb{A}_\infty) \rtimes K \rtimes K^\times) \cong \Lambda^*(\Gamma)$.

c) $\#\{v_{\mathbb{R}}\} \geq 2$ even:

Again, let $K^\times = \mu \times \Gamma$ and choose a \mathbb{Z} -basis $\{p, p_1, p_2, \dots\}$ of Γ , with $p \in \mathbb{Z}_{>0}$. We can arrange that $\#\{v_{\mathbb{R}} \mid v_{\mathbb{R}}(p_1) < 0\}$ is odd and $\#\{v_{\mathbb{R}} \mid v_{\mathbb{R}}(p_i) < 0\}$ is even for all $i > 1$. Let $\Gamma_m = \langle p, \dots, p_m \rangle$ and $\Gamma'_m = \langle p, p_2, \dots, p_m \rangle$. As above, we can show that $K_j(C_0(\mathbb{A}_\infty) \rtimes K \rtimes (\mu \times \Gamma'_m)) \cong \mathbb{Z}^{2^m}$ for $j = 0, 1$ ($[K : \mathbb{Q}]$ is even by assumption).

Lemma 6.4. *The following holds:*

$$(\hat{\beta}_{p_1})_* = \begin{pmatrix} 1 & & & & \\ & -1 & & 0 & \\ & & \ddots & & \\ & & & 1 & \\ & 0 & & & -1 \end{pmatrix}$$

on $K_j(C_0(\mathbb{A}_\infty) \rtimes K \rtimes (\mu \times \Gamma'_m)) \cong \mathbb{Z}^{2^m}$ for $j = 0, 1$ with respect to an appropriate \mathbb{Z} -basis.

Proof. Note that it is equivalent to prove our claim on $K_j(C_0(\mathbb{A}_\infty) \rtimes (\mu \times \Gamma'_m))$ because a refinement of Lemma 6.3 yields that $i : C_0(\mathbb{A}_\infty) \hookrightarrow C_0(\mathbb{A}_\infty) \rtimes K$ induces isomorphisms $i_* : K_*(C_0(\mathbb{A}_\infty) \rtimes (\mu \times \Gamma'_m)) \cong K_*(C_0(\mathbb{A}_\infty) \rtimes K \rtimes (\mu \times \Gamma'_m))$ for $m \in \mathbb{Z}_{\geq 0}$.

Now we extend the multiplicative action of K^\times on $C_0(\mathbb{A}_\infty)$ to \mathbb{A}_∞^\times and denote the corresponding action by $\hat{\beta}$ as before. We have $\hat{\beta}_{p_1} \sim_h \hat{\beta}_{(v(p_1)/|v(p_1)|)_{v|\infty}}$ and $\hat{\beta}_{(v(p_1)/|v(p_1)|)_{v|\infty}} = \hat{\beta}_{(1,-1,1,1,\dots)} \circ \hat{\beta}_{p'}$ where $\#\{v_{\mathbb{R}} \mid v_{\mathbb{R}}(p') < 0\}$ is even. Thus, $(\hat{\beta}_{p'})_* = \text{id}$ on $K_*(C_0(\mathbb{A}_\infty) \rtimes (\mu \times \Gamma'_m))$ and it remains to show the claim for $\hat{\beta}_{(1,-1,1,1,\dots)}$. Here, $(1, -1, 1, 1, \dots)$ is the element in \mathbb{A}_∞ given by 1, -1 on the first two real places ($\#\{v_{\mathbb{R}}\} \geq 2$) and by 1 on the remaining places (real and complex ones). We proceed inductively:

To get started, we need to determine $\hat{\beta}_{(1,-1)}$ explicitly on $K_0(C_0(\mathbb{R}^2) \rtimes \mu)$. To do so, it is helpful to consider the concrete situation where $K = \mathbb{Q}[\sqrt{2}]$ since we can explicitly write down generators for the K-groups in this special case. For $\mathbb{Q}[\sqrt{2}]$, we have $\mathfrak{o} = \mathbb{Z} + \mathbb{Z}\sqrt{2}$. We would like to show $(\hat{\beta}_{\sqrt{2}})_* = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$ on $K_0(C_0(\mathbb{A}_\infty) \rtimes \mu)$. By Theorem 4.1 and the Pimsner–Voiculescu sequence applied to $C_0(\mathbb{A}_\infty) \rtimes (\mu \times \langle p \rangle)$, it suffices to show that $\text{Ad}(s_{\sqrt{2}})$ induces $\begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$ on $K_j(C^*(\{u^a\}, s_{-1}, \{e_b\}, s_p))$ ($j = 0, 1$).

To determine $\text{Ad}(s_{\sqrt{2}})$, we compute $(e_{0,0} \otimes \text{id})_*^{-1}(\varphi_{\sqrt{2}})_*(\iota_{\sqrt{2}})_*$. We have

$$(v_{\sqrt{2}})_*([u^1]_1 \times [u^{\sqrt{2}}]_1) = [u^{\sqrt{2}}]_1 \times [u^2]_1 = -2[u^1]_1 \times [u^{\sqrt{2}}]_1.$$

Thus $\text{Ad}(s_{\sqrt{2}})$ induces -1 on the second copy of \mathbb{Z} in $K_0(C^*(\{u^a\}, s_{-1}, \{e_b\}))$. Here we used that (14) also holds for $d = \sqrt{2}$.

Now take the set of representatives $\mathcal{R}_{\sqrt{2}} = \{0, 1\}$ for $\mathfrak{o}/(\sqrt{2})$ in \mathfrak{o} as in the proof of Lemma 5.2. Then, under similar identifications as in Lemma 5.2,

$$(\varphi_{\sqrt{2}}) \circ (\iota_{\sqrt{2}}) \left(\frac{1}{2}(1 + s_{-1}) \right) = \begin{pmatrix} \frac{1}{2}(1 + s_{-1}) & \\ & \frac{1}{2}(1 + u^{-\sqrt{2}}s_{-1}) \end{pmatrix}$$

and

$$(\varphi_{\sqrt{2}}) \circ (\iota_{\sqrt{2}}) \left(\frac{1}{2}(1 + u^1s_{-1}) \right) = \frac{1}{2} \begin{pmatrix} 1 & s_{-1} \\ s_{-1} & 1 \end{pmatrix}.$$

Thus,

$$\begin{aligned} (e_{0,0} \otimes \text{id})_*^{-1} \circ (\varphi_{\sqrt{2}})_* \circ (\iota_{\sqrt{2}})_* \left(\left[\frac{1}{2}(1 + s_{-1}) \right]_0 - \left[\frac{1}{2}(1 + u^1s_{-1}) \right]_0 \right) \\ = \left[\frac{1}{2}(1 + s_{-1}) \right]_0 - \left([1]_0 - \left[\frac{1}{2}(1 + u^{-\sqrt{2}}s_{-1}) \right]_0 \right). \end{aligned}$$

In the inductive limit, we get that $\text{Ad}(s_{\sqrt{2}})$ induces $\text{id}_{\mathbb{Z}}$ on the first copy of \mathbb{Z} in $K_0(C^*(\{u^a\}, s_{-1}, \{e_b\}))$ by the analogue of (5).

Thus, $(\hat{\beta}_{(1,-1)})_* = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$ on $K_0(C_0(\mathbb{A}_\infty) \rtimes \mu) = K_0(C_0(\mathbb{R}^2) \rtimes \mu)$ as $K \ni \sqrt{2} \mapsto (\sqrt{2}, -\sqrt{2}) \sim_h (1, -1) \in \mathbb{A}_\infty$. But this already implies $(\hat{\beta}_{(1,-1,1,1,\dots)})_* =$

$\begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$ on $K_0(C_0(\mathbb{A}_\infty) \rtimes \mu)$ for any number field with $\mu = \{\pm 1\}$: Let B be the Bott element in $\text{KK}^\mu(\mathbb{C}, C_0(\mathbb{R}^{n-2}))$. Then $1_{C_0(\mathbb{R}^2)} \otimes B$ is an invertible element of $\text{KK}^\mu(C_0(\mathbb{R}^2), C_0(\mathbb{R}^2) \otimes C_0(\mathbb{R}^{n-2}))$. By [Kas], Theorem 2.14 8),

$$(\hat{\beta}_{(1,-1)})_* \times (1_{C_0(\mathbb{R}^2)} \otimes B) \times \sigma_* = (B \otimes 1_{C_0(\mathbb{R}^2)}) \times (1_{C_0(\mathbb{R}^{n-2})} \otimes (\hat{\beta}_{(1,-1)})_*)$$

where σ is the flip $C_0(\mathbb{R}^2) \otimes C_0(\mathbb{R}^{n-2}) \cong C_0(\mathbb{R}^{n-2}) \otimes C_0(\mathbb{R}^2)$. This implies that

$$\begin{aligned} (\hat{\beta}_{(1,-1)})_* \times (1_{C_0(\mathbb{R}^2)} \otimes B) &= (B \otimes 1_{C_0(\mathbb{R}^2)}) \times (1_{C_0(\mathbb{R}^{n-2})} \otimes (\hat{\beta}_{(1,-1)})_*) \times \sigma_*^{-1} \\ &= (B \otimes 1_{C_0(\mathbb{R}^2)}) \times \sigma_*^{-1} \times ((\hat{\beta}_{(1,-1)})_* \otimes 1_{C_0(\mathbb{R}^{n-2})}) \\ &= (1_{C_0(\mathbb{R}^2)} \otimes B) \times ((\hat{\beta}_{(1,-1)})_* \otimes 1_{C_0(\mathbb{R}^{n-2})}). \end{aligned}$$

Thus, using $1_{C_0(\mathbb{R}^2)} \otimes B$ to identify $K_0(C_0(\mathbb{A}_\infty) \rtimes \mu)$ with $\mathbb{Z}^2 \cong K_0(C_0(\mathbb{R}^2) \rtimes \mu)$, $(\hat{\beta}_{(1,-1,1,1,\dots)})_*$ is given by $\begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$. This proves our claim for $m = 1$, where we applied the Pimsner–Voiculescu sequence to $\Gamma'_0 = \langle p \rangle \cong \mathbb{Z}$.

To go from m to $m + 1$, we apply the induction hypothesis together with the Pimsner–Voiculescu sequence to find a \mathbb{Z} -basis for $K_j(C_0(\mathbb{A}_\infty) \rtimes (\mu \times \Gamma'_{m+1}))$ for $j = 0, 1$ such that

$$(\hat{\beta}_{(1,-1,1,1,\dots)})_* = \left(\begin{array}{ccc|ccc} 1 & & & & & \\ & -1 & 0 & & & \\ & & \ddots & & & \\ & 0 & & 1 & & * \\ \hline & & & & -1 & \\ & & & & & 1 \\ & 0 & & & -1 & 0 \\ & & & & & \ddots \\ & & & & 0 & 1 \\ & & & & & -1 \end{array} \right).$$

Now the Pimsner–Voiculescu sequence implies that the torsion-free part of $K_j(C_0(\mathbb{A}_\infty) \rtimes (\mu \times \Gamma'_{m+1}))$ is $\mathbb{Z}^{2^{m+1}}$. Using $(\hat{\beta}_{(1,-1,1,1,\dots)})_*^2 = \text{id}$, we deduce that $K_j(C_0(\mathbb{A}_\infty) \rtimes (\mu \times \Gamma'_{m+1}))/\text{im}(\text{id} - (\hat{\beta}_{(1,-1,1,1,\dots)})_*)$ can only contain torsion of order 2. Thus, $K_j(C_0(\mathbb{A}_\infty) \rtimes (\mu \times \Gamma'_{m+1}))/\text{im}(\text{id} - (\hat{\beta}_{(1,-1,1,1,\dots)})_*) \cong \mathbb{Z}^{2^m} \oplus (\mathbb{Z}/2\mathbb{Z})^{2^m}$ by the Pimsner–Voiculescu sequence. But this result, together with the elementary divisor theorem, tells us that we can modify the first chosen \mathbb{Z} -basis for $K_j(C_0(\mathbb{A}_\infty) \rtimes (\mu \times \Gamma'_{m+1}))$ so that our claim holds. \square

Hence, applying the Pimsner–Voiculescu sequence iteratively gives

$$K_*(C_0(\mathbb{A}_\infty) \rtimes K \rtimes (\mu \times \Gamma_m)) \cong \Lambda^*(\Gamma'_m) \oplus (\mathbb{Z}/2\mathbb{Z} \otimes_{\mathbb{Z}} \Lambda^*(\Gamma'_m))$$

and thus

$$K_*(\mathfrak{A}) \cong \Lambda^*(\Gamma) \oplus ((\mathbb{Z}/2\mathbb{Z}) \otimes_{\mathbb{Z}} \Lambda^*(\Gamma)).$$

Remark 6.5. At this point, it also becomes clear that we cannot treat the general case (without the restriction $\mu = \{\pm 1\}$) for the following reasons: It is not clear how to prove analogous statements as Lemma 6.1 and Lemma 6.3 in general. Once these two problems are solved, it should be possible to determine the K-theory without further assumptions on the number fields.

Remark 6.6. With a similar idea as in Theorem 4.1, we can treat the case of the full adèle ring with the action of the $ax + b$ -group: Let K be any global field. As K is a discrete subgroup of \mathbb{A} , it acts freely and properly on \mathbb{A} so that

$$C_0(\mathbb{A}) \rtimes K \sim_M C(\mathbb{A}/K) \cong C(\widehat{K}) \cong C^*(K).$$

$C^*(K)$ is the group C*-algebra of $(K, +)$. Moreover, it turns out that this Morita equivalence can be chosen equivariantly (in the sense of [CMW]) with respect to the multiplicative action of K^\times . Thus, by [CMW], we get

$$C_0(\mathbb{A}) \rtimes P_K = C_0(\mathbb{A}) \rtimes K \rtimes K^\times \sim_M C^*(K \rtimes K^\times) = C^*(P_K). \quad (21)$$

This means that the crossed product is Morita equivalent to the C*-algebra of the $ax + b$ -group over K . We note that this group C*-algebra is also the ring C*-algebra associated with the field K .

(21) can be used to compute the K-theory. For example, in the case $K = \mathbb{Q}$, we get

$$K_*(C_0(\mathbb{A}) \rtimes \mathbb{Q} \rtimes \mathbb{Q}^\times) \cong K_0(C^*(\{\pm 1\})) \otimes_{\mathbb{Z}} \Lambda^*(\mathbb{Q}_{>0}).$$

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