

Simply Laced Extended Affine Weyl Groups (A Finite Presentation)

By

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Abstract

Extended affine Weyl groups are the Weyl groups of extended affine root systems. Finite presentations for extended affine Weyl groups are known only for nullities ≤ 2 , where for nullity 2 there is only one known such presentation. We give a finite presentation for the class of simply laced extended affine Weyl groups. Our presentation is nullity free if $\text{rank} > 1$ and for $\text{rank} = 1$ it is given for nullities ≤ 3 . The generators and relations are given uniformly for all types, and for a given nullity they can be read from the corresponding finite Cartan matrix and the semilattice involved in the structure of the root system.

§0. Introduction

Extended affine Weyl groups are the Weyl groups of *extended affine root systems* which are higher nullity generalizations of affine and finite root systems. In 1985, K. Saito [S] introduced axiomatically the notion of an extended affine root system, he was interested in possible applications to the study of singularities.

Extended affine root systems also arise as the root systems of a class of infinite dimensional Lie algebras called *extended affine Lie algebras*. A systematic study of extended affine Lie algebras and their root systems is given in

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[AABGP], in particular a set of axioms, different from those given by Saito [S], is extracted from algebras for the corresponding root systems. In [A2], the relation between axioms of [S] and [AABGP] for extended affine root system's is clarified.

Let R be an extended affine root system of nullity ν (see Definition 2.2) and \mathcal{V} be the real span of R which by definition is equipped with a positive semi-definite bilinear form I . We consider R as a subset of a hyperbolic extension $\tilde{\mathcal{V}}$ of \mathcal{V} , where $\tilde{\mathcal{V}}$ is equipped with a non-degenerate extension \tilde{I} of I (see Section 1). Then the extended affine Weyl group \mathcal{W} of R is by definition the subgroup of the orthogonal group of $\tilde{\mathcal{V}}$ generated by reflections based on the set of non-isotropic roots R^\times of R .

This work is the first output of a three steps project on the presentations of extended affine Weyl groups and its application to the study of extended affine Lie algebras. In the first step, we study finite presentations for extended affine Weyl groups, where in this work we restrict ourself to the simply laced cases. In the second step the results of the current work will apply to investigate the existence of the so called a *presentation by conjugation* for the simply laced extended affine Weyl groups (see [Kr] and [A3]). Finally, in the third step, we will apply the results of the second step to investigate validity of certain classical results for the class of simply laced extended affine Lie algebras. There is only a little known about the presentations of extended affine Weyl groups. In fact if $\nu > 2$, there is no known finite presentation for this class and for $\nu = 2$ there is only one known finite presentation called the *generalized Coxeter presentation* (see [ST]).

We give a finite presentation, for simply laced extended affine Weyl groups, which is nullity free if rank > 1 and for rank $= 1$ it is given for nullities ≤ 3 (see Theorem 4.1). Our presentation highly depends on the classification of semilattices (see Definition 2.1), up to similarity, which appears in the structure of extended affine root systems (see (2.5)). Since for types A_ℓ ($\ell \geq 2$), D_ℓ , E_ℓ for arbitrary nullity and for type A_1 for nullity ≤ 3 this classification is known (see [AABGP, Chapter II]), our presentation is explicit for the mentioned types and nullities.

The paper is arranged as follows. In Section 1, we obtain several results regarding the structure of certain reflection groups. The results of this section are similar but more general than those in [MS] and [A4], and are applicable to a wide range of reflection groups including extended affine Weyl groups.

In Section 2, we introduce a notion of *supporting class* for the semilattices involved in the structure of extended affine root systems. This notion plays a

crucial role in our work. Also we study an intrinsic subgroup \mathcal{H} of an extended affine Weyl group \mathcal{W} which we call it a *Heisenberg-like group*. The center of \mathcal{H} is fully analyzed in terms of the supporting class of the root system. This a basic achievement which distinguishes the results of this section from those in [A4] and [MS] (See Corollary 2.2). In particular it, together with other results, provides a unique expression of Weyl group elements in terms of the elements of a finite Weyl group, certain well-known linear transformations belonging to the corresponding Heisenberg-like group and central elements (see Proposition 2.3). We encourage the reader to compare this with its similar results in [MS] and [A4]. The main results are given in Sections 3 and 4 where we give our explicit presentations for the extended affine Weyl group \mathcal{W} and the Heisenberg-like group \mathcal{H} .

The presentation is obtained as follows. First by analyzing the semilattice involved in the structure of R , we obtain a finite presentation for \mathcal{H} . The generators and relations depend on nullity, the supporting class of the involved semilattice and the Cartan matrix of the corresponding finite type. Next using the fact that $\mathcal{W} = \dot{\mathcal{W}} \rtimes \mathcal{H}$, where $\dot{\mathcal{W}}$ is a finite Weyl group of the same type of R , we obtain our presentation for \mathcal{W} . So this presentation consists of three parts, a presentation for \mathcal{H} , the Coxeter presentation for the finite Weyl group $\dot{\mathcal{W}}$ and the relations imposed by the semidirect product (see Theorems 3.1 and 4.1).

For a systematic study of extended affine Lie algebras and their root systems we refer the reader to [AABGP]. For the study of extended affine Weyl group we refer the reader to [S], [MS], [A1,A2,A3,A4], [ST] and [T].

§1. Reflections Groups

Let \mathcal{V} be a finite dimensional real vector space equipped with a non-trivial symmetric bilinear form $I = (\cdot, \cdot)$ of nullity ν . An element α of \mathcal{V} is called *non-isotropic (isotropic)* if $(\alpha, \alpha) \neq 0$ ($(\alpha, \alpha) = 0$). We denote the set of nonisotropic elements of a subset A with A^\times . If α is non-isotropic, we set $\alpha^\vee = 2\alpha/(\alpha, \alpha)$. Let \mathcal{V}^0 be the radical of the form and $\hat{\mathcal{V}}$ be a fixed complement of \mathcal{V}^0 in \mathcal{V} of dimension ℓ . Throughout this section we fix a basis $\{\alpha_1, \dots, \alpha_\ell\}$ of $\hat{\mathcal{V}}$, a basis $\{\sigma_1, \dots, \sigma_\nu\}$ of \mathcal{V}^0 and a basis $\{\lambda_1, \dots, \lambda_\nu\}$ of $(\mathcal{V}^0)^*$. We enlarge the space \mathcal{V} to a $\ell + 2\nu$ -dimensional vector space as follows. Set

$$(1.1) \quad \tilde{\mathcal{V}} := \mathcal{V} \oplus (\mathcal{V}^0)^*,$$

where $(\mathcal{V}^0)^*$ is the dual space of \mathcal{V}^0 . Now we extend the form (\cdot, \cdot) on \mathcal{V} to a non-degenerate form, denoted again by (\cdot, \cdot) , on $\tilde{\mathcal{V}}$ as follows:

$$(1.2) \quad \begin{aligned} & \bullet (\cdot, \cdot)|_{\mathcal{V} \times \mathcal{V}} := (\cdot, \cdot), \\ & \bullet (\tilde{\mathcal{V}}, (\mathcal{V}^0)^*) = ((\mathcal{V}^0)^*, (\mathcal{V}^0)^*) := 0, \\ & \bullet (\sigma_r, \lambda_s) := \delta_{r,s}, \quad 1 \leq r, s \leq \nu \end{aligned}$$

The pair $(\tilde{\mathcal{V}}, I)$ is called a *hyperbolic extension* of (\mathcal{V}, I) .

Let $O(\tilde{\mathcal{V}}, I)$ be the orthogonal subgroup of $GL(\tilde{\mathcal{V}})$, with respect to $I = (\cdot, \cdot)$. We also set

$$FO(\tilde{\mathcal{V}}, I) = \{w \in O(\tilde{\mathcal{V}}, I) \mid w(\delta) = \delta \text{ for all } \delta \in \mathcal{V}^0\}.$$

For $\alpha \in \mathcal{V}^\times$, the element $w_\alpha \in FO(\tilde{\mathcal{V}}, I)$ defined by

$$w_\alpha(u) = u - (u, \alpha^\vee)\alpha, \quad (u \in \tilde{\mathcal{V}}),$$

is called the *reflection based on α* . It is easy to check that

$$(1.3) \quad ww_\alpha w^{-1} = w_{w(\alpha)} \quad (w \in O(\tilde{\mathcal{V}}, I)).$$

For a subset A of \mathcal{V} , the group

$$(1.4) \quad \mathcal{W}_A = \langle w_\alpha \mid \alpha \in A^\times \rangle,$$

is called the *reflection group in $FO(\tilde{\mathcal{V}}, I)$ based on A* .

For $\alpha \in \mathcal{V}$ and $\sigma \in \mathcal{V}^0$, we define $T_\alpha^\sigma \in \text{End}(\tilde{\mathcal{V}})$ by

$$(1.5) \quad T_\alpha^\sigma(u) := u - (\sigma, u)\alpha + (\alpha, u)\sigma - \frac{(\alpha, \alpha)}{2}(\sigma, u)\sigma \quad (u \in \tilde{\mathcal{V}}).$$

The basic properties of the linear maps T_α^σ are listed in the following lemma. The terms of the form $[x, y]$ appearing in the lemma denotes the commutator $x^{-1}y^{-1}xy$ of two elements x, y in a group.

Lemma 1.1. *Let $\alpha, \beta, \gamma \in \mathcal{V}$, $\sigma, \delta, \tau \in \mathcal{V}^0$ and $w \in FO(\tilde{\mathcal{V}}, I)$. Then*

- (i) $T_\alpha^{r\sigma} = T_{r\alpha}^\sigma$, $r \in \mathbb{R}$,
- (ii) $T_{\alpha+\delta}^{\sigma+\delta} = T_\alpha^\sigma T_\alpha^\delta T_\delta^{(\alpha, \alpha)\frac{\sigma}{2}}$,
- (iii) $T_{\alpha+\beta}^\sigma = T_\alpha^\sigma T_\beta^\sigma$,
- (iv) $[T_\alpha^\sigma, T_\beta^\delta] = T_\sigma^{(\alpha, \beta)\delta}$,
- (v) $[T_\alpha^\sigma T_\beta^\delta, T_\gamma^\tau] = [T_\alpha^\sigma, T_\gamma^\tau][T_\beta^\delta, T_\gamma^\tau]$,
- (vi) $wT_\alpha^\sigma w^{-1} = T_{w(\alpha)}^\sigma$,
- (vii) $T_{\alpha^\vee}^\sigma = w_{\alpha+\sigma} w_\alpha$, $\alpha \in \mathcal{V}^\times$,
- (viii) $(T_\sigma^\delta)^{-1} = T_\delta^\sigma$.

Proof. The proof of each statement can be seen using the definition of the maps T_α^σ and straightforward computations. \square

Part (vii) of Lemma 1.1 has been in fact our motivation for defining the maps T_α^σ . If we denote by $Z(G)$ the center of a group G , then we have from parts (iii), (vi) and (vii) of Lemma 1.1 that for $\alpha \in \mathcal{V}$, $\sigma, \delta \in \mathcal{V}^0$,

$$(1.6) \quad T_\alpha^\sigma \in \text{FO}(\tilde{\mathcal{V}}, I) \quad \text{and} \quad T_\delta^\sigma \in Z(\text{FO}(\tilde{\mathcal{V}}, I)).$$

Lemma 1.2. *Let $\alpha \in \mathcal{V}$, $n_r \in \mathbb{R}$ and $1 \leq r \leq \nu$. Then*

$$T_\alpha^{\sum_{r=1}^\nu n_r \sigma_r} = \prod_{r=1}^\nu T_\alpha^{n_r \sigma_r} \prod_{1 \leq r < s \leq \nu} T_{\sigma_s}^{n_r n_s \frac{(\alpha, \alpha)}{2} \sigma_r}.$$

Proof. For $1 \leq r \leq \nu$, set $\delta_r := \sum_{i=r}^\nu n_i \sigma_i$. Using Lemma 1.1 (ii)–(iii) and (1.6), we have

$$\begin{aligned} T_\alpha^{\sum_{r=1}^\nu n_r \sigma_r} &= T_\alpha^{n_1 \sigma_1 + \delta_2} = T_\alpha^{n_1 \sigma_1} T_{\delta_2}^{\delta_2} T_\alpha^{n_1 \frac{(\alpha, \alpha)}{2} \sigma_1} \\ &= T_\alpha^{n_1 \sigma_1} T_\alpha^{n_2 \sigma_2} T_{\delta_3}^{\delta_3} T_\alpha^{n_2 \frac{(\alpha, \alpha)}{2} \sigma_2} T_{\delta_2}^{n_1 \frac{(\alpha, \alpha)}{2} \sigma_1} \\ &\quad \vdots \\ &= \prod_{r=1}^\nu T_\alpha^{n_r \sigma_r} \prod_{r=1}^{\nu-1} T_{\delta_{r+1}}^{n_r \frac{(\alpha, \alpha)}{2} \sigma_r} \\ &= \prod_{r=1}^\nu T_\alpha^{n_r \sigma_r} \prod_{r=1}^{\nu-1} \prod_{s=r+1}^\nu T_{\sigma_s}^{n_r n_s \frac{(\alpha, \alpha)}{2} \sigma_r} \\ &= \prod_{r=1}^\nu T_\alpha^{n_r \sigma_r} \prod_{1 \leq r < s \leq \nu} T_{\sigma_s}^{n_r n_s \frac{(\alpha, \alpha)}{2} \sigma_r}. \end{aligned}$$

\square

For a subset A of \mathcal{V} , consider the subgroup

$$(1.7) \quad \mathcal{H}(A) := \langle w_{\alpha+\sigma} w_\alpha \mid \alpha \in A^\times, \sigma \in \mathcal{V}^0, \alpha + \sigma \in A \rangle.$$

of \mathcal{W}_A . We note that $\mathcal{H}(A) \leq \mathcal{H}(\mathcal{V}) = \langle w_{\alpha+\sigma} w_\alpha \mid \alpha \in \mathcal{V}^\times, \sigma \in \mathcal{V}^0 \rangle$.

We recall that a group H is called *two-step nilpotent* if the commutator $[H, H]$ is contained in the center $Z(H)$ of H . We also recall that in a two-step nilpotent group H , the commutator is bi-multiplicative, that is

$$(1.8) \quad \left[\prod_{i=1}^n x_i, \prod_{j=1}^m y_j \right] = \prod_{i=1}^n \prod_{j=1}^m [x_i, y_j]$$

for all $x_i, y_j \in H$.

Lemma 1.3. *Let A be a subset of \mathcal{V} . Then*

(i) $\mathcal{H}(A)$ is a two-step nilpotent group.

(ii) If X is a generating subset of $\mathcal{H}(A)$, then $[X, X]$ generates the commutator $[\mathcal{H}(A), \mathcal{H}(A)]$ of $\mathcal{H}(A)$.

Proof. (i) If $w_\alpha w_{\alpha+\sigma}$ and $w_\beta w_{\beta+\delta}$ are two generators of $\mathcal{H}(A)$, $\alpha, \beta \in A^\times$, $\alpha + \sigma, \beta + \delta \in A$, then by Lemma 1.1(vii),(iv), we have

$$[w_{\alpha+\sigma} w_\alpha, w_{\beta+\delta} w_\beta] = [T_{\alpha^\vee}^\sigma, T_{\beta^\vee}^\delta] = T_\sigma^{(\alpha^\vee, \beta^\vee)\delta}.$$

Now the result follows from (1.6). (ii) is an immediate consequence of (i) and (1.8). □

Lemma 1.4.

$$\mathcal{H}(\mathcal{V}) = \langle T_{\alpha_i}^{n_{i,r}\sigma_r}, T_{\sigma_r}^{m_{r,s}\sigma_s} \mid 1 \leq i \leq \ell; 1 \leq r, s \leq \nu; n_{i,r}, m_{r,s} \in \mathbb{R} \rangle.$$

Proof. From parts (iii) and (vii) of Lemma 1.1 and Lemma 1.2 we see that $\mathcal{H}(\mathcal{V})$ is a subset of the right hand side. Conversely, it follows from Lemma 1.1(vii) and (iv) that the right hand side is a subset of $\mathcal{H}(\mathcal{V})$. □

Lemma 1.5. *Let $h = \prod_{r=1}^\nu \prod_{i=1}^\ell T_{\alpha_i}^{n_{i,r}\sigma_r} \prod_{1 \leq r < s \leq \nu} T_{\sigma_r}^{m_{r,s}\sigma_s}$, where $n_{i,r}, m_{r,s}$ are in \mathbb{R} . If $\beta_j = \sum_{i=1}^\ell n_{i,j}\alpha_i$, $1 \leq j \leq \nu$, then*

$$h(\lambda_j) = \lambda_r - \beta_j - \frac{(\beta_j, \beta_j)}{2} \sigma_j - \sum_{1 \leq r \leq j-1} m_{r,j} \sigma_r + \sum_{j+1 \leq s \leq \nu} m_{j,s} \sigma_s.$$

Proof. Let $1 \leq j \leq \nu$ and $\dot{\alpha} \in \dot{\mathcal{V}}$. Then from (1.5) and (1.2), it follows that

$$T_{\dot{\alpha}}^{\sigma_j}(\lambda_j) = \lambda_j - \dot{\alpha} - \frac{(\dot{\alpha}, \dot{\alpha})}{2} \sigma_j \quad \text{and} \quad T_{\sigma_r}^{\sigma_s}(\lambda_j) = \lambda_j - \delta_{s,j} \sigma_r + \delta_{r,j} \sigma_s, \quad 1 \leq r, s \leq \nu,$$

and so using (1.6) and Lemma 1.1(ii)–(iii), we have

$$\begin{aligned} h(\lambda_j) &= \prod_{r=1}^\nu \prod_{i=1}^\ell T_{\alpha_i}^{n_{i,r}\sigma_r} \prod_{1 \leq r < s \leq \nu} T_{\sigma_r}^{m_{r,s}\sigma_s}(\lambda_j) \\ &= \prod_{r=1}^\nu \prod_{i=1}^\ell T_{\alpha_i}^{n_{i,r}\sigma_r} \prod_{j+1 \leq s \leq \nu} T_{\sigma_j}^{m_{j,s}\sigma_s} \prod_{1 \leq r \leq j-1} T_{\sigma_r}^{m_{r,j}\sigma_j} \prod_{\{1 \leq r < s \leq \nu \mid r, s \neq j\}} T_{\sigma_r}^{m_{r,s}\sigma_s}(\lambda_j) \\ &= \prod_{r=1}^\nu \prod_{i=1}^\ell T_{\alpha_i}^{n_{i,r}\sigma_r} T_{\sigma_j}^{\sum_{j+1 \leq s \leq \nu} m_{j,s}\sigma_s} T_{\sum_{1 \leq r \leq j-1} m_{r,j}\sigma_r}^{\sigma_j}(\lambda_j) \end{aligned}$$

$$\begin{aligned}
 &= \prod_{r=1}^{\nu} \prod_{i=1}^{\ell} T_{\alpha_i}^{n_{i,r}\sigma_r} T_{\sigma_j}^{\sum_{j+1 \leq s \leq \nu} m_{j,s}\sigma_s} \left(\lambda_j - \sum_{1 \leq r \leq j-1} m_{r,j}\sigma_r \right) \\
 &= \prod_{r=1}^{\nu} \prod_{i=1}^{\ell} T_{\alpha_i}^{n_{i,r}\sigma_r} \left(\lambda_j - \sum_{1 \leq r \leq j-1} m_{r,j}\sigma_r + \sum_{j+1 \leq s \leq \nu} m_{j,s}\sigma_s \right) \\
 &= \prod_{r=1}^{\nu} \prod_{i=1}^{\ell} T_{\alpha_i}^{n_{i,r}\sigma_r} (\lambda_j) - \sum_{1 \leq r \leq j-1} m_{r,j}\sigma_r + \sum_{j+1 \leq s \leq \nu} m_{j,s}\sigma_s \\
 &= \prod_{i=1}^{\ell} T_{\alpha_i}^{n_{i,j}\sigma_j} (\lambda_j) - \sum_{1 \leq r \leq j-1} m_{r,j}\sigma_r + \sum_{j+1 \leq s \leq \nu} m_{j,s}\sigma_s \\
 &= T_{\beta_j}^{\sigma_j} (\lambda_j) - \sum_{1 \leq r \leq j-1} m_{r,j}\sigma_r + \sum_{j+1 \leq s \leq \nu} m_{j,s}\sigma_s \\
 &= \lambda_j - \beta_j - \frac{(\beta_j, \beta_j)}{2} \sigma_j - \sum_{1 \leq r \leq j-1} m_{r,j}\sigma_r + \sum_{j+1 \leq s \leq \nu} m_{j,s}\sigma_s.
 \end{aligned}$$

□

Lemma 1.6. Each element $h \in \mathcal{H}(\mathcal{V})$ has a unique expression in the form

$$(1.9) \quad h = h(n_{i,r}, m_{r,s}) := \prod_{r=1}^{\nu} \prod_{i=1}^{\ell} T_{\alpha_i}^{n_{i,r}\sigma_r} \prod_{1 \leq r < s \leq \nu} T_{\sigma_r}^{m_{r,s}\sigma_s} \quad (n_{i,r}, m_{r,s} \in \mathbb{R}).$$

Proof. Let $h \in \mathcal{H}$. From (1.6), Lemmas 1.4 and 1.1(iv) it follows that h has an expression in the form (1.9). Now let $h(n'_{i,r}, m'_{r,s})$ be another expression of h in the form (1.9). Then by acting these two expressions of h on λ_j 's, $1 \leq j \leq \nu$, we get from Lemma 1.5 that $n_{i,r} = n'_{i,r}$ and $m_{r,s} = m'_{r,s}$ for all $1 \leq i \leq \ell$ and $1 \leq r < s \leq \nu$. □

Lemma 1.7. (i) $Z(\mathcal{H}(\mathcal{V})) = \langle T_{\sigma_r}^{m_{r,s}\sigma_s} \mid 1 \leq r < s \leq \nu, m_{r,s} \in \mathbb{R} \rangle$, if $\nu \geq 2$.

(ii) For any fixed nonzero real numbers $m_{r,s}$, $1 \leq r < s \leq \nu$, the group $\langle T_{\sigma_r}^{m_{r,s}\sigma_s} \mid 1 \leq r < s \leq \nu \rangle$ is free abelian of rank $\frac{\nu(\nu-1)}{2}$.

(iii) $\mathcal{H}(\mathcal{V})$ is a torsion free group.

Proof. (i) By (1.6), Lemmas 1.1(vii) and 1.4, it is clear that the right hand side in the statement is a subset of the left hand side. To show the reverse inclusion, let $h \in Z(\mathcal{H}(\mathcal{V}))$. Consider an expression $h(n_{i,r}, m_{r,s})$ of h in the form (1.9). We must show that $n_{i,r} = 0$, for all $1 \leq i \leq \ell$ and $1 \leq r \leq \nu$.

Since $\mathcal{H}(\mathcal{V})$ is a two-step nilpotent group, we have from (1.8) and Lemmas 1.4 and 1.1(iv) that for all $1 \leq j \leq \ell$ and $1 \leq s \leq \nu$,

$$\begin{aligned} 1 = [h, T_{\alpha_j}^{\sigma_s}] &= \prod_{r=1}^{\nu} \prod_{i=1}^{\ell} [T_{\alpha_i}^{n_{i,r}\sigma_r}, T_{\alpha_j}^{\sigma_s}] \\ &= \prod_{r=1}^{\nu} \prod_{i=1}^{\ell} T_{\sigma_r}^{n_{i,r}(\alpha_i, \alpha_j)\sigma_s} = \prod_{r=1}^{\nu} T_{\sigma_r}^{(\sum_{i=1}^{\ell} n_{i,r}\alpha_i, \alpha_j)\sigma_s}. \end{aligned}$$

Therefore by Lemma 1.6, $\sum_{i=1}^{\ell} n_{i,r}(\alpha_i, \alpha_j) = 0$, for all $1 \leq j \leq \ell$ and $1 \leq r \leq \nu$. But $\dot{\mathcal{V}} = \sum_{i=1}^{\ell} \mathbb{R}\alpha_i$ and the form restricted to $\dot{\mathcal{V}}$ is non-degenerate, so $n_{i,r} = 0$ for all $1 \leq i \leq \ell, 1 \leq r \leq \nu$.

(ii) We show that $\{T_{\sigma_r}^{m_{r,s}\sigma_s} \mid 1 \leq r \leq s \leq \nu\}$ is a free basis for the group under consideration. Let $\prod_{1 \leq r < s \leq \nu} T_{\sigma_r}^{n_{r,s}m_{r,s}\sigma_s} = 1, n_{r,s} \in \mathbb{Z}$ for $1 \leq r < s \leq \nu$. Then by Lemma 1.6, $n_{r,s}m_{r,s} = 0$ for all r, s . The result now follows as $m_{r,s}$'s are nonzero.

(iii) Let $h \in \mathcal{H}(\mathcal{V})$ and assume $h^n = 1$ for some $n \in \mathbb{N}$. Let $h = h(n_{i,r}, m_{r,s})$ (see (1.9)). Since $\mathcal{H}(\mathcal{V})$ is two-step nilpotent, we have

$$1 = h^n = \prod_{r=1}^{\nu} \prod_{i=1}^{\ell} T_{\alpha_i}^{nn_{i,r}\sigma_r} c \prod_{1 \leq r < s \leq \nu} T_{\sigma_r}^{nm_{r,s}\sigma_s},$$

where c is a central element. By part (i), $1 = h^n = h^n(nn_{i,r}, m'_{r,s})$ for some real numbers $m'_{r,s}$. By Lemma 1.6, $n_{i,r} = 0$ for all i, r . Thus h is central and so $c = 1$. Now it follows again from Lemma 1.6 that $m_{r,s} = 0$ for all r, s . \square

Corollary 1.1. *For any subset A of \mathcal{V} , $\mathcal{H}(A)$ is a torsion free group.*

Recall that we have fixed a complement $\dot{\mathcal{V}}$ of \mathcal{V}^0 in \mathcal{V} . Now for a subset A of \mathcal{V} , we set

$$\dot{A} := \{\alpha \in \dot{\mathcal{V}} \mid \alpha + \sigma \in A \text{ for some } \sigma \in \mathcal{V}^0\}.$$

Proposition 1.1. *Let A be a subset of \mathcal{V} such that $w_{\alpha}(A) \subseteq A$ for all $\alpha \in A^{\times}$ and $\mathcal{H}(A) = \langle w_{\alpha+\sigma}w_{\alpha} \mid \alpha \in \dot{A}^{\times}, \sigma \in \mathcal{V}^0, \alpha + \sigma \in A \rangle$. Then $\mathcal{W}_A = \mathcal{W}_{\dot{A}} \times \mathcal{H}(A)$.*

Proof. Let $\alpha \in A, \sigma \in \mathcal{V}^0, \alpha + \sigma \in A$ and $w \in \mathcal{W}_A$. By assumption $w(\alpha)$ and $w(\alpha + \sigma) = w(\alpha) + \sigma$ are elements of A . Thus, $\mathcal{H}(A)$ is a normal subgroup of \mathcal{W}_A , by (1.3). Now we show that $\mathcal{H}(A) \cap \mathcal{W}_A = \{1\}$. Since $\mathcal{H}(A) \subseteq \mathcal{H}(\mathcal{V})$ and $\mathcal{W}_{\dot{A}} \subseteq \mathcal{W}_{\dot{\mathcal{V}}}$, it is enough to show that $\mathcal{H}(\dot{\mathcal{V}}) \cap \mathcal{W}_{\dot{\mathcal{V}}} = \{1\}$. Let $h \in \mathcal{W}_{\dot{\mathcal{V}}} \cap \mathcal{H}(\mathcal{V})$

and consider an expression of h in the form (1.9). Since $h \in \mathcal{W}_{\dot{\nu}}$, we have from (1.2) that $h(\lambda_j) = \lambda_j$, $1 \leq j \leq \nu$ and so it follows from Lemma 1.5 that $h = 1$.

To complete the proof, we must show $\mathcal{W}_A = \mathcal{W}_{\dot{A}}\mathcal{H}(A)$. Let $\alpha \in \dot{A}$ and $\sigma \in \mathcal{V}^0$ such that $\dot{\alpha} + \sigma \in A$. By assumption, $w_{\dot{\alpha}+\sigma}w_{\dot{\alpha}} \in \mathcal{H}(A) \subseteq \mathcal{W}_A$ and $w_{\dot{\alpha}+\sigma} \in \mathcal{W}_A$, therefore $w_{\dot{\alpha}} = w_{\dot{\alpha}+\sigma}w_{\dot{\alpha}+\sigma}w_{\dot{\alpha}} \in \mathcal{W}_A$ and so $\mathcal{W}_{\dot{A}} \subseteq \mathcal{W}_A$. This shows that $\mathcal{W}_{\dot{A}}\mathcal{H}(A) \subseteq \mathcal{W}_A$. To see the reverse inclusion, let w_{α} , $\alpha \in A^\times$, be a generator of \mathcal{W}_A . Then $\alpha = \dot{\alpha} + \sigma$ where $\dot{\alpha} \in \dot{A}^\times$ and $\sigma \in \mathcal{V}^0$. Since $w_{\dot{\alpha}} \in \mathcal{W}_{\dot{A}}$ and since by assumption $w_{\dot{\alpha}}w_{\dot{\alpha}+\sigma} \in \mathcal{H}(A)$, we have $w_{\alpha} = w_{\dot{\alpha}+\sigma} = w_{\dot{\alpha}}(w_{\dot{\alpha}}w_{\dot{\alpha}+\sigma}) \in \mathcal{W}_{\dot{A}}\mathcal{H}(A)$. This completes the proof. \square

§2. Extended Affine Weyl Groups

In this section, we study Weyl groups of simply laced extended affine root systems. We are mostly interested in finding a particular finite set of generators for such a Weyl group and its center (see Proposition 2.2). Since the semilattice involved in the structure of an extended affine root system plays a crucial role in our study, we start this section with recalling the definition of a semilattice from [AABGP, II.§1] and introducing a notion of *supporting class* for semilattices. For the theory of extended affine root systems the reader is referred to [AABGP]. In particular, we will use the notation and concepts introduced there without further explanations.

Definition 2.1. A *semi-lattice* is a subset S of a finite dimensional real vector space \mathcal{V}^0 such that $0 \in S$, $S \pm 2S \subseteq S$, S spans \mathcal{V}^0 and S is discrete in \mathcal{V}^0 . The *rank* of S is defined to be the dimension ν of \mathcal{V}^0 . Note that the replacement of $S \pm 2S \subseteq S$ by $S \pm S \subseteq S$ in the definition gives one of the equivalent definitions for a *lattice* in \mathcal{V}^0 . Semilattices S and S' in \mathcal{V}^0 are said to be similar if there exist $\psi \in GL(\mathcal{V}^0)$ so that $\psi(S) = S' + \sigma'$ for some $\sigma' \in S'$.

Let S be a semilattice in \mathcal{V}^0 of rank ν . The \mathbb{Z} -span Λ of S is a lattice in \mathcal{V}^0 , a free abelian group of rank ν which has an \mathbb{R} -basis of \mathcal{V}^0 as its \mathbb{Z} -basis. By [AABGP, II.1.11], S contains a subset $B = \{\sigma_1, \dots, \sigma_\nu\}$ of S which forms a basis for Λ . We call such a set B , a *basis* for S . Then

$$\Lambda = \langle S \rangle = \sum_{i=1}^{\nu} \mathbb{Z}\sigma_r \quad \text{with } \sigma_r \in S \text{ for all } r.$$

Consider $\tilde{\Lambda} := \Lambda/2\Lambda$ as a \mathbb{Z}_2 -vector space with ordered basis \tilde{B} , the image of B in $\tilde{\Lambda}$. For $\sigma \in \Lambda$ and $1 \leq r \leq \nu$, let $\sigma(r) \in \{0, 1\}$ be the unique integer such that $\tilde{\sigma} = \sum_{r=1}^{\nu} \sigma(r)\tilde{\sigma}_r$. Then we set

$$\text{supp}_B(\sigma) := \{1 \leq r \leq \nu \mid \sigma(r) = 1\}.$$

Then $\sigma = \sum_{r \in \text{supp}_B(\sigma)} \sigma_r \pmod{2\Lambda}$. By [AABGP, II.1.6], S can be written in the form

$$S = \bigcup_{j=0}^m (\delta_j + 2\Lambda),$$

where $\delta_0 = 0$ and δ_j 's are distinct coset representatives of 2Λ in Λ . The integer m is called the *index* of S and is denoted by $\text{ind}(S)$. The collection

$$(2.1) \quad \text{supp}_B(S) := \{ \text{supp}_B(\delta_j) \mid 0 \leq j \leq m \}$$

is called the *supporting class* of S , with respect to B . Since $\delta_j = \sum_{r \in \text{supp}_B(\delta_j)} \sigma_r \pmod{2\Lambda}$, the supporting set determines S uniquely. Therefore, we may write

$$(2.2) \quad S = \biguplus_{J \in \text{supp}_B(S)} (\tau_J + 2\Lambda) \quad \text{where} \quad \tau_J := \sum_{r \in J} \sigma_r.$$

(By convention we have $\tau_\emptyset := \sum_{r \in \emptyset} \sigma_r = 0$). By [A5, Proposition 1.12], if $\nu \leq 3$, then the index determines uniquely, up to similarity, the semilattices in Λ . So by [AABGP, Table II.4.5], up to similarity, the semilattices of rank ≤ 3 in Λ are listed in the following table, according to their supporting classes:

Table 2.3. The supporting classes of semilattices, up to similarity, for $\nu \leq 3$.

ν	index	$\text{supp}_B(S)$
0	0	$\{\emptyset\}$
1	1	$\{\emptyset, \{1\}\}$
2	2	$\{\emptyset, \{1\}, \{2\}\}$
	3	$\{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$
3	3	$\{\emptyset, \{1\}, \{2\}, \{3\}\}$
	4	$\{\emptyset, \{1\}, \{2\}, \{3\}, \{2, 3\}\}$
	5	$\{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 3\}, \{2, 3\}\}$
	6	$\{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}\}$
	7	$\{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$

Next we recall the definition of an extended affine root system.

Definition 2.2. A subset R of a non-trivial finite dimensional real vector space \mathcal{V} , equipped with a positive semi-definite symmetric bilinear form (\cdot, \cdot) , is called an *extended affine root system* if R satisfies the following 8 axioms:

- R1) $0 \in R$,

- R2) $-R = R$,
- R3) R spans \mathcal{V} ,
- R4) $\alpha \in R^\times \implies 2\alpha \notin R$,
- R5) R is discrete in \mathcal{V} ,
- R6) For $\alpha \in R^\times$ and $\beta \in R$, there exist non-negative integers d, u such that $\beta + n\alpha \in R$, $n \in \mathbb{Z}$, if and only if $-d \leq n \leq u$, moreover $(\beta, \alpha^\vee) = d - u$,
- R7) If $R = R_1 \cup R_2$, where $(R_1, R_2) = 0$, then either $R_1 = \emptyset$ or $R_2 = \emptyset$,
- R8) For any $\sigma \in R^0$, there exists $\alpha \in R^\times$ such that $\alpha + \sigma \in R$.

The dimension ν of the radical \mathcal{V}^0 of the form is called the *nullity* of R , and the dimension ℓ of $\bar{\mathcal{V}} := \mathcal{V}/\mathcal{V}^0$ is called the *rank* of R . Sometimes, we call R a ν -extended affine root system.

Corresponding to the integers ℓ and ν , we set

$$J_\ell = \{1, \dots, \ell\} \quad \text{and} \quad J_\nu = \{1, \dots, \nu\}.$$

Let R be a ν -extended affine root system. It follows that the form restricted to $\bar{\mathcal{V}}$ is positive definite and that \bar{R} , the image of R in $\bar{\mathcal{V}}$, is an irreducible finite root system (including zero) in $\bar{\mathcal{V}}$ ([AABGP, II.2.9]). The *type* of R is defined to be the type of the finite root system \bar{R} . In this work *we always assume that R is an extended affine root system of simply laced type*, that is it has one of the types $X_\ell = A_\ell, D_\ell, E_6, E_7$ or E_8 .

According to [AABGP, II.2.37], we may fix a complement $\dot{\mathcal{V}}$ of \mathcal{V}^0 in \mathcal{V} such that

$$(2.4) \quad \dot{R} := \{\dot{\alpha} \in \dot{\mathcal{V}} \mid \dot{\alpha} + \sigma \in R \text{ for some } \sigma \in \mathcal{V}^0\}$$

is a finite root system in $\dot{\mathcal{V}}$, isometrically isomorphic to \bar{R} , and that R is of the form

$$(2.5) \quad R = R(X_\ell, S) = (S + S) \cup (\dot{R} + S)$$

where S is a semilattice in \mathcal{V}^0 . Here X_ℓ denotes the type of \dot{R} . It is known that if $\ell > 1$, then S is a lattice in \mathcal{V}^0 .

Throughout our work, we fix two sets

$$\Pi = \{\alpha_1, \dots, \alpha_\ell\} \quad \text{and} \quad B = \{\sigma_1, \dots, \sigma_\nu\},$$

where Π is a fundamental system for \dot{R} with $(\alpha_i, \alpha_i) = 2$ for all $i \in J_\ell$, and B is a basis for S . In particular, S has the expression as in (2.2). Since B is fixed, we write $\text{supp}(S)$ instead of $\text{supp}_B(S)$. From $S \pm 2S \subseteq S$, it follows that $\mathbb{Z}\sigma_r \subseteq S$ for $r \in J_\nu$, and so we have from (2.5) that

$$(2.6) \quad \alpha_i + \mathbb{Z}\sigma_r \subseteq R, \quad (i, r) \in J_\ell \times J_\nu$$

As in Section 1, let $\tilde{\mathcal{V}} = \dot{\mathcal{V}} \oplus \mathcal{V}^0 \oplus (\mathcal{V}^0)^*$, where $\dot{\mathcal{V}}$ is the real span of \dot{R} . With respect to the basis B , we extend the form (\cdot, \cdot) on \mathcal{V} to a non-degenerate form, denoted again by (\cdot, \cdot) , on $\tilde{\mathcal{V}}$ by (1.2).

We recall from (1.4) and (1.7) that

$$(2.7) \quad \mathcal{W}_R = \langle w_\alpha \mid \alpha \in R^\times \rangle$$

is a subgroup of $\text{FO}(\tilde{\mathcal{V}}, I)$ and

$$\mathcal{H}(R) = \langle w_{\alpha+\sigma}w_\alpha \mid \alpha \in R^\times, \sigma \in \mathcal{V}^0, \alpha + \sigma \in R \rangle$$

is a subgroup of W_R .

Definition 2.3. The groups \mathcal{W}_R and $\mathcal{H}(R)$ are called the *extended affine Weyl group* and the *Heisenberg-like group* of R , respectively. Since $\dot{R} \subseteq R$, we may identify the finite Weyl group of \dot{R} with the subgroup $\dot{\mathcal{W}} = \langle w_\alpha \mid \alpha \in \dot{R}^\times \rangle$ of \mathcal{W}_R . When there is no confusion we simply write \mathcal{W} and \mathcal{H} instead of \mathcal{W}_R and $\mathcal{H}(R)$ respectively.

Lemma 2.1. $\mathcal{H} = \langle T_\alpha^\sigma \mid \dot{\alpha} \in \dot{R}, \sigma \in S \rangle$.

Proof. For $\dot{\alpha} \in \dot{R}$ and $\sigma \in S$, we have from Lemma 1.1(vii) and (2.5) that $T_\alpha^\sigma = w_{\dot{\alpha}+\sigma}w_{\dot{\alpha}} \in \mathcal{H}$. Also if $\alpha \in R^\times$, $\sigma \in \mathcal{V}^0$ and $\alpha + \sigma \in R$, then $\alpha = \dot{\alpha} + \tau$, where $\dot{\alpha} \in \dot{R}$ and by (2.5), $\tau, \sigma, \tau + \sigma \in S$. Then $w_{\alpha+\sigma}w_\alpha = w_{\dot{\alpha}+\tau+\sigma}w_{\dot{\alpha}}w_{\dot{\alpha}+\tau} = T_\alpha^{\tau+\sigma}T_{-\dot{\alpha}}^\tau$. \square

We next want to find certain finite sets of generators for both \mathcal{W} and \mathcal{H} and their centers. For $r, s \in J_\nu$, we set

$$(2.8) \quad c_{r,s} := T_{\sigma_r}^{\sigma_s} \quad \text{and} \quad C := \langle c_{r,s} \mid 1 \leq r < s \leq \nu \rangle.$$

Then by (1.6) and Lemma 1.1(viii) for all $r, s \in J_\nu$, we have

$$(2.9) \quad c_{r,r} = c_{s,r}c_{r,s} = 1 \quad \text{and} \quad C \leq Z(\text{FO}(\tilde{\mathcal{V}}, I)).$$

Moreover from (2.8) and Lemma 1.7(ii), it follows that

$$(2.10) \quad C \text{ is a free abelian group of rank } \nu(\nu - 1)/2.$$

Also for $(i, r) \in J_\ell \times J_\nu$, we set

$$(2.11) \quad t_{i,r} := T_{\alpha_i}^{\sigma_r}.$$

From (2.6) and parts (iv) and (vii) of Lemma 1.1 for all $r, s \in J_\nu$ and $i, j \in J_\ell$, one can see easily that

$$(2.12) \quad t_{i,r} = w_{\alpha_i + \sigma_r} w_{\alpha_i} \in \mathcal{H} \quad \text{and} \quad [t_{i,r}, t_{j,s}] = c_{r,s}^{(\alpha_i, \alpha_j)} \in \mathcal{H}.$$

Lemma 2.2. $w_{\alpha_i} t_{j,r} w_{\alpha_i} = t_{j,r} t_{i,r}^{- (\alpha_i, \alpha_j)}$, for $i, j \in J_\ell, r \in J_\nu$.

Proof. Let $i, j \in J_\ell, r \in J_\nu$ and $\alpha = w_{\alpha_i}(\alpha_j) = \alpha_j - (\alpha_j, \alpha_i)\alpha_i$. We have from Lemma 1.1(vi) that $w_{\alpha_i} t_{j,r} w_{\alpha_i} = T_{\alpha}^{\sigma_r} = t_{j,r} t_{i,r}^{- (\alpha_i, \alpha_j)}$. \square

In order to describe the centers $Z(\mathcal{H})$ and $Z(\mathcal{W})$, we use the notion of supporting class of S (with respect to B) by assigning a subgroup of C to S as follows. We set

$$(2.13) \quad F(S) := \langle z_J \mid J \subseteq J_\nu \rangle \leq C,$$

where

$$(2.14) \quad z_J := \begin{cases} \prod_{\{r,s \in J \mid r < s\}} c_{r,s} & \text{if } J \in \text{supp}(S) \\ c_{r,s}^2 & \text{if } J = \{r, s\} \notin \text{supp}(S), \\ 1, & \text{otherwise.} \end{cases}$$

(Here we interpret the product on an empty index set to be 1.) Our goal is to prove that $Z(\mathcal{W}) = Z(\mathcal{H}) = F(S)$. Note that if $\{r, s\} \subseteq \text{supp}(S)$, then $z_{\{r,s\}} = c_{r,s}$. In particular, if S is a lattice then the second condition in the definition of z_J is surplus and so $F(S) = \langle z_{\{r,s\}} \mid 1 \leq r < s \leq \nu \rangle = C$. Also from the way $z_{\{r,s\}}$ is defined and (2.12) we note that

$$(2.15) \quad [t_{i,r}, t_{j,s}] = c_{r,s}^{(\alpha_i, \alpha_j)} \in \langle z_{\{r,s\}} \rangle, \quad i, j \in J_\ell, \quad r, s \in J_\nu.$$

Lemma 2.3. Let $\alpha = \sum_{i=1}^\ell m_i \alpha_i \in \dot{R}$ and $\sigma = \sum_{r=1}^\nu n_r \sigma_r \in \Lambda$. Then

$$T_\alpha^\sigma = \prod_{r=1}^\nu \prod_{i=1}^\ell t_{i,r}^{m_i n_r} \prod_{1 \leq r < s \leq \nu} c_{s,r}^{n_r n_s}.$$

Proof. Using Lemmas 1.2 and 1.1(iii), we have

$$T_\alpha^\sigma = \prod_{r=1}^\nu T_\alpha^{n_r \sigma_r} \prod_{1 \leq r < s \leq \nu} (T_{\sigma_s}^{\sigma_r})^{n_s n_r} = \prod_{r=1}^\nu \prod_{i=1}^\ell t_{i,r}^{m_i n_r} \prod_{1 \leq r < s \leq \nu} c_{s,r}^{n_r n_s}.$$

\square

For a subset $J = \{i_1, \dots, i_n\}$ of J_ν with $i_1 < i_2 < \dots < i_n$ and a group G we make the convention

$$\prod_{i \in J} a_i = a_{i_1} a_{i_2} \cdots a_{i_n} \quad (a_i \in G).$$

Proposition 2.1. $\mathcal{H} = \langle t_{i,r}, z_J \mid (i, r) \in J_\ell \times J_\nu, J \subseteq J_\nu \rangle.$

Proof. Let T be the group in the right hand side of the equality. We proceed with the proof in the following two steps.

(1) $T \subseteq \mathcal{H}$. By (2.12), it is enough to show that $z_J \in \mathcal{H}$ for any set $J \subseteq J_\nu$. First, let $J \in \text{supp}(S)$. Then by the definition of z_J , we have $z_J = \prod_{\{r,s \in J \mid r < s\}} c_{r,s}$. Now it follows from Lemma 1.2 that

$$T_{\alpha_i}^{\tau_J} = T_{\alpha_i}^{\sum_{r \in J} \sigma_r} = \prod_{\{r,s \in J \mid r < s\}} T_{\sigma_s}^{\sigma_r} \prod_{r \in J} T_{\alpha_i}^{\sigma_r} = \prod_{\{r,s \in J \mid r < s\}} c_{s,r} \prod_{r \in J} T_{\alpha_i}^{\sigma_r} = z_J^{-1} \prod_{r \in J} t_{i,r}.$$

But since $\alpha_i + \tau_J \in R$ (by (2.2) and (2.5)), it follows from Lemma 2.1 and (2.12) that

$$z_J = (T_{\alpha_i}^{\tau_J})^{-1} \prod_{r \in J} t_{i,r} \in \mathcal{H}.$$

Finally, suppose $J = \{r, s\} \notin \text{supp}(S)$ where $1 \leq r < s \leq \nu$. Then from the definition of z_J and (2.12) we have

$$z_J = c_{r,s}^2 = [t_{i,r}, t_{i,s}] \in \mathcal{H}.$$

This completes the proof of step (1).

(2) $\mathcal{H} \subseteq T$. We have from Lemmas 2.1, 1.1(ii) and (2.2) that

$$\begin{aligned} \mathcal{H} &= \langle T_\alpha^\sigma \mid \alpha \in \dot{R}, \sigma \in S \rangle \\ &= \langle T_\alpha^\sigma \mid \alpha \in \dot{R}, \sigma \in \cup_{J \in \text{supp}(S)} (\tau_J + 2\Lambda) \rangle \\ &= \langle T_\alpha^\sigma \mid \alpha \in \dot{R}, \sigma \in \tau_J + 2\Lambda, J \in \text{supp}(S) \rangle \\ &= \langle T_\alpha^{\tau_J + \sigma} \mid \alpha \in \dot{R}, \sigma \in 2\Lambda, J \in \text{supp}(S) \rangle \\ &= \langle T_\alpha^{\tau_J} T_\alpha^\sigma T_\alpha^{\tau_J} \mid \alpha \in \dot{R}, \sigma \in 2\Lambda, J \in \text{supp}(S) \rangle. \end{aligned}$$

We get from Lemma 2.1 and the facts that $2\Lambda \subseteq S$ and $\tau_J \in S$ for $J \in \text{supp}(S)$, that

$$\mathcal{H} = \langle T_\alpha^\sigma, T_\alpha^{\tau_J}, T_\alpha^{\tau_J} \mid \alpha \in \dot{R}, \sigma \in 2\Lambda, J \in \text{supp}(S) \rangle.$$

Now we show that each generator of \mathcal{H} of the form $T_\alpha^\sigma, T_\alpha^{\tau_J}, T_\alpha^{\tau_J}$ belongs to T . Let $\alpha = \sum_{i=1}^\ell m_i \alpha_i \in \dot{R}, \sigma = \sum_{s=1}^\nu 2n_s \sigma_s \in 2\Lambda, n_s \in \mathbb{Z}$ and $J \in \text{supp}(S)$.

Then it follows from Lemmas 1.2 and 2.3, the definition of z_J and the fact that $c_{r,s}^2 \in \langle z_{\{r,s\}} \rangle \subseteq T$, for all $1 \leq r, s \leq \nu$ that

$$T_{\tau_J}^\sigma = T_{\sum_{r \in J} \sigma_r}^{\sum_{s=1}^\nu 2n_s \sigma_s} = \prod_{r \in J} \prod_{s=1}^\nu (c_{r,s}^2)^{n_s} \in T$$

and

$$T_\alpha^\sigma = \prod_{r=1}^\nu \prod_{i=1}^\ell (t_{i,r})^{2m_i n_r} \prod_{1 \leq r < s \leq \nu} (c_{s,r}^2)^{2n_r n_s} \in T.$$

Finally,

$$T_\alpha^{\tau_J} = T_\alpha^{\sum_{r \in J} \sigma_r} = \prod_{r \in J} \prod_{i=1}^\ell t_{i,r}^{m_i} \prod_{\{r,s \in J | r < s\}} c_{s,r} = \prod_{r \in J} \prod_{i=1}^\ell t_{i,r}^{m_i} z_J^{-1} \in T.$$

From steps (1)–(2), the result follows. □

Corollary 2.1. *If $\ell \geq 2$ or S is a lattice, then*

$$\mathcal{H} = \langle t_{i,r}, c_{r,s} \mid 1 \leq i \leq \ell, 1 \leq r \leq s \leq \nu \rangle.$$

Proof. This is an immediate consequence of Proposition 2.1 and the fact $F(S) = C$. □

The remaining results of this section are new only for type A_1 . In fact for types different from A_1 , one can find essentially the same results in [MS] and [A4]. However, for completeness we provide a short proof of them, where the proofs now are easy consequences of our results in Section 1.

Proposition 2.2.

- (i) $\mathcal{W} = \dot{\mathcal{W}} \rtimes \mathcal{H}$.
- (ii) If $\ell = 1$, then $\mathcal{W} = \langle w_{\alpha_1}, t_{1,r}, z_J \mid r \in J_\nu, J \subseteq J_\nu \rangle$.
- (iii) If $\ell > 1$, then $\mathcal{W} = \langle w_{\alpha_i}, t_{i,r}, c_{r,s} \mid i \in J_\ell, r \in J_\nu, 1 \leq r < s \leq \nu \rangle$.
- (iv) \mathcal{H} is a torsion free group.
- (v) \mathcal{H} is a two-step nilpotent group.
- (vi) $Z(\mathcal{W}) = Z(\mathcal{H}) = F(S)$, if $\nu \geq 2$.
- (vii) $F(S)$ is a free abelian group of rank $\nu(\nu - 1)/2$.

Proof. (i) is an immediate consequence of Proposition 1.1 and Lemma 2.1. From (i), Corollary 2.1, Proposition 2.1 and the fact that $\dot{\mathcal{W}}$ is generated by

$w_{\alpha_i}, \dots, w_{\alpha_\ell}$, it follows that (ii) and (iii) hold. (iv) and (v) follow from Lemma 1.3 and Corollary 1.1. From (2.13) and Proposition 2.1 we have $F(S) \subseteq Z(\mathcal{W})$. So to prove (vi) it is enough to show that $Z(\mathcal{W}) \subseteq Z(\mathcal{H}) \subseteq F(S)$. Let $w \in Z(\mathcal{W})$. By (i), $w = \dot{w}h$ for some $\dot{w} \in \dot{\mathcal{W}}$ and $h \in \mathcal{H}$. Since for all $(i, r) \in J_\ell \times J_\nu$, $t_{i,r} \in \mathcal{H}$ we have from Lemma 1.1 that

$$1 = wt_{i,r}^{-1}w^{-1}t_{i,r} = wT_{-\alpha_i}^{\sigma_r}w^{-1}T_{\alpha_i}^{\sigma_r} = T_{-w(\alpha_i)}^{\sigma_r}T_{\alpha_i}^{\sigma_r} = T_{\alpha_i-w(\alpha_i)}^{\sigma_r}.$$

This gives $\alpha_i - w(\alpha_i) \in \mathcal{V}^0$. Since $w = \dot{w}h$ and $h(\alpha_i) = \alpha_i \pmod{\mathcal{V}^0}$, it follows that $\dot{w}(\alpha_i) = \alpha_i$ for all $i \in J_\ell$. Thus $\dot{w} = 1$ and so $w = h \in \mathcal{H} \cap Z(\mathcal{W})$. This gives $Z(\mathcal{W}) \subseteq Z(\mathcal{H})$. Next let $h \in Z(\mathcal{H})$. From Proposition 2.1 and (2.15), it follows that

$$h = z \prod_{r=1}^\nu \prod_{i=1}^\ell t_{i,r}^{m_{i,r}}, \quad (z \in F(S), m_{i,r} \in \mathbb{Z}).$$

Then by (v), (1.8) and (2.12) we have that

$$1 = [h, t_{j,s}] = \prod_{r=1}^\nu \prod_{i=1}^\ell [t_{i,r}^{m_{i,r}}, t_{j,s}] = \prod_{r=1}^\nu \prod_{i=1}^\ell c_{r,s}^{m_{i,r}(\alpha_i, \alpha_j)} = \prod_{r=1}^\nu c_{r,s}^{(\sum_{i=1}^\ell m_{i,r} \alpha_i, \alpha_j)}.$$

Therefore from (2.10), it follows that $(\sum_{i=1}^\ell m_{i,r} \alpha_i, \alpha_j) = 0$ for all $j \in J_\ell$. Since the form on \mathcal{V} restricted to $\dot{\mathcal{V}} = \sum_{i=1}^\ell \mathbb{R}\alpha_i$ is positive definite we get $m_{i,r} = 0$ for all $i \in J_\ell$ and $r \in J_\nu$. Then $h = z \in F(S)$ and so (vi) holds. By (2.15) and the fact that $c_{r,s}^2 = [t_{1,r}, t_{1,s}]$, we see that the group in the statement is squeezed between two groups $\langle c_{r,s}^2 : 1 \leq r < s \leq \nu \rangle$ and C . Since C is free abelian on generators $c_{r,s}$, $1 \leq r < s \leq \nu$, then (vii) follows. \square

The following important type-dependent result gives explicitly the center $Z(\mathcal{W}) = Z(\mathcal{H})$ in terms of the generators $c_{r,s}$.

Corollary 2.2. (i) *If $X_\ell = A_1$, then*

$$Z(\mathcal{H}) = \langle c_{r,s}^2, z_J \mid 1 \leq r < s \leq \nu, J \in \text{supp}(S) \rangle.$$

In particular if S is a lattice, then $Z(\mathcal{H}) = \langle c_{r,s} \mid 1 \leq r < s \leq \nu \rangle$.

(ii) *If $X_\ell = A_\ell (\ell \geq 2)$, D_ℓ or E_ℓ , then $Z(\mathcal{H}) = \langle c_{r,s} \mid 1 \leq r < s \leq \nu \rangle$.*

Proof. Both (i) and (ii) follow immediately from (2.14) and Proposition 2.2. \square

Proposition 2.3.

(i) If $X_\ell = A_1$, then each element w in \mathcal{W} has a unique expression in the form

$$(2.16) \quad w = w(n, m_r, z) := w_{\alpha_1}^n \prod_{r=1}^{\nu} t_{1,r}^{m_r} z \quad (n \in \{0, 1\}, m_r \in \mathbb{Z}, z \in F(S)).$$

(ii) If $X_\ell = A_\ell (\ell \geq 2)$, D_ℓ or E_ℓ , then each element w in \mathcal{W} has a unique expression in the form

$$(2.17) \quad w = w(\dot{w}, m_{i,r}, m_{r,s}) := \dot{w} \prod_{r=1}^{\nu} \prod_{i=1}^{\ell} t_{i,r}^{m_{i,r}} \prod_{1 \leq r < s \leq \nu} c_{r,s}^{m_{r,s}},$$

where $\dot{w} \in \dot{\mathcal{W}}$, and $m_{i,r}, m_{r,s} \in \mathbb{Z}$.

Proof. (i) First we can express each element $w \in \mathcal{W}$ in terms of generators given in Proposition 2.2(ii). Next we can reorder the appearance of generators in any such expression using (2.15), Proposition 2.2(vi), Corollary 2.2(i), Lemmas 2.2 and the fact that $w_{\alpha_1}^2 = 1$. Now to complete the proof it is enough to show that the expression of w in the form (2.16) is unique. Let $w(n', m'_r, z')$ be another expression of w in the form (2.16). Then from Proposition 2.2(i) and Lemma 1.6, it follows that $n = n' n_r = n'_r$ for all $r \in J_\nu$ and $z = z'$.

(ii) Let $w \in \mathcal{W}$. By parts (iii) and (vi) of Proposition 2.2, Corollary 2.2(ii), Lemma 2.2 and (2.15), w can be written in the form (2.17). Let $w(\dot{w}', n'_{i,r}, m'_{r,s})$ be another expression of w in the form (2.17). Then from Proposition 2.2(i) and Lemma 1.6, it follows that $\dot{w} = \dot{w}'$ and $n_{i,r} = n'_{i,r}$, $(i, r) \in J_\ell \times J_\nu$ and $m_{r,s} = m'_{r,s}$ for all $r, s \in J_\nu$. □

§3. A Presentation for Heisenberg-Like Group

We keep all the notations as in Section 2. In particular, R is a simply laced extended affine root system and \mathcal{H} is its Heisenberg-like group.

We recall from Proposition 2.2 that $F(S) = Z(\mathcal{H})$. For $1 \leq r < s \leq \nu$, we define

$$(3.1) \quad n(r, s) = \min\{n \in \mathbb{N} : c_{r,s}^n \in F(S)\}.$$

Remark 1. It is important to notice that we may consider $C = \langle c_{r,s} : 1 \leq r < s \leq \nu \rangle$ as an abstract free abelian group (see Lemma 1.7(ii)) and $F(S)$ as a subgroup whose definition depends only on the semilattice S . It follows that the integers $n(r, s)$ are uniquely determined by S (and so by R).

We note from (2.14) that if $1 \leq r < s \leq \nu$, then depending on either $\{r, s\}$ is in $\text{supp}(S)$ or not we have $z_{\{r,s\}} = c_{r,s}$ or $z_{\{r,s\}} = c_{r,s}^2$. Thus we have the following lemma.

Lemma 3.1. (i) $n(r, s) \in \{1, 2\}$, $1 \leq r < s \leq \nu$.

(ii) If $\{r, s\} \in \text{supp}(S)$ for all $1 \leq r < s \leq \nu$, then $n(r, s) = 1$ for all such r, s and

$$F(S) = \langle c_{r,s} \mid 1 \leq r < s \leq \nu \rangle.$$

In particular, this holds if S is a lattice.

(iii) If $\text{supp}(S) \subseteq \{\emptyset, \{r, s\} \mid 1 < r \leq s \leq \nu\}$, then

$$F(S) = \langle c_{r,s}^{n(r,s)} \mid 1 \leq r < s \leq \nu \rangle$$

where

$$(3.2) \quad n(r, s) = \begin{cases} 1, & \text{if } \{r, s\} \in \text{supp}(S), \\ 2, & \text{if } \{r, s\} \notin \text{supp}(S). \end{cases}$$

Let $A = (a_{i,j})_{1 \leq i,j \leq \ell}$ be the Cartan matrix of type X_ℓ , that is

$$a_{i,j} = (\alpha_i, \alpha_j), \quad i, j \in J_\ell.$$

Lemma 3.2. $n(r, s) \mid a_{i,j}$, $i, j \in J_\ell$, $1 \leq r < s \leq \nu$

Proof. By (2.15), $c_{r,s}^{a_{i,j}} = c_{r,s}^{(\alpha_i, \alpha_j)} \in F(S)$ for all $r, s \in J_\nu$ and $i, j \in J_\ell$. Now the result follows by the way $n(r, s)$ is defined. \square

By Lemma 3.2, we have $a_{i,j}n(r, s)^{-1} \in \mathbb{Z}$, $i, j \in J_\ell$, $1 \leq r < s \leq \nu$. So from (2.12) we have

$$(3.3) \quad [t_{i,r}, t_{j,s}] = (c_{r,s}^{n(r,s)})^{a_{i,j}n(r,s)^{-1}}.$$

Note that the integer $n(r, s)$ appears only in type A_1 as in other types $n(r, s) = 1$.

Theorem 3.1. Let $R = R(X_\ell, S)$ be a simply laced extended affine root system of type X_ℓ and nullity ν with Heisenberg-like group \mathcal{H} . Let $A = (a_{i,j})$ be the Cartan matrix of type X_ℓ , $n(r, s)$'s be the unique integers defined by (3.1) and $m = \text{ind}(S)$. If

$$(3.4) \quad F(S) \text{ is generated by elements } c_{r,s}^{n(r,s)}, \quad 1 \leq r < s \leq \nu,$$

then \mathcal{H} is isomorphic to the group $\widehat{\mathcal{H}}$ defined by generators

$$(3.5) \quad \begin{cases} y_{i,r} & 1 \leq i \leq \ell, \quad 1 \leq r \leq \nu, \\ z_{r,s} & 1 \leq r < s \leq \nu, \end{cases}$$

and relations

$$(3.6) \quad \mathcal{R}_{\widehat{\mathcal{H}}} := \begin{cases} [z_{r,s}, z_{r',s'}], \\ [y_{i,r}, z_{r',s'}], [y_{i,r}, y_{j,r}], \\ [y_{i,r}, y_{j,s}] = z_{r,s}^{a_{i,j} n(r,s)^{-1}}, \quad r < s, \end{cases}$$

where if $\ell > 1$ or $\ell = 1$ and $\nu \leq 3$, the condition (3.4) is automatically satisfied. Moreover, if $\ell > 1$, $n(r, s) = 1$ for all r, s and if $\ell = 1$ and $\nu \leq 3$, $n(r, s)$'s are given by the following table:

ν	m	$n(1, 2)$	$n(1, 3)$	$n(2, 3)$
0	0	-	-	-
1	1	-	-	-
2	2	2	-	-
	3	1	-	-
3	3	2	2	2
	4	2	2	1
	5	2	1	1
	6	1	1	1
	7	1	1	1

Proof. By Propositions 2.1, 2.2(vi), (2.13) and assumption (3.4), we have

$$\mathcal{H} = \langle t_{i,r} \mid (i, r) \in J_\ell \times J_\nu \rangle F(S) = \langle t_{i,r}, c_{r,s}^{n(r,s)} \mid 1 \leq i \leq \ell, 1 \leq r < s \leq \nu \rangle.$$

From (3.3) and the fact that $c_{r,s}^{n(r,s)} \in Z(\mathcal{H})$, for $1 \leq r < s \leq \nu$, it is clear that there exists a unique epimorphism $\varphi : \widehat{\mathcal{H}} \rightarrow \mathcal{H}$ such that $\varphi(y_{i,r}) = t_{i,r}$ and $\varphi(z_{r,s}) = c_{r,s}^{n(r,s)}$. We now prove that φ is a monomorphism. Let $\hat{h} \in \widehat{\mathcal{H}}$ and $\varphi(\hat{h}) = 1$. By (3.6), \hat{h} can be written as

$$\hat{h} = \prod_{r=1}^{\nu} \prod_{i=1}^{\ell} y_{i,r}^{m_{i,r}} \prod_{1 \leq r < s \leq \nu} z_{r,s}^{n_{r,s}} \quad (m_{i,r}, n_{r,s} \in \mathbb{Z}).$$

Then

$$1 = \varphi(\hat{h}) = \prod_{r=1}^{\nu} \prod_{i=1}^{\ell} t_{i,r}^{m_{i,r}} \prod_{1 \leq r < s \leq \nu} c_{r,s}^{n(r,s)n_{r,s}}.$$

Now it follows from Proposition 2.3 and (2.10) that $m_{i,r} = n_{r,s} = 0$, for all i, r, s and so $\hat{h} = 1$.

Next let $\ell > 1$. By [AABGP, Proposition II.4.2] the involved semilattice S in the structure of R is a lattice and so by Lemma 3.1(ii), $n(r, s) = 1$ for all r, s . Finally, let $\ell = 1$. According to [AABGP, Proposition II.4.2], any extended affine root system of type A_1 and nullity ≤ 3 is isomorphic to an extended affine root system of the form $R = R(A_1, S)$ where $\text{supp}(S)$ is given in Table 2.3. The result now follows immediately from this table and Lemma 3.1(ii)–(iii). □

§4. A Presentation for Extended Affine Weyl Groups

We keep the same notation as in the previous sections. As before $R = R(X_\ell, S)$ is a simply laced extended affine root system of nullity ν , \mathcal{W} is its extended affine Weyl group and \mathcal{H} is its Heisenberg-like group. Using the Coxeter presentation for the finite Weyl group $\dot{\mathcal{W}}$, Theorem 3.1 and the semidirect product $\dot{\mathcal{W}} \rtimes \mathcal{H}$, we obtain a finite presentation for \mathcal{W} . Let $A = (a_{i,j})_{1 \leq i,j \leq \ell}$ be the Cartan matrix of type X_ℓ .

We recall from [Ka, Proposition 3.13] that $\dot{\mathcal{W}}$ is a Coxeter group with generators $w_{\alpha_1}, \dots, w_{\alpha_\ell}$ and relations

$$(4.1) \quad w_{\alpha_i}^2 \quad \text{and} \quad (w_{\alpha_i} w_{\alpha_j})^{a_{i,j}+2} \quad (i \neq j).$$

Theorem 4.1. *Let $R = R(X_\ell, S)$ be a simply laced extended affine root system of type X_ℓ and nullity ν with extended affine Weyl group \mathcal{W} . Let $A = (a_{i,j})$ be the Cartan matrix of type X_ℓ and $n(r, s)$'s be the unique integers defined by (3.1). If*

$$(4.2) \quad F(S) \text{ is generated by elements } c_{r,s}^{n(r,s)}, \quad 1 \leq r < s \leq \nu,$$

then \mathcal{W} is isomorphic to the group $\widehat{\mathcal{W}}$ defined by generators

$$\begin{cases} x_i, & 1 \leq i \leq \ell, \\ y_{i,r}, & 1 \leq i \leq \ell, 1 \leq r \leq \nu, \\ z_{r,s}, & 1 \leq r < s \leq \nu \end{cases}$$

and relations

$$\mathcal{R}_{\widehat{\mathcal{W}}} := \begin{cases} x_i^2, (x_i x_j)^{a_{i,j}+2}, & (i \neq j), \\ x_i y_{j,r} x_i = y_{j,r} y_{i,r}^{-a_{i,j}}, \\ [z_{r,s}, z_{r',s'}], [x_i, z_{r,s}], [y_{i,r}, z_{r,s}], [y_{i,r}, y_{j,r}], \\ [y_{i,r}, y_{j,s}] = z_{r,s}^{a_{i,j} n(r,s)-1}, & 1 \leq r < s \leq \nu. \end{cases}$$

Moreover if $\ell > 1$ then $n(r, s) = 1$ for all r, s , (in particular, the assumption (4.2) holds). Furthermore, if $\ell = 1$ and $\nu \leq 3$ then the assumption (4.2) is automatically satisfied and the relations $\mathcal{R}_{\widehat{\mathcal{W}}}$ reduces to the relations

$$\mathcal{R}_{\widehat{\mathcal{W}}} := \begin{cases} x^2, & xy_r x = y_r^{-1}, \\ [z_{r,s}, z_{r',s'}], [x, z_{r,s}], [y_r, z_{r,s}], \\ [y_r, y_s] = z_{r,s}^{2n(r,s)^{-1}}, & 1 \leq r < s \leq \nu, \end{cases}$$

where $n(r, s)$'s are given explicitly by (3.7) (depending on $m = \text{ind}(S)$).

Proof. From parts (ii), (iii) and (vi) of Proposition 2.2, (2.13) and assumption (4.2), it follows that

$$(4.3) \quad \begin{aligned} \mathcal{W} &= \langle w_{\alpha_i}, t_{i,r} \mid (i, r) \in J_\ell \times J_\nu \rangle F(S) \\ &= \langle w_{\alpha_i}, t_{i,r}, c_{r,s}^{n(r,s)} \mid 1 \leq i \leq \ell, 1 \leq r < s \leq \nu \rangle. \end{aligned}$$

By (3.3), Lemma 2.2, (4.1) and the fact that $c_{r,s}^{n(r,s)} \in Z(\mathcal{H})$, for $1 \leq r < s \leq \nu$, the assignment $x_i \mapsto w_{\alpha_i}$, $y_{i,r} \mapsto t_{i,r}$ and $z_{r,s} \mapsto c_{r,s}^{n(r,s)}$ induces a unique epimorphism ψ from $\widehat{\mathcal{W}}$ onto \mathcal{W} . Also by Lemma (4.1), the restriction of ψ to $\widehat{\mathcal{W}} := \langle x_i \mid 1 \leq i \leq \ell \rangle$ induces the isomorphism

$$(4.4) \quad \widehat{\mathcal{W}} \stackrel{\psi}{\cong} \mathcal{W}.$$

We now show that ψ is injective. Let $\psi(\hat{w}) = 1$, for some $\hat{w} \in \widehat{\mathcal{W}}$. From the defining relations for $\widehat{\mathcal{W}}$, it is easy to see that \hat{w} can be written in the form

$$\hat{w} = \hat{w} \prod_{r=1}^{\nu} \prod_{i=1}^{\ell} y_{i,r}^{m_{i,r}} \prod_{1 \leq r < s \leq \nu} z_{r,s}^{n_{r,s}} \quad (\hat{w} \in \widehat{\mathcal{W}}, n_{r,s}, m_{i,r} \in \mathbb{Z}).$$

Then

$$1 = \psi(\hat{w}) = \psi(\hat{w}) \prod_{r=1}^{\nu} \prod_{i=1}^{\ell} t_{i,r}^{m_{i,r}} \prod_{1 \leq r < s \leq \nu} c_{r,s}^{n_{r,s} n_{r,s}}.$$

Therefore from (4.4) and Propositions 2.3, it follows that $m_{i,r} = 0$, $n_{r,s} = 0$ for all i, r, s and $\hat{w} = 1$. Thus $\hat{w} = 1$ and so $\mathcal{W} \cong \widehat{\mathcal{W}}$. Now an argument similar to the last paragraph of the proof of Theorem 3.1 completes the proof. \square

We close this section with the following remark.

Remark 2. In Section 2, we fixed a choice of a hyperbolic extension $(\tilde{\mathcal{V}}, I)$ of (\mathcal{V}, I) , determined by extended affine root system R and then we defined the extended affine Weyl group \mathcal{W} as a subgroup of $O(\tilde{\mathcal{V}}, I)$. As Remark 1 and Theorem 4.1 suggest, the definition of \mathcal{W} is independent of the choice of this particular hyperbolic extension, if condition (3.4) holds.

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