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Remarks on Framed Bordism Classes of Classical Lie Groups

By

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Introduction

We write (G, R) for a compact connected Lie group G of dimension $d > 0$ regarded as a framed manifold with the right invariant framing R and denote by $[G, R]$ its bordism class determined in π_d^S via the Thom-Pontrjagin construction. For this it is known [7] that $72[G, R] = 0$ and more previously in [2] it is conjectured that $[G, R] = 0$ if $\operatorname{rank} G \geq 10$ or so. We have $[SO(2n), R] = 0$ $(n \geq 2)$ already in [3] and so we are interested in such a conjecture for the cases $G = SO(n), SU(n)$ or $Sp(n)$.

In this note we consider a slight modification of Proposition 5.3 of [1] which describes the behavior of framings of G and using this we show the following partial results:

$$[SO(8n+1), R] = 0, \ [SO(32n+3), R] = 0, \ [Sp(8n), R] = 0 \text{ and}$$
$$[SU(8n+1), R] = 0 \quad \text{for } n \geq 1$$

And also we give a direct proof of the result of [6] about the 3-component of $[SO(2n+1), R]$ and so of $[Sp(n), R]$. In particular, two third parts of it follow immediately from this modification.

§1. A Formula for Null Bordism Classes

In this section we will reconsider a result of Proposition 5.3 of [1] and give a proof of it in order to grope for a little improvement. Let G be a compact connected Lie group and $H \subset G$ a closed subgroup isomorphic to S^s where

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$s = 1$ or 3. Let us assume that H is identified with S^s and denote by σ the usual inclusion $H \hookrightarrow SO(s+1)$.

Let η be the vector bundle associated via σ with the principal H-bundle $\pi : G \to G/H$ and denote by $p : D(\eta) \to G/H$ its disk bundle. Here G/H means the space of right cosets of H in G as usual. Putting $W = D(\eta)$ its tangent bundle $T(W)$ can be decomposed as

$$T(W) \cong p^*(T(G/H) \oplus \eta) \tag{1.1}$$

as vector bundles over W where $T(G/H)$ denotes the tangent bundle of G/H (see [2] and also [4]). Clearly the associated sphere bundle $S(\eta) \to G/H$ of η is isomorphic to the principal H-bundle $\pi : G \to G/H$. Therefore we see that the restriction of the equation of (1.1) to $S(\eta)$ becomes an isomorphism

$$T(G) \oplus (G \times \boldsymbol{R}) \cong \pi^*(T(G/H)) \oplus \pi^*(\eta)$$

of vector bundles over G where $T(G)$ denotes the tangent bundle of G. Note that $\pi^*(\eta)$ has a natural cross-section so that it can be decomposed as $\pi^*(\eta) \cong \eta_0 \oplus (G \times \boldsymbol{R})$. Here η_0 coincides with the bundle $T_H(G)$ consisting of tangents along the fibres of the principal H-bundle $\pi : G \to G/H$. This has a natural right action of H. So we have an isomorphism

$$T(G) \oplus (G \times \boldsymbol{R}) \cong \pi^*(T(G/H) \oplus T_H(G)/H) \oplus (G \times \boldsymbol{R}) \tag{1.2}$$

of vector bundles over G.

Analogously as in (1.1) we have a decomposition

$$T(G) \cong \pi^*(T(G/H) \oplus T_H(G)/H). \tag{1.3}$$

It is easily checked that the equation of (1.2) is just a stabilization of this isomorphism. Now dividing the equation of (1.3) by the right action of H yields an isomorphism

$$T(G)/H \cong T(G/H) \oplus T_H(G)/H$$

of vector bundles over G/H. Denote by Ad_G (resp. Ad_H) the adjoint representation of G (resp. H). Then the restriction of Ad_G to H is decomposed as

$$\mathrm{Ad}_G|H = \mathrm{Ad}_{(G,H)} \oplus \mathrm{Ad}_H$$

since $\mathrm{Ad}_G|H$ contains Ad_H as a subrepresentation. Let H act via $\mathrm{Ad}_G|H$ on the tangent space $T_e(G)$ at the identity element e of G. Then $T_e(G)$ is also decomposed as $T_e(G) = V \oplus T_e(H)$ corresponding to the decomosition of $\mathrm{Ad}_G|H$ mentioned above. Consider the isomorphism $L : T(G) \to G \times T_e(G)$ of vector bundles given by $L(v) = (g, L_{g^{-1}*}(v))$ for $v \in T_g(G)$ where $L_{g^{-1}}$ denotes left multiplication by g^{-1} and $T_g(G)$ the tangent space at $g \in G$. Since this isomorphism becomes compatible with the right action of H, we have $T(G)/H \cong G \times_H T_e(G)$. Similarly we obtain $T_H(G)/H \cong G \times_H T_e(H)$ so that we can deduce from the equation following (1.3)

$$T(G/H) \cong G \times_H V. \tag{1.4}$$

Let us identify the preceding two isomorphisms and write $g \times_H u$ for the element of $G \times_H V$ (resp. $G \times_H T_e(H)$) represented by $(g, u) \in G \times V$ (resp. $G \times T_e(H)$). If furthermore an element $v \in T_e(G)$ is decomposed as

$$v = v_b + v_f$$

where $v_b \in V$ and $v_f \in T_e(H)$, then we see that the isomorphism of vector bundles of (1.3) is given by the assignment

$$L_{g*}(v) \mapsto (g \times_H v_b) + (g \times_H v_f). \tag{1.5}$$

Here we recall the definition of framings of tangent bundles. Identifying $\boldsymbol{R}^d$ with $T_e(G)$ in a orientation preserving way, the right invariant framing $R : T(G) \to G \times \boldsymbol{R}^d$ of $T(G)$ is given by $R(v) = (g, R_{g^{-1}*}(v))$ for $v \in T_g(G)$ where $R_{g^{-1}}$ denotes right multiplication by g^{-1}. We note here that the left invariant framing L defined above and this one are transformed into each other with the change of orientation by $(-1)^d$ in degree under the map $t : G \to G$ given by $t(g) = g^{-1}$ for $g \in G$, i.e. it holds that $[G, R] = (-1)^d [G, L]$.

Given a map $\varphi : G \to SO(n)$, we have an automorphism of the trivial bundle $G \times \boldsymbol{R}^n$ given by $(g, w) \mapsto (g, \varphi(g)^{-1}(w))$. Then the twisted framing R^φ of R by φ is defined as a direct sum of R and this automorphism. It is easy to see that the determination of $[G, R^\varphi]$ depends essentially on the element $\beta(\varphi)$ of $\widetilde{KO}^{-1}(G^+)$ represented by φ where G^+ denotes the G adjoined a disjoint base point. In fact by definition ([10], Proposition 8.14) we see that $[G, R^\varphi]$ can be represented as a Kronecker product of $J(\beta(\varphi)) \in \pi_S^0(G^+)$ and the homotopy fundamental class $\sigma(G, R) \in \pi_d^S(G^+)$ of (G, R), i.e.

$$[G, R^\varphi] = \langle J(\beta(\varphi)), \sigma(G, R) \rangle \tag{1.6}$$

where J denotes the J-map $\widetilde{KO}^{-1}(G^+) \to \pi_S^0(G^+)$. Henceforth we assume that φ is identified with $\beta(\varphi)$. Also we abbreviate (G, R^φ) to (G, φ) and write $[G, R^\varphi] = [G, \varphi]$ so that we have $[G, 0] = [G, R]$.

We turn now to the equation of (1.1). Suppose that there is a real representation f of G such that

$$f|H = \mathrm{Ad}_{(G,H)} \oplus \sigma \oplus \ell$$

where the integer ℓ denotes the ℓ-dimensional trivial representation. Applying f to the equation of (1.1) in the usual fashion under the identification of (1.4) yields an isomorphism

$$F : T(W) \oplus (W \times \boldsymbol{R}^{\ell}) \to W \times \boldsymbol{R}^{d+\ell+1}$$

of vector bundles over W, so that W becomes a framed manifold with F as a framing. And further considering the assignment of (1.5) we find that the restriction

$$F' : T(G) \oplus (G \times \boldsymbol{R}^{\ell+1}) \to G \times \boldsymbol{R}^{d+\ell+1}$$

of F to the restriction of $T(W) \oplus (W \times \boldsymbol{R}^{\ell}) \to W$ to $G \subset W$ is given by

$$(v, (g, w)) \mapsto (g, f(g)(v + w))$$

for $v \in T_g(G)$ and $w \in \boldsymbol{R}^{\ell+1}$. This implies that the twisted framing of R by $-f$ equals F', so that it can be extended over W. So $(G, -f)$ becomes a framed boundary of (W, F).

Note that this result is slightly generalized as follows. Let ρ_1, ρ_2 be n-dimensional real representations of G such that $\rho_1|H = \rho_2|H$. Define a map $\bar{\varphi} : G/H \to SO(n)$ by $\bar{\varphi}(gH) = \rho_1(g)\rho_2(g)^{-1}$ for $g \in G$ and put $\varphi = \bar{\varphi} \circ p : W \to SO(n)$. Then we see that the restriction of the twisted framing F^{φ} of W to G equals the twisted framing of R by $-f + \rho_1 - \rho_2$, so that $(G, -f + \rho_1 - \rho_2)$ also becomes a framed boundary of (W, F^{φ}).

In general there holds the following formula. For any real representations ρ_1, ρ_2 of G we have

$$[G, \mathrm{Ad}_G - \rho_1 + \rho_2] = (-1)^d [G, \rho_1 - \rho_2]. \tag{1.7}$$

This is an easy modification of Lemma 4 of [9]. The proof can be done also using the map $t : G \to G$ as above. In fact t transforms $(G, \mathrm{Ad}_G - \rho_1 + \rho_2)$ into $(G, \rho_1 - \rho_2)$ and then changes the orientation by $(-1)^d$ in degree. Using (1.7) we therefore have that $(G, \mathrm{Ad}_G + f - \rho_1 + \rho_2)$ is framed null-bordant. Thus we get the following.

Proposition 1.8 (cf. [1], Proposition 5.3). *Let $H \subset G$ be a closed subgroup isomorphic to either S^1 or S^3. Suppose that there exists a real representation f of G such that $f|H = \mathrm{Ad}_{(G,H)} \oplus \sigma$ up to trivial representations* *and given two real representations* ρ_1, ρ_2 *of* G *satisfying* $\rho_1|H = \rho_2|H$*. Then*

$$[G, -f + \rho_1 - \rho_2] = 0$$

(*hence equivalently* $[G, \mathrm{Ad}_G + f - \rho_1 + \rho_2] = 0$).

§2. Formulas for Classical Lie Groups

By (1.6) we have $[G, \varphi] = \langle J(\varphi),\ \sigma(G, R)\rangle$. Hence putting $\kappa(x) = \langle x, \sigma(G, R)\rangle$ for $x \in \pi_S^0(G^+)$ we can write as $\kappa(J(\varphi)) = [G, \varphi]$ and in particular $\kappa(1) = [G, R]$ since $J(0) = 1$. For any prime p we denote by $x_{(p)}$ the p-component of elements x of the relevant groups and further by $x_{(odd)}$ the odd-component of x. Let ρ denote the identity map representation of $SO(n)$ or the realifications of those of $SU(n)$ and $Sp(n)$. Then from Proposition 1.8 we have the following proposition.

Proposition 2.1 (cf. [1], Proposition 5.2)**.**

a) $[SO(n), -(n-1-3k)\rho]_{(odd)} = 0\ (n \geq 2)$,

$[SO(n), -(n-3-3k)\rho]_{(odd)} = 0\ (n \geq 4)$,

$[SO(n), -(n-1-8k)\rho]_{(2)} = 0\ (n \geq 2)$,

$[SO(n), -(n-3-32k)\rho]_{(2)} = 0\ (n \geq 4)$,

b) $[SU(n), -(n-1-3k)\rho]_{(odd)} = 0\ (n \geq 2)$,

$[SU(n), -(n-1-8k)\rho]_{(2)} = 0\ (n \geq 2)$,

c) $[Sp(n), -(n-3k)\rho]_{(odd)} = 0\ (n \geq 1)$,

$[Sp(n), -(n-8k)\rho]_{(2)} = 0\ (n \geq 1)$

for integer k*.*

Proof. a) Choose $SO(2) \times I_{n-2}$ for the subgroup H of $G = SO(n)$ in Proposition 1.8 where I_t denotes the unit matrix of degree t. Then we can take $(n-1)\rho$ for the representation f required in Proposition 1.8 because $\mathrm{Ad}_G = \lambda^2\rho$, $\mathrm{Ad}_H = 1$ and $\rho|H = \sigma \oplus (n-2)$. Moreover, for any $k \geq 0$, if we take $\rho_1 = k(\rho^2 \oplus (n^2 - 3n))$ and $\rho_2 = k(\psi^2\rho \oplus (2n-4)\rho)$ where ψ^2 denotes the 2nd Adams operation, then we see that ρ_1 coincides with ρ_2 on H. Hence we have from Proposition 1.8

$$\kappa(J(-(n-1)\rho - k(\psi^2\rho - \rho^2 + (2n-4)\rho))) = 0.$$

for any $k \geq 0$. But interchange ρ_1 and ρ_2 shows that this equality is valid in the case where $k < 0$. Now the solution of the Adams conjecture [8] shows that $J(\rho - \psi^2\rho)_{(odd)} = 1$ and also $J(\rho^2 - 2n\rho) = 1$ since $\beta(\rho^2) = 2n\beta(\rho)$ in $\widetilde{KO}^{-1}(G^+)$. Hence by substituting these two equalities into the above one using the multiplicative formula $J(x+y) = J(x)J(y)$, we get for any integer k

$$\kappa(J(-(n-1-3k)\rho))_{(odd)} = 0$$

which shows that $[SO(n), -(n-1-3k)\rho]_{(odd)} = 0$.

To prove the second formula we set $H = SU(2) \times I_{n-4} \subset G = SO(n)$. Then we can take $f = (n-3)\rho$ and also it can be verified that $2\psi^2\rho \oplus (2n-8)\rho$ coincides with $\rho^2 \oplus (n^2 - 6n)$ on H. So by applying Proposition 1.8 again we have

$$\kappa(J(-(n-3)\rho - k((1+p^N)/2)(2\psi^2\rho - \rho^2 + (2n-8)\rho))) = 0$$

for any integer k and $N \geq 0$ where p denotes an odd prime. Hence by the same reason as above we have

$$\kappa(J(-(n-3-3k)\rho + 3kp^N\rho))_{(odd)} = 0.$$

Now we know that $J(x) \in \pi_S^0(G^+) = \boldsymbol{Z} \oplus \pi_S^0(G)$ can be written as $J(x) = 1 + \tilde{J}(x)$ where $\tilde{J}(x) \in \pi_S^0(G)$ and any elment of $\pi_S^0(G)$ is nilpotent and has finite order. So if N is taken to be sufficiently large, then we see that there holds $J(3kp^N\rho)_{(p)} = 1$. Therefore it follows from the above equality that $\kappa(J(-(n-3-3k)\rho)_{(p)} = 0$ for any odd prime p, so that we have

$$\kappa(J(-(n-3-3k)\rho))_{(odd)} = 0$$

and hence $[SO(n), -(n-3-3k)\rho]_{(odd)} = 0$ for any integer k.

As for the succeeding two formulas about their 2-components it suffices to use the 3rd Adams operation ψ^3 instead of ψ^2 in the arguments similar to those in the above cases.

b), c) The proofs of these two cases are quite parallel to that of a). Here it suffices to choose $SU(2) \times I_{n-2}$ (resp. $Sp(1) \times I_{n-1}$) for the required subgroup H of $G = SU(n)$ (resp. $Sp(n)$). Then f can be taken to be $(n-1)\rho$ (resp. $n\rho$). □

Proposition 2.2.

a) $[SO(3n+1), R]_{(odd)} = 0$,

$[SO(3n+3), R]_{(odd)} = 0,$

$[SO(8n+1), R]_{(2)} = 0,$

$[SO(32n+3), R]_{(2)} = 0,$

b) $[SU(3n+1), R]_{(odd)} = 0,$

$[SU(8n+1), R]_{(2)} = 0,$

c) $[Sp(3n), R]_{(odd)} = 0,$

$[Sp(8n), R]_{(2)} = 0$

for $n \geq 1$.

Proof. These follow by substituting adequate integers for n and k in the formulas of Proposition 2.1. For example, the first formula is just the first one of Proposition 2.1 with $3n+1$ instead of n and $k = n$. □

The following is also an immediate corollary of Proposition 2.2.

Corollary 2.3.

$[SO(24n+1), R] = 0,$

$[SO(96n+3), R] = 0,$

$[SU(24n+1), R] = 0,$

$[Sp(24n), R] = 0$

for $n \geq 1$.

As noted in the introduction we know in [7] that $[G, R]$ has at most 2- and 3-components. But we have $[G, R]_{(3)} = 0$ for $G = SO(2n+1), Sp(n)$ $(n \geq 3, n \neq 5, 7, 11)$ and $SU(n)$ $(n \geq 3)$ by [6] and [5] respectively. Hence Corollary 2.3 can be improved as follows.

Corollary 2.4.

$[SO(8n+1), R] = 0,$

$[SO(32n+3), R] = 0,$

$[SU(8n+1), R] = 0,$

$$[Sp(8n), R] = 0$$

for $n \geq 1$.

§3. Remarks for $SO(2n+1)$ and $Sp(n)$

In this section we will give a direct proof of the following result of [6] using Proposition 2.1.

(3.1) $[SO(2n+1), R]_{(3)} = 0$, equivalently $[Sp(n), R]_{(3)} = 0$ for $n \geq 3, n \neq 5$.

Here we find that this assertion holds in the cases $n = 7$ and 11 which remain undecided in [6]. But it is regrettable that the case $n = 5$ does so yet. Since $[SO(2n+1), R]_{(3)}$ and $[Sp(n), R]_{(3)}$ have the same order by Lemma 2.6 of [6] we consider only the case $G = SO(2n+1)$ below.

Especially the cases where $n \equiv 0, 1 \mod 3$ are straightforward from the first and second formulas of Proposition 2.1, a). In fact by substituting $n = 3\ell, k = 2\ell$ and $n = 3\ell+1, k = 2\ell$ into them with $2n+1$ instead of n respectively we can get $[G, R]_{(3)} = 0$ immediately.

We next consider the case $n \equiv 2 \mod 3$, i.e. $n = 3\ell + 2$ $(\ell \geq 2)$. Putting $\mu = \tilde{J}(\rho)_{(3)}$ where ρ is as above we have from the first and second formulas of Proposition 2.1, a)

$$(3.2) \qquad \kappa((1+\mu)^{3k+1}) = 0 \quad \text{and} \quad \kappa((1+\mu)^{3k+2}) = 0 \quad \text{for} \quad k \geq 0.$$

Let $H = SO(2) \times I_{2n-1} \subset G$ and let G and $T(G)$ be provided with the natural right actions of H. Then the twisted framing R^{φ} which occurs in (1.6) becomes an H-equivariant isomorphism. So we can apply Lemma 2.2 of [7] to this equivariant framed manifold (G, R^{φ}) and hence we have $24[G, \varphi] = 0$ by the same argument as in Theorem 1.1 of [7]. So taking as φ the one used in the proof of Proposition 2.1, a) we have $3\kappa((1+\mu)^{3k}) = 0$ for any $k \geq 0$. Combining this with (3.2) yields $3\kappa((1+\mu)^{k}) = 0$ for any $k \geq 0$ and so by induction on k we have

$$(3.3) \qquad 3\kappa(\mu^k) = 0 \quad \text{for } k \geq 0.$$

Using this Theorem (5.3) of [2] gives

$$(3.4) \qquad [G, R]_{(3)} = (-1)^{\ell}\kappa(\mu^{3\ell+2}) \quad \text{and} \quad \mu^{3\ell+3} = 0.$$

From now on we work modulo 3 due to (3.3). Again from (3.2) by induction on k we have

$$(3.5) \qquad \kappa(\mu^{3k} + \mu^{3k+1}) = 0 \quad \text{and} \quad \kappa(\mu^{3k+1} + \mu^{3k+2}) = 0 \quad \text{for} \quad k \geq 0.$$

Put $\alpha = \tilde{J}(\mathrm{Ad}_G)_{(3)}$. Then it can be deduced from (1.7) that

$$\kappa((1+\alpha)\mu^k) = (-1)^\ell \kappa(R^k) \text{ and } \kappa((1+\alpha)R^k) = (-1)^\ell \kappa(\mu^k) \text{ for } k \geq 0 \tag{3.6}$$

where R is the formal power series given by $1+R = (1+\mu)^{-1}$. As is seen above it holds that $J(\rho - \psi^2\rho)_{(3)} = 1$ so that $J(2\lambda^2\rho)_{(3)} = J((12\ell+9)\rho)_{(3)}$. Since $\lambda^2\rho = \mathrm{Ad}_G$ this yields

$$1+\alpha = 1-\ell\mu^3-\ell(\ell+1)\mu^6+(\ell+1)(\ell^2-\ell+3)/3\mu^9-\ell^2(\ell+1)(\ell^2-\ell+3)/3\mu^{12}+h_{15}$$

where h_{15} denotes the sum of the higher terms with degrees above 15. From (3.5) it follows immediately that

$$\kappa(\mu^{3\ell+2}) = -\kappa(\mu^{3\ell+1}) = \kappa(\mu^{3\ell}),\ \kappa(\mu^{3\ell-1}) = -\kappa(\mu^{3\ell-2}) = \kappa(\mu^{3\ell-3}),$$

$$\kappa(\mu^{3\ell-4}) = -\kappa(\mu^{3\ell-5}) = \kappa(\mu^{3\ell-6}),\ \kappa(\mu^{3\ell-7}) = -\kappa(\mu^{3\ell-8}) = \kappa(\mu^{3\ell-9}).$$

The calculations below are done taking account of these equalities. By calculating both of the equalities of (3.6) with $k = 3\ell-6$ we obtain

$$\kappa(-\mu^{3\ell-3} + (\ell+1)\mu^{3\ell}) = 0 \quad \text{and} \quad \kappa((\ell-1)\mu^{3\ell-3} + (\ell^2+1)\mu^{3\ell}) = 0.$$

These show that $\kappa(\mu^{3\ell+2}) = 0$. Hence by virtue of (3.4) we see that if ℓ is prime to 3 then $[G,R]_{(3)} = 0$. Now suppose that $\ell \equiv 0 \mod 3$ and put $\ell = 3s$. Then calculating the first equalities of (3.6) with $k = 3\ell-7, 3\ell-8$ yields

$$\kappa(\mu^{3\ell-7} - (s-1)\mu^{3\ell+2}) = 0 \quad \text{and} \quad \kappa(-\mu^{3\ell-7} + s\mu^{3\ell+2}) = 0.$$

By adding these equations, we have $\kappa(\mu^{3\ell+2}) = 0$ which completes the proof of (3.1).

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