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An Interpolation Theorem on Cycle Spaces for Functions Arising as Integrals of $\bar{\partial}$-Closed Forms

By

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Abstract

It is shown that an interpolation theorem with L^1-growth condition holds on the cycle spaces of q-complete manifolds with respect to those holomorphic functions arising as integrals of $\bar{\partial}$-closed $(q-1, q-1)$-forms. The proof is based on the L^2-method of Andreotti and Vesentini.

Introduction

Let M be a (connected and paracompact) complex manifold of dimension n and let $\mathcal{C}_k(M)$ be the *k-th* cycle space of M. $\mathcal{C}_k(M)$ consists of formal linear combinations of finitely many compact and irreducible k-dimensional complex analytic subsets of M with positive integral coefficients.

$\mathcal{C}_k(M)$ is canonically endowed with the structure of a reduced complex analytic space (cf. [B-2]).

The notion of cycle spaces has its origin in the classical function theory of Abel and Jacobi, and was first generally formulated by D. Barlet [B-1, 2] after the fundamental work of A. Douady [D].

In virtue of the basic works of A. Andreotti, F. Norguet, Y.-T. Siu and Barlet, it turned out that $\mathcal{C}_k(M)$ is a Stein space if M is $(k+1)$-complete in the sense of Andreotti-Grauert [A-G] (cf. [A-N-1, 2], [N-S], [B-1, 2]). Recall that M is said to be q-complete if there exists a C^2 exhaustion function φ :

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$M \longrightarrow [0,\infty)$ such that its Levi form has everywhere at least $n-q+1$ positive eigenvalues on M. We say φ is a q-convex exhaustion function on M.

The analytic sheaf cohomology theory of [A-G] was applied in [A-N-1, 2] and [N-S] to study the space, say $A_k(= A_{k,M})$ of holomorphic functions on $\mathcal{C}_k(M)$ arising as integrals of $C^\infty \bar\partial$-closed (k,k)-forms over k-cycles, especially when M is $(k+1)$-complete.

On the other hand, Barlet showed in [B-2] that the elements of A_k are actually holomorphic with respect to the canonical complex structure of $\mathcal{C}_k(M)$.

Although not explicitly mentioned in the literature, after the establishment of the general notion of cycle spaces, it is concluded immediately by this approach that the followings hold for every component Ω of $\mathcal{C}_k(M)$, whenever M is $(k+1)$-complete and admits a Kähler metric.

(1) Ω is A_k-convex, i.e. the set

$$\Omega \setminus \bigcup_{f\in A_k} \left\{x\in\Omega \mid |f(x)| > \sup_K |f|\right\}$$

is compact for every $K \subset\subset \mathcal{C}_k(M)$.

(2) Any two points of Ω can be separated by some element of A_k.

Moreover, it is known from the works of A. Fujiki [F] and T. Nishino [N] that the A_{n-1}-convexity of $\Omega \subset \mathcal{C}_{n-1}(M)$ is true if M is either compact or pseudoconvex.

Therefore, a significant relationship exists between the geometry of M and the analysis of A_k. The purpose of the present article is to clarify a quantitative aspect of this relation by establishing the following.

Theorem 0.1. *Let M be a $(k+1)$-complete manifold with a $(k+1)$-convex exhaustion function φ and let $\gamma_\mu (\mu = 1,2,\dots)$ be a sequence in $\mathcal{C}_k(M)$. Suppose that one can choose points x_μ from the supports of γ_μ in such a way that x_μ do not accumulate to any point of M. Then there exist a complete Hermitian metric g on M and a C^∞ convex increasing function τ: $\mathbb{R} \longrightarrow \mathbb{R}$ and $K>0$ such that, for any sequence $c = \left\{c_\mu\right\}_{\mu=1}^\infty \subset \mathbb{C}$ satisfying*

$$\|c\|_\tau := \sum_{\mu=1}^{\infty} |c_\mu| \exp(-\tau(\varphi(x_\mu))) < \infty$$

one can find a C^∞ $\bar\partial$-closed (k,k)-form ω on M satisfying

$$\|\omega\|_\tau := \int_M |\omega|_g \exp(-\tau\cdot\varphi)dv_g \le K\|c\|_\tau$$

and

$$\int_{\gamma_\mu} \omega = c_\mu (\mu = 1, 2, \ldots).$$

Here $|\cdot|_g$ *and* dV_g *denote respectively the length and the volume form with respect to* g.

In case M admits a Kähler metric and $\{\gamma_\mu\}$ is contained in a connected component of $\mathcal{C}_k(M)$, the assumption on γ_μ is equivalent to the discreteness of γ_μ in $\mathcal{C}_k(M)$.

It is likely that there exist sharper variants of Theorem 0.1 with respect to the growth conditions on ω and that they are useful in specific situations of complex geometry.

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§1. Preliminary — A Theorem of Andreotti-Vesentini

Let us recall a basic method of solving the $\bar{\partial}$-equation on k-complete manifolds after preparing some notations. Everything in this section is well known and routine.

Let M be a k-complete manifold of dimension n, let φ be a C^∞ k-convex exhaustion function on M, and let g be a C^∞complete Hermitian metric on M.

Let $C_0^{p,q}(M)$ be the space of $C^\infty(p,q)$-forms on M with compact support. For $u, v \in C_0^{p,q}(M)$, let $(u,v)_\varphi = (u,v)_{g,\varphi}$ denote the inner product of u and v with respect to g and φ, i.e.

$$(u,v)_\varphi = \int_M e^{-\varphi} \langle u, v \rangle dV,$$

where $\langle u, v \rangle$ denotes the pointwise inner product of u and v with respect to g and dV denotes the volume form with respect to g.

We put

$$\|u\|_\varphi = (u,u)_\varphi^{1/2}$$

Let ω_g be the fundamental form of g, let $e(\omega_g)$ be the exterior multiplication by ω_g, and let $\Lambda = \Lambda_g$ be the pointwise adjoint of $e(\omega_g)$ with respect to the inner product $\langle u, v \rangle$.

Let $L_{(2)}^{p,q}(M)_\varphi (= L_{(2)}^{p,q}(M)_{g,\varphi})$ be the completion of the prehilbert space $(C_0^{p,q}(M), \| \|_\varphi)$.

Similarly as above, for any holomorphic Hermitian vector bundle (E, h) over M, we consider the space of C^∞ compactly supported E-valued (p, q)-forms and their completions, which we denote respectively by $(C_0^{p,q}(M, E), \| \|_\varphi)$ and by $L_{(2)}^{p,q}(M, E)_\varphi$, by abbreviating g and h.

Recall that $(C_0^{p,q}(M, E), \| \ \|_\varphi)$ is naturally isometrically equivalent to $(C_0^{n,q}(M, \bigwedge^n T^{1,0}M \otimes \bigwedge^p (T^{1,0}M)^* \otimes E, \| \ \|_\varphi)$. Here $T^{1,0}M$ (resp. $(T^{1,0}M)^*$) denotes the holomorphic tangent (resp. cotangent) bundle of M, and the norm $\| \ \|_\varphi$ is with respect to g and the fiber metric on $\bigwedge^n T^{1,0}M \otimes \bigwedge^p (T^{1,0}M)^* \otimes E$ induced from g and h.

Let $\bar{\partial}$(resp. ∂) be the complex exterior derivative of type $(0,1)$ (resp. (1, 0)) acting on the space of currents on M.

The maximal closed extension of $\bar{\partial} \,|\, C_0^{p,q}(M)$ as an operator from $L_{(2)}^{p,q}(M)_\varphi$ to $L_{(2)}^{p,q+1}(M)_\varphi$ will be denoted by the same symbol as $\bar{\partial}$. Recall that the domain of $\bar{\partial}$ in $L_{(2)}^{p,q}(M)_\varphi$, say Dom$\bar{\partial}$, is defined as the set of elements u for which $\bar{\partial}u$, as a $(p, q+1)$-current, belongs to $L_{(2)}^{p,q+1}(M)_\varphi$.

Then, as a closed operator from $L_{(2)}^{p,q}(M)_\varphi$ to $L_{(2)}^{p,q+1}(M)_\varphi$, $\bar{\partial}$ coincides with the minimal closed extension of $\bar{\partial} \mid C_0^{p,q}(M)$, for the metric g is complete (cf. [A-V]).

Let ϑ_φ(resp. $\bar{\vartheta}$) be the formal adjoint of $\bar{\partial}$(resp. ∂) with respect to $(,)_\varphi$ (resp. $(,)_0$), and put $\partial_\varphi = \partial - e(\partial\varphi)$, where $e(\partial\varphi)u := \partial\varphi \wedge u$. Then ∂_φ is the formal adjoint of $\bar{\vartheta}$ with respect to $(,)_\varphi$.

The operators $\bar{\partial}, \bar{\vartheta}$ and ϑ_φ naturally act on $C_0^{p,q}(M, E)$. By an abuse of notation, we shall denote the formal adjoint of $\bar{\vartheta}$: $C_0^{p,q}(M, E) \longrightarrow C_0^{p-1,q}(M, E)$ by ∂_φ.

We recall basic formulas satisfied by $\bar{\partial}, \bar{\vartheta}, \vartheta_\varphi$ and ∂_φ.

First we quote two formulas from [O, Theorem 1.3] as follows.

$$\partial_\varphi \Lambda - \Lambda \partial_\varphi = -\sqrt{-1}(\vartheta_\varphi + T_1) \tag{1}$$

$$\bar{\partial}\Lambda - \Lambda\bar{\partial} = \sqrt{-1}(\bar{\vartheta} + T_2). \tag{2}$$

Here T_1 and T_2 are independent of φ, and they are of order 0, i.e. $T_j(fu) = fT_j(u)$ hold for any C^∞ function f and for any current u. As is well known, $T_1 = T_2 = 0$ if $d\omega_g = 0$.

Let $x \in M$ and let $\sigma_1, \ldots, \sigma_n$ be a basis of the holomorphic cotangent space of M at x such that

$$\omega_g \,|_x = \frac{\sqrt{-1}}{2} \sum_{j=1}^{n} \sigma_j \wedge \bar{\sigma}_j$$

and

$$\partial\bar{\partial}\varphi\Big|_x = \frac{1}{2}\sum_{j=1}^{n}\gamma_j(x)\sigma_j \wedge \bar{\sigma}_j$$

$$\gamma_1(x) \le \gamma_2(x) \le \cdots \le \gamma_n(x)$$

hold. Note that γ_j are continuous functions on M. By assumption $\gamma_k(x) > 0$ everywhere.

Once for all we shall fix φ and choose the metric g in such a way that

$$\gamma_1 + \gamma_2 + \cdots + \gamma_k > 0 \tag{3}$$

holds at any point of M.

We shall say that φ is a hyper-k-convex function with respect to g if (3) holds everywhere.

Let Θ_h denote the curvature form of h. Then

$$\bar{\partial}\partial_\varphi + \partial_\varphi\bar{\partial} = e(Id_E \otimes \partial\bar{\partial}\varphi + \Theta_h) \tag{4}$$

where $e(\cdot)$ denotes the exterior multiplication from the left hand side.

Hence, combining (1), (2) and (3), we obtain

$$\begin{aligned} &\bar{\partial}\vartheta_\varphi + \vartheta_\varphi\bar{\partial} - \bar{\vartheta}\partial_\varphi - \partial_\varphi\bar{\vartheta} \\ &= [\sqrt{-1}e(Id_E \otimes \partial\bar{\partial}_\varphi + \Theta_h), \Lambda] - \bar{\partial}T_1 - T_1\bar{\partial} + T_2\partial_\varphi + \partial_\varphi T_2. \end{aligned} \tag{5}$$

Here $[,]$ denotes the commutator.

From (5) we deduce the following.

Proposition 1.1. *Let (M, g) be a complete Hermitian manifold of dimension n, let φ be a hyper-q-convex exhaustion function on M, and let (E, h) be a holomorphic Hermitian vector bundle over M. Then there exists a C^∞ convex increasing function $\rho : \mathbb{R} \longrightarrow \mathbb{R}$ such that, for any C^∞ convex increasing function $\tau : \mathbb{R} \longrightarrow \mathbb{R}$, the estimate*

$$\|u\|_\psi^2 \le \|\bar{\partial}u\|_\psi^2 + \|\vartheta_\psi u\|_\psi^2 \tag{6}$$

holds for $\psi = \rho \circ \varphi + \tau \circ \varphi$ and for any $u \in \mathrm{Dom}\bar{\partial} \cap \mathrm{Dom}\vartheta_\psi \cap L^{n,q}_{(2)}(M, E)_\psi$.

Accordingly, applying Proposition 1.1 to $E = \bigwedge^n T^{1,0}M \otimes \bigwedge^p (T^{1,0}M)^*$ and combining (6) with the Hahn-Banach's theorem we obtain

Theorem 1.2 (cf. [A-V]). *Let (M, φ) be a q-complete manifold of dimension n. Then there exists a complete Hermitian metric g on M and a C^∞*

convex increasing function $\rho : \mathbb{R} \longrightarrow \mathbb{R}$ *such that, if one puts* $\psi_\tau = \rho \circ \varphi + \tau \circ \varphi$, *the inclusion*

$$\mathrm{Ker}\bar{\partial} \cap L^{p,q}_{(2)}(M)_{\psi_\tau} \subset \bar{\partial}(L^{p,q-1}_{(2)}(M)_{\psi_\tau}) \tag{7}$$

holds true for any C^∞ *convex increasing function* τ *and for any* $p \geq 0$. *Moreover, one can choose* ρ *in such a way that, for any* τ *and* p *as above and for any* $v \in \mathrm{Ker}\bar{\partial} \cap L^{p,q}_{(2)}(M)_{\psi_\tau}$, *there exists* $u \in L^{p,q-1}_{(2)}(M)_{\psi_\tau}$ *satisfying* $\bar{\partial}u = v$ *and* $\|u\|_{\psi_\tau} \leq \|v\|_{\psi_\tau}$.

§2. An Interpolation Theorem

Before going into the proof of Theorem 0.1, we prepare an elementary lemma.

Lemma 2.1. *Let* D *be a domain in* $\mathbb{C}^n$ *containing the origin* $0 \in \mathbb{C}^n$, *and let* X *be a closed complex analytic subset of* D *satisfying* $X \ni 0$, *and let* φ *be a* C^2 q*-convex function on* D. *Suppose that* $\varphi \mid X$ *attains its maximum at* 0. *Then there exist a neighbourhood* $U \ni 0$, *a closed* $n-q+1$ *dimensional complex submanifold* Y *of* U *containing* 0, *and a holomorphic function* f *on* U *satisfying* $X \cap Y = \{0\}$, $f(0) = e^{\varphi(0)}$ *and* $|f(z)| < e^{\varphi(z)}$ *for all* $z \in Y \setminus \{0\}$.

Proof. By a local holomorphic change of coordinate, it suffices to prove the assertion by assuming that

$$\varphi(z) = \mathrm{Re}Q(z) + \sum_{i=q}^{n} |z_i|^2 + \sum_{i=1}^{q-1} a_i |z_i|^2 + \sigma(\|z\|^2)$$

for some quadratic holomorphic polynomial $Q(z)$.

Replacing φ by $\varphi - \varphi(0)$ if necessary, we may assume that $\varphi(0) = 0$.

Let U be a neighbourhood of 0 such that

$$(\varphi - \mathrm{Re}Q) \mid \left\{z \in U \setminus \{0\} \mid z_1 = \cdots = z_{q-1} = 0\right\} > 0$$

and put

$$Y = \left\{z \in U \mid z_1 = \cdots = z_{q-1} = 0\right\}.$$

Since $\varphi|Y$ is strictly plurisubharmonic and $\varphi|X$ takes its maximum at 0, the origin is isolated in $X \cap Y$. Hence we may assume that $X \cap Y = \{0\}$ by shrinking U is necessary.

Then it suffices to put $f(z) = e^{Q(z)}$ and shrink U again if necessary. □

Now let (M, φ) be a $(k+1)$-complete manifold of dimension n, and let $\gamma_\mu(\mu = 1, 2, \ldots)$ be a sequence in $\mathcal{C}_k(M)$ such that one can find x_μ in the support $|\gamma_\mu|$ of γ_μ so that $\{x_\mu\}_{\mu=1}^{\infty}$ has no accumulation points.

Once for all we fix a complete Hermitian metric g on M for which φ is hyper $(k+1)$-convex (see §1).

Proof of Theorem 0.1. Since φ is an exhaustion function by the assumption on γ_μ one can find $x_\mu \in |\gamma_\mu|$ in such a way that

$$\varphi(x_\mu) = \max\{\varphi(x) \mid x \in |\gamma_\mu|\}.$$

Let $a_\mu(\mu = 1, 2, \ldots)$ be a strictly increasing sequence of real numbers such that

$$\{a_\mu \mid \mu = 1, 2, \ldots\} = \{\varphi(x_\mu) \mid \mu = 1, 2, \ldots\}.$$

For simplicity we shall prove the assertion only for the case $a_\mu = \varphi(x_\mu)$, since the proof given below can be easily modified to prove the general case.

In this situation, we shall construct an increasing sequence $\tau_0, \tau_1, \ldots, \tau_\mu, \ldots$ of convex increasing functions on $\mathbb{R}$ and a sequence of $\bar{\partial}$-closed (k, k)-forms $\omega_0, \omega_1, \ldots, \omega_\mu, \ldots$ in such a way that the limit τ of τ_μ exists and $(\tau, \{\omega_\mu\}_{\mu=1}^{\infty})$ satisfies the following.

$$\int_{\gamma_\mu} \omega_\mu = 1 \quad \text{for all} \quad \mu. \tag{8}$$

(9) $\displaystyle\sum_{\mu=1}^{\infty} \omega_\mu \exp(\tau(\varphi(x_\mu)))$ converges on compact subsets of M absolutely with respect to Sobolev norms of any order.

$$\left| \int_{\gamma_\nu} \omega_\mu \right| \leq \exp(\tau(\varphi(x_\nu)) - \tau(\varphi(x_\mu)) - \nu) \quad \text{if} \quad \mu \neq \nu \tag{10}$$

$$\int_M |\omega_\mu| \exp(\tau(\varphi(x_\mu)) - \tau \circ \varphi) dv < 2^{-\mu} \quad \text{for all} \quad \mu. \tag{11}$$

Then, by (8) and (10), the correspondence defined by

$$c = \{c_\mu\}_{\mu=1}^{\infty} \longmapsto \left\{ c_\mu - \int_{\gamma_\mu} \sum_{\nu=1}^{\infty} c_\nu \omega_\nu \right\}_{\mu=1}^{\infty}$$

say T, will become a strictly norm-decreasing map from the space of sequences $\{c \mid \|c\|_\tau < \infty\}$ into itself.

$Id - T$ will then be invertible. Combining this with (9) and (11), we shall obtain the desired conclusion.

To find such $\{\tau_\mu\}$ and $\{\omega_\mu\}$ inductively, we first put $\tau_0 = \rho(t)$ and $\omega_0 = 0$. Here $\rho(t)$ is a C^∞ convex increasing function as in Theorem 1.2 for $q = k+1$.

We choose $\rho(t)$ in such a way that

$$\int_M e^{-\rho\circ\varphi} dV < \infty$$

in order to attain (11) from L^2 estimates for ω_μ.

Suppose that $\tau_0, \dots, \tau_{\mu-1}$ and $\omega_0, \dots, \omega_{\mu-1}$ have already been found in such a way that

(12) $\tau_\nu = \tau_{\nu-1} + \sigma_\nu$ for some nonnegative convex increasing function σ_ν,

$$\int_{\gamma_\nu} \omega_\nu = 1 \quad \text{for} \quad 1 \le \nu \le \mu - 1, \tag{13}$$

(14) $\sup\{ \left| \nabla^j \omega_\nu \big|_x \right| \varphi(x) < \varphi(x_{\nu-1}),\ 1 \le j \le \nu \} < \exp(-\tau_\nu(\varphi(x_\nu)) - \nu)$
for $1 \le \nu \le \mu$, where ∇ denotes the covariant derivative,

$$\left| \int_{\gamma_\nu} \omega_\kappa \right| < \exp(\tau_\nu(\varphi(x_\nu)) - \tau_\kappa(\varphi(x_\kappa)) - \nu) \quad \text{for} \quad 1 \le \kappa \ne \nu \le \mu - 1. \tag{15}$$

and

$$\int_M |\omega_\nu| \exp(-\tau_\nu \circ \varphi) dV < 2^{-\nu} \quad \text{for} \quad 1 \le \nu \le \mu - 1. \tag{16}$$

Then we define τ_μ and ω_μ as in the following.

Let σ^* be a convex increasing function satisfying $\sigma^*(t) = 0$ on $(-\infty, \tau_{\mu-1}(\varphi(x_{\mu-1})))$ and

$$\left| \int_{\gamma_\mu} \omega_\kappa \right| \le \exp(\tau_{\mu-1}(\varphi(x_\nu)) + \sigma^*(\varphi(x_\mu)) - \tau_\kappa(\varphi(x_\kappa))) \tag{17}$$

We put $\tau^*_{\mu-1} = \tau_{\mu-1} + \sigma^*$.

By Lemma 2.1, there exists a coordinate neighbourhood $U \ni x_\mu$, a holomorphic local coordinate $z = (z_1, \dots, z_n)$ around x_μ on U, and a holomorphic function f on U such that

$$U \cap |\gamma_\nu| = \emptyset \quad \text{for} \quad 1 \leq \nu \leq \mu - 1, \tag{18}$$

$$f(x_\mu)\exp(-\tau^*_{\mu-1}(\varphi(x_\mu))) = 1, \tag{19}$$

$$|f(x)|\exp(-\tau^*_{\mu-1}(\varphi(x))) < 1 \text{ on } \{x \in U \mid z_1(x) = \cdots z_k(x) = 0\} - \{x_\mu\} \tag{20}$$

and

$$\{x \in U \cap |\gamma_\mu| \mid z_1(x) = \cdots = z_k(x) = 0\} = \{x_\mu\}. \tag{21}$$

We put $z' = (z_1, \ldots, z_k)$ and $Y = \{x \in U \mid z'(x) = 0\}$.

Let us fix a C^∞ function $J : \mathbb{R} \longrightarrow \mathbb{R}$ such that $J'(t) \leq 0$, $J''(t) \geq 0$, $J(t) = 0$ on $(-\infty, 0)$ and $J(t) = t - 1$ on $(2, \infty)$.

Then, for any $\varepsilon > 0$ we put

$$\omega_\varepsilon = (\sqrt{-1})^k \bigwedge^k \partial\bar{\partial} J(\log \|z'\| - \log \varepsilon). \tag{22}$$

The values of ω_ε will be put to be zero outside U. Note that $\omega_\varepsilon = 0$ if $\|z'\| \geq e^2\varepsilon$ or $\|z'\| \leq \varepsilon$.

Obviously

$$d_\mu := \lim_{\varepsilon \to 0} \int_{\gamma_\mu} \omega_\varepsilon > 0,$$

so we put $\hat{\omega}_\varepsilon = \omega_\varepsilon / d_\mu$.

Let V and W be neighbourhoods of x_μ satisfying $W \subset\subset V \subset\subset U$, Since $f(x_\mu)\exp(-\tau^*_{\mu-1}(\varphi(x_\mu))) = 1$ and $|\gamma_\mu| \cap Y \cap \bar{V} = \{x_\mu\}$, it is clear that

$$\begin{aligned} &\lim_{\epsilon \to 0} \int_{\gamma_\mu} (f \exp(-\tau^*_{\mu-1}(\varphi(x_\mu)))^{[-A \log \epsilon]} \hat{\omega}_\epsilon \\ &= \lim_{\epsilon \to 0} \int_{\gamma_\mu} \hat{\omega}_\epsilon = 1 \end{aligned} \tag{23}$$

holds for any $A > 0$. Here $[\beta]$ denotes the largest integer which does not exceed β.

We choose $\delta_0 > 0$ so that $f(x)\exp(-\tau^*_{\mu-1}(\varphi(x_\mu))) < 1$ holds if $x \in \bar{V} \cap Y \setminus W$ and $\tau^*_{\mu-1}(\varphi(x)) < \tau^*_{\mu-1}(\varphi(x_\mu)) + \delta_0$. This is possible because (20) holds. To be more explicit, one can argue as follows: There exist $0 < b < 1, \delta_0 > 0$, such that $|f(x)|\exp(-\tau_{\mu-1}(\varphi(x_\mu)) < b$ on $Y \cap (\bar{V} \setminus W) \cap \{\tau_{\mu-1} \circ \varphi < \tau_{\mu-1} \circ \varphi(x_\mu)$ $+\delta_0\}$. Then the corresponding statement for $\tau^*_{\mu-1}$ follows automatically.

Then for any $\alpha \in (0, 1)$ we choose $\varepsilon_0 > 0$ and $A > 0$ so that

$$\left| f(x)\exp(-\tau^*_{\mu-1}(\varphi(x_\mu))) \right|^{[-A \log \varepsilon]} < \varepsilon^{2k+\alpha+1} \tag{24}$$

holds if $0 < \varepsilon < \varepsilon_0, x \in \bar{V} \setminus W, \|z'(x)\| < \varepsilon^\alpha$ and $\tau^*_{\mu-1}(\varphi(x)) < \tau^*_{\mu-1}(\varphi(x_\mu)) + \delta_0$.

We put

$$\omega^*_\varepsilon = (f \exp(-\tau^*_{\mu-1}(\varphi(x_\mu))))^{[-A\log\varepsilon]}\hat{\omega}_\varepsilon. \tag{25}$$

From (23) we obtain

$$\lim_{\varepsilon\longrightarrow 0}\int_{\gamma_\mu}\omega^*_\varepsilon = 1.$$

We note also that the supremum of $|\omega^*_\varepsilon(x)|$ on the set

$$(\bar{V}\setminus W)\cap\{x \mid \tau^*_{\mu-1}(\varphi(x) < \tau^*_{\mu-1}(\varphi(x_\mu) + \delta_0\}$$

tends to 0 as $\varepsilon \longrightarrow 0$.

Let $\eta : \mathbb{R} \longrightarrow [0,1]$ be a C^∞ function such that $\eta(t) = 1$ on $(-\infty, 3/4)$ and $\eta(t) = 0$ on $[1,\infty)$, and let $\xi : M \longrightarrow [0,1]$ be a C^∞ function satisfying $\xi|W = 1$ and $\xi|U\setminus V = 0$.

Then we put for any $0 < \varepsilon < \varepsilon_0$ and $0 < \delta < \delta_0$,

$$w_{\varepsilon,\delta} = \xi(x)\eta((\tau^*_{\mu-1}(\varphi(x)) - \tau^*_{\mu-1}(\varphi(x_\mu))/\delta)\cdot\eta(\|z'\|/\varepsilon^\alpha)\omega^*_\varepsilon \tag{26}$$

and $v_{\varepsilon,\delta} = \bar{\partial}w_{\varepsilon,\delta}$.

Let σ be a C^∞ convex increasing function satisfying $\sigma(t) = 0$ for $t \le \varphi(x_\mu) + 2\delta/3$, and put $\psi_\sigma = \tau^*_{\mu-1}\circ\varphi + \sigma\circ\varphi$.

Then by Theorem 1.2, there exist $u_{\varepsilon,\delta,\sigma}$ satisfying

$$\bar{\partial}u_{\varepsilon,\delta,\sigma} = v_{\varepsilon,\delta} \quad \text{and} \quad \|u_{\varepsilon,\delta,\sigma}\|_{\psi_\sigma} \le \|v_{\varepsilon,\delta}\|_{\psi_\sigma}. \tag{27}$$

It is clear from (26) that, for any $m < 1/\alpha$, there exist $s_0 > 0$ such that, for any $s, \varepsilon, \delta \in (0, s_0)$, one can find σ satisfying

$$\|\nabla^j v_{\varepsilon,\delta}\|_{\psi_\sigma} < \varepsilon^s \quad \text{for all} \quad 0 \le j \le m. \tag{28}$$

Thus, by the strong ellipticity of the Laplacian, if we choose $\varepsilon, \delta, \sigma$ and $u_{\varepsilon,\delta,\sigma}$, after fixing $s \in (0, s_0)$, in such a way that $\vartheta_{\psi_\sigma}u_{\varepsilon,\delta,\sigma} = 0$ and $\delta = |\log\varepsilon|^{-1}$, the Sobolev norm of $u_{\varepsilon,\delta,\sigma}$ of order $[\alpha^{-1}] - 1$ tends to 0 as $\varepsilon \longrightarrow 0$ on the set

$$(\tau^*_{\mu-1}\circ\varphi)^{-1}((-\infty, \tau^*_{\mu-1}(\varphi(x_\mu)) + \delta/2)).$$

Therefore, if $\alpha = (n+2+\mu)^{-1}$, by Sobolev's embedding theorem applied after a suitable scale change, it follows that $u_{\varepsilon,\delta,\sigma} \longrightarrow 0$ in the C^0-norm as $\varepsilon \longrightarrow 0$. Hence, for sufficiently small ε, for $\delta = |\log\varepsilon|^{-1}$, and for a sufficiently rapidly increasing function σ as above, the functions $\tau_1, \tau_2, \ldots, \tau_\mu = \tau_{\mu-1}^* + \sigma$ and the (k,k)-forms $\omega_1, \ldots, \omega_\mu = \beta_\mu(w_{\varepsilon,\delta} - u_{\varepsilon,\delta,\sigma})$ for a suitable constant β_μ satisfy

$$\int_{\gamma_\nu} \omega_\nu = 1 \quad \text{for} \quad 1 \le \nu \le \mu \tag{29}$$

$$\begin{aligned} &\sup\left\{\left|\nabla^j \omega_\nu|_x\right| \varphi(x) < \varphi(x_{\nu-1}), \quad 0 \le j \le \nu\right\} \\ &\quad < \exp(-\tau_\nu(\varphi(x_\nu)) - \nu) \quad \text{for} \quad 1 \le \nu \le \mu \end{aligned} \tag{30}$$

$$\left|\int_{\gamma_\kappa} \omega_\nu\right| < \exp(\tau_\kappa(\varphi(x_\kappa)) - \tau_\nu(\varphi(x_\nu)) - \mu \quad \text{if} \quad 1 \le \kappa \ne \nu \le \mu \tag{31}$$

and

$$\int_M |\omega_\nu| \exp(-\tau_\nu \circ \varphi) dV < 2^{-\nu}, \quad 1 \le \nu \le \mu. \tag{32}$$

This was what we planned to show.

References

[A-G] A. Andreotti and H. Grauert, Théorème de finitude pour la cohomologie des espaces complexes, Bull. Soc. Math. France **90** (1962), 193–259.

[A-N-1] A. Andreotti and F. Norguet, Problème de Levi et convexité holomorphe pour les classes de cohomologie, Ann. Scuola Norm. Sup. Pisa (3) **20** (1966), 197–241.

[A-N-2] ———, La convexité holomorphe dans l'espace analytique des cycles d'une variété algébrique, Ann. Scuola Norm. Sup. Pisa (3) **21** (1967), 31–82.

[A-V] A. Andreotti and E. Vesentini, Carleman estimates for the Laplace-Beltrami equation on complex manifolds, Inst. Hautes Études Sci. Publ. Math. No. 25 (1965), 81–130.

[B-1] D. Barlet, Espace analytique réduit des cycles analytiques complexes compacts d'un espace analytique complexe de dimension finie, in *Fonctions de plusieurs variables complexes, II (Sém. François Norguet, 1974–1975)*, 1–158. Lecture Notes in Math., 482, Springer, Berlin.

[B-2] ———, Familles analytiques de cycles et classes fondamentales relatives, C. R. Acad. Sci. Paris Sér. A-B **280** (1975), Ai, A1443–A1446.

[B-3] ———, Convexité de l'espace des cycles, Bull. Soc. Math. France **106** (1978), no. 4, 373–397.

[B-K] D. Barlet and V. Koziarz, Fonctions holomorphes sur l'espace des cycles: la méthode d'intersection, Math. Res. Lett. **7** (2000), no. 5-6, 537–549.

[D] A. Douady, Le problème des modules pour les sous-espaces analytiques compacts d'un espace analytique donné, Ann. Inst. Fourier (Grenoble) **16** (1966), fasc. 1, 1–95.

[F] A. Fujiki, Projectivity of the space of divisors on a normal compact complex space, Publ. Res. Inst. Math. Sci. **18** (1982), no. 3, 1163–1173.

[N] T. Nishino, L'existence d'une fonction analytique sur une variété analytique complexe à dimension quelconque, Publ. Res. Inst. Math. Sci. **19** (1983), no. 1, 263–273.

[N-S] F. Norguet and Y. T. Siu, Holomorphic convexity of spaces of analytic cycles, Bull. Soc. Math. France **105** (1977), no. 2, 191–223.

[O] T. Ohsawa, Isomorphism theorems for cohomology groups of weakly 1-complete manifolds, Publ. Res. Inst. Math. Sci. **18** (1982), no. 1, 191–232.