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A Note on Embeddings of S_4 and A_5 into the Two-dimensional Cremona Group and Versal Galois Covers

By

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Abstract

In this article, we prove that two versal Galois covers for S_4 and A_5 introduced in [17], [18] and [19] are birationally distinct to each other. As a corollary, we obtain two non-conjugate embeddings of S_4 and A_5 into $\mathrm{Cr}_2(\mathbb{C})$.

Introduction

Let X and Y be normal projective varieties defined over $\mathbb{C}$, the field of complex numbers. A finite surjective morphism $\pi : X \to Y$ is called Galois, if the induced field extension $\mathbb{C}(X)/\mathbb{C}(Y)$ of the field of rational functions is Galois. Given a finite group G, we simply call $\pi : X \to Y$ a G-cover if it is Galois and $\mathrm{Gal}(\mathbb{C}(X)/\mathbb{C}(Y)) \cong G$. In [17] and [19], a notion called "versal Galois covers" is introduced, of which the definition is as follows:

Definition 0.1. Let G be a finite group. A G-cover $\varpi : X \to Y$ is called a versal Galois cover for G or a versal G-cover if it satisfies the following property:

For any G-cover $\pi : W \to Z$, there exists a G-equivariant rational map $\mu : W \dashrightarrow X$ such that

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$$\mu(W) \not\subset \mathrm{Fix}(X, G),$$

where $\mathrm{Fix}(X, G) := \{x \in X \mid \text{the stabilizer group at } x,\ G_x \neq \{1\}\}$.

Remark. The rational map μ induces a rational map $\overline{\mu} : Z \dashrightarrow Y$. Concerning this rational map $\overline{\mu}$, there exists a Zariski open set U such that (i) $U \subset \mathrm{dom}(\overline{\mu})$, $\mathrm{dom}(\bullet)$ being the domain of a rational map $\bullet$, and (ii) $\pi^{-1}(U)$ is birationally equivalent to $U \times_Y X$ over U. (see [18], Proposition 1.2).

The notion of versal G-covers implicitly appeared in [12] and [13] as the "pull-back" construction of G-covers, where Namba showed that there exists a versal G-cover of dimension $\sharp(G)$ for any finite group G. Namba's model, however, has too large dimension for practical use.

For a finite subgroup G in $\mathrm{GL}(n, \mathbb{Z})$, Bannai and Tsuchihashi construct versal G-covers of dimension n by using toric geometry in [1] and [19].

In [5], the notion of the essential dimension, $\mathrm{ed}_{\mathbb{C}}(G)$, of G is introduced and it is known that the following equality holds (see [5] and [18]):

$$\mathrm{ed}_{\mathbb{C}}(G) = \min\{\dim X \mid \ \varpi : X \to Y \text{ is a versal } G\text{-cover}\}.$$

By Theorem 6.2 in [5], $\mathrm{ed}_{\mathbb{C}}(G) = 1$ if and only if G is either a cyclic group or a dihedral group of order $2n$ (n: odd). As a next step, in [17], [18] and [19], we study the case of $\mathrm{ed}_{\mathbb{C}}(G) = 2$ and give some explicit examples.

Among explicit examples in [17], [18], two different versal G-covers, $\varpi_{G,1} : X_1 \to Y_1$ and $\varpi_{G,2} : X_2 \to Y_2$ are given for the cases when G is S_4, the symmetric group of 4-letters and A_5, the alternating group of 5-letters (see §1 for description of X_1 and X_2). Here X_1 and X_2 are del-Pezzo surfaces which are known to be rational. Moreover, by the definition of versal G-covers, there exist G-equivariant rational maps $\mu_1 : X_1 \dashrightarrow X_2$ and $\mu_2 : X_2 \dashrightarrow X_1$ such that $\mu_1(X_1) \not\subset \mathrm{Fix}(X_2, G)$ and $\mu_2(X_2) \not\subset \mathrm{Fix}(X_1, G)$. Under these circumstances, it may be natural to raise a question as follows:

Question 0.1. Let G be either S_4 or A_5. Let $\varpi_{G,1} : X_1 \to Y_1$ and $\varpi_{G,2} : X_2 \to Y_2$ be versal G-covers as above. Does there exist any G-equivariant birational map from X_1 to X_2?

In this note, we consider Question 0.1 and prove the following:

Theorem 0.1. *There exists no G-equivariant birational map from X_1 to X_2*

Since both X_1 and X_2 are rational, their birational automorphism group is the 2-dimensional Cremona group $\mathrm{Cr}_2(\mathbb{C})$. For $G = S_4, A_5$, we have two different embeddings $\eta_i : G \to \mathrm{Cr}_2(\mathbb{C})$ $(i = 1, 2)$ via $G \subset \mathrm{Aut}(X_i) \subset \mathrm{Cr}_2(\mathbb{C})(i = 1, 2)$. Our theorem implies that $\eta_1(G)$ is *not* conjugate to $\eta_2(G)$ in $\mathrm{Cr}_2(\mathbb{C})$. Combining Proposition 0.3 (i) in [18], we have the following corollary:

Corollary 0.1. *Both S_4 and A_5 have at least 3 non-conjugate embeddings into $\mathrm{Cr}_2(\mathbb{C})$.*

Our results could be found in old literatures such as [10] and [20], but we would like to emphasize that our question comes from the study of versal G-covers, which is a rather new notion. Also conjugacy classes of finite subgroups of $\mathrm{Cr}_2(\mathbb{C})$ have been studied by several mathematicians ([2], [3], [4], [6], [8]). The notion of versal G-covers may add another interest to this subject.

This article goes as follows. We first give a detailed description of the versal G-covers $\varpi_{G,i} : X_i \to Y_i$ $(i = 1, 2)$ in §1. In §2, we explain our main tool, "Noether's inequality," which plays an important role in [8] and [9]. We prove Theorem 0.1 in §3. In §4, we consider rational maps between X_1 and X_2 in the case of $G = S_4$.

§1. Versal S_4- and A_5-covers: Two Examples

§1.1. Versal S_4-covers

Let S_4 be the symmetric group of 4-letters. Put $\sigma = (12), \tau = (123), \lambda_1 = (13)(24), \lambda_2 = (12)(34)$

Let $\rho : S_4 \to \mathrm{GL}(3, \mathbb{C})$ be a faithful irreducible representation as follows:

$$\sigma \mapsto \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \tau \mapsto \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

$$\lambda_1 \mapsto \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \lambda_2 \mapsto \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Versal S_4-cover $\varpi_{S_4,1} : X_1 \to Y_1$

Let X_1 be a surface in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ defined by the equation

$$x_0 y_0 z_0 - x_1 y_1 z_1 = 0,$$

where $([x_0, x_1], [y_0, y_1], [z_0, z_1])$ denotes the homogeneous coordinates. Put $x = x_1/x_0, y = y_1/y_0, z = z_1/z_0$. Define an S_4-action on $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ as follows:

$$\begin{aligned}\sigma(x,y,z) &= (x,y,z)\rho(\sigma^{-1}) = (y,x,z),\\ \tau(x,y,z) &= (x,y,z)\rho(\tau^{-1}) = (z,x,y),\\ \lambda_1(x,y,z) &= (x,y,z)\rho(\lambda_1^{-1}) = (-x,y,-z),\\ \lambda_2(x,y,z) &= (x,y,z)\rho(\lambda_2^{-1}) = (-x,-y,z).\end{aligned}$$

The defining equation of X_1 is invariant under this S_4-action. Hence S_4 acts on X_1. Put $Y_1 = X_1/G$ and denote the quotient morphism by $\varpi_{S_4,1} : X_1 \to Y_1$. By [17] and [19], $\varpi_{S_4,1} : X_1 \to Y_1$ is a versal S_4-cover.

We look into some properties of X_1 with respect to this S_4-action for later use. We first remark that X_1 is a del-Pezzo surface of degree 6, i.e., X_1 is obtained by blowing-up at distinct 3 points of $\mathbb{P}^2$.

Lemma 1.1. *The divisor of X_1 given by $x_0y_0z_0 = 0$ is a cycle of rational curves $C_1, C_2, \dots, C_6$. Each C_i is a smooth rational curve with $C_i^2 = -1$.*

Proof. Let $p_{12} : \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1 \times \mathbb{P}^1$ be the projection to the product of the first two factors. By its defining equation, we infer that the restriction of p_{12} to X_1 is the blowing-up of $\mathbb{P}^1 \times \mathbb{P}^1$ at $([1,0],[0,1])$ and $([0,1],[1,0])$. Our statement easily follows from this observation.

Lemma 1.2. *Let* $\mathrm{Pic}(X_1)$ *be the Picard group of X_1. Then the S_4 invariant part* $\mathrm{Pic}^{S_4}(X_1) = \mathbb{Z}(-K_{X_1})$.

Proof. $-K_{X_1} \sim \sum_{i=1}^{6} C_i$ where $\sim$ denotes linear equivalence, and one can easily check that the divisor class in the right hand generates $\mathrm{Pic}^{S_4}(X_1)$.

For $x \in X_1$, we put $d_x = \sharp O_{S_4}(x)$, where $O_{S_4}(x)$ denotes the orbit of x. For later use, we study points with $d_x < 6$.

Lemma 1.3. (i) *There are no points with $d_x = 1, 2, 5$.*
(ii) *There are exactly* 12 *points with $d_x = 4$ as follows*:

$$\begin{array}{llll} R_{11}(1,1,1), & R_{12}(1,-1,-1), & R_{13}(-1,-1,1), & R_{14}(-1,1,-1),\\ R_{21}(\omega,\omega,\omega), & R_{22}(\omega,-\omega,-\omega), & R_{23}(-\omega,-\omega,\omega), & R_{24}(-\omega,\omega,-\omega),\\ R_{31}(\omega^2,\omega^2,\omega^2), & R_{32}(\omega^2,-\omega^2,-\omega^2), & R_{33}(-\omega^2,-\omega^2,\omega^2), & R_{34}(-\omega^2,\omega^2,-\omega^2),\end{array}$$

where the coordinates mean the affine coordinates (x, y, z) *and* $\omega = \exp(2\pi\sqrt{-1}/3)$. *These* 12 *points are divided into three* S_4*-orbits.*

(iii) *There are exactly* 6 *points with* $d_x = 3$ *as follows*:

$$P_1([0,1],[1,0],[0,1]),\ P_2([1,0],[0,1],[0,1]),\ P_3([0,1],[0,1],[1,0]),$$
$$Q_1([1,0],[1,0],[0,1]), Q_2([1,0],[0,1],[1,0]), Q_3([0,1],[1,0],[1,0]).$$

These 6 *points are divided into two* S_4*-orbits.*

Proof. Note that τ acts on the divisor $x_0y_0z_0 = 0$ freely and the subgroup $\langle\lambda_1, \lambda_2\rangle$ has no fixed points on the affine surface $xyz = 1$. Taking these observation into account, we can easily check the above statement by direct computation.

Lemma 1.4. *The divisors on* X_1 *given by the equations* $x_1 = \omega^i x_0$ $(i = 0, 1, 2)$ *are rational curves with self-intersection number* 0.

Proof. By the proof of Lemma 1.1, we infer that the divisors as above come from those in $\mathbb{P}^1 \times \mathbb{P}^1$ with self-intersection number 0 and all of these divisors in $\mathbb{P}^1 \times \mathbb{P}^1$ do not pass through $([1,0],[0,1])$ and $([0,1],[1,0])$. This implies our statement.

Versal S_4-cover $\varpi_{S_4,2} : X_2 \to Y_2$

Let $[t_0, t_1, t_2]$ be homogeneous coordinates of $\mathbb{P}^2$. Define a S_4 action on $\mathbb{P}^2$ by $g([t_0, t_1, t_2]) = [t_0, t_1, t_2]\rho(g^{-1})$, $g \in S_4$. By Proposition 4.1 (ii) in [17], we have a versal S_4-cover $\mathbb{P}^2 \to \mathbb{P}^2/S_4$. Put $X_2 = \mathbb{P}_2$, $Y_2 = \mathbb{P}^2/S_4$ and let $\varpi_{S_4,2} \colon X_2 \to Y_2$ be the quotient morphism.

§1.2. Versal A_5-covers

We first start with the following lemma.

Lemma 1.5. *Let* S *be a smooth projective surface on which* A_5 *acts faithfully on* S. *Let* d_x *be the number of points of* $O_{A_5}(x)$. *Then there exists no point* x *on* S *with* $d_x < 5$.

Proof. Case $d_x = 1$. Assume that there exists a point x with $d_x = 1$. Then we have a non-trivial homomorphism $\eta : A_5 \to \mathrm{GL}(T_xS)$, where T_xS is the tangent plane at x. Since A_5 is simple, η is injective. This contradicts the non-existance of 2-dimensional faithful representations.

Case $d_x = 2, 3$ or 4. Assume that such a point exists. Then we have a non-trivial homomorphism from A_5 to the symmetric group of either $2, 3$ or 4 letters. The kernel of this homomorphism is a non-trivial normal subgroup, which is a contradiction.

Versal A_5-cover $\varpi_{A_5,1} : X_1 \to Y_1$

Let $\tilde{X} = \mathbb{P}^1 \times \cdots \times \mathbb{P}^1$ be the product of five copies of $\mathbb{P}^1$. Put $p_i = [p_0^i, p_1^i] \in \mathbb{P}^1$. We define an S_5-action on $\tilde{X}$ by permutation of coordinates as follows:

$$\sigma \cdot (p_1, \dots, p_5) := (p_{\sigma(1)}, \dots, p_{\sigma(5)})$$

for a point $(p_1, \dots, p_5) \in \tilde{X}$ and $\sigma \in S_5$. Note that S_5 acts on $\{1, 2, 3, 4, 5\}$ from the right. Let $\tilde{\varpi} : \tilde{X} \to \tilde{X}/S_5$ be the quotient morphism.

Lemma 1.6. *$\tilde{\varpi} : \tilde{X} \to \tilde{X}/S_5$ is a versal S_5-cover.*

Proof. Let $\pi : Z \to W$ be an arbitrary S_5-cover. Since $\mathbb{C}(Z)$ can be regarded as a splitting field of a certain algebraic equation of degree 5 over $\mathbb{C}(W)$, there exist rational functions $\varphi_1, \dots, \varphi_5$ such that $\varphi_i^\sigma (:= \varphi_i \circ \sigma) = \varphi_{\sigma(i)}$ for $\sigma \in S_5$ (Note that $\varphi_i^{\sigma\tau} = (\varphi_i^\sigma)^\tau = \varphi_{\sigma(i)}^\tau = \varphi_{\tau(\sigma(i))} = \varphi_{\tau\sigma(i)}$). Define a rational map $\mu_{Z/\tilde{X}} : Z \dashrightarrow \tilde{X}$ by $p \in Z \mapsto (\varphi_1(p), \dots, \varphi_5(p))$. For $\sigma \in S_5$, we have

$$\begin{aligned}
(\mu_{Z/\tilde{X}} \circ \sigma)(p) &= (\varphi_1^\sigma(p), \dots, \varphi_5^\sigma(p)) \\
&= (\varphi_{\sigma(1)}(p), \dots, \varphi_{\sigma(5)}(p)) \\
&= \sigma \cdot (\varphi_1(p), \dots, \varphi_5(p)) \\
&= \sigma \cdot \mu_{Z/\tilde{X}}(p).
\end{aligned}$$

Hence $\mu_{Z/\tilde{X}}$ is S_5-equivariant. Since $\pi : Z \to W$ is an S_5-cover, if we choose a point p in general, the S_5-orbit of $(\varphi_1(p), \dots, \varphi_5(p))$ has 120 distinct points. This means $\mu_{Z/\tilde{X}}(Z) \not\in \mathrm{Fix}(\tilde{X}, S_5)$.

Let ψ_1 and ψ_2 be rational functions on $\tilde{X}$ given by

$$\begin{cases}
\psi_1 = \dfrac{(x_4 - x_1)(x_2 - x_3)}{(x_4 - x_3)(x_2 - x_1)} \\
\psi_2 = \dfrac{(x_5 - x_1)(x_2 - x_3)}{(x_5 - x_3)(x_2 - x_1)}
\end{cases}$$

where $x_i = p_1^i / p_0^i$.

We can check

$$\psi_1^{(12)} = -\psi_1 + 1, \quad \psi_2^{(12)} = -\psi_2 + 1$$
$$\psi_1^{(12345)} = \frac{\psi_2 - 1}{\psi_2 - \psi_1}, \quad \psi_2^{(12345)} = \frac{1}{\psi_1},$$

where $\psi_i^{\sigma}(p_1, \dots, p_5) = \psi_i(\sigma \cdot (p_1, \dots, p_5)) = \psi_i(p_{\sigma(1)}, \dots, p_{\sigma(5)})$. The subfield $\mathbb{C}(\psi_1, \psi_2)$ of $\mathbb{C}(\tilde{X})$ is S_5-invariant and the S_5 action induced on $\mathbb{C}(\psi_1, \psi_2)$ by that on $\mathbb{C}(\tilde{X})$ is faithful. Using this action, we have a birational S_5 action on $\mathbb{P}^2$. Explicitly the birational maps σ_1 and σ_2 induced by (12) and (12345) are given as follows:

$$\begin{aligned}
\sigma_1 = (12) \quad &: [s_0, s_1, s_2] \mapsto [s_0, s_0 - s_1, s_0 - s_2] \\
\sigma_2 = (12345) \quad &: [s_0, s_1, s_2] \mapsto [s_1(s_2 - s_1), s_1(s_2 - s_0), s_0(s_2 - s_1)], \\
\sigma_2^{-1} = (15432) &: [s_0, s_1, s_2] \mapsto [s_2(s_0 - s_1), s_0(s_0 - s_1), s_0(s_2 - s_1)]
\end{aligned}$$

where $[s_0, s_1, s_2]$ denotes a homogeneous coordinate of $\mathbb{P}^2$ and we put $\psi_1 = s_1/s_0$ and $\psi_2 = s_2/s_0$. As $\{(12), (12345)\}$ are generators of S_5, the birational S_5 action on $\mathbb{P}^2$ as above is given by some compositions of σ_1 and σ_2. Note that σ_1 is an automorphism of $\mathbb{P}^2$. σ_2 has three base points $[1, 0, 0]$, $[0, 0, 1]$ and $[1, 1, 1]$. σ_2^{-1} also has three base points $[0, 1, 0]$, $[0, 0, 1]$ and $[1, 1, 1]$.

Let X_1 be the surface obtained by blowing up $\mathbb{P}^2$ at $[1, 0, 0]$, $[0, 1, 0]$, $[0, 0, 1]$ and $[1, 1, 1]$. As σ_1 and σ_2 are lifted to automorphisms on X_1, the birational action on $\mathbb{P}^2$ as above induces an S_5-action on X_1. By restricting this action to the subgroup A_5, the alternating group of 5 letters, we also have an A_5 action on X_1. Let $Y_1 = X_1/A_5$ and let $\varpi_{A_5,1} : X_1 \to Y_1$ be the quotient morphism. Since $\mathrm{ed}_{\mathbb{C}}(A_5) = 2$, by Proposition 1.4 in [18] and the lemma below, $\varpi_{A_5,1} : X_1 \to Y_1$ is a versal A_5-cover.

Lemma 1.7. *Let G be a finite group, let $\varphi_1 : X' \to Y'$ be a versal G-cover, and let X be a normal projective variety of dimension $\mathrm{ed}_{\mathbb{C}}(G)$ on which G acts faithfully. If there exists a G-equivariant dominant rational map $\gamma : X' \dashrightarrow X$, then the quotient morphism $\varphi_2 : X \to X/G$ with respect to the G-action gives rise to another versal G-cover.*

Proof. Let V_{reg} be a vector space with the G-action given by the left regular representation, i.e.,

$$h\left(\sum_{g \in G} a_g g\right) := \sum_{g \in G} a_g hg, \quad \sum_{g \in G} a_g g \in V_{reg}, \quad h \in G.$$

Put $N = \sharp(G)$. One can can consider V_{reg} as an affine open subset of the projective space $\mathbb{P}^N = \mathbb{P}(\mathbb{C} \oplus V_{reg})$. As the G-action on V_{reg} canonically extends to $\mathbb{P}^N$, we have a G-cover $\mathbb{P}^N \to \mathbb{P}^N/G$. Hence there exists a G-equivariant rational map $\mu_{reg} : \mathbb{P}^N \dashrightarrow X'$ such that $\mu_{reg}(\mathbb{P}^N) \not\subset \mathrm{Fix}(X', G)$. The restriction μ_{reg} to V_{reg} gives rise to a G-equivariant rational map from V_{reg} to X'. We denote it by μ'. Thus we have a G-equvariant rational map $\gamma \circ \mu' : V_{reg} \dashrightarrow X$. By Theorem 3.2 in [5] and since $\dim X = \mathrm{ed}_{\mathbb{C}}(G)$, $\gamma \circ \mu'$ is dominant. Choose a point $a \in V_{reg}$ such that

- $\gamma \circ \mu'$ is defined at a and
- the G-orbit of $\gamma \circ \mu'(a)$ has N distinct points.

Let $\pi : Z \to W$ be an arbitrary G-cover. By Lemma 3.4 in [5], there exist an affine subvariety Y of V_{reg} such that the G-action of V_{reg} induces a faithful G-action on Y and a G-equivariant dominant rational map $g : Z \dashrightarrow Y$. Now choose a point $\tilde{a} \in Z$ such that

- g is defined at $\tilde{a}$ and
- the G-orbit of $g(\tilde{a})$ has N distinct points.

By Lemma 3.2 (a) in [5], there exists a G-equivariant morphism $\alpha : V_{reg} \to V_{reg}$ such that $\alpha(g(\tilde{a})) = a$. Consider the rational map $\mu_{Z/X} := \gamma \circ \mu' \circ \alpha \circ g : Z \dashrightarrow X$. Then (i) $\mu_{Z/X}$ is G-equivariant and (ii) the G-orbit of $\mu_{Z/X}(\tilde{a})$ has N distinct points, i.e., $\mu_{Z/X}(Z) \not\subset \mathrm{Fix}(X, G)$.

Versal A_5-cover $\varpi_{A_5,2} : X_2 \to Y_2$

Let $\rho' : A_5 \to \mathrm{GL}(3, \mathbb{C})$ be any faithful irreducible representation. Define a A_5 action on $\mathbb{P}^2$ by $g([t_0, t_1, t_2]) = [t_0, t_1, t_2]\rho'(g^{-1})$, $g \in A_5$. By Proposition 4.1 (ii) in [17], we have a versal A_5-cover $\mathbb{P}^2 \to \mathbb{P}^2/A_5$. Put $X_2 = \mathbb{P}^2$, $Y_2 = \mathbb{P}^2/A_5$ and let $\varpi_{A_5,2}\colon X_2 \to Y_2$ be the quotient morphism.

§2. Noether's Inequality

In this section we explain Noether's inequality in our setting. The proof is identical to the proof of the general form of Noether's inequality given in [9]. We only need to keep in mind that we are using G-invariant linear systems.

Let X and X' be smooth projective surfaces with G-action. Let $\mathcal{K}_X$ (resp. $\mathcal{K}_{X'}$) be the canonical linear system of X (resp. X'). Let $\Phi\colon X \dashrightarrow X'$ be a G-equivariant birational map. Let $\mathcal{H}_{X'}$ be a G-invariant variable linear system of divisors on X' which does not have any fixed components. Let $\mathcal{H}_X = \Phi^{-1}(\mathcal{H}_{X'})$ be the proper inverse image of $\mathcal{H}_{X'}$. Note that χ is G-equivariant, so $\mathcal{H}_X$ is also G-invariant.

Let $\eta\colon X_N \to X$ be the G-equivariant resolution of indeterminacies of [14]. It is a composition of G-equivariant blow-ups along smooth centers, which are blow-ups along 0-dimensional G-orbits $O_G(x)$ in our case. Let $\psi = \Phi \circ \eta$.

$$\eta\colon X_N \xrightarrow{\eta_{N,N-1}} X_{N-1} \xrightarrow{\eta_{N-1,N-2}} \cdots \xrightarrow{\eta_{2,1}} X_1 \xrightarrow{\eta_{1,0}} X_0 = X$$

$$\begin{array}{ccc} X_N & & \\ \big\downarrow{\scriptstyle \eta} & \searrow^{\psi} & \\ X & \underset{\Phi}{\dashrightarrow} & X' \end{array}$$

$\eta_{i+1,i}$ is a blow-up along a 0-dimensional G-orbit $O(x_i)$. Let $\eta_{j,i} = \eta_{j,j-1} \circ \cdots \circ \eta_{i+1,i}$ $(N \geq j > i+1 \geq 1)$, $\eta_{N,N} = \mathrm{id}_{X_N}$. Let $\mathcal{H}_{X_N}$ be the proper transform of $\mathcal{H}_{X'}$ on X_N. Let $H_\bullet$ and $K_\bullet$ be a member of $\mathcal{H}_\bullet$ and $\mathcal{K}_\bullet$ respectively, where $\bullet = X$, X_N, and X'. Then we have

$$H_{X_N} = \eta^* H_X - \sum_{i=0}^{N-1} r(x_i)\eta^*_{N,i+1}(E_{i+1})$$

$$K_{X_N} = \eta^* K_X + \sum_{i=0}^{N-1} \eta^*_{N,i+1}(E_{i+1})$$

where $r(x_i)$ is the multiplicity of a base point $x_i \in O(x_i)$ (a point in the center $O(x_i)$ of the blow-up $\eta_{i+1,i}$) of $\mathcal{H}_X$, and E_i is the exceptional divisor of $\eta_{i,i-1}$. We note that E_i is a disjoint union of (-1)-curves corresponding to the points in $O(x_i)$, and $r(x_i) = r(x_j)$ if $O(x_i) = O(x_j)$ since $\mathcal{H}_X$ is G-invariant.

Definition 2.1. Given a linear system $\mathcal{H}$ and an integer m, x is called a maximal singularity of $\mathcal{H} + m\mathcal{K}$ if x is a base point of $\mathcal{H}$ with multiplicity $r(x) > m$.

Lemma 2.1. [*Noether's Inequality*] *Under the notation above,*

(i) *Suppose that $\mathcal{H}_{X'} + m\mathcal{K}_{X'} = \emptyset$ then either there exists a 0-dimensional G-orbit $O_G(x)$ consisting of maximal singularities, or the adjoint linear system $\mathcal{H}_X + m\mathcal{K}_X$ is empty on X.*

(ii) *If there exists a variable family of curves $\mathcal{C}'$ such that $(H_{X'} + mK_{X'})C' < 0$ then either there exists a 0-dimensional G-orbit of maximal singularities, or else there is a curve $C \subset X$ such that $(H_X + mK_X)C < 0$.*

Proof. (i) We have

$$H_{X_N} + mK_{X_N} = \eta^*(H_X + mK_X) + \sum_{i=0}^{N-1}(m - r(x_i))\eta^*_{N,i+1}(E_{i+1}) \tag{2.1}$$

Then by applying ψ_* to both sides, we have

$$\begin{aligned} H_{X'} + mK_{X'} &= \psi_*(H_{X_N} + mK_{X_N}) \\ &= \psi_*\eta^*(H_X + mK_X) + \psi_*\left(\sum_{i=0}^{N-1}(m - r(x_i))\eta^*_{N,i+1}(E_{i+1})\right) \end{aligned}$$

Since $\mathcal{H}_{X'} + m\mathcal{K}_{X'} = \emptyset$ by hypothesis the right hand side cannot be an effective divisor, hence $r(x_i) > m$ for at least one i, or else $\mathcal{H}_X + m\mathcal{K}_X = \emptyset$.

(ii) $\psi^*(H_{X'} + mK_{X'}) = (H_{X_N} + mK_{X_N}) + F$ where F is the exceptional divisor of ψ. Then $\psi^*\mathcal{C}'F = 0$. Then we have $(H_{X_N} + mK_{X_N})\psi^*\mathcal{C}' < 0$. Suppose that $r(x_i) \leq m$ for all i. Then by intersecting both sides of (2.1) with $C \in \psi^*\mathcal{C}'$ we find that $\eta^*(H_X + mK_X)\psi^*\mathcal{C}' < 0$. Hence $(H_X + mK_X)\eta_*\psi^*\mathcal{C}' < 0$. A general member C' of $\eta_*\psi^*\mathcal{C}'$ may be reducible but we have $(H_X + mK_X)C < 0$ for at least one irreducible component of C'. □

§3. Proof of Theorem 0.1

§3.1. The case of S_4

Suppose that there exists an S_4-equivariant rational map $\Phi : X_1 \dashrightarrow X_2 (= \mathbb{P}^2)$. Let Λ be the complete linear system given by the class of line L on X_2, and let $\Phi^{-1}(\Lambda)$ be the proper inverse image of Λ. Since the map Φ is given by $\Phi^{-1}(\Lambda)$, $\Phi^{-1}(\Lambda)$ has no fixed components. Also $\Phi^{-1}(\Lambda)$ is S_4-invariant. Hence any element $H \in \Phi^{-1}(\Lambda)$ is linearly equivalent to $-aK_{X_1}$ for some $a \geq 1$. Now apply Lemma 2.1 to $\Lambda + a\mathcal{K}_{X_2}$ and $\Phi^{-1}(\Lambda) + a\mathcal{K}_{X_1}$. Then $\Phi^{-1}(\Lambda) + a(K_{X_1})$ must have an S_4-orbit consisting of maximal singularities. Let r be the multiplicity of the points of $O(x)$ in $\Phi^{-1}(\Lambda)$. As any element in $\Phi^{-1}(\Lambda)$ passes through $O_{S_4}(x)$ with multiplicity r, we have $a^2K^2_{X_1} \geq r^2 d$, d being $\sharp(O_{S_4}(x))$; and we have $d < K^2_{X_1} = 6$. Hence $O_{S_4}(x)$ is one of the orbits described in Lemma 1.3.

Lemma 3.1. *The points in the orbit $O_{S_4}(x)$ with $d = 4$ can not be maximal singularities of $\Phi^{-1}(\Lambda) + a\mathcal{K}_{X_1}$.*

Proof. Let E_i be the divisor on X_1 given by $x_1 = \omega^i x_0$ $(i = 0, 1, 2)$ as in Lemma 1.4. Suppose that $O((\omega^i, \omega^i, \omega^i))$ are maximal singularities, and let

$q : \hat{X}_1 \to X_1$ be the blowing-up at $O((\omega^i, \omega^i, \omega^i))$. Then the linear system $q^*(\Phi^{-1}(\Lambda)) - r(R_{i1} + R_{i2} + R_{i3} + R_{i4})$ does not have any fixed components (we identify R_{ij} $(j = 1, 2, 3, 4)$ with the exceptional curves). Let $\bar{E}_i$ be the proper transform of E_i. Then

$$\left(-aq^*K_{X_1} - r\sum_{j=1}^{4} R_{ij}\right)\bar{E}_i = 2a - 2r < 0.$$

This means that $\bar{E}_i$ is a fixed component of $q^*(\Phi^{-1}(\Lambda)) - r(R_{i1} + R_{i2} + R_{i3} + R_{i4})$.

Lemma 3.2. *The points in the orbit $O_{S_4}(x)$ with $d = 3$ can not be maximal singularities of $\Phi^{-1}(\Lambda) + a\mathcal{K}_{X_1}$.*

Proof. Suppose that $O(P_1) = \{P_1, P_2, P_3\}$ are maximal singularities. We may assume that the irreducible component C_1 in the divisor $x_0y_0z_0 = 0$ passes through P_1. Let $q : \hat{X}_1 \to X_1$ be the blowing-up at $O(P_1)$. Then the linear system $q^*(\Phi^{-1}(\Lambda)) - r(P_1 + P_2 + P_3)$ does not have any fixed components (we identify P_j $(j = 1, 2, 3)$ with the exceptional curves). Let $\bar{C}_1$ be the proper transform of C_1. Then

$$\left(-aq^*K_{X_1} - r\sum_{j=1}^{3} P_j\right)\bar{C}_1 = a - r < 0.$$

This means that $\bar{C}_1$ is a fixed component of $q^*(\Phi^{-1}(\Lambda)) - r(P_1 + P_2 + P_3)$.

By Lemmas 3.1 and 3.2, Theorem 0.1 for S_4 follows.

§3.2. The case of A_5

By the same argument as in the previous case, the existence of Φ implies the existence of an A_5-orbit $O_{A_5}(x)$, $x \in X_1$ with $\sharp(O_{A_5}(x)) < 5$. This contradicts Lemma 1.5.

§4. A Remark for Versal S_4-covers $\varpi_{S_4,1} : X_1 \to Y_1$ and $\varpi_{S_4,2} : X_2 \to Y_2$

By the definition of versality, there exist S_4-equivariant rational maps $\mu_1 : X_1 \dashrightarrow X_2$ and $\mu_2 : X_2 \dashrightarrow X_1$ such that $\mu_1(X_1) \not\subset \mathrm{Fix}(X_2, G)$ and $\mu_2(X_2) \not\subset \mathrm{Fix}(X_1, G)$. Note that both of μ_i $(i = 1, 2)$ are dominant as there exists no

1-dimensional versal S_4-cover. In this section, we give examples of such μ_i ($i = 1, 2$) such that

(i) both field extensions $\mathbb{C}(X_1)/\mathbb{C}(X_2)$ and $\mathbb{C}(X_2)/\mathbb{C}(X_1)$ induced by μ_1 and μ_2, respectively, are cyclic extension of degree 3, and

(ii) the field extension $\mathbb{C}(X_2)/(\mu_2 \circ \mu_1)^*(\mathbb{C}(X_2))$ is Galois and its Galois group is ismorphic to $(\mathbb{Z}/3\mathbb{Z})^{\oplus 2}$.

Let $([x_0, x_1], [y_0, y_1], [z_0, z_1])$ be homogeneous coordinates for $X_1 \subset \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. $\mathbb{C}(X_1) = \mathbb{C}(y, z)$ where $y = y_1/y_0$ and $z = z_1/z_0$. Let $[t_0, t_1, t_2]$ be homogeneous coordinates for $X_2 = \mathbb{P}^2$. $\mathbb{C}(X_2) = \mathbb{C}(u, v)$ where $u = t_1/t_0$ and $v = t_2/t_0$. We construct μ_1 and μ_2 as follows.

Define $\mu_2 \colon X_2 \dashrightarrow \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ by

$$\mu_2([t_0, t_1, t_2]) = ([t_0t_1t_2, t_0^3], [t_0t_1t_2, t_1^3], [t_0t_1t_2, t_2^3])$$

It can be checked immediately that μ_2 is an S_4-equivariant rational map, $\mu_2(X_2) \subset X_1$ and $\mu_2(X_2) \not\subset \mathrm{Fix}(X_1, S_4)$. We have $\mu_2^*(y) = u^2/v$, $\mu_2^*(z) = v^2/u$. Let $\theta = u/v$. Then $\mathbb{C}(X_2) = \mu_2^*(\mathbb{C}(X_1))(\theta)$ and $\theta^3 = \mu_2^*(y)/\mu_2^*(z) \in \mu_2^*(\mathbb{C}(X_1))$. Hence $[\mathbb{C}(X_2) : \mu_2^*(\mathbb{C}(X_1))] = 3$. This means that μ_2 is a rational map of degree 3 as desired.

Define $\mu_1 \colon X_1 \dashrightarrow X_2$ by

$$\mu_1([x_0, x_1], [y_0, y_1], [z_0, z_1]) = [x_1/x_0, y_1/y_0, z_1/z_0]$$

It can be checked immediately that μ_1 is an S_4-equivariant rational map and $\mu_1(X_1) \not\subset \mathrm{Fix}(X_2, S_4)$. We have $(\mu_1 \circ \mu_2)^*(u) = u^3$ and $(\mu_1 \circ \mu_2)^*(v) = v^3$. This implies that $\mathbb{C}(X_2)/(\mu_1 \circ \mu_2)^*(\mathbb{C}(X_2))$ is Galois, $[\mathbb{C}(X_2) : (\mu_1 \circ \mu_2)^*(\mathbb{C}(X_2))] = 9$ and $\mathrm{Gal}(\mathbb{C}(X_2)/(\mu_1 \circ \mu_2)^*(\mathbb{C}(X_2))) = (\mathbb{Z}/3\mathbb{Z})^{\oplus 2}$. Hence $[\mathbb{C}(X_1) : \mu_1^*(\mathbb{C}(X_2))] = 3$. This means that μ_1 is a rational map of degree 3 as desired.

Remark. It may be an interesting question to consider if there exists a simple relation between X_1 and X_2 in the case of A_5 as above.

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