Some remarks on the orthogonality of generalized eigenfunctions for singular second-order differential equations

By

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Let us consider a differential equation of second order

$$\frac{d^2u}{dx^2} - q(x)u(x) + \lambda u(x) = 0, \qquad (0 < x < \infty). \tag{0. 1}$$

Here $q(x)$ is a real-valued function which is locally summable in $(0, \infty)$.

In the case where $x=0$ is a regular point of the equation, M. Matsuda has proved that any pseudo-spectral measure in the limit point case at $x=\infty$ is the Weyl spectral measure (Matsuda [6]).

In this paper we try to extend this result to the case where $x=0$ may be a singular point of the equation.

We take a linearly independent system of solutions $(\varphi_1(x, l), \varphi_2(x, l))$ of the equation (0. 1) which satisfies

$$\varphi_i(1, l) = \eta_i(l), \quad \frac{\partial \varphi_i}{\partial x}(1, l) = \zeta_i(l)$$

for $i=1, 2$ where $\eta_i(l)$ and $\zeta_i(l)$ are entire functions of l which satisfy $\eta_2(l)\zeta_1(l) - \eta_1(l)\zeta_2(l) = 1$ for every complex number l.

M. H. Stone, E. C. Titchmarsh and K. Kodaira proved that there exists a spertral measure matrisx $\boldsymbol{P}(\lambda) = (\rho_{ij}(\lambda))_{i,j=1,2}$ which satisfies the following three conditions (Kodaira [3], [4]):

(A) $\boldsymbol{P}(\lambda)$ is a positive semi-definite measure matrix on $(-\infty, \infty)$.

(B) Denote by $L_2(-\infty, \infty; d\boldsymbol{P}(\lambda))$ the Hilbert space with the norm

$$||\boldsymbol{v}||^2 = \int_{-\infty}^{\infty} \tilde{\boldsymbol{v}}(\lambda) d\boldsymbol{P}(\lambda) \bar{\boldsymbol{v}}(\lambda),$$

where $\boldsymbol{v}(\lambda)$ is a vector-valued function on $(-\infty, \infty)$

Received June 1, 1966

$$\boldsymbol{v}(\lambda) = \begin{pmatrix} v_1(\lambda) \\ v_2(\lambda) \end{pmatrix},$$

$\tilde{\boldsymbol{v}}(\lambda)$ is the transpose of $\boldsymbol{v}(\lambda)$, and $\bar{\alpha}$ means the conjugate complex number of α. Then the generalized Fourier transformation

$$\mathcal{F}_P\colon f(x) \to \int_0^\infty f(x)\boldsymbol{y}(x, \lambda)dx\,, \qquad \boldsymbol{y}(x, l) = \begin{pmatrix} \varphi_1(x, l) \\ \varphi_2(x, l) \end{pmatrix}$$

from $L_2(0, \infty; dx)$ into $L_2(-\infty, \infty; d\boldsymbol{P}(\lambda))$ is isometric.

(C) $\mathcal{F}_P$ transforms $L_2(0, \infty; dx)$ onto $L_2(-\infty, \infty; d\boldsymbol{P}(\lambda))$.

In Theorem 1 we shall prove that if both $x=0$ and $x=\infty$ belong to the limit point case, then the measure matrix which satisfies (A), (B) and (C) is unique.

We shall prove in Theorem 2 that if both $x=0$ and $x=\infty$ belong to the limit point case, then any measure matrix which satisfies (A) and (B) satisfies (C).

Aknowledgements. I took up this problem, suggested by Mr. M. Matsuda and had a lot of useful discussions with him in the course of writing this paper. I would like to express my gratitude.

§ 1 The measure matrix in the eigenfunction expansion for singular differential equations.

Let us consider a differential equation of the second order

$$\frac{d^2u}{dx^2} - q(x)u(x) + \lambda u(x) = 0\,, \qquad (0<x<\infty)\,, \tag{1.1}$$

where $q(x)$ is a locally summable function in $(0, \infty)$. We assume that $x=0$ is a singular point of the equation. Moreover we assume that the equation (1. 1) is of the limit point type both at 0 and at ∞.

Let $(\varphi_1(x, l), \varphi_2(x, l))$ be a linearly independent system of solutions of (1. 1) which satisfies

$$\varphi_i(1, l) = \eta_i(l)\,, \quad \frac{\partial \varphi_i}{\partial x}(1, l) = \zeta_i(l)$$

for $i=1, 2$, where $\eta_i(l)$ and $\zeta_i(l)$ are entire functions of l such that

$$\eta_2(l)\zeta_1(l) - \eta_1(l)\zeta_2(l) = 1$$

for every complex number l. Then for $(\varphi_1\ \varphi_2)$ there exists a matrix function on $(-\infty, \infty)$

$$\boldsymbol{P}(\lambda) = \begin{pmatrix} \rho_{11}(\lambda) & \rho_{12}(\lambda) \\ \rho_{21}(\lambda) & \rho_{22}(\lambda) \end{pmatrix}, \quad \rho_{12}(\lambda) \equiv \rho_{21}(\lambda),$$

which satisfies the following three conditions (A), (B) and (C) (Kodaira [3]):

(A) Each $\rho_{ij}(\lambda)$ is a function of bounded variation on every finite interval in $(-\infty, \infty)$, and $\boldsymbol{P}(\lambda)$ is a positive semi-definite measure on $(-\infty, \infty)$. Namely for every finite interval Δ and for every pair of continuous functions

$$\boldsymbol{v}_0(\lambda) = \begin{pmatrix} v_1^0(\lambda) \\ v_2^0(\lambda) \end{pmatrix}$$

we have the inequality

$$\int_\Delta \tilde{\boldsymbol{v}}_0(\lambda) d\boldsymbol{P}(\lambda) \overline{\boldsymbol{v}}_0(\lambda) = \sum_{i,j=1,2} \int_\Delta v_i^0(\lambda) \overline{v}_j^0(\lambda) d\rho_{ij}(\lambda) \geqq 0,$$

where $\tilde{\boldsymbol{v}}_0(\lambda)$ is the transpose of $\boldsymbol{v}_0(\lambda)$.

(B) The generalized Fourier transformation from $L_2(0, \infty; dx)$ into $L_2(-\infty, \infty; d\boldsymbol{P}(\lambda))$

$$\mathcal{F}_P\colon f(x) \to \int_0^\infty f(x) \boldsymbol{y}(x, \lambda) dx, \quad \boldsymbol{y}(x, l) = \begin{pmatrix} \varphi_1(x, l) \\ \varphi_2(x, l) \end{pmatrix}$$

is isometric. Here the element of $L_2(-\infty, \infty; d\boldsymbol{P}(\lambda))$ is a pair of measurable functions

$$\boldsymbol{v}(\lambda) = \begin{pmatrix} v_1(\lambda) \\ v_2(\lambda) \end{pmatrix}$$

such that

$$||\boldsymbol{v}(\lambda)||^2 \equiv \int_{-\infty}^\infty \tilde{\boldsymbol{v}}(\lambda) d\boldsymbol{P}(\lambda) \overline{\boldsymbol{v}}(\lambda) < \infty.$$

(C) $\mathcal{F}_P$ transforms $L_2(0, \infty; dx)$ onto $L_2(-\infty, \infty; d\boldsymbol{P}(\lambda))$.

We shall prove the following two theorems.

Theorem 1. *Let the equation* (1. 1) *be of the limit point type both at* 0 *and at* ∞. *Then the measure matrix which satisfies* (A), (B) *and* (C) *is uniquely determined by* $\boldsymbol{y}(x, l)$.

Theorem 2. *Let the equation* (1. 1) *be of the limit point type both at* 0 *and at* ∞. *Then if a measure matrix satisfies* (A) *and* (B) *with respect to* $\boldsymbol{y}(x, l)$, *then it satisfies* (C).

To prove Theorem 1 and Theorem 2 we prepare the following lemma.

Lemma 1. *Let* $\boldsymbol{P}(\lambda)$ *be a measure matrix which only satisfies* (A) *and* (B) *with respect to* $\boldsymbol{y}(x, l)$ *and put*

$$E_P(x, y; \Delta) = \int_\Delta \tilde{\boldsymbol{y}}(x, \lambda) d\boldsymbol{P}(\lambda) \boldsymbol{y}(y, \lambda). \tag{1.2}$$

Then

(i) $E_P(x,y; \Delta)$ *is a symmetric kernel of Carleman type such that*

$$\int_0^\infty (E_P(x, y; \Delta))^2 dx \leqq \int_\Delta \tilde{\boldsymbol{y}}(y, \lambda) d\boldsymbol{P}(\lambda) \boldsymbol{y}(y, \lambda), \tag{1.3}$$

and

$$\int_0^\infty E_P(x, y; \Delta) f(y) dy = \int_\Delta \tilde{\mathfrak{F}}_P f(\lambda) d\boldsymbol{P}(\lambda) \boldsymbol{y}(x, \lambda) \tag{1.4}$$

hold for every $f(x)$ *in* $L_2(0, \infty; dx)$.

(ii) *Let* $E_P(\Delta)$ *be a linear transformation defined by*

$$E_P(\Delta) f(x) = \int_0^\infty E_P(x, y; \Delta) f(y) dy \tag{1.5}$$

for $f(x)$ *in* $L_2(0, \infty; dx)$. *Then* $E_P(\Delta)$ *is a bounded symmetric operator on* $L_2(0, \infty; dx)$ *and we have*

$$||E_P(\Delta)|| \leqq 1, \quad \lim_{\Delta \to (-\infty, \infty)} E_P(\Delta) = identity,$$

$$\langle E_P(\Delta) f, u \rangle = \int_\Delta \tilde{\mathfrak{F}}_P f(\lambda) d\boldsymbol{P}(\lambda) \mathfrak{F}_P u(\lambda) \tag{1.6}$$

for every pair of $f(x)$, $u(x)$ *in* $L_2(0, \infty; dx)$.

(iii) *For y fixed* $\dfrac{\partial E_P(x, y; \Delta)}{\partial y}$ *belongs* $L_2(0, \infty; dx)$ and we have

$$\int_0^\infty \left(\frac{\partial E_P(x, y; \Delta)}{\partial y}\right)^2 dx \leqq \int_\Delta \frac{\partial \tilde{\boldsymbol{y}}(y, \lambda)}{\partial y} d\boldsymbol{P}(\lambda) \frac{\partial \boldsymbol{y}(y, \lambda)}{\partial y}. \tag{1.7}$$

In the case where $x=0$ is a regular point, we have a corresponding fact as follows:

Lemma 2. *Let $\rho(\lambda)$ be a speudo-spectral measure*[1] *for the equation*

$$\frac{d^2u}{dx^2}-q(x)u(x)+\lambda u(x)=0\,, \qquad (0\leqq x<\infty)\,,$$

with respect to the solution $\varphi(x, l)$ and put

$$E_\rho(x, y;\Delta)=\int_\Delta \varphi(x, \lambda)\varphi(y, \lambda)d\rho(\lambda)\,, \tag{1.8}$$

where Δ is a finite interval and $x, y\geqq 0$. Then

(i) *$E_\rho(x, y;\Delta)$ is a bounded symmetric kernel of Carleman type such that*

$$\int_0^\infty (E_\rho(x, y;\Delta))^2 dx \leqq \int_\Delta \varphi^2(y, \lambda)d\rho(\lambda)\,, \tag{1.9}$$

and

$$\int_0^\infty E_\rho(x, y;\Delta)f(y)dy=\int_\Delta \varphi(x, \lambda)\mathcal{F}_\rho f(\lambda)d\rho(\lambda) \tag{1.10}$$

hold for $f(x)$ in $L_2(0, \infty; dx)$.

(ii) *Let $E_\rho(\Delta)$ be a linear transformation defined by*

$$E_\rho(\Delta)f(x)=\int_0^\infty E_\rho(x, y;\Delta)f(y)dy$$

for $f(x)$ in $L_2(0, \infty; dx)$. Then $E_\rho(\Delta)$ is a bounded symmetric operator on $L_2(0, \infty; dx)$ and we have

$$||E_\rho(\Delta)||\leqq 1\,, \qquad \lim_{\Delta\to(-\infty,\infty)} E_\rho(\Delta)=\textit{identity},$$

$$\langle E_\rho(\Delta)f, u\rangle=\int_\Delta \mathcal{F}_\rho f(\lambda)\overline{\mathcal{F}_\rho u}(\lambda)d\rho(\lambda) \tag{1.11}$$

for every pair of $f(x), u(x)$ in $L_2(0, \infty; dx)$.

(iii) *For y fixed, $\dfrac{\partial E_\rho(x, y;\Delta)}{\partial y}$ belongs to $L_2(0, \infty; dx)$ and we have*

$$\int_0^\infty\left(\frac{\partial E_\rho(x, y;\Delta)}{\partial y}\right)^2 dx\leqq\int\left(\frac{\partial\varphi(y\ \lambda)}{\partial y}\right)^2 d\rho(\lambda)\,. \tag{1.12}$$

We only prove Lemma 2, because the proof of Lemma 1 is similar.

Proof of (i). Consider a linear functional on $L_2(0, \infty; dx)$

(1) Matsuda [6].

$$l_{x_0,\Delta}(f) = \int_\Delta \varphi(x_0, \lambda)\mathcal{F}_\rho f(\lambda)d\rho(\lambda) \tag{1.13}$$

for x_0 and Δ fixed. Then we have

$$\begin{aligned} |l_{x_0,\Delta}(f)| &\leqq \left[\int_\Delta \varphi_2(x_0, \lambda)d\rho(\lambda)\right]^{1/2}\left[\int_\Delta |\mathcal{F}_\rho f(\lambda)|^2 d\rho(\lambda)\right]^{1/2} \\ &\leqq \left[\int_\Delta \varphi^2(x_0, \lambda)d\rho(\lambda)\right]^{1/2} \|\mathcal{F}_\rho f\|_\rho \\ &= M_{x_0,\Delta}\|f\|, \qquad \left(M_{x_0,\Delta} = \left[\int_\Delta \varphi^2(x_0, \lambda)d\rho(\lambda)\right]^{1/2}\right). \end{aligned} \tag{1.14}$$

This shows that $l_{x_0,\Delta}$ is a bounded linear functional. And hence by Riesz theorem we can find a function $e_{x_0,\Delta}(x)$ in $L_2(0, \infty; dx)$ such that

$$l_{x_0,\Delta}(f) = \int_\Delta \varphi(x_0, \lambda)\mathcal{F}_\rho f(\lambda)d\rho(\lambda) = \int_0^\infty e_{x_0,\Delta}(x)f(x)dx\,. \tag{1.15}$$

On the other hand we have

$$l_{x_0,\Delta}(f_0) = \int_0^\infty E(x_0, x; \Delta)f_0(x)dx \tag{1.16}$$

for $f_0(x)$ in $L_2(0, \infty; dx)$ which has a compact carrier. It follows from (1. 15) and (1. 16) that

$$E(x_0, x; \Delta) = e_{x_0,\Delta}(x)\,,$$

and hence $E(x, y; \Delta)$ is a kernel of Carleman type and we have (1. 10) by (1. 15) and (1. 16). (1. 9) follows from (1. 14).

Proof of (ii). Putting $f_\Delta(x) = E_\rho(\Delta) f(x)$, we have

$$\begin{aligned} \left|\int_0^\infty f_\Delta(x)\overline{u(x)}dx\right| &= \left|\int_0^\infty \left[\int_\Delta \varphi(x, y)\mathcal{F}_\rho f(\lambda)d\rho(\lambda)\right]\overline{u(x)}\,dx\right| \\ &= \left|\int_\Delta \mathcal{F}_\rho f(\lambda)\overline{\mathcal{F}_\rho u(\lambda)}d\rho(\lambda)\right| \\ &\leqq \left[\int_\Delta |\mathcal{F}_\rho f(\lambda)|^2 d\rho\right]^{1/2}\left[\int_\Delta |\mathcal{F}_\rho u(\lambda)|^2 d\rho\right]^{1/2} \\ &\leqq \|f\|\,\|u\| \end{aligned} \tag{1.17}$$

for $u(x)$ in $L_2(0, \infty; dx)$ which has a compact carrier. For a positive N and a finite interval Δ, let us define $u_{N,\Delta}(x)$ by

$$u_{N,\Delta}(x) = \begin{cases} f_\Delta(x)\,, & 0 \leqq x \leqq N \\ 0\,, & x > N\,. \end{cases}$$

Then $u_{N,\Delta}(x)$ belongs to $L_2(0, \infty; dx)$ and so (1. 17) implies

$$\int_0^N |f_\Delta(x)|^2 dx \leqq \left[\int_0^N |f_\Delta(x)|^2 dx\right]^{1/2} \|f\|,$$

namely

$$\left[\int_0^N |f_\Delta(x)|^2 dx\right]^{1/2} \leqq \|f\|. \tag{1. 18}$$

Since N is arbitrary, (1. 18) implies

$$\|E_\rho(\Delta)f\| \leqq \|f\|.$$

Proof of (iii). Consider a linear functional on $L_2(0, \infty; dx)$

$$k_{x_0,\Delta}(f) = \int_\Delta \frac{\partial \varphi(x_0, \lambda)}{\partial x} \mathcal{F}_\rho f(\lambda) d\rho(\lambda)$$

for x_0 and Δ fixed. Then we have

$$|k_{x_0,\Delta}(f)| \leqq \tilde{M}_{x_0,\Delta}\|f\|, \quad \left((\tilde{M}_{x_0,\Delta} = \left[\int_\Delta \left(\frac{\partial\varphi(x_0, \Delta)}{\partial x}\right)^2 d\rho(\lambda)\right]^{1/2}\right) \tag{1. 19}$$

as in the proof of (i). By the method used in the proof of (i), we can show that $\dfrac{\partial E_\rho(x, y; \Delta)}{\partial y}$ belongs to $L_2(0, \infty; dx)$ and that (1. 12) holds.

Proof of Theorem 1. Let $\boldsymbol{P}_1(\lambda)$ and $\boldsymbol{P}_2(\lambda)$ satisfy (A), (B) and (C). We shall denote by $\mathcal{D}_\infty$ the space of all functions $u(x)$ in $L_2(0, \infty; dx)$ that satisfy the following conditions:

i) $u(x) \in L_2(0, \infty; dx)$.

ii) $u(x)$ is differentiable in the open interval $(0, \infty)$.

iii) $\dfrac{du}{dx}$ is absolutely continuous in every closed subinterval $[a, b]$ $(0<a<b<\infty)$ in $(0, \infty)$.

iv) $u(x)$ has a compact carrier in $(0, \infty)$.

v) $-\dfrac{d^2u}{d^2x}+q(x)u(x) \in L_2(0, \infty; dx)$.

Define an operator L_∞ which transforms $u(x) \in \mathcal{D}_\infty$ to

$$L_\infty u(x) = -\frac{d^2u}{dx^2}+q(x)u(x).$$

By the assumption of Theorem 1, if we denote the closure of L_∞ by L, L is a self-adjoint operator. Let l be a complex number with $I_m l \neq 0$ and $\boldsymbol{L}_l$ be the resolvent $(l-L)^{-1}$. We have for $u(x)$ in $\mathcal{D}_\infty$

$$\mathcal{F}_{P_k}(l-L)u(\lambda) = (l-\lambda)\mathcal{F}_{P_k}u(\lambda), \qquad (k=1, 2).$$

Therefore we obtain

$$\begin{aligned}\langle \boldsymbol{L}_l(l-L)u, f\rangle &= \langle u, f\rangle = \langle \mathcal{F}_{P_k}u, \mathcal{F}_{P_k}f\rangle_{P_k} \\ &= \left\langle \frac{\mathcal{F}_{P_k}(l-L)u}{l-\lambda}, \mathcal{F}_{P_k}f\right\rangle_{P_k}, \qquad (k=1, 2)\end{aligned}$$

for $u(x)$ in $\mathcal{D}_\infty$ and $f(x)$ in $L_2(0, \infty; dx)$. Since the family of functions $\{(l-L)u(x)/u(x)\in\mathcal{D}_\infty\}$ is dense in $L_2(0, \infty; dx)$, we have

$$\text{(1. 20)} \qquad \langle \boldsymbol{L}_l f, h\rangle = \int_{-\infty}^{\infty} \frac{\tilde{\mathcal{F}}_{P_k}f(\lambda)d\boldsymbol{P}_k(\lambda)\overline{\mathcal{F}_{P_k}h}(\lambda)}{\lambda - l}, \qquad (k=1, 2)$$[1]

for every pair of $f(x)$ and $h(x)$ in $L_2(0, \infty; dx)$.

Let $E_{P_1}(\Delta)$ and $E_{P_2}(\Delta)$ be the operators in Lemma 1 with respect to $\boldsymbol{P}_1(\Delta)$ and $\boldsymbol{P}_2(\Delta)$. Then making use of the inversion formula for Stieltjes transformation[2] we have from (1. 20)

$$\text{(1. 21)} \qquad \langle E_{P_1}(\Delta)f, h\rangle = \langle E_{P_2}(\Delta)f, h\rangle$$

for every finite interval Δ in $(-\infty, \infty)$. From (1. 21) and (1. 5) we get

$$\text{(1. 22)} \qquad E_{P_1}(x, y; \Delta) = E_{P_2}(x, y; \Delta),$$

namely

$$\text{(1. 23)} \qquad \int_\Delta \tilde{\boldsymbol{y}}(x, \lambda)d\boldsymbol{P}_1(\lambda)\boldsymbol{y}(y, \lambda) = \int_\Delta \tilde{\boldsymbol{y}}(x, \lambda)d\boldsymbol{P}_2(\lambda)\boldsymbol{y}(y, \lambda).$$

Let $\boldsymbol{y}_0(x, l)$ be a system of solutions

$$\boldsymbol{y}_0(x, l) = \begin{pmatrix}\varphi_1^{(0)}(x, l)\\ \varphi_2^{(0)}(x, l)\end{pmatrix}$$

such that

$$\text{(1. 24)} \qquad \begin{cases}\varphi_1^{(0)}(1, l) = 0, & \dfrac{\partial\varphi_1^{(0)}}{\partial x}(1, l) = 1, \\[2mm] \varphi_2^{(0)}(1, l) = 1, & \dfrac{\partial\varphi_2^{(0)}}{\partial x}(1, l) = 0.\end{cases}$$

(1) This formula is due to M. Matsuda.
(2) Neumark [7], Anhang.

Then there exists a matrix

$$A(l)=\begin{pmatrix}\alpha(l) & \gamma(l)\\ \beta(l) & \delta(l)\end{pmatrix},\quad \alpha,\beta,\gamma,\delta \text{ being entire functions of } l$$

such that

$$y(x, l) = A(l)y_0(x, l). \tag{1.25}$$

Define two density matrices $dP_1^{(0)}(\lambda)$ and $dP_2^{(0)}(\lambda)$ by

$$dP_k^{(0)}(\lambda) = \widetilde{A(\lambda)}dP_k(\lambda)A(\lambda), \qquad k=1, 2. \tag{1.26}$$

Then the measure matrices $P_k^{(0)}(\lambda)$ $(k=1, 2)$ will satisfy (A), (B) and (C) with respect to $y_0(x, \lambda)$.

To prove $P_1(\Delta)=P_2(\Delta)$ it is sufficient to prove $P_1^{(0)}(\Delta)=P_2^{(0)}(\Delta)$. From (1. 2), (1. 25) and (1. 26) we obtain for $k=1, 2$

$$E_{P_k}(x, y; \Delta) = \int_\Delta \tilde{y}_0(x, \lambda)dP_k^{(0)}(\lambda)y_0(y, \lambda),$$

and hence (1. 22) implies

$$\int_\Delta \tilde{y}_0(x, \lambda)dP_1^{(0)}(\lambda)y_0(y, \lambda) = \int_\Delta \tilde{y}_0(x, \lambda)dP_2^{(0)}(\lambda)y_0(y, \lambda). \tag{1.27}$$

We differentiate (1. 27) with respect to x or y to obtain

$$\int_\Delta \frac{\partial\tilde{y}_0(x, \lambda)}{\partial x}dP_1^{(0)}(\lambda)y_0(y, \lambda) = \int_\Delta \frac{\partial\tilde{y}_0(x, \lambda)}{\partial x}dP_2^{(0)}(\lambda)y_0(y, \lambda). \tag{1.28}$$

and

$$\int_\Delta \frac{\partial\tilde{y}_0(x, \lambda)}{\partial x}dP_1^{(0)}(\lambda)\frac{\partial y_0(y, \lambda)}{\partial y} = \int_\Delta \frac{\partial\tilde{y}_0(x, \lambda)}{\partial x}dP_2^{(0)}(\lambda)\frac{\partial y_0(y, \lambda)}{\partial y}. \tag{1.29}$$

Set

$$P_k^{(0)}(\Delta)=\begin{pmatrix}\rho_{11}^{(k)}(\Delta) & \rho_{12}^{(k)}(\Delta)\\ \rho_{21}^{(k)}(\Delta) & \rho_{22}^{(k)}(\Delta)\end{pmatrix}, \qquad (k=1, 2).$$

Then putting $x=y=1$ in (1. 27), (1. 28) and (1. 29) we have

$$\rho_{22}^{(1)}(\Delta) = \rho_{22}^{(2)}(\Delta), \quad \rho_{22}^{(1)}(\Delta) = \rho_{12}^{(2)}(\Delta), \quad \rho_{11}^{(1)}(\Delta) = \rho_{11}^{(2)}(\Delta)$$

respectively, which completes the proof.

Proof of Theorem 2. Let $P(\lambda)$ be a measure matrix which satisfies (A) and (B). Then we have

(1.30) $$\langle \boldsymbol{L}_l f, h\rangle = \int_{-\infty}^{\infty} \frac{\tilde{\mathcal{F}}_P f(\lambda) d\boldsymbol{P}(\lambda) \overline{\mathcal{F}_P h}(\lambda)}{l-\lambda},$$

and $E_P(\Delta)$ becomes a resolution of the identity[1].

Let $\boldsymbol{y}_0(x, \lambda)$ be a system of solutions which satisfies the initial conditions (1.24). Putting

$$d\boldsymbol{P}_0(\lambda) = \tilde{\boldsymbol{A}}(\lambda) d\boldsymbol{P}(\lambda) \boldsymbol{A}(\lambda)$$

for $\boldsymbol{A}(l)$ satisfying (1.25), we have a resolution of the identity $E_{P_0}(\Delta)$.

Defining $\boldsymbol{u}_0(x, \Delta)$ by

(1.31) $$\boldsymbol{u}_0(x, \Delta) = \int_\Delta d\boldsymbol{P}_0(\lambda) \boldsymbol{y}_0(x, \lambda) = \begin{pmatrix} u_1^{(0)}(x, \Delta) \\ u_2^{(0)}(x, \Delta) \end{pmatrix},$$

we shall prove

(1.32) $$\boldsymbol{P}_0(\Delta \cap \Delta_1) = \int_0^\infty \boldsymbol{u}_0(x, \Delta) \tilde{\boldsymbol{u}}_0(x, \Delta_1) dx$$

for every pair of intervals Δ and Δ_1.

Since $E_{P_0}(\Delta)$ is a resolution of the identity, we have

(1.33) $$\int_0^\infty E_{P_0}(s, x; \Delta) E_{P_0}(s, y; \Delta_1) ds = E_{P_0}(x, y; \Delta \cap \Delta_1).$$

By (iii) of Lemma 1 we can differentiate (1.33) with respect to x or y to obtain

(1.34) $$\int_0^\infty \frac{\partial E_{P_0}(s, x; \Delta)}{\partial x} E_{P_0}(s, y; \Delta_1) ds = \frac{\partial E_{P_0}(x, y; \Delta \cap \Delta_1)}{\partial x},$$

and

(1.35) $$\int_0^\infty \frac{\partial E_{P_0}(s, x; \Delta)}{\partial x} \frac{\partial E_{P_0}(s, y; \Delta_1)}{\partial y} ds = \frac{\partial^2 E_{P_0}(x, y; \Delta \cap \Delta_1)}{\partial x \partial y}.$$

Setting $x = y = 1$ in (1.33) and (1.34) and (1.35), we have

(1.36) $$\int_0^\infty u_i^{(0)}(s, \Delta) u_j^{(0)}(s, \Delta_1) ds = \rho_{ij}^{(0)}(\Delta \cap \Delta_1), \qquad (i, j = 1, 2),$$

where

$$\boldsymbol{P}_0(\Delta) = \begin{pmatrix} \rho_{11}^{(0)}(\Delta) & \rho_{12}^{(0)}(\Delta) \\ \rho_{21}^{(0)}(\Delta) & \rho_{22}^{(0)}(\Delta) \end{pmatrix}.$$

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(1) See the proof of Theorem 1 of Matsuda [6].

Thus the identity (1. 32) is proved.

For $\boldsymbol{y}(x, l)$ and $\boldsymbol{P}(\Delta)$ putting

$$\boldsymbol{u}(x, \Delta) = \int_{\Delta} d\boldsymbol{P}(\lambda)\boldsymbol{y}(x, y) \tag{1. 37}$$

we have by (1. 36)

$$\boldsymbol{P}(\Delta \cap \Delta_1) = \int_0^{\infty} \boldsymbol{u}(x, \Delta)\tilde{\boldsymbol{u}}(x, \Delta_1)dx\,. \tag{1. 38}$$

Define a transformation $\mathcal{F}_P^*$ from $L_2(-\infty, \infty; d\boldsymbol{P}(\lambda))$ onto $L_2(0, \infty; dx)$ by

$$\mathcal{F}_P^*\colon \boldsymbol{v}(\lambda) \to \int_{-\infty}^{\infty} \boldsymbol{y}(x, y)d\boldsymbol{P}(\lambda)\boldsymbol{v}(\lambda)\,.$$

Then $\mathcal{F}_P^* \cdot \mathcal{F}_P$ proves to be the identity operator.(1)

By (1. 38), we can prove that $\mathcal{F}_P^*$ is an isometric transformation from $L_2(-\infty, \infty; d\boldsymbol{P}(\lambda))$ onto $L_2(0, \infty; dx)$ (Kodaira [3] 2, [5]). $\mathcal{F}_P$ is therefore surjective, and the proof is completed.

Remark. If we assume the existence of the measure matrix $\boldsymbol{P}_*(\lambda)$ which satisfies (A), (B) and (C), calculated by Titchmarsh-Kodaira's spectral formula, the proof of Theorem 2 will be easier (Kodaira [3], [4]).

In fact, let $\boldsymbol{P}(\lambda)$ be a measure matrix satisfying (A) and (B). Then we have

$$\langle \boldsymbol{L}_l f, h\rangle = \int_{-\infty}^{\infty} \frac{\mathcal{F}_{P_*}f(\lambda)d\boldsymbol{P}_*(\lambda)\mathcal{F}_{P_*}h(\lambda)}{l-\lambda} = \int_{-\infty}^{\infty} \frac{\mathcal{F}_P f(\lambda)d\boldsymbol{P}(\lambda)\mathcal{F}_P h(\lambda)}{l-\lambda}\,.$$

Using Lemma 1 we have

$$E_{P_*}(x, y; \Delta) = E_P(x, y; \Delta)\,.$$

We obtain $\boldsymbol{P}_*(\Delta)=\boldsymbol{P}(\Delta)$ by the method used in the proof of Theorem 1. Therefore $\boldsymbol{P}(\Delta)$ satisfies (C).

If the equation (1. 1) is of the linit cirdle type at 0, the situation is essentially the same as in the case where $x=0$ is a regular point.

§ 2. The spectrum in the limit circle case at infinity.

In the case where $x=0$ is a regular point of the equation (1. 1),

(1) See Proposition 1 of Matsuda [6].

M. Matsuda has proved that the spectrum is unbounded below in the limit circle case at $x=\infty$ (M. Matsuda [6]).

In this section we assume that the equation (1. 1) belongs to the limit circle case at $x=\infty$. Then by setting some boundary conditions at ∞ and also at 0 if necessary, we obtain a self-adjoint operator L which is a symmetric extension of the L_∞ in §1. The spectrum of this operator is simple(1).

Then we shall prove the following theorem:

Theorem 3. *Let the equation* (1. 1) *belong to the limit circle case at* ∞. *Then the self-adjoint operator* L *is unbounded below.*

In fact, let $I_1=[1, \infty)$ and $I_2=(0, 1]$. Setting some boundary conditions at $x=1$, we obtain L_1 and L_2 which are the restricitions of L to I_1 and to I_2 respectively. Then L is bounded below if and only if L_1 and L_2 are both bounded below(2). L_1 is unbounded below by virtue of Theorem 2 of Matsuda [6], and hence L is also unbounded below.

Using Weyl's classification of the limit point case and the limit circle case, we can see that Theorem 3 is equivalent to the following fact:

Let $q(x)$ be locally summable in $(0, \infty)$. Then if L_∞ in §1 is bounded below, L_∞ is essentially self-adjoint.

Let us make a remark on this fact. In the m-dimensional case, the following result is known (Wienholtz [8], Kato [2]):

Let L_0 be a partial differential operator

$$L_0 = -\Delta + q(x),$$

where $q(x)$ has a following property: there exists a constant $\alpha(0<\alpha<1)$ such that

$$M(x) = \int_{|x-y|\leqq 1} |x-y|^{\mu(m,\alpha)} |q(y)|^2 dy, \quad \mu(m, \alpha) = \begin{cases} 0, & m\leqq 3 \\ -m+4-\alpha, & m\geqq 4 \end{cases}$$

is locally bounded. The domain of L_0 consists of C^∞-functions of compact carrier. Then if L_0 is bounded below, L_0 is essentially self-adjoint.

By slight modification of their method we can replace the local

(1) Kodaira [3].
(2) Dunford-Schwartz [1]. p. 1455.

boundedness of $M(x)$ with the local summability of $q(x)$ to prove the fact we obtained above. However, our method seems to be of some interest in that we derived this in the scheme of the inverse problem of Gelfand-Levitan.

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