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Boundaries for Spaces of Holomorphic Functions on $\mathcal{C}(K)$

By

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Abstract

We consider the Banach space $\mathcal{A}_u(X)$ of holomorphic functions on the open unit ball of a (complex) Banach space X which are uniformly continuous on the closed unit ball, endowed with the supremum norm. A subset $\mathcal{B}$ of the unit ball of X is a boundary for $\mathcal{A}_u(X)$ if for every $F \in \mathcal{A}_u(X)$, the norm of F is given by $\|F\| = \sup_{x \in \mathcal{B}} |F(x)|$. We prove that for every compact K, the subset of extreme points in the unit ball of $\mathcal{C}(K)$ is a boundary for $\mathcal{A}_u(\mathcal{C}(K))$. If the covering dimension of K is at most one, then every norm attaining function in $\mathcal{A}_u(\mathcal{C}(K))$ must attain its norm at an extreme point of the unit ball of $\mathcal{C}(K)$. We also show that for any infinite K, there is no Shilov boundary for $\mathcal{A}_u(\mathcal{C}(K))$, that is, there is no minimal closed boundary, a result known before for K scattered.

§1. Introduction

A classical result by Šilov [Lo, p. 80] states that if $\mathcal{A}$ is a separating algebra of continuous functions on a compact Hausdorff space K, then there is a smallest closed subset $F \subset K$ with the property that every function of $\mathcal{A}$ attains its maximum absolute value at some point of F. Bishop [Bi] proved that for every compact metrizable Hausdorff space K, any separating Banach algebra $A \subset \mathcal{C}(K)$ has a minimal boundary, that is, there is $M \subset K$ such that every element in A attains its norm at M and M is minimal with such a property. For the non compact case, Globevnik [Glo] introduced the corresponding concept

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of boundary for a subalgebra of the space of bounded continuous functions on a Hausdorff space T (not necessarily compact). Given an algebra $A \subset \mathcal{C}_b(T)$, a subset $F \subset T$ is a boundary of A if

$$\|f\| = \sup_{x \in F} |f(x)|, \qquad \forall f \in A.$$

Globevnik also described the boundaries of $\mathcal{A}_u(c_0)$, the space of complex valued functions which are holomorphic on the open unit ball of c_0 and uniformly continuous on the closed unit ball. He proved that there is no a minimal closed boundary (the Shilov boundary) in this case. Aron, Choi, Lourenço and Paques [ACLP] showed that the Shilov boundary for $\mathcal{A}_u(\ell_p)$ $(1 \leq p < \infty)$ is the unit sphere of ℓ_p and it does not exist for ℓ_∞.

Moraes and Romero [MoRo] gave the corresponding description for a predual of a Lorentz sequence space G of the boundaries of $\mathcal{A}_u(G)$ and, as a consequence, they also obtained the non-existence of a minimal closed boundary in this case. In some of the mentioned papers the role played by the subset of the $\mathbb{C}$-extreme points of the unit ball seems to be essential.

Choi, García, Kim and Maestre showed that any function $T \in \mathcal{A}_u(\mathcal{C}(K, \mathbb{C}))$ attaining its norm at a function that does not vanish, in fact attains the norm at an extreme point of the unit ball of $\mathcal{C}(K, \mathbb{C})$ [CGKM, Theorem 2.8]. If the dimension of K is at most one, then they obtain that the previous statement is always satisfied [CGKM, Theorem 2.9]. The same authors also prove that for any scattered compact K, and for every function $T \in \mathcal{A}_u(\mathcal{C}(K, \mathbb{C}))$, the norm of T is the supremum of the evaluations at the extreme points of the unit ball of $\mathcal{C}(K, \mathbb{C})$ [CGKM, Theorem 3.3]. In the case that K is scattered and infinite, they show that there is no minimal closed boundary for $\mathcal{A}_u(\mathcal{C}(K, \mathbb{C}))$ [CGKM, Theorem 3.4].

On the other hand, it has been studied for many Banach spaces how the unit ball can be described in terms of a rich extremal structure. More precisely, Aron and Lohman [ArLo] introduced the so-called λ-property. A Banach space has the λ-property if every element in the closed unit ball can be expressed as a convex series of extreme points. For instance, ℓ_1 clearly satisfies this condition. As a consequence, the norm of any (bounded and linear) functional is the supremum of the evaluations on the extreme points of the unit ball. Also, every norm attaining functional on a Banach space satisfying the λ-property, attains its norm at an extreme point of the unit ball. Several authors studied the λ-property in Banach spaces such as Aron, Bogachev, Jiménez-Vargas, Lohman, Mena-Jurado and Navarro-Pascual (see, for instance [ArLo, BMN, JMN1, JMN2, MeNa]).

Our intention here is found at some average of the two kind of ideas we mentioned before. We plan to find non-linear versions of results stated for spaces satisfying the λ-property. In such versions we will use certain holomorphic functions instead of linear functionals and the maximum modulus principle will play the role of convexity. Along this line we got somehow surprising results in the sense that holomorphic functions behave somehow as if they were linear.

In Section 2 we consider norm attaining holomorphic functions. We prove that in the case that a function $F \in \mathcal{A}_u(\mathcal{C}(K,X))$ attains its norm at a function in $\mathcal{C}(K,X)$ that does not vanish, then F attains its norm at a function whose evaluation at any point has norm one. As a consequence, if the pair (K,X) has the extension property and X is $\mathbb{C}$-rotund, any norm attaining function $F \in \mathcal{A}_u(\mathcal{C}(K,X))$ attains its norm at a $\mathbb{C}$-extreme point of the unit ball of $\mathcal{C}(K,X)$. Examples of spaces satisfying the previous assumption are $(K,\mathbb{C})$, for K scattered or $[a,b] \subset \mathbb{R}$. If X is infinite-dimensional, then (K,X) has the extension property for any compact K.

In Section 3 we give results stating that (under some conditions) it is enough to know the evaluations of a function $F \in \mathcal{A}_u(\mathcal{C}(K,X))$ on the extreme points in the unit ball of $\mathcal{C}(K,X)$ in order to compute the norm of F. We obtain that in the case that the set of continuous functions from K to X that do not vanish is dense in $\mathcal{C}(K,X)$, then the norm of any element $F \in \mathcal{A}_u(\mathcal{C}(K,X))$ is given by

$$\|F\| = \sup\{|F(f)| : f \in \mathcal{C}(K,X), \|f(t)\| = 1, \quad \forall t \in K\}.$$

As a consequence, if X is finite-dimensional and $1 + \dim K \leq \dim X$ or X is infinite-dimensional, then the above statement is satisfied. If we also assume that X is $\mathbb{C}$-rotund, then the subset of $\mathbb{C}$-extreme points in the unit ball of $\mathcal{C}(K,X)$ is a boundary for $\mathcal{A}_u(\mathcal{C}(K,X))$.

Last Section contains examples of spaces for which $\mathcal{A}_u(\mathcal{C}(K,X))$ does not have a minimal closed boundary. In the vector-valued case, we show under the same assumptions used in Section 3, that there are two closed boundaries disjoint for the subset of polynomials on Y which are weakly sequentially continuous on the unit ball of Y. As a consequence, if Y has also the Dunford-Pettis property, we obtain that there is no Shilov boundary for $\mathcal{A}_u(\mathcal{C}(K,X))$.

In the case of complex-valued functions, we can show the same result without any restriction on K. For $Y = \mathcal{C}(K)$ (any infinite compact K) we give examples of two closed boundaries whose intersection is empty. Therefore $\mathcal{A}_u(\mathcal{C}(K))$ has no a minimal closed boundary without any extra assumption on K.

§2. Holomorphic Functions Attaining Their Norms at Extreme Points

In the following, we will write B_X for the closed unit ball of a Banach space X, S_X for the unit sphere. If X is a complex Banach space, $\mathcal{A}_\infty(X)$ will be the Banach space of all functions $T : B_X \longrightarrow \mathbb{C}$ which are holomorphic in the open unit ball and continuous and bounded on the closed unit ball, endowed with the supremum norm. $\mathcal{A}_u(X)$ will be the Banach space of all functions in $\mathcal{A}_\infty(X)$ which are holomorphic in the open unit ball and uniformly continuous on the closed unit ball. A function $T \in \mathcal{A}_\infty(X)$ attains its norm if for some element x_0 in the unit ball of X, it holds that

$$|Tx_0| = \|T\|.$$

The following result is an abstract version of [CGKM, Lemma 2.6].

Lemma 2.1. *Let X be a complex Banach space and assume that the element $T \in \mathcal{A}_\infty(X)$ attains its norm at $x_0 \in B_X$. If for some $y \in X$ it is satisfied that*

$$\|x_0 + zy\| \leq 1, \quad \forall z \in \mathbb{C}, |z| \leq 1,$$

then $\|T\| = |T(x_0 + y)|$.

Proof. Let D be the open unit disk in $\mathbb{C}$ and consider the function $f : \overline{D} \longrightarrow \mathbb{C}$ given by

$$f(z) = T(x_0 + zy), \qquad (z \in \overline{D}).$$

Since $T \in \mathcal{A}_\infty(X)$, then f is holomorphic on D and continuous on $\overline{D}$, since f is the uniform limit of the sequence of functions

$$f_n(z) = T(r_n(x_0 + zy)) \quad (z \in \overline{D}),$$

where $\{r_n\}$ is a sequence in $]0, 1[$ converging to 1.

Since T attains its norm at x_0, then

$$\|T\| = |Tx_0| = |f(0)| \leq \max\{|f(z)| : z \in \overline{D}\} \leq \|T\|,$$

and, as a consequence of the maximum modulus principle, f is constant on $\overline{D}$, and so

$$|T(x_0 + y)| = |f(1)| = |f(0)| = \|T\|,$$

that is, T also attains its norm at $x_0 + y$. □

Now we use the same argument of [CGKM, Theorem 2.8] for the vector valued case. There is only a small difference: we skip an approximation argument used in the proof and apply directly the previous lemma.

Proposition 2.1. *Let K be a compact Hausdorff topological space. Assume that X is a complex Banach space and $T \in \mathcal{A}_\infty(\mathcal{C}(K,X))$ is a function attaining its norm at an element $f_0 \in B_{\mathcal{C}(K,X)}$ such that*

$$f_0(t) \neq 0, \qquad \forall t \in K.$$

If we define the continuous function g by

$$g(t) := \frac{f_0(t)}{\|f_0(t)\|} \qquad (t \in K),$$

then T also attains its norm at g.

Proof. We can assume that T is normalized. By the assumption we know that

$$\|f_0\| \leq 1, \quad 1 = \|T\| = |Tf_0|.$$

We will use the previous lemma for the functions f_0 and $g - f_0$ playing the role of x_0 and y. For an element $z \in \overline{D}$, we have that for any $t \in K$, it is satisfied

$$\begin{aligned}
&\|f_0(t) + z(g(t) - f_0(t))\| \leq \\
&\quad \leq \|f_0(t)\| + \|g(t) - f_0(t)\| = \\
&\quad = \|f_0(t)\| + \|f_0(t)\| \left| \frac{1}{\|f_0(t)\|} - 1 \right| = \\
&\quad = \|f_0(t)\| + (1 - \|f_0(t)\|) = 1.
\end{aligned}$$

We checked that $f_0 + z(g - f_0)$ is an element in the unit ball of $\mathcal{C}(K,X)$. By Lemma 2.1, T attains its norm at g, as we wanted to show. □

We will introduce a topological condition on the pair (K,X) in order that the assumption of the previous proposition is satisfied.

Definition 2.1 [BMN]**.** Let T be a topological space and X a normed space. We say that the pair (T,X) has the extension property if for every closed subset $C \subset T$, every function $f : C \longrightarrow S_X$ which is the restriction of a continuous function from T to the unit ball of X, admits a continuous extension $\tilde{f} : T \longrightarrow S_X$.

Theorem 2.1. *Let K be a compact Hausdorff topological space and X a complex Banach space such that (K,X) has the extension property. If $T \in \mathcal{A}_\infty(\mathcal{C}(K,X))$ attains its norm, then T attains its norm at a function g satisfying that*

$$\|g(t)\| = 1, \qquad \forall t \in K.$$

Proof. Since T attains its norm by assumption, by using the maximum modulus Theorem, we can assume that there is a function $f_0 \in S_{\mathcal{C}(K,X)}$ such that

$$1 = \|T\| = |Tf_0|.$$

In the case that $f_0(t) \neq 0$ for every t, the above proposition gives us the desired statement. Assume that $0 \in f_0(K)$. Define the subset

$$C := \left\{ t \in K : \|f_0(t)\| = \frac{1}{4} \right\}.$$

If C is not empty, let us consider the function $f : C \longrightarrow S_{\mathcal{C}(K,X)}$ given by

$$f(t) = 4f_0(t), \qquad \forall t \in C.$$

It is satisfied that f is the restriction to C of the continuous function from T to the closed unit ball of $\mathcal{C}(K,X)$ given by

$$\begin{array}{ll} t \mapsto 4f_0(t) & \text{if } \|f_0(t)\| \leq \dfrac{1}{4} \\ t \mapsto \dfrac{f_0(t)}{\|f_0(t)\|} & \text{if } \|f_0(t)\| > \dfrac{1}{4}. \end{array}$$

Since we are assuming that (K,X) has the extension property, and C is a closed set, there is a continuous function $\hat{f} : K \longrightarrow S_X$ such that

$$\hat{f}(t) = f(t), \qquad \forall t \in C.$$

Now we proceed as in [CGKM, Theorem 2.9], and define

$$h(t) = \begin{cases} f_0(t) & \text{if } \|f_0(t)\| \geq \dfrac{1}{4}, \\ \\ \dfrac{1}{4}\hat{f}(t) & \text{if } \|f_0(t)\| < \dfrac{1}{4} \end{cases} \qquad (t \in K).$$

Since $\hat{f}$ is a continuous extension of f to K, then h is continuous. $\hat{f}$ takes values on the unit sphere of $\mathcal{C}(K,X)$ and so, $0 \notin h(K)$. Finally, if z belongs to $\overline{D}$, we have that

$$\|f_0(t) + z(h(t) - f_0(t))\| \leq \begin{cases} \|f_0(t)\| & \text{if } \|f_0(t)\| \geq \dfrac{1}{4} \\ \dfrac{3}{4} & \text{if } \|f_0(t)\| < \dfrac{1}{4}, \end{cases}$$

and hence $\|f_0 + z(h - f_0)\| \leq 1$ for every $z \in \overline{D}$. Since T attains its norm at f_0, by Lemma 2.1, T also attains its norm at h, and now, it is sufficient to use Proposition 2.1 in order to get the announced statement.

If $C = \emptyset$, it is sufficient to fix an element x_0 in X with $\|x_0\| = \dfrac{1}{4}$ and define

$$h(t) = \begin{cases} f_0(t) & \text{if } \|f_0(t)\| > \dfrac{1}{4}, \\ \\ x_0 & \text{if } \|f_0(t)\| < \dfrac{1}{4} \end{cases} \qquad (t \in K).$$

By using the same argument as in the previous case, T attains its norm at the continuous function h and the use of Proposition 2.1 finishes the proof. □

Definition 2.2. Given a complex Banach space X, an element $x_0 \in B_X$ is called a $\mathbb{C}$-extreme point of B_X if it satisfies that

$$(y \in X, \quad \|x_0 + zy\| \leq 1, \quad \forall z \in \mathbb{C}, \ |z| \leq 1) \ \Rightarrow \ y = 0$$

A Banach space X is called $\mathbb{C}$-rotund if all the points in the unit sphere of X are $\mathbb{C}$-extreme points of the unit ball of X.

Since a continuous function $f : K \longrightarrow X$ satisfying that for every $t \in K$, $f(t)$ is a $\mathbb{C}$-extreme point of the unit ball of X, is a $\mathbb{C}$-extreme point of the unit ball of $\mathcal{C}(K, X)$, then, by using the previous theorem we obtain the following result:

Corollary 2.1. *Let K be a compact topological space and X a complex Banach space. Assume that (K, X) has the extension property and X is $\mathbb{C}$-rotund. If $T \in \mathcal{A}_\infty(\mathcal{C}(K, X))$ attains its norm, then T attains its norm at a $\mathbb{C}$-extreme point of $B_{\mathcal{C}(K,X)}$.*

Examples of pairs (K, X) satisfying the extension property are the following:

- (K, X), for every infinite-dimensional Banach space X [Dug, Theorem 6.2].
- In the case that X is finite-dimensional, the pair (K, X) has the extension property if, and only if, $1 + \dim K \leq \dim X$, where $\dim K$ is the covering dimension of the topological space K (see [Smi, Theorem 9t]).

Let us mention that scattered compact topological spaces are 0-dimensional [PeSe, Theorem 2, p. 214]. Since $\mathbb{C}$ is $\mathbb{C}$-rotund and in fact all the points in the unit sphere are extreme points, the previous result generalizes the scalar version given in [CGKM, Theorem 1]. If we do not assume any restriction on

K, the statement in Corollary 2.1 is not true. For instance, if K is the closed unit disk of $\mathbb{C}$, the subset

$$E = \{f \in \mathcal{C}(K) : |f(t)| = 1, \ \forall t \in K\}$$

does not even satisfy that every norm attaining functional on $\mathcal{C}(K)$ attains its norm at an element of E (see [Ai, Example 5]). In fact, Aizpuru showed that if K is metrizable and every norm attaining functional on $\mathcal{C}(K, X)$ attains its norm at an extreme point of the unit ball, then the pair (K, X) has the extension property [Ai, Theorem 7]. Therefore, Corollary 2.1 can be read, in fact, as a characterization in the case that the compact is metrizable.

§3. Holomorphic Functions for Which the Subset of Extreme Points of the Unit Ball is a Boundary

In the previous section, in order to apply the Maximum modulus Principle (proof of Proposition 2.1), it is essential that the holomorphic mapping attains its norm. Here we will use a different perturbation in order to assert that the subset of functions $f \in \mathcal{C}(K, X)$ satisfying that

$$\|f(t)\| = 1, \quad \forall t \in K$$

are enough to compute the norm of any element in $\mathcal{A}_\infty(\mathcal{C}(K, X))$.

Lemma 3.1. *For every $\lambda \in \mathbb{C}$ satisfying $0 < |\lambda| < 1$, the complex-valued mapping given by*

$$h(z) = \frac{z + \lambda}{1 + \overline{\lambda} z} \qquad \left(z \in \mathbb{C}, \ |z| < \frac{1}{|\lambda|}\right)$$

is a holomorphic mapping satisfying the following conditions:

i) $h(0) = \lambda$,

ii) $|z| < 1 \Rightarrow |h(z)| < 1$,

iii) $|z| = 1 \Rightarrow |h(z)| = 1$,

iv) $h(z) = (\overline{\lambda})^{-1} + \big(\lambda - (\overline{\lambda})^{-1}\big) \sum_{n=0}^{\infty} (-1)^n (\overline{\lambda} z)^n \quad \left(|z| < \frac{1}{|\lambda|}\right)$.

Proof. h is just the restriction of a Möbius transformation that is holomorphic on the open disk of radius $\frac{1}{|\lambda|}$ and clearly satisfies (i). Since

$$1 = |h(1)| = |h(-1)| = |h(i)|,$$

then h preserves the unit sphere. Also $|h(0)| = |\lambda| < 1$, and so h preserves the open unit disk.

Finally, for $|z| < \frac{1}{|\lambda|}$, the Taylor series of h at zero is given by

$$\begin{aligned} h(z) &= (\overline{\lambda})^{-1} + \Big(\frac{|\lambda|^2 - 1}{\overline{\lambda}}\Big)\frac{1}{1+\overline{\lambda}z} = \\ &= (\overline{\lambda})^{-1} + \Big(\lambda - (\overline{\lambda})^{-1}\Big)\sum_{n=0}^{\infty}(-1)^n(\overline{\lambda}z)^n. \end{aligned}$$

□

Globevnik introduced the definition of boundary for the noncompact case.

Definition 3.1. Let $\mathcal{A} \subset \mathcal{A}_\infty(X)$ be a subset, we will say that $\mathcal{B} \subset B_X$ is a boundary for $\mathcal{A}$ if for every $F \in \mathcal{A}$ it is satisfied that

$$\sup_{x \in B_X} |F(x)| := \|F\| = \sup_{x \in \mathcal{B}} |F(x)|.$$

The Shilov boundary for $\mathcal{A}$ is a boundary for $\mathcal{A}$ which is closed and minimal under these two conditions.

Theorem 3.1. *Let X be a complex Banach space, $Y = \mathcal{C}(K, X)$ and assume that the subset*

$$\{f \in B_{\mathcal{C}(K,X)} : f(t) \neq 0, \ \forall t \in K\}$$

is a boundary for $\mathcal{A}_\infty(Y)$. Then the subset of elements f in $\mathcal{C}(K, X)$ satisfying that

$$\|f(t)\| = 1, \ \forall t \in K$$

is also a boundary for $\mathcal{A}_\infty(Y)$. The same statement also holds for $\mathcal{A}_u(Y)$.

Proof. Let $F \in \mathcal{A}_\infty(Y)$ and $\varepsilon > 0$. By assumption there is a function $f \in B_Y$ such that

$$f(t) \neq 0, \ \forall t \in K \quad \text{and} \quad |F(f)| > \|F\| - \varepsilon.$$

Since F is continuous we can also assume that $r := \|f\| < 1$.

We define the mapping $G : \overline{D} \longrightarrow Y$ given by

$$G(z)(t) = \frac{z + \|f(t)\|}{1 + \|f(t)\|z} \, \frac{f(t)}{\|f(t)\|} \qquad (|z| \leq 1, t \in K)$$

$G(0) = f$, G is continuous and, in fact, by using Lemma 3.1, we know that

$$G(z)(t) = \frac{f(t)}{\|f(t)\|^2} + \Big(1 - \frac{1}{\|f(t)\|^2}\Big) \sum_{n=0}^{\infty} \|f(t)\|^n (-1)^n z^n f(t), \qquad \forall |z| \leq 1.$$

Since $\|f(t)\| \leq r < 1$ for every t, the above series converges uniformly on the closed unit disk and so, G is holomorphic on the open disk and continuous on $\overline{D}$, and also, by Lemma 3.1, satisfies

$$\|G(z)(t)\| = \Big| \frac{z + \|f(t)\|}{1 + \|f(t)\| z} \Big| < 1, \qquad \forall t \in K, |z| < 1,$$

that is, G applies the open unit disk on the open unit ball of Y.

We consider the composition $H : \overline{D} \longrightarrow \mathbb{C}$ given by

$$H(z) = F(G(z)) \qquad (|z| \leq 1).$$

Since G is holomorphic on D, $G(D)$ is contained in the open unit ball of Y and $F \in \mathcal{A}_\infty(Y)$, then H is holomorphic on D. Also H is continuous on the closed unit disk.

The maximum modulus of H on the closed unit disk is attained at some element z_0 in the unit sphere, and so,

$$\|F\| - \varepsilon < |F(f)| = |H(0)| \leq |H(z_0)| = |F(G(z_0))|.$$

Finally, let us observe that in view of Lemma 3.1, the function $G(z_0)$ verifies that

$$\|G(z_0)(t)\| = \Big| \frac{z_0 + \|f(t)\|}{1 + \|f(t)\| z_0} \Big| = 1, \qquad \forall t \in K,$$

and so the set of elements in $\mathcal{C}(K, X)$ such that every evaluation has norm one is a boundary for $\mathcal{A}_\infty(Y)$.

The same proof also works for $\mathcal{A}_u(Y)$ □

Corollary 3.1. *Assume that K is a compact topological space and X is a $\mathbb{C}$-rotund Banach space, such that the set of continuous functions from K to X that do not vanish is dense in $\mathcal{C}(K, X)$. Then the subset of $\mathbb{C}$-extreme points of $B_{\mathcal{C}(K,X)}$ is a boundary for $\mathcal{A}_\infty(\mathcal{C}(K, X))$.*

We mentioned before that scattered compact are 0-dimensional. Compact intervals of the real line are 1-dimensional. In both cases, by the results mentioned at the end of the previous section, the pair $(K, \mathbb{C})$ has the extension property. This condition implies the denseness in $\mathcal{C}(K)$ of the set of continuous functions that do not vanish (see [BMN, Lemma 7]). Hence, by using a simpler proof, we obtained an improvement of [CGKM, Theorem 3.3]:

Corollary 3.2. *For $K = [a, b] \subset \mathbb{R}$ or for a scattered compact topological space, the subset of extreme points in $\mathcal{C}(K)$ is a boundary for $\mathcal{A}_\infty(C(K))$.*

As we mentioned before, if X is finite-dimensional, the pair (K, X) has the extension property if, and only if,

$$1 + \dim K \leq \dim X.$$

If X is infinite-dimensional, the assumption of the denseness of the set of continuous functions not vanishing is satisfied, since, for any $f \in \mathcal{C}(K, X)$, $f(K)$ is a compact subset and it is sufficient to choose an element $x_0 \in \varepsilon B_X \backslash f(K)$ and define $g := f - x_0$. The function g does not vanish and $\|f - g\| \leq \varepsilon$.

In any of the previous cases, the set of continuous functions from K to X that do not vanish is dense in $\mathcal{C}(K, X)$ [BMN, Lemma 7]. Therefore, by the above comments, Theorem 3.1 and Corollary 3.1 we obtain:

Corollary 3.3. *Let X be a (complex) Banach space and K a compact Hausdorff topological space. Assume that one of the following conditions is satisfied:*

i) *X is finite-dimensional and $1 + \dim K \leq \dim X$.*

ii) *X is infinite-dimensional.*

Then, for every $F \in \mathcal{A}_\infty(\mathcal{C}(K, X))$, it is satisfied that

$$\|F\| = \sup\{|F(f)| : f \in \mathcal{C}(K, X), \|f(t)\| = 1 \ \forall t \in K\}.$$

As a consequence, if we assume also that X is $\mathbb{C}$-rotund, then

$$\|F\| = \sup\{|F(f)| : f \text{ is a } \mathbb{C}\text{-extreme point in } B_{\mathcal{C}(K,X)}\},$$

for any $F \in \mathcal{A}_\infty(\mathcal{C}(K, X))$.

In fact, as a consequence of a beautiful result due to Harris (see [Ha2, Proposition 2 and Theorem 9]), if X is a C^*-algebra, then the space $Y = C(K, X)$ satisfies the statement given in Corollary 3.3, without any restriction on the compact topological space K. Therefore, the above result holds, for instance, if we take $\mathbb{C}^n$ (endowed with the maximum norm) as X. On the other hand, the result appearing in Corollary 3.3 for infinite-dimensional spaces does not require any special algebraic structure in X.

§4. There is No Shilov Boundary for $\mathcal{A}_u(\mathcal{C}(K))$

For the kind of spaces we are considering, first Globevnik proved that there is no Shilov boundary for $\mathcal{A}_u(c_0)$ [Glo, Theorem 1.8]. Aron, Choi, Lourenço and Paques proved the same result for ℓ_∞ [ACLP, Theorems 1 and 3]. Choi, García, Kim and Maestre showed that for any infinite scattered compact K, the Shilov boundary for $\mathcal{A}_u(\mathcal{C}(K))$ does not exist [CGKM, Theorem 3.4]. We will prove that under the assumptions of Corollary 3.3, the same result also holds in a more general setting.

Theorem 4.1. *Assume that K is an infinite compact Hausdorff topological space and $X \neq 0$ is a complex Banach space. Suppose that one of the following conditions is satisfied*:

1) $1 + \dim K \leq \dim X$.

2) *X is infinite-dimensional.*

Then there is no Shilov boundary for $\mathcal{P}_{wsc}(\mathcal{C}(K,X))$, the subset of complex-valued polynomials on $\mathcal{C}(K,X)$ that are weakly sequentially continuous.

Proof. Let us write $Y = \mathcal{C}(K,X)$. Consider the set C given by

$$C = \{\delta_t \otimes x^* : t \in K, x^* \in B_{X^*}\},$$

that is a subset of Y^*, acting as

$$(\delta_t \otimes x^*)(y) = x^*(y(t)) \qquad (y \in Y).$$

It is clear that the subset C is norming for $\mathcal{C}(K,X)$ and it is weak*-closed. We check the last assertion; if we assume that a net $\{\delta_{t_\lambda} \otimes x^*_\lambda\}$ of elements in C is w^*-convergent to y^*, then for every element $x \in X$, by applying the above net to the function in $\mathcal{C}(K,X)$ constant and equal to x, then

$$y^*(x) = \lim_\lambda\, (\delta_{t_\lambda} \otimes x^*_\lambda)(x) = \lim_\lambda x^*_\lambda(x).$$

As a consequence, the net $\{x^*_\lambda\}$ converges in the w^*-topology to an element x^* in the unit ball of X^*. Finally, if we assume that $x^* \neq 0$ (otherwise our argument will give $y^* = 0$), and we choose $x_0 \in X$ satisfying that $x^*(x_0) = 1$, then for any $f \in \mathcal{C}(K)$, we consider the element $fx_0 \in Y$, then

$$y^*(fx_0) = \lim_\lambda\, (\delta_{t_\lambda} \otimes x^*_\lambda)(fx_0) = \lim_\lambda f(t_\lambda)$$

that is, the net $\{t_\lambda\}$ converges in K to an element t. Since $\{\delta_{t_\lambda}\} \overset{w^*}{\to} \delta_t$ in $\mathcal{C}(K)^*$ and $\{x^*_\lambda\} \overset{w^*}{\to} x^*$ in X^*, we know that for any $f \in \mathcal{C}(K)$ and $x \in X$ it is satisfied

$$y^*(fx) = \lim_\lambda (\delta_{t_\lambda} \otimes x^*_\lambda)(fx) = f(t)x^*(x).$$

Since $Y = \mathcal{C}(K, X) = \mathcal{C}(K) \hat{\otimes}_\varepsilon X$ [DeFl, p. 48], the above convergence implies that $y^* = \delta_t \otimes x^*$, since the subspace generated by

$$\{fx : f \in \mathcal{C}(K), x \in X\}$$

is dense in $\mathcal{C}(K) \hat{\otimes}_\varepsilon X$. Until now we checked that C is weak*-closed in Y^*, C is norming and so, by the reversed Krein-Milman Theorem, C contains the subset of extreme points of B_{Y^*}.

Since K is infinite, then $\mathcal{C}(K)$ contains an isometric copy of c_0. In fact, there is a sequence $\{t_n\}$ in the compact K and a sequence of functions $\{f_n\}$ in $B_{\mathcal{C}(K)}$ such that

$$0 \le f_n \le 1, \quad f_n(t_n) = 1, \quad \text{and} \quad \operatorname{supp} f_n \cap \operatorname{supp} f_m = \emptyset \quad \text{for} \quad m \ne n.$$

In such a case, the closed linear span of $\{f_n : n \in \mathbb{N}\}$ is isometric to c_0 and $\{f_n\}$ is equivalent to the usual Schauder basis of c_0, hence $\{f_n\} \overset{w}{\to} 0$.

If we fix a bounded sequence $\{z_n\}$ in Y, and we define

$$y_n(t) = f_n(t) z_n(t) \qquad (t \in K),$$

then we will check that $\{y_n\}$ is a weak-null sequence in $\mathcal{C}(K, X)$. Since $\{f_n\} \overset{w}{\to} 0$ in $\mathcal{C}(K)$, for any bounded sequence $\{z_n\}$ in Y we know that

$$\{(\delta_t \otimes x^*)(y_n)\} = \{f_n(t)\ x^*(z_n(t))\} \to 0, \qquad \forall t \in K,\ \forall x^* \in B_{X^*}.$$

In view of Rainwater's Theorem [Die, p. 155], this implies that $\{y_n\}$ is a weakly-null sequence in Y.

Let us consider the sets

$$\mathcal{N} = \{g \in B_Y :\ g(t_n) = 0 \ \text{ for some } \ n\},$$
$$\mathcal{B} = \{h \in B_Y :\ \|h(t)\| = 1,\ \forall t \in K\}.$$

We know that $\mathcal{B}$ is a boundary for $\mathcal{A}_\infty(Y)$ (Corollary 3.3), and as a consequence, it is also a boundary for $\mathcal{P}_{wsc}(Y)$. It is clear that

$$\|g - h\| \ge 1, \qquad \forall g \in \mathcal{N}, h \in \mathcal{B}$$

and so $\overline{\mathcal{N}} \cap \mathcal{B} = \emptyset$. We will show that $\mathcal{N}$ is also a boundary for $\mathcal{P}_{wsc}(Y)$. Since $\mathcal{B}$ is a boundary for $\mathcal{P}_{wsc}(Y)$ and the elements in $\mathcal{P}_{wsc}(Y)$ are weakly sequentially continuous, then we consider an element $h \in \mathcal{B}$. The sequence $\{g_n\}$ given by

$$g_n(t) = h(t)(1 - f_n(t)) \qquad (t \in K)$$

satisfies that $\|g_n\| \leq 1$ since $0 \leq f \leq 1$ and $\|h\| \leq 1$ and also $g_n(t_n) = 0$, so $g_n \in \mathcal{N}$. On the other hand, we know that $\{g_n\} \overset{w}{\to} h$ and so, for any $P \in \mathcal{P}_{wsc}(Y)$, it is satisfied that

$$\{P(g_n)\} \to P(h).$$

Therefore, $\mathcal{B}$ and $\overline{\mathcal{N}}$ are closed boundaries satisfying that $\mathcal{B} \cap \overline{\mathcal{N}} = \emptyset$. □

In the case that the Banach space Y has the Dunford-Pettis property, it also has the polynomial Dunford-Pettis property ([Ry] or [Din, Proposition 2.34]). This means that under this assumption $\mathcal{P}_{wsc}(Y)$ coincides with all polynomials. Examples of spaces having the Dunford-Pettis property are $\mathcal{C}(K_1, \mathcal{C}(K_2)), \mathcal{C}(K, L_1(\mu))$ and $\mathcal{C}(K, X)$, where X is a space having the Schur property (see [Die1, p. 47 and 48]).

By using that any function $F \in \mathcal{A}_\infty(Y)$ is the uniform limit of polynomials on any ball rB_Y $(0 < r < 1)$, we obtain the following result.

Corollary 4.1. *Let K be an infinite compact topological space and let Y be one of the following spaces*:

a) $\mathcal{C}(K, \mathcal{C}(K_1))$, *for a compact topological space* K_1.

b) $\mathcal{C}(K, L_1(\mu))$, *where μ is any measure.*

c) $\mathcal{C}(K, X)$, *where X is a finite-dimensional space and* $1 + \dim K \leq \dim X$.

Then there is no Shilov boundary for $\mathcal{A}_\infty(Y)$.

In the case that the continuous functions are $\mathbb{C}$-valued, then we will get that the set of extreme points of the unit ball is a boundary for $\mathcal{A}_\infty(\mathcal{C}(K, \mathbb{C}))$ without any restriction on K, which improves the result that we obtained in Section 3.

Theorem 4.2. *For any compact topological space K, the set of extreme points in the unit ball of $\mathcal{C}(K)$ is a boundary for $\mathcal{A}_\infty(\mathcal{C}(K))$.*

Proof. We will use a similar trick to the one appearing in the proof of Theorem 4.1, which is based on a idea by Harris (see [Ha1] or [Ha2, Example 1, §3]). For $F \in \mathcal{A}_\infty(\mathcal{C}(K))$ and $\varepsilon > 0$, we choose a function $F \in B_{\mathcal{C}(K)}$ such that

$$|F(f)| > \|F\| - \varepsilon.$$

Since F is continuous on the closed unit ball, we can assume that $\|f\| < 1$.

We consider the holomorphic function $g : \overline{D}(0,1) \longrightarrow \mathcal{C}(K)$ given by

$$g(z)(t) = \frac{z + f(t)}{1 + \overline{f(t)}z} \qquad (|z| \leq 1, t \in K),$$

which is well-defined, holomorphic on $D(0,1)$ and continuous on the closed unit disk. By Lemma 3.1, we have $g(\overline{D}(0,1)) \subset B_{\mathcal{C}(K)}$ and $g(D(0,1))$ is contained in the open unit ball of $\mathcal{C}(K)$. Hence, the function $H : \overline{D}(0,1) \longrightarrow \mathbb{C}$ given by

$$H(z) = F(g(z)) \qquad (|z| \leq 1)$$

is holomorphic in the open unit disk and continuous on $\overline{D}(0,1)$. By the Maximum modulus Principle, there is $z_0 \in \mathbb{C}$ with $|z_0| = 1$ such that

$$|H(z_0)| \geq |H(z)|, \qquad \forall z \in \overline{D}(0,1).$$

Since $H(0) = F(g(0)) = F(f)$, then

$$|F(g(z_0))| = |H(z_0)| \geq |H(0)| = |F(f)| > \|F\| - \varepsilon.$$

Since $|z_0| = 1$, then, by Lema 3.1, $|g(z_0)(t)| = 1$ for every $t \in K$ and $|F(g(z_0))| \geq \|F\| - \varepsilon$, then $g(z_0)$ is an extreme point of $B_{\mathcal{C}(K)}$ and we proved that the set of extreme points in the unit ball of $\mathcal{C}(K)$ is a boundary for $\mathcal{A}_\infty(\mathcal{C}(K))$. □

Proposition 4.1. *If $\mathcal{B}, \mathcal{S} \subset B_{\mathcal{C}(K)}$, $\mathcal{B}$ is a boundary for $\mathcal{A}_\infty(\mathcal{C}(K))$ and $\mathcal{S}$ is weakly sequentially dense in $\mathcal{B}$ and balanced, then $\mathcal{S}$ is also a boundary. As a consequence, if K is infinite, there are two closed boundaries for $\mathcal{A}_\infty(\mathcal{C}(K))$ whose intersection is empty, hence there is no Shilov boundary. The same statements also hold for $\mathcal{A}_u(\mathcal{C}(K))$.*

Proof. Let us fix $F \in \mathcal{A}_\infty(\mathcal{C}(K))$ and $\varepsilon > 0$. Since $\mathcal{B}$ is a boundary for $\mathcal{A}_\infty(\mathcal{C}(K))$, there is $f \in \mathcal{B}$ such that $|F(f)| > \|F\| - \varepsilon$. Since $\mathcal{S}$ is weakly sequentially dense in $\mathcal{B}$, we can find a sequence $\{g_n\} \xrightarrow{w} f$ such that $g_n \in \mathcal{S}$, for each n. Since F is continuous, there is $\delta > 0$ such that

$$g \in B_{\mathcal{C}(K)}, \ \|g - f\| \leq \delta \quad \Rightarrow \quad |F(g) - F(f)| < \varepsilon.$$

By using that F is holomorphic in the open ball of $\mathcal{C}(K)$, there is a continuous polynomial P on $\mathcal{C}(K)$ such that

$$|P(g) - F(g)| \leq \varepsilon, \qquad \forall g \in (1-\delta)B_{\mathcal{C}(K)}.$$

Since $\mathcal{C}(K)$ has the Dunford-Pettis property, it also has the polynomial Dunford-Pettis property ([Ry] or [Din, Proposition 2.34]). Hence, for n large enough we will have

$$|P((1-\delta)g_n) - P((1-\delta)f)| < \varepsilon.$$

Finally, for n large enough we obtain

$$\begin{aligned} |F(f) - F((1-\delta)g_n)| &\leq |F(f) - F((1-\delta)f)| + \\ &+ |F((1-\delta)f) - P((1-\delta)f)| + |P((1-\delta)f) - P((1-\delta)g_n)| + \\ &+ |P((1-\delta)g_n) - F((1-\delta)g_n)| \leq 4\varepsilon. \end{aligned}$$

Hence

$$|F((1-\delta)g_n)| \geq |F(f)| - 4\varepsilon \geq \|F\| - 5\varepsilon,$$

for n large enough and so by the Maximum modulus Theorem there is a scalar λ_n with $|\lambda_n| = 1$, satisfying that $|F(\lambda_n g_n)| \geq \|F\| - 5\varepsilon$. Since $g_n \in \mathcal{S}$ and $\mathcal{S}$ is balanced, then $\lambda_n g_n \in \mathcal{S}$ and we proved that $\mathcal{S}$ is also a boundary for $\mathcal{A}_\infty(\mathcal{C}(K))$.

We proved in Theorem 4.2 that

$$\mathcal{B} = \{f \in \mathcal{C}(K) : |f(t)| = 1, \qquad \forall t \in K\}$$

is a (closed) boundary for $\mathcal{A}_\infty(\mathcal{C}(K))$. For any infinite compact K, we can follow the same argument appearing in the proof of Theorem 4.1 by fixing a sequence $\{f_n\}$ in $\mathcal{C}(K)$ which is equivalent to the c_0-basis, satisfying that $f_n(t_n) = 1$ for some $t_n \in K$ and $0 \leq f_n \leq 1$. The subset

$$\mathcal{S} = \{g \in B_{\mathcal{C}(K)} : g(t_n) = 0 \text{ for some } n\}$$

is balanced and weakly sequentially dense in $\mathcal{B}$, since for any $h \in \mathcal{B}$, the sequence $\{h(1-f_n)\}$ converges weakly to h and $h(1-f_n) \in \mathcal{S}$. By the assertion we proved before, $\mathcal{S}$ is also a boundary for $\mathcal{A}_\infty(\mathcal{C}(K))$. It is satisfied that $\overline{\mathcal{S}} \cap \mathcal{B} = \emptyset$ since $\|h - g\| \geq 1$ for every $h \in \mathcal{B}, g \in \mathcal{S}$.

The same proof also works for $\mathcal{A}_u(\mathcal{C}(K))$. □

Aron, Choi, Lourenço and Paques proved that $\mathcal{A}_\infty(\ell_\infty)$ has no Shilov boundary [ACLP, Proposition 4]. We followed their scheme to obtain the same result for every $\mathcal{C}(K)$ instead of ℓ_∞.

Remark. By modifying a little bit the above argument, it can be shown that any subset $\mathcal{S}$ which is weakly sequentially dense in a boundary of $\mathcal{A}_u$ $(\mathcal{C}(K))$, is also a boundary of the same algebra.

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