Publ. RIMS, Kyoto Univ.
42 (2006), 323–359

Computations of Nambu-Poisson Cohomologies: Case of Nambu-Poisson Tensors of Order 3 on $\mathbb{R}^4$

By

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Abstract

We compute Nambu-Poisson cohomology for Nambu-Poisson tensors of order three which are defined on $\mathbb{R}^4$. In particular, we prove that Nambu-Poisson cohomology of exact Nambu-Poisson tensors is equivalent to relative cohomology.

§1. Introduction

A Nambu-Poisson structure was given by L. Takhtajan [14] in 1994 in order to extend Nambu mechanics defined on $\mathbb{R}^3$ to Nambu-Poisson mechanics defined on an n-dimensional manifold, $n \geq 3$. One of the main objects of Nambu-Poisson geometry is to study Nambu-Poisson cohomology and its related topics. The notion of Nambu-Poisson cohomology was first introduced by R. Ibáñez *et al.* [7], and it is an extension of Poisson cohomology (or Lichnerowicz-Poisson cohomology) on a Poisson manifold. Let (M, η) be an m-dimensional Nambu-Poisson manifold. (See Definition 2.1 for the precise definition.) Whenever we mention a Nambu-Poisson manifold, m is assumed to be $m \geq 3$. Then a Nambu-Poisson tensor η defines the so-called *characteristic foliation*, which is, in general, a singular foliation on M. In case that η is a Nambu-Poisson tensor, then the set of *Hamiltonian vector fields* becomes a Lie subalgebra of $\chi(M)$,

Communicated by K. Saito. Received February 2004. Revised July 29, 2004, December 9, 2004.
2000 Mathematics Subject Classification(s): 53D, 17.
Key words: Nambu-Poisson tensor, Nambu-Poisson cohomology, relative cohomology.
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the Lie algebra of all vector fields on M. This Lie subalgebra will be denoted by $\mathcal{H}$.

Let $\Omega^k(M)$ be the space of k-forms on M, and let the order of η be n. (i.e. $\eta \in \Gamma(\Lambda^n TM)$, where $\Gamma(\Lambda^n TM)$ is the space of cross-sections $M \longrightarrow \Lambda^n TM$.) Here $m \geq n \geq 3$, and $n \geq k$. We define a mapping

$$\sharp_k : \Omega^k(M) \longrightarrow \Gamma(\Lambda^{n-k} TM)$$

by $\sharp_k(\alpha) = i(\alpha)\eta$ for $\alpha \in \Omega^k(M)$. If $k = n-1$, $\Omega^{n-1}(M)$ has a structure of Leibniz algebra, which is defined by

$$\{\alpha, \beta\} = \mathcal{L}_{\sharp_{n-1}(\alpha)}\beta + (-1)^n \sharp_n(d\alpha)\beta, \ \ \alpha, \beta \in \Omega^{n-1}(M),$$

where $\mathcal{L}$ stands for the Lie derivative. The image of $\sharp_{n-1}$, which is denoted by $\mathfrak{g}$, becomes a Lie subalgebra of $\chi(M)$. (See Proposition 3.1 and its explanation.) It is clear that $\mathcal{H}$ is contained in $\mathfrak{g}$. Nambu-Poisson cohomology is a cohomology group of a Lie algebra $\mathfrak{g}$ having $C^\infty(M, \mathbb{R})$ as its representation space, which is also called Chevalley-Eilenberg cohomology of $\mathfrak{g}$. It will be denoted by H^*_{NP}. It is easy to see that H^0_{NP} is equal to the space of invariant functions of $\mathfrak{g}$. Moreover H^1_{NP} is deeply related to the *modular class* of (M, η) [7]. It will be expected that other cohomologies H^*_{NP} have also some geometric meanings.

If η does not vanish anywhere on M, it is said to be *regular*. Then R. Ibáñez *et al.* computed Nambu-Poisson cohomology of a regular Nambu-Poisson manifold (M, η) [7]. If η has some singularities, it is quite difficult to compute its Nambu-Poisson cohomology. As an example of a singular Nambu-Poisson manifold, they also considered $(\mathbb{R}^3, \eta = (x^2+y^2+z^2)\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y} \wedge \frac{\partial}{\partial z})$, and they proved that the first Nambu-Poisson cohomology group $H^1_{NP}(\mathbb{R}^3, \eta)$ is isomorphic to $\mathbb{R}$.

On the other hand, P. Monnier [9] computed Nambu-Poisson cohomology for germs at 0 of n-vectors $\eta = f\frac{\partial}{\partial x_1} \wedge \cdots \wedge \frac{\partial}{\partial x_n}$ on $\mathbb{K}^n (\mathbb{K} = \mathbb{R}$ or $\mathbb{C})$, with the assumption that f is a quasihomogeneous polynomial of finite codimension. His results contain the result of R. Ibáñez *et al.*, (at least in the formal case).

As the next step, it is natural to consider the case that the order of a Nambu-Poisson tensor η is smaller than the dimension of a space on which η is defined. In the present paper, along this concept, we will compute Nambu-Poisson cohomology for the following three cases.

(a) Exact Nambu-Poisson tensors η of order 3 defined on $\mathbb{R}^4(x, y, z, u)$. A Nambu-Poisson tensor η is called *exact* if there is a function f such that $i(\eta)\Omega = df$ for $\Omega = dx \wedge dy \wedge dz \wedge du$.

(b) Linear Nambu-Poisson tensors of order 3 defined on $\mathbb{R}^4(x, y, z, u)$.

(c) A quadratic Nambu-Poisson tensor $\eta = (x^2 + y^2 + z^2 + u^2)\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y} \wedge \frac{\partial}{\partial z}$ of order 3 defined on $\mathbb{R}^4(x, y, z, u)$.

The computation for the case (a) naturally leads us to the notion of *relative cohomology* which was studied by C. A. Roche [13]. In this case, we know that $H^k_{NP} = H^k_{rel}$ for $0 \leq k \leq 2$. In computing Nambu-Poisson cohomology of the case (b), we will use the classification theorem of linear Nambu-Poisson tensors which was proved by J-P. Dufour and N. T. Zung [3]. A part of this case is also discussed in (a). In treating the case (c), we will take advantage of the results of P. Monnier [9].

Here we computed Nambu-Poisson cohomology only for the case $(\mathbb{R}^4, \eta)$, where the order of η is three. But it is not so difficult to extend all the results we have obtained here to more general situations. In fact let us consider a Nambu-Poisson manifold $(\mathbb{R}^n, \eta)$, where the order of η is n'. We can easily see that if $n - n' > 1$, then spaces of cohomologies are, in general, greater than those of the case $n - n' = 1$. This is because that the space of $\mathfrak{g}$-invariant functions becomes greater if $n - n' > 1$.

The author would like to express his deep thanks to Professors T.Fukuda, H. Sato, Y. Agaoka and G. Ishikawa for helpful and stimulating discussions. He is also grateful to P. Monnier for valuable communications.

§2. Reviews of Nambu-Poisson Manifolds

We will review some useful results of geometry of Nambu-Poisson manifolds. Details are referred to [7],[10] and [14]. Let M be an m-dimensional C^∞-manifold, and $\mathcal{F}$ its algebra of real valued C^∞-functions on M. We denote by $\Gamma(\Lambda^n TM)$ the space of global cross-sections $\eta : M \longrightarrow \Lambda^n TM$. Then for each $\eta \in \Gamma(\Lambda^n TM)$, there corresponds the bracket defined by

$$\{f_1, ..., f_n\} = \eta(df_1, ..., df_n), \quad f_1, ..., f_n \in \mathcal{F}.$$

This bracket operation is an n-linear skew-symmetric map from $\mathcal{F}^n$ to $\mathcal{F}$ which satisfies the Leibniz rule:

$$\{f_1, ..., f_{n-1}, g_1 \cdot g_2\} = \{f_1, ..., f_{n-1}, g_1\} \cdot g_2 + g_1 \cdot \{f_1, ..., f_{n-1}, g_2\},$$

for all $f_1, ..., f_{n-1}, g_1, g_2 \in \mathcal{F}$.

Let $A = \sum f_{i_1} \wedge \cdots \wedge f_{i_{n-1}}$, $f_{i_j} \in \mathcal{F}$. Since the bracket operation satisfies the Leibniz rule, we can define a vector field X_A corresponding to A by the following equation:

$$X_A(g) = \sum \{f_{i_1}, ..., f_{i_{n-1}}, g\}, \quad g \in \mathcal{F}.$$

Such a vector field is called a *Hamiltonian vector field.* The space of Hamiltonian vector fields is denoted by $\mathcal{H}$.

Definition 2.1. $\eta \in \Gamma(\Lambda^n TM)$ is called a Nambu-Poisson tensor of order n if it satisfies $\mathcal{L}_{X_A}\eta = 0$ for all $X_A \in \mathcal{H}$, where $\mathcal{L}$ is the Lie derivative. Then a Nambu-Poisson manifold is a pair (M, η).

Let $\eta(p) \neq 0,\ p \in M$. Then we say that η is *regular* at p. Now we can state the following local structure theorem for Nambu-Poisson tensors [5],[10].

Theorem 2.1. *Let $\eta \in \Gamma(\Lambda^n TM),\ n \geq 3$. If η is a Nambu-Poisson tensor of order n, then for any regular point p, there exists a coordinate neighborhood U with local coordinates $(x_1, ..., x_n, x_{n+1}, ..., x_m)$ around p such that*

$$\eta = \frac{\partial}{\partial x_1} \wedge \cdots \wedge \frac{\partial}{\partial x_n}$$

on U, and vice versa.

Let (M, η) be a Nambu-Poisson manifold with volume form Ω, and $m \geq n \geq 3$. Put $\omega = i(\eta)\Omega$, where the right hand side is the interior product of η and Ω. Hence ω is an $(m-n)$-form. The following theorem gives a necessary and sufficient condition for η to be a Nambu-Poisson tensor. For the proof, see [11].

Theorem 2.2. *Let $\eta \in \Gamma(\Lambda^n TM)$. Then η is a Nambu-Poisson tensor if and only if η satisfies the following two conditions around each regular point*:

(a) *ω is (locally) decomposable, and*

(b) *there exists a locally defined 1-form θ such that $d\omega = \theta \wedge \omega$.*

§3. Nambu-Poisson Cohomology

Let (M, η) be a Nambu-Poisson manifold of order n and let k be an integer with $k \leq n$. Denote by $\Omega^k(M)$ the space of k-forms on M. If $\Lambda^k(T^*M)$ (respectively, $\Lambda^{n-k}(TM)$) denotes the vector bundle of the k-forms (respectively, $(n-k)$-vectors) then η induces a homomorphism of vector bundles $\sharp_k : \Lambda^k(T^*M) \to \Lambda^{n-k}(TM)$ by defining

$$\sharp_k(\beta) = i(\beta)\eta(x)$$

for $\beta \in \Lambda^k(T_x^*M)$ and $x \in M$, where $i(\beta)$ is the contraction by β. Denote also by $\sharp_k$ the homomorphism of $\mathcal{F}$-modules from the space $\Omega^k(M)$ into the space $\Gamma(\Lambda^{n-k}TM)$ given by

$$\sharp_k(\alpha)(x) = \sharp_k(\alpha(x))$$

for all $\alpha \in \Omega^k(M)$ and $x \in M$.

Next we define a *Leibniz algebra structure* on $\Omega^{n-1}(M)$. The Leibniz algebra on $\Omega^{n-1}(M)$ attached to M is the bracket of $(n-1)$-forms $\{,\} : \Omega^{n-1}(M) \times \Omega^{n-1}(M) \to \Omega^{n-1}(M)$ defined by

$$\{\alpha, \beta\} = \mathcal{L}_{\sharp_{n-1}(\alpha)}\beta + (-1)^n \sharp_n(d\alpha)\beta$$

for all $\alpha, \beta \in \Omega^{n-1}(M)$. In particular, we have that

$$\sharp_{n-1}(\{\alpha, \beta\}) = [\sharp_{n-1}(\alpha), \sharp_{n-1}(\beta)]$$

for all $\alpha, \beta \in \Omega^{n-1}(M)$.

Using Theorem 2.1, the following proposition was proved by R. Ibáñez *et al.* [7].

Proposition 3.1. *Let (M, η) be an m-dimensional Nambu-Poisson manifold of order n, with $n \geq 3$. Then the center of the algebra $(\Omega^{n-1}(M), \{,\})$ is the $\mathcal{F}$-module*

$$\ker \sharp_{n-1} = \{\alpha \in \Omega^{n-1}(M) \mid \sharp_{n-1}(\alpha) = 0\}.$$

By the above proposition, we know that $\Omega^{n-1}(M)/\ker \sharp_{n-1}$ is isomorphic to a Lie subalgebra of $\chi(M)$. This Lie algebra is often denoted by $\mathfrak{g}$. And $\mathcal{F}$ is a $(\Omega^{n-1}(M)/\ker \sharp_{n-1})$-module relative to the representation:

$$\Omega^{n-1}(M)/\ker \sharp_{n-1} \times \mathcal{F} \longrightarrow \mathcal{F}, \quad ([\alpha], f) \mapsto [\alpha]f = (\sharp_{n-1}(\alpha))(f).$$

According to [7], one can define *the skew symmetric-cochain complex*

$$\Big(C^*(\Omega^{n-1}(M)/\ker \sharp_{n-1}; \mathcal{F}) = \bigoplus_k C^k(\Omega^{n-1}(M)/\ker \sharp_{n-1}; \mathcal{F}), \partial\Big)$$

where the space of the k-cochains $C^k(\Omega^{n-1}(M)/\ker \sharp_{n-1}; \mathcal{F})$ consists of skew-symmetric $\mathcal{F}$-linear mappings

$$c^k : (\Omega^{n-1}(M)/\ker \sharp_{n-1}) \times \cdots \times (\Omega^{n-1}(M)/\ker \sharp_{n-1}) \to \mathcal{F}$$

and the coboundary operator ∂ is given by

$$\partial c^k([\alpha_0],...,[\alpha_k]) = \sum_{i=0}^{k}(-1)^i(\sharp_{n-1}(\alpha_i))(c^k([\alpha_0],...,\widehat{[\alpha_i]},...,[\alpha_k])) + \sum_{0\leq i<j\leq k}(-1)^{i+j}c^k([\{\alpha_i,\alpha_j\}],[\alpha_0],...,\widehat{[\alpha_i]},...,\widehat{[\alpha_j]},...,[\alpha_k])$$

for all $c^k \in C^k(\Omega^{n-1}(M)/\ker\sharp_{n-1};\mathcal{F})$, and $[\alpha_0],...,[\alpha_k] \in \Omega^{n-1}(M)/\ker\sharp_{n-1}$. Then we have $\partial\circ\partial = 0$. The cohomology of this complex is called *Nambu-Poisson cohomology* and denoted by $H^*_{NP}(M,\eta)$.

Remark 3.1. Since a Nambu-Poisson tensor η satisfies $[\eta,\eta] = 0$ (Schouten bracket), we can define three cohomology spaces $H^0_\eta(M), H^1_\eta(M)$ and $H^2_\eta(M)$ as in the case of usual Poisson manifolds. We see that these three spaces appear as parts of Nambu-Poisson cohomology spaces. (See [9].)

The first attempt at the computation of singular Nambu-Poisson cohomology was carried out by R. Ibáñez *et al.* In [7], they considered a Nambu-Poisson manifold $\{\mathbb{R}^3, \eta = (x^2+y^2+z^2)\frac{\partial}{\partial x}\wedge\frac{\partial}{\partial y}\wedge\frac{\partial}{\partial z}\}$. They obtained that $H^1_{NP}(\mathbb{R}^3,\eta)\cong\mathbb{R}$.

In [9], P. Monnier studied Nambu-Poisson cohomology from slightly more general viewpoint, which includes the case of R. Ibáñez *et al.* [7]. That is to say, he computed Nambu-Poisson cohomology of Nambu-Poisson manifolds of the form $(\mathbb{R}^n, \eta = f\frac{\partial}{\partial x_1}\wedge\cdots\wedge\frac{\partial}{\partial x_n})$, where f is a *quasihomogeneous* polynomial of *finite codimension.* Using his results, we compute Nambu-Poisson cohomology of $(\mathbb{R}^4, \eta = (x^2+y^2+z^2+u^2)\frac{\partial}{\partial x}\wedge\frac{\partial}{\partial y}\wedge\frac{\partial}{\partial z})$ in the last section.

§4. Computation of Nambu-Poisson Cohomology: Exact Case

§4.1. Notation and general remarks

Let $\mathcal{F}$ be the space of C^∞-functions on $\mathbb{R}^4$. Throughout this section, we suppose that $\mathcal{F}\ni f$ satisfies $f(0)=0$, and is of finite codimension, which means that $\mathcal{F}/\langle f\rangle$ ($\langle f\rangle$ is the ideal spanned by f_x,f_y,f_z,f_u) is a finite dimensional vector space. Here we simply write, for example, f_x for $\frac{\partial f}{\partial x}$.

Let η be a Nambu-Poisson tensor of order 3 on $\mathbb{R}^4(x,y,z,u)$. η is said to be *exact* if η satisfies $i(\eta)\Omega = df$, where $\Omega = dx\wedge dy\wedge dz\wedge du$. Then η is written as follows.

$$\eta = -f_x\frac{\partial}{\partial y}\wedge\frac{\partial}{\partial z}\wedge\frac{\partial}{\partial u} + f_y\frac{\partial}{\partial x}\wedge\frac{\partial}{\partial z}\wedge\frac{\partial}{\partial u} - f_z\frac{\partial}{\partial x}\wedge\frac{\partial}{\partial y}\wedge\frac{\partial}{\partial u} + f_u\frac{\partial}{\partial x}\wedge\frac{\partial}{\partial y}\wedge\frac{\partial}{\partial z}.$$

A Lie subalgebra $\mathfrak{g} = \sharp_2(\Omega^2(\mathbb{R}^4))$ of $\chi(\mathbb{R}^4)$ is spanned over $\mathcal{F}$ by the following six vector fields.

$$\begin{cases} X_1 = f_x \frac{\partial}{\partial y} - f_y \frac{\partial}{\partial x}, & X_2 = f_x \frac{\partial}{\partial z} - f_z \frac{\partial}{\partial x}, \quad X_3 = f_x \frac{\partial}{\partial u} - f_u \frac{\partial}{\partial x}, \\ X_4 = f_y \frac{\partial}{\partial z} - f_z \frac{\partial}{\partial y}, & X_5 = f_y \frac{\partial}{\partial u} - f_u \frac{\partial}{\partial y}, \quad X_6 = f_z \frac{\partial}{\partial u} - f_u \frac{\partial}{\partial z}. \end{cases}$$

It is easy to see that $\Lambda^4\mathfrak{g} = 0$. Hence $H^k_{NP} = 0$, for $k \geq 4$.

§4.2. Relative cohomology

In this subsection, we show that Nambu-Poisson cohomology of exact Nambu-Poisson structure is equivalent to *relative cohomology* which was studied by C. A. Roche [13].

In the first half of this subsection, all objects are considered on $\mathbb{R}^s$. And we simply write Ω^k for $\Omega^k(\mathbb{R}^s)$. Suppose that $C^\infty(\mathbb{R}^s) \ni f$ satisfies $f(0) = 0$ and is of finite codimension. That is to say, an ideal generated by coefficients of df is of finite codimension in $C^\infty(\mathbb{R}^s)$.

First note that $df \wedge \Omega^k$ is compatible with the exterior differential d: *i.e.,* $d(df \wedge \Omega^{k-1}) \subset df \wedge \Omega^k$. Hence the linear mapping

$$d_{rel} : \Omega^k/df \wedge \Omega^{k-1} \longrightarrow \Omega^{k+1}/df \wedge \Omega^k$$

is well-defined.

Definition 4.1. The following sequence defined on $\mathbb{R}^s$ is called relative complex of f.

$$0 \longrightarrow \Omega^0 \xrightarrow{d_{rel}} \Omega^1/df \wedge \Omega^0 \xrightarrow{d_{rel}} \Omega^2/df \wedge \Omega^1 \xrightarrow{d_{rel}} \cdots \xrightarrow{d_{rel}} \Omega^s/df \wedge \Omega^{s-1} \longrightarrow 0.$$

The cohomology of complex defined above is called relative cohomlogy of f, and is denoted by $H^*_{rel}(f)$ or H^*_{rel}. In the above sequence, if we put $\mathcal{I}\cdot\Omega^k$ into Ω^k, then we have flat relative cohomology $H^k_{\infty rel}$, where $\mathcal{I}$ denotes the space of flat functions of $\mathcal{F}$ at the origin. Moreover if we consider formal differential k-forms instead of Ω^k, we have formal relative cohomology $\hat{H}^k_{rel}$.

To state the structure of $H^k_{\infty rel}$ it is convenient to introduce the following notations: For a positive small number c,

$b^k_+ = \dim H^k(X_{+c}, \mathbb{R})$, $b^k_- = \dim H^k(X_{-c}, \mathbb{R})$,

$m^\infty(1) =$ the space of flat functions at the origin of 1-variable,

$m^\infty_\pm = \{h \in m^\infty(1) \mid h(\mathbb{R}^\mp) = 0\}$,

$X_{\pm c} = f^{-1}(\pm c) \cap B$, where B is a small ball centered at the origin.

Then C. A. Roche [13] proved the following theorems. All objects are considered on $\mathbb{R}^s$.

Theorem 4.1. *The $m^\infty(1)$-module $H^k_{\infty rel}$ is isomorphic to $(m^\infty_+)^{b^k_+} \times (m^\infty_-)^{b^k_-}$.*

Theorem 4.2. *There are the following mutual relations among three cohomologies.*

$$H^k_{rel} \cong H^k_{\infty rel} \text{ if } 0 < k < s-1$$
$$H^0_{rel}/H^0_{\infty rel} \cong \mathcal{F}(1), \ \ H^{s-1}_{rel}/H^{s-1}_{\infty rel} \cong \hat{H}^{s-1}_{rel} \cong \mathcal{F}(1)^\mu,$$

where $\mathcal{F}(1)$ is the space of formal functions of 1-variable, and $\mu = codim\ f$. $\mathcal{F}(1)^\mu$ denotes the free $\mathcal{F}(1)$-module of rank μ.

In the latter half of this subsection, let us return to the case of $\mathbb{R}^4$. We simply write Ω^k for $\Omega^k(\mathbb{R}^4)$.

Definition 4.2. We define the subspace I^k of Ω^k by

$$I^k = \{c \in \Omega^k | c(\overbrace{\mathfrak{g}, ..., \mathfrak{g}}^{k}) = 0\},$$

for $1 \leq k \leq 4$. Put $I^0 = 0$.

It is clear that $I^4 = \Omega^4$ since $\Lambda^4 \mathfrak{g} = 0$. In the rest of this subsection, we give a characterization of I^k for $k = 1, 2, 3$.

Proposition 4.3. $I^k = \{c \in \Omega^k | c \wedge df = 0\}$, *for* $0 \leq k \leq 4$.

Proof. In case of $k = 1$, put $c = Adx + Bdy + Cdz + Ddu \in \Omega^1$. Then $c(\mathfrak{g}) = 0$ implies that $f_x B = f_y A,\ f_x C = f_z A,\ f_x D = f_u A,\ f_y C = f_z B,\ f_z D = f_u C$, and $f_y D = f_u B$. On the other hand,

$$\begin{aligned} c \wedge df &= (Adx + Bdy + Cdz + Ddu) \wedge (f_x dx + f_y dy + f_z dz + f_u du) \\ &= (f_y A - f_x B) dx \wedge dy + (f_z A - f_x C) dx \wedge dz + (f_u A - f_x D) dx \wedge du \\ &\quad + (f_z B - f_y C) dy \wedge dz + (f_u B - f_y D) dy \wedge du + (f_u C - f_z D) dz \wedge du. \end{aligned}$$

Thus we have that $c(\mathfrak{g}) = 0$ if and only if $c \wedge df = 0$.

For cases of $k \geq 2$, we can prove in the same way as the case of $k = 1$. □

Now let us recall G. de Rham's division lemma [2]. We will explain this lemma in the general situation, s-dimensional Euclidean space $\mathbb{R}^s$. (Our case is, of course, $s = 4$.)

Definition 4.3. An element ω of Ω^1 is said to possess the property of division in Ω^* if for any $\alpha \in \Omega^p,\ 1 \leq p \leq s-1$, which satisfies $\omega \wedge \alpha = 0$, there exists $\beta \in \Omega^{p-1}$ such that $\alpha = \omega \wedge \beta$.

Definition 4.4. Let $\omega \in \Omega^1$ and let $I(\omega)$ be the ideal of $\Omega^0 = C^\infty(\mathbb{R}^s)$ spanned by the coefficients of ω. Then 0 is said to be algebraically isolated zero of ω if $\Omega^0/I(\omega)$ is a finite dimensional vector space over $\mathbb{R}$.

Lemma 4.4. *Let ω be an element of Ω^1. If 0 is algebraically isolated zero of ω, then ω possesses the property of division.*

Since f is of finite codimension in our situation, $\omega = df$ satisfies the condition of Lemma 4.4. Hence by Proposition 4.3, we know that $I^k = df \wedge \Omega^{k-1}$ for $1 \leq k \leq 3$.

Recall that a k-th cochain $c \in C^k$ is $\mathcal{F}$-linear skew-symmetric mapping from $\mathfrak{g} \times \cdots \times \mathfrak{g}$ to $\mathcal{F}$. The natural inclusion $\iota : \mathfrak{g} \hookrightarrow \chi(\mathbb{R}^4)$ induces the surjective mapping $\phi : \Omega^k \longrightarrow C^k$ as the dual mapping of the natural inclusion ι. Note that $\ker \phi = I^k$ for $1 \leq k \leq 3$. Then it is easy to obtain the following proposition.

Proposition 4.5. $C^k \cong \Omega^k/I^k \cong \Omega^k/df \wedge \Omega^{k-1}$, *for* $1 \leq k \leq 3$. *For* $k = 0$, $C^0 = \Omega^0 = \mathcal{F}$, *and for* $k = 4$, $C^4 = 0$.

Now by Proposition 4.5, we have obtained the following commutative diagram. In particular, note that $d_{rel} : \Omega^k/I^k \to \Omega^{k+1}/I^{k+1}$ coincides with $\partial : C^k \to C^{k+1}$ for $0 \leq k \leq 2$.

$$
\begin{array}{ccccccccccccc}
0 & \longrightarrow & \Omega^0 & \xrightarrow{d} & \Omega^1 & \xrightarrow{d} & \Omega^2 & \xrightarrow{d} & \Omega^3 & \xrightarrow{d} & \Omega^4 & \longrightarrow & 0 \\
 & & \| & & {\scriptstyle\pi}\downarrow & & {\scriptstyle\pi}\downarrow & & {\scriptstyle\pi}\downarrow & & {\scriptstyle\pi}\downarrow & & \\
0 & \longrightarrow & \Omega^0 & \xrightarrow{d_{rel}} & \Omega^1/I^1 & \xrightarrow{d_{rel}} & \Omega^2/I^2 & \xrightarrow{d_{rel}} & \Omega^3/I^3 & \xrightarrow{d_{rel}} & \Omega^4/df \wedge \Omega^3 & \longrightarrow & 0 \\
 & & \| & & \| & & \| & & \| & & \downarrow & & \\
0 & \longrightarrow & \Omega^0 & \xrightarrow{\partial} & C^1 & \xrightarrow{\partial} & C^2 & \xrightarrow{\partial} & C^3 & \longrightarrow & 0 & &
\end{array}
$$

Using the above commutative diagram, we can get the following theorem.

Theorem 4.6. *Let η be the exact Nambu-Poisson tensor corresponding to $f \in \mathcal{F}$ defined on $\mathbb{R}^4$, where f is of finite codimension. Then*

$$H^k_{NP} \cong H^k_{rel} \ \textit{for} \ 0 \le k \le 2,$$
$$H^3_{NP} \cong H^3_{rel} \oplus \Omega^4/df \wedge \Omega^3,$$
$$H^k_{NP} = 0 \ \textit{for} \ 4 \le k.$$

To compute some examples of exact Nambu-Poisson cohomology, let us recall the results of C. A. Roche [13]. (See Theorem 4.1 and Theorem 4.2.)

Examples. Let $f = x^k + y^2 + z^2 + u^2,\ \ k \ge 3$. Then if k is an odd positive integer, both X_{+c} and X_{-c} are homeomorphic to D^3, where D^3 denotes a three dimensional ball. Hence by Theorem 4.1 and Theorem 4.2, we have

$$H^0_{\infty rel} \cong m^\infty_+ \times m^\infty_-,\ \ H^1_{\infty rel} = 0,\ \ H^2_{\infty rel} = 0,\ \ H^3_{\infty rel} = 0.$$
$$H^0_{rel} \cong C^\infty(\mathbb{R}^+) \times C^\infty(\mathbb{R}^-),\ \ H^1_{rel} = 0,\ \ H^2_{rel} = 0,\ \ H^3_{rel} \cong \mathcal{F}(1)^{k-1}.$$

Moreover if we use Theorem 4.6, we have

$$H^0_{NP} \cong C^\infty(\mathbb{R}^+) \times C^\infty(\mathbb{R}^-),\ \ H^1_{NP} = 0,\ \ H^2_{NP} = 0,\ \ H^3_{NP} \cong \mathcal{F}(1)^{k-1} \oplus \mathbb{R}^{k-1}.$$

On the other hand, if k is an even positive integer, then X_{+c} is homeomorphic to S^3 and $X_{-c} = \phi$. Hence we have

$$H^0_{\infty rel} \cong m^\infty_+,\ \ H^1_{\infty rel} = 0,\ \ H^2_{\infty rel} = 0,\ \ H^3_{\infty rel} \cong m^\infty_+.$$
$$H^0_{rel} \cong C^\infty(\mathbb{R}^+),\ \ H^1_{rel} = 0,\ \ H^2_{rel} = 0,\ \ H^3_{rel} \cong (C^\infty(\mathbb{R}^+))^{k-1}.$$

Moreover if we use Theorem 4.6, we have

$$H^0_{NP} \cong C^\infty(\mathbb{R}^+),\ \ H^1_{NP} = 0,\ \ H^2_{NP} = 0,\ \ H^3_{NP} \cong (C^\infty(\mathbb{R}^+))^{k-1} \oplus \mathbb{R}^{k-1}.$$

§5. Computation of Nambu-Poisson Cohomology: Linear Case

§5.1. Notation and general remarks

In this section we consider linear Nambu-Poisson tensors which are of order 3 on $\mathbb{R}^4(x, y, z, u)$. By the classification theorem of linear Nambu-Poisson structures [3],[6], we know that there are the following four types of linear Nambu-Poisson tensors.

(I) $\eta = -f_x \frac{\partial}{\partial y} \wedge \frac{\partial}{\partial z} \wedge \frac{\partial}{\partial u} + f_y \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial z} \wedge \frac{\partial}{\partial u} - f_z \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y} \wedge \frac{\partial}{\partial u} + f_u \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y} \wedge \frac{\partial}{\partial z}$, where f is a homogeneous quadratic function on $\mathbb{R}^4$.

(II) $\eta = \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y} \wedge \{(a_{11}z + a_{12}u)\frac{\partial}{\partial z} + (a_{21}z + a_{22}u)\frac{\partial}{\partial u}\}$, $(a_{ij} \in \mathbb{R})$.

(III) $\eta_\phi = \phi\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y} \wedge \frac{\partial}{\partial z}$, where ϕ is any linear function on $\mathbb{R}^4$.

(IV) $\eta_\psi = \{px + (q-1)y - b_3z - b_4u\}\frac{\partial}{\partial y} \wedge \frac{\partial}{\partial z} \wedge \frac{\partial}{\partial u} - \{(q+1)x + ry + a_3z + a_4u\}\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial z} \wedge \frac{\partial}{\partial u}$, where $p, q, r, a_3, a_4, b_3, b_4 \in \mathbb{R}$. Put $\alpha = d\psi + (x + a_3z + a_4u)dy - (y + b_3z + b_4u)dx$, where $\psi = \frac{1}{2}px^2 + qxy + \frac{1}{2}ry^2$. Then η_ψ is defined by $i(\eta_\psi)dx \wedge dy \wedge dz \wedge du = \alpha$.

In (IV), recall that η_ψ becomes a Nambu-Poisson tensor if and only if $\alpha \wedge d\alpha = 0$. Thus seven constants must satisfy $a_3b_4 = a_4b_3, a_3p + b_3(q+1) = 0, a_3(q-1) + b_3r = 0, a_4p + b_4(q+1) = 0, a_4(q-1) + b_4r = 0$.

In considering type (II), since a matrix (a_{ij}) can be chosen to be in Jordan form, there are five classes with nondegenerate Jordan forms ($\eta_1 \sim \eta_5$) and two classes with degenerate Jordan forms ($\eta_6 \sim \eta_7$) as follows.

(i) $\eta_1 = \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y} \wedge \{(\alpha z + u)\frac{\partial}{\partial z} + (\alpha u)\frac{\partial}{\partial u}\}$, $\alpha \neq 0$,

(ii) $\eta_2 = \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y} \wedge \{(\alpha z)\frac{\partial}{\partial z} + (\beta u)\frac{\partial}{\partial u}\}$, $\alpha \neq 0$, $\beta \neq 0$, $\alpha \neq \beta$,

(iii) $\eta_3 = \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y} \wedge \{(\alpha z - \beta u)\frac{\partial}{\partial z} + (\beta z + \alpha u)\frac{\partial}{\partial u}\}$, $\alpha \neq 0$ $\beta \neq 0$,

(iv) $\eta_4 = \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y} \wedge \alpha(z\frac{\partial}{\partial z} + u\frac{\partial}{\partial u})$, $\alpha \neq 0$,

(v) $\eta_5 = \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y} \wedge \beta(u\frac{\partial}{\partial z} - z\frac{\partial}{\partial u})$, $\beta \neq 0$,

(vi) $\eta_6 = \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y} \wedge (\alpha z)\frac{\partial}{\partial z}$, $\alpha \neq 0$,

(vii) $\eta_7 = \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y} \wedge u\frac{\partial}{\partial z}$.

A linear Nambu-Poisson tensor of type (I) is one of exact Nambu-Poisson tensors. And this case was already considered in the previous section. Hence in this section we will only give the results concerning *nondegenerate* Nambu-Poisson tensors (i.e. $f = \pm x^2 \pm y^2 \pm z^2 \pm u^2$) for type (I). And here we will mainly study the computation for type (II).

Throughout this section, we will use the following notations:

- $\mathcal{F}$ is the algebra of real-valued C^∞ functions on $\mathbb{R}^4(x, y, z, u)$;
- $\tilde{\mathcal{G}}$ is the algebra of real-valued C^∞ functions on $\mathbb{R}^3(y, z, u)$;
- $\tilde{\mathcal{F}}$ is the algebra of real-valued C^∞ functions on $\mathbb{R}^2(z, u)$;
- $\mathcal{F}(1)$ is the algebra of formal functions of one variable;
- $\chi(\mathbb{R}^4)$ is the Lie algebra of all vector fields on $\mathbb{R}^4$;
- Ω^k is the space of k-forms on $\mathbb{R}^4$.

§5.2. Computation of Nambu-Poisson cohomology of type (I)

In this subsection, we confine ourselves to *nondegenerate* linear Nambu-Poisson tensors of type (I). This means that $f = \pm x^2 \pm y^2 \pm z^2 \pm u^2$ and it is clear that f is of finite codimension. We get the following results by using Theorem 4.1 of C. A. Roche [13]. We use the same notations as those of the previous section. Let η be a linear Nambu-Poisson tensor of type (I) defined by $i(\eta)\Omega = df$. Then we get the following flat relative cohomology. In Table 1, D^i denotes an i-dimensional ball.

Table 1. Flat Relative Cohomology

f	X_{+c}	X_{-c}	$H^0_{\infty rel}$	$H^1_{\infty rel}$	$H^2_{\infty rel}$	$H^3_{\infty rel}$
$x^2+y^2+z^2+u^2$	S^3	ϕ	m_+^∞	0	0	m_+^∞
$x^2+y^2+z^2-u^2$	$S^2\times D^1$	$S^0\times D^3$	$m_+^\infty\times m_-^\infty\times m_-^\infty$	0	m_+^∞	0
$x^2+y^2-z^2-u^2$	$S^1\times D^2$	$S^1\times D^2$	$m_+^\infty\times m_-^\infty$	$m_+^\infty\times m_-^\infty$	0	0
$x^2-y^2-z^2-u^2$	$S^0\times D^3$	$S^2\times D^1$	$m_+^\infty\times m_+^\infty\times m_-^\infty$	0	m_-^∞	0
$-x^2-y^2-z^2-u^2$	ϕ	S^3	m_-^∞	0	0	m_-^∞

Combining the results in Table 1 with Theorem 4.2 and Theorem 4.6, we can compute cohomology of type (I). In computing H^3_{NP}, note that $\Omega^4/df\wedge\Omega^3 \cong \mathbb{R}$, and $\mu = 1$. We collect the results in the following table.

Table 2. Exact Nambu-Poisson Cohomology

f	H^0_{NP}	H^1_{NP}	H^2_{NP}	H^3_{NP}
$x^2+y^2+z^2+u^2$	$C^\infty(\mathbb{R}^+)$	0	0	$C^\infty(\mathbb{R}^+)\oplus\mathbb{R}$
$x^2+y^2+z^2-u^2$	$C^\infty(\mathbb{R}^+)\times C^\infty(\mathbb{R}^-)\times C^\infty(\mathbb{R}^-)$	0	m_+^∞	$\mathcal{F}(1)\oplus\mathbb{R}$
$x^2+y^2-z^2-u^2$	$C^\infty(\mathbb{R}^+)\times C^\infty(\mathbb{R}^-)$	$m_+^\infty\times m_-^\infty$	0	$\mathcal{F}(1)\oplus\mathbb{R}$
$x^2-y^2-z^2-u^2$	$C^\infty(\mathbb{R}^+)\times C^\infty(\mathbb{R}^+)\times C^\infty(\mathbb{R}^-)$	0	m_-^∞	$\mathcal{F}(1)\oplus\mathbb{R}$
$-x^2-y^2-z^2-u^2$	$C^\infty(\mathbb{R}^-)$	0	0	$C^\infty(\mathbb{R}^-)\oplus\mathbb{R}$

§5.3. Computation of Nambu-Poisson cohomology of type (II)

In this subsection, we compute Nambu-Poisson cohomology of type (II). Denote by $\mathfrak{g}_i$ the Lie algebra corresponding to η_i, $i = 1, 2, ..., 7$. Recall that $\mathfrak{g}_i$ is defined by $\mathfrak{g}_i = i(\Omega^2)\eta_i$. Then each $\mathfrak{g}_i$ is spanned over $\mathcal{F}$ by several vector fields as follows.

$$\begin{aligned}
\mathfrak{g}_1 &= \langle z\frac{\partial}{\partial x}, u\frac{\partial}{\partial x}, z\frac{\partial}{\partial y}, u\frac{\partial}{\partial y}, (\alpha z + u)\frac{\partial}{\partial z} + \alpha u\frac{\partial}{\partial u}\rangle;\\
\mathfrak{g}_2 &= \langle z\frac{\partial}{\partial x}, u\frac{\partial}{\partial x}, z\frac{\partial}{\partial y}, u\frac{\partial}{\partial y}, \alpha z\frac{\partial}{\partial z} + \beta u\frac{\partial}{\partial u}\rangle;\\
\mathfrak{g}_3 &= \langle z\frac{\partial}{\partial x}, u\frac{\partial}{\partial x}, z\frac{\partial}{\partial y}, u\frac{\partial}{\partial y}, (\alpha z - \beta u)\frac{\partial}{\partial z} + (\beta z + \alpha u)\frac{\partial}{\partial u}\rangle;\\
\mathfrak{g}_4 &= \langle z\frac{\partial}{\partial x}, u\frac{\partial}{\partial x}, z\frac{\partial}{\partial y}, u\frac{\partial}{\partial y}, z\frac{\partial}{\partial z} + u\frac{\partial}{\partial u}\rangle;\\
\mathfrak{g}_5 &= \langle z\frac{\partial}{\partial x}, u\frac{\partial}{\partial x}, z\frac{\partial}{\partial y}, u\frac{\partial}{\partial y}, u\frac{\partial}{\partial z} - z\frac{\partial}{\partial u}\rangle;\\
\mathfrak{g}_6 &= \langle z\frac{\partial}{\partial x}, z\frac{\partial}{\partial y}, z\frac{\partial}{\partial z}\rangle;\\
\mathfrak{g}_7 &= \langle u\frac{\partial}{\partial x}, u\frac{\partial}{\partial y}, u\frac{\partial}{\partial z}\rangle.
\end{aligned}$$

As is easily seen, we know that

$$\Lambda^4\mathfrak{g}_i = 0, \text{ for } 1 \le i \le 7.$$

Denote by $H^k_{NP}(\eta_i)$ the k-th cohomology group corresponding to the Nambu-Poisson tensor η_i. Then for $1 \le i \le 7$, $H^k_{NP}(\eta_i) = 0$ if $4 \le k$.

For $0 \le k \le 4$, $I^k \subset \Omega^k$ is similarly defined as in the previous section (see Definition 4.2). Then we also have $C^k \cong \Omega^k/I^k$. First let us determine explicit forms of all I^k. They are summarized in the following lemma.

Lemma 5.1. *Let A, B, C, D, E, F be elements of $\mathcal{F}$.*

(a) *In case of η_1,*

$$\begin{aligned}
I^1 &= \{Cdz + Ddu \mid (\alpha z + u)C + \alpha uD = 0\},\\
I^2 &= \{Bdx\wedge dz + Cdx\wedge du + Ddy\wedge dz + Edy\wedge du\\
&\qquad + Fdz\wedge du \mid (\alpha z + u)B + \alpha uC = 0, (\alpha z + u)D + \alpha uE = 0\},\\
I^3 &= \{Adx\wedge dy\wedge dz + Bdx\wedge dy\wedge du + Cdx\wedge dz\wedge du\\
&\qquad + Ddy\wedge dz\wedge du \mid (\alpha z + u)A + \alpha uB = 0\},\\
I^4 &= \Omega^4.
\end{aligned}$$

(b) *In case of* η_2,

$$\begin{aligned} I^1 &= \{Cdz + Ddu \mid \alpha zC + \beta uD\}, \\ I^2 &= \{Bdx \wedge dz + Cdx \wedge du + Ddy \wedge dz + Edy \wedge du \\ &\quad + Fdz \wedge du \mid \alpha zB + \beta uC = 0, \alpha zD + \beta uE = 0\}, \\ I^3 &= \{Adx \wedge dy \wedge dz + Bdx \wedge dy \wedge du + Cdx \wedge dz \wedge du \\ &\quad + Ddy \wedge dz \wedge du \mid \alpha zA + \beta uB = 0\}, \\ I^4 &= \Omega^4. \end{aligned}$$

(c) *In case of* η_3,

$$\begin{aligned} I^1 &= \{Cdz + Ddu \mid (\alpha z - \beta u)C + (\beta z + \alpha u)D = 0\}, \\ I^2 &= \{Bdx \wedge dz + Cdx \wedge du + Ddy \wedge dz + Edy \wedge du \\ &\quad + Fdz \wedge du \mid (\alpha z - \beta u)B + (\beta z + \alpha u)C = 0, \\ &\quad (\alpha z - \beta u)D + (\beta z + \alpha u)E = 0\}, \\ I^3 &= \{Adx \wedge dy \wedge dz + Bdx \wedge dy \wedge du + Cdx \wedge dz \wedge du \\ &\quad + Ddy \wedge dz \wedge du \mid (\alpha z - \beta u)A + (\beta z + \alpha u)B = 0\}, \\ I^4 &= \Omega^4. \end{aligned}$$

(d) *In case of* η_4,

$$\begin{aligned} I^1 &= \{Cdz + Ddu \mid zC + uD = 0\}, \\ I^2 &= \{Bdx \wedge dz + Cdx \wedge du + Ddy \wedge dz + Edy \wedge du \\ &\quad + Fdz \wedge du \mid zB + uC = 0, zD + uE = 0\}, \\ I^3 &= \{Adx \wedge dy \wedge dz + Bdx \wedge dy \wedge du + Cdx \wedge dz \wedge du \\ &\quad + Ddy \wedge dz \wedge du \mid zA + uB = 0\}, \\ I^4 &= \Omega^4. \end{aligned}$$

(e) *In case of* η_5,

$$\begin{aligned} I^1 &= \{Cdz + Ddu \mid zD - uC = 0\}, \\ I^2 &= \{Bdx \wedge dz + Cdx \wedge du + Ddy \wedge dz + Edy \wedge du \\ &\quad + Fdz \wedge du \mid uB - zC = 0, uD - zE = 0\}, \\ I^3 &= \{Adx \wedge dy \wedge dz + Bdx \wedge dy \wedge du + Cdx \wedge dz \wedge du \\ &\quad + Ddy \wedge dz \wedge du \mid uA - zB = 0\}, \\ I^4 &= \Omega^4. \end{aligned}$$

(f) *In cases of* η_6 *and* η_7,

$$\begin{aligned}
I^1 &= \mathcal{F}du,\\
I^2 &= \mathcal{F}dx \wedge du + \mathcal{F}dy \wedge du + \mathcal{F}dz \wedge du,\\
I^3 &= \mathcal{F}dx \wedge dy \wedge du + \mathcal{F}dx \wedge dz \wedge du + \mathcal{F}dy \wedge dz \wedge du,\\
I^4 &= \Omega^4.
\end{aligned}$$

Proof. Straightforward computation. □

In linear cases, we also have the following commutative diagram which is similar to that of relative cases. (Its proof is obtained as a direct consequence of the definition of the operator ∂.)

$$\begin{array}{ccccccccccccc}
0 & \longrightarrow & \Omega^0 & \xrightarrow{d} & \Omega^1 & \xrightarrow{d} & \Omega^2 & \xrightarrow{d} & \Omega^3 & \xrightarrow{d} & \Omega^4 & \longrightarrow & 0\\
 & & \| & & \downarrow{\scriptstyle\pi} & & \downarrow{\scriptstyle\pi} & & \downarrow{\scriptstyle\pi} & & \downarrow{\scriptstyle\pi} & & \\
0 & \longrightarrow & \Omega^0 & \xrightarrow{\partial} & C^1 & \xrightarrow{\partial} & C^2 & \xrightarrow{\partial} & C^3 & \xrightarrow{\partial} & C^4 & \longrightarrow & 0
\end{array}$$

Now let us compute Nambu-Poisson cohomology for Nambu-Poisson tensors η_i, $1 \leq i \leq 7$. Recall that $I^4 = \Omega^4$. This means that $C^4 = 0$ in the above commutative diagram. Hence we have only to compute $H^k_{NP}(\eta_i)$ for $0 \leq k \leq 3$.

We denote by Z^k the space of cocycles and by B^k the space of coboundaries in C^k. Clearly it holds that $B^k \subset Z^k \subset C^k$, and by definition, $H^k_{NP} = Z^k/B^k$.

Definition 5.1. We define the subspaces $\tilde{Z}^k$ and $\tilde{B}^k$ of Ω^k as follows.

$$\begin{aligned}
\tilde{Z}^k &= \{c \in \Omega^k | dc \in I^{k+1}\},\\
\tilde{B}^k &= d\Omega^{k-1}.
\end{aligned}$$

Note that it holds $I^k \subset \tilde{Z}^k$.

Proposition 5.2. $H^k_{NP} \cong \tilde{Z}^k/(\tilde{B}^k + I^k)$ *for* $1 \leq k \leq 3$.

Proof. We first prove that $\pi^{-1}(Z^k) = \tilde{Z}^k$. For $c \in \pi^{-1}(Z^k)$, we have $0 = \partial(\pi c) = \pi(dc)$. Hence $dc \in I^{k+1}$ and this implies $c \in \tilde{Z}^k$. The converse is clear. Hence the linear mapping $\pi : \tilde{Z}^k \longrightarrow Z^k$ is surjective. Since $\ker \pi = I^k$, we have $Z^k \cong \tilde{Z}^k/I^k$. Next note that $B^k = \partial C^{k-1} = \partial(\pi\Omega^{k-1}) = \pi(d\Omega^{k-1}) = \pi\tilde{B}^k$. Hence $\pi^{-1}(B^k) = \tilde{B}^k + I^k$, and $B^k \cong (\tilde{B}^k + I^k)/I^k$. Finally we have

$$H^k_{NP} = Z^k/B^k \cong (\tilde{Z}^k/I^k)/((\tilde{B}^k + I^k)/I^k) \cong \tilde{Z}^k/(\tilde{B}^k + I^k).$$

□

To compute Nambu-Poisson cohomology for linear Nambu-Poisson tensors, the following lemma is useful. After the preparation of this paper, T. Fukuda informed me that J. Mather [8] and T. Fukuda and S. Janeczko [4] had already proved an analogous kind of result in a more general situation. So we omit the proof.

Lemma 5.3. *Let $f(x,y,z,u)$ and $g(x,y,z,u)$ be C^∞-functions on $\mathbb{R}^4$ (x,y,z,u), and let $A(z,u)$ and $B(z,u)$ be linear functions of two variables z,u such that $\partial(A,B)/\partial(z,u) \neq 0$. If $f(x,y,z,u)$ and $g(x,y,z,u)$ satisfy the condition:*

$$A(z,u)\cdot f(x,y,z,u) = B(z,u)\cdot g(x,y,z,u), \tag{1}$$

then there exists a function $h(x,y,z,u) \in C^\infty(\mathbb{R}^4)$ such that

$$\begin{cases} f(x,y,z,u) = B(z,u)\cdot h(x,y,z,u), \\ g(x,y,z,u) = A(z,u)\cdot h(x,y,z,u). \end{cases} \tag{2}$$

Let us begin with computing Nambu-Poisson cohomology for Nambu-Poisson tensor $\eta_1 = \frac{\partial}{\partial x}\wedge\frac{\partial}{\partial y}\wedge\{(\alpha z+u)\frac{\partial}{\partial z}+\alpha u\frac{\partial}{\partial u}\}$, where $\alpha \neq 0$. Then the corresponding Lie algebra $\mathfrak{g}_1$ is spanned by $\langle z\frac{\partial}{\partial x}, u\frac{\partial}{\partial x}, z\frac{\partial}{\partial y}, u\frac{\partial}{\partial y}, (\alpha z+u)\frac{\partial}{\partial z}+\alpha u\frac{\partial}{\partial u}\rangle$ over $\mathcal{F}$. It is clear that $H^k_{NP}(\eta_1) = 0$ for $k \geq 4$ since $\Lambda^4\mathfrak{g}_1 = 0$.

Lemma 5.4.

(a) *Put $c = Adx + Bdy + Cdz + Ddu$. Then $c \in \tilde{Z}^1$ if and only if*

$$\begin{cases} B_x = A_y, \\ (\alpha z+u)(C_x - A_z) = \alpha u(A_u - D_x), \\ (\alpha z+u)(C_y - B_z) = \alpha u(B_u - D_y). \end{cases}$$

(b) *Put $c = Adx\wedge dy + Bdx\wedge dz + Cdx\wedge du + Ddy\wedge dz + Edy\wedge du + Fdz\wedge du$. Then $c \in \tilde{Z}^2$ if and only if $(\alpha z+u)(A_z - B_y + D_x) = -\alpha u(A_u - C_y + E_x)$.*

(c) *$\tilde{Z}^3 = \Omega^3$.*

Proof. We have only to recall that $c \in \tilde{Z}^k$ if and only if $dc \in I^{k+1}$. Then direct computation shows the above results. □

Theorem 5.5. *Let* $\eta_1 = \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y} \wedge \{(\alpha z + u)\frac{\partial}{\partial z} + \alpha u \frac{\partial}{\partial u}\}$. *Then we have*

$$\begin{aligned} &H^0_{NP}(\eta_1) \cong \mathbb{R}, \\ &H^1_{NP}(\eta_1) \cong \tilde{\mathcal{F}}/\tilde{\mathcal{F}}_1 \cong \mathcal{I}_{\mathbb{R}^2}/\tilde{\mathcal{F}}_1 \cap \mathcal{I}_{\mathbb{R}^2}, \\ &\qquad \text{where } \tilde{\mathcal{F}}_1 = \{(\alpha z + u)\tilde{h}_z + \alpha u \tilde{h}_u + 2\alpha \tilde{h} \mid \tilde{h} \in \tilde{\mathcal{F}}\}, \\ &H^2_{NP}(\eta_1) \cong \tilde{\mathcal{G}}/\tilde{\mathcal{G}}_1 \cong \tilde{\mathcal{I}}_{\mathbb{R}^3}/\tilde{\mathcal{G}}_1 \cap \tilde{\mathcal{I}}_{\mathbb{R}^3}, \\ &\qquad \text{where } \tilde{\mathcal{G}}_1 = \{(\alpha z + u)\tilde{g}_z + \alpha u \tilde{g}_u + 2\alpha \tilde{g} \mid \tilde{g} \in \tilde{\mathcal{G}}\}, \\ &H^k_{NP}(\eta_1) = 0 \text{ for } k \geq 3. \end{aligned}$$

In the above results, $\tilde{\mathcal{I}}_{\mathbb{R}^2}$ *(resp.* $\tilde{\mathcal{I}}_{\mathbb{R}^3}$*) stands for the space of functions defined on* $\mathbb{R}^2(z,u)$ *(resp.* $\mathbb{R}^3(y,z,u)$*) which are flat at the origin.*

Proof. Let f be an element of $H^0_{NP}(\eta_1)$. Then $f = f(z,u)$, and it holds that $(\alpha z + u)f_z + \alpha u f_u = 0$. The solution is $f(z,u) = \phi(\frac{\alpha z - u \log u}{u})$, where ϕ is any C^∞-function of 1-variable. Hence smooth solutions $f(z,u)$ are only constants.

For the computation of $H^1_{NP}(\eta_1)$, put $c = Adx + Bdy + Cdz + Ddu \in \tilde{Z}^1$. Then by Lemma 5.4(a), there exists a function $h \in \mathcal{F}$ such that $A = h_x, B = h_y$. Then the last two equations in (a) can be rewritten as follows.

$$\begin{aligned} \alpha u(h_u - D)_x =& (\alpha z + u)(C - h_z)_x, \\ \alpha u(h_u - D)_y =& (\alpha z + u)(C - h_z)_y. \end{aligned}$$

Hence by Lemma 5.3, there exist $k, l \in \mathcal{F}$, such that $(C - h_z)_x = \alpha u k$, $(h_u - D)_x = (\alpha z + u)k$, $(C - h_z)_y = \alpha u l$, $(h_u - D)_y = (\alpha z + u)l$. Then we have

$$\begin{aligned} C - h_z =& \alpha u \int k dx + \phi_1(y,z,u) = \alpha u \int l dy + \phi_2(x,z,u), \\ h_u - D =& (\alpha z + u) \int k dx + \psi_1(y,z,u) = (\alpha z + u) \int l dy + \psi_2(x,z,u). \end{aligned}$$

By the integrability condition, it holds that $k_y = l_x$. And we have $(C - h_z)_y = \alpha u \int k_y dx + (\phi_1)_y = \alpha u \int l_x dx + (\phi_1)_y = \alpha u(l - \bar{\phi_1}(y,z,u)) + (\phi_1)_y$ for some function $\bar{\phi_1}(y,z,u)$. On the other hand, since $(C - h_z)_y = \alpha u l$, we must have $(\phi_1)_y = \alpha u \bar{\phi_1}(y,z,u)$, and hence $\phi_1(y,z,u) = \alpha u \int \bar{\phi_1}(y,z,u)dy + \tilde{\phi_1}(z,u)$ for some function $\tilde{\phi_1}(z,u)$. By the same way as above, we have $(h_u - D)_y = (\alpha z + u)\int k_y dx + (\psi_1)_y = (\alpha z + u)\int l_x dx + (\psi_1)_y = (\alpha z + u)(l - \bar{\psi_1}(y,z,u)) + (\psi_1)_y = (\alpha z + u)l$. Hence we have $\psi_1(y,z,u) = (\alpha z + u)\int \bar{\psi_1} dy + \tilde{\psi_1}(z,u)$

for some functions $\bar{\psi_1}(y,z,u)$ and $\tilde{\psi_1}(z,u)$. Now C and D can be written as follows.

$$C = h_z + \alpha u \int k dx + \alpha u \int \bar{\phi_1} dy + \tilde{\phi_1}(z,u),$$
$$D = h_u - (\alpha z + u) \int k dx - (\alpha z + u) \int \bar{\psi_1} dy - \tilde{\psi_1}(z,u).$$

Then we have

$$\begin{aligned}\alpha u(B_u - D_y) &= \alpha u \left(h_{yu} + (\alpha z + u) \int k_y dx + (\alpha z + u)\bar{\psi_1} - h_{yu} \right) \\ &= \alpha u(\alpha z + u) \left(\int k_y dx + \bar{\psi_1} \right), \\ (\alpha z + u)(C_y - B_z) &= (\alpha z + u) \left(h_{yz} + \alpha u \int k_y dx + \alpha u \bar{\phi_1} - h_{yz} \right) \\ &= \alpha u(\alpha z + u) \left(\int k_y dx + \bar{\phi_1} \right).\end{aligned}$$

Since $\alpha u(B_u - D_y) = (\alpha z + u)(C_y - B_z)$, we get $\bar{\phi_1} = \bar{\psi_1}$. Thus $c \in \tilde{Z}^1$ has the following expression.

$$\begin{aligned}c &= Adx + Bdy + Cdz + Ddu \\ &= h_x dx + h_y dy + \Big(h_z + \alpha u \int k dx + \alpha u \int \bar{\phi_1}(y,z,u) dy + \tilde{\phi_1}(z,u) \Big) dz \\ &\quad + \Big(h_u - (\alpha z + u) \int k dx - (\alpha z + u) \int \bar{\psi_1}(y,z,u) dy - \tilde{\psi_1}(z,u) \Big) du \\ &= dh + \alpha u \Big(\int k dx + \int \bar{\phi_1} dy \Big) dz + \tilde{\phi_1}(z,u) dz \\ &\quad - (\alpha z + u) \Big(\int k dx + \int \bar{\phi_1} dy \Big) du - \tilde{\psi_1}(z,u) du.\end{aligned}$$

Note that $dh + \alpha u\Big(\int k dx + \int \bar{\phi_1} dy\Big) dz - (\alpha z + u)\Big(\int k dx + \int \bar{\phi_1} dy\Big) du$ is contained in $\tilde{B}^1 + I^1$. Hence by Proposition 5.2, we can consider $H^1_{NP}(\eta_1)$ as $\{\tilde{\phi_1}(z,u)dz - \tilde{\psi_1}(z,u)du \mid \tilde{\phi_1}, \tilde{\psi_1} \in \tilde{\mathcal{F}}\}$ modulo $\tilde{B}^1 + I^1$. Let A_1 be the space of 1-forms on $\mathbb{R}^2(z,u)$, A_2 be the space of 2-forms on $\mathbb{R}^2(z,u)$, and B_1 be the space of exact 1-forms on $\mathbb{R}^2(z,u)$. It is clear that $A_1/B_1 \cong A_2 \cong \tilde{\mathcal{F}}$. We also define the subspace C_1 of A_1 by

$$C_1 = \{\alpha u \tilde{h} dz - (\alpha z + u)\tilde{h} du \mid \tilde{h} \in \tilde{\mathcal{F}}\}.$$

Note that $B_1 \subset \tilde{B}^1$ and $C_1 \subset I^1$. Then we have

$$\begin{aligned} H^1_{NP}(\eta_1) &\cong A_1/(B_1+C_1) \\ &\cong (A_1/B_1)/((B_1+C_1)/B_1) \\ &\cong (A_1/B_1)/(C_1/B_1\cap C_1). \end{aligned}$$

Let $\alpha u\tilde{h}dz - (\alpha z+u)\tilde{h}du$ be any element of $B_1\cap C_1$. Then $\tilde{h}$ has the form $(\alpha z+u)\tilde{h}_z + \alpha u\tilde{h}_u = -2\alpha\tilde{h}$. Any solution has the form $\tilde{h} = u^{-2}\phi(\frac{\alpha z - u\log u}{u})$ with an arbitrary function ϕ of 1-variable. Hence C^∞-solution is only $\tilde{h}=0$, and $B_1\cap C_1 = 0$. This means that $C_1 \cong dC_1$, and we know that C_1 is isomorphic to the space

$$\tilde{\mathcal{F}}_1 = \{(\alpha z+u)\tilde{h}_z + \alpha u\tilde{h}_u + 2\alpha\tilde{h} \mid \tilde{h}\in\tilde{\mathcal{F}}\}.$$

Let F be the space of formal functions on $\mathbb{R}^2(z,u)$. Each element $[\tilde{h}]$ of F is obtained by the formal Taylor expansion of $\tilde{h}\in\tilde{\mathcal{F}}$ at the origin. It is easy to see that the mapping $T:\tilde{h}\longrightarrow[\tilde{h}]$ is linear and surjective. The kernel of T will be denoted by $\mathcal{I}_{\mathbb{R}^2}$. Put

$$[\tilde{h}] = \sum_{i,j\geq 0} a_{ij}z^i u^j \in F.$$

Then we have

$$\begin{aligned} &[(\alpha z+u)\tilde{h}_z + \alpha u\tilde{h}_u + 2\alpha\tilde{h}] \\ &\qquad = \sum_{i,j\geq 0}\alpha(i+j+2)a_{ij}z^iu^j \\ &\qquad + \sum_{i\geq 0, j\geq 1}(i+1)a_{i+1,j-1}z^iu^j, \end{aligned}$$

and we know that $T(\tilde{\mathcal{F}}) = T(\tilde{\mathcal{F}}_1) = F$. This means that for any $f\in\tilde{\mathcal{F}}$, there exists $g\in\tilde{\mathcal{F}}_1$ such that $f-g\in\mathcal{I}_{\mathbb{R}^2}$, and it holds that $\tilde{\mathcal{F}} = \tilde{\mathcal{F}}_1 + \mathcal{I}_{\mathbb{R}^2}$. Thus we obtain that

$$H^1_{NP}(\eta_1) \cong (A_1/B_1)/C_1 \cong \tilde{\mathcal{F}}/\tilde{\mathcal{F}}_1 \cong \mathcal{I}_{\mathbb{R}^2}/\tilde{\mathcal{F}}_1\cap\mathcal{I}_{\mathbb{R}^2}.$$

For the computation of $H^2_{NP}(\eta_1)$, let $\gamma = Adx\wedge dy + Bdx\wedge dz + Cdx\wedge du + Ddy\wedge dz + Edy\wedge du + Fdz\wedge du$ be any element of $\tilde{Z}^2$. Then by Lemma 5.3 and by Lemma 5.4(b), there exists a function $k(x,y,z,u)\in\mathcal{F}$ such that

$$\begin{aligned} A_z - B_y + D_x &= \alpha uk, \\ A_u - C_y + E_x &= -(\alpha z+u)k. \end{aligned}$$

Then there exist two functions ϕ_1 and ϕ_2 of $\tilde{\mathcal{G}}$ such that D and E have the following expressions.

$$D = \alpha u \int kdx + \int B_y dx - \int A_z dx + \phi_1(y,z,u),$$
$$E = -(\alpha z + u)\int kdx + \int C_y dx - \int A_u dx + \phi_2(y,z,u).$$

Define a 1-form ϖ by $\varpi = Pdx + Qdy + Rdz + Sdu$. If we put $Q = \int Adx,\ \ R = \int Bdx,\ \ S = \int Cdx$, then

$$\begin{aligned} d\varpi =&(A - P_y)dx \wedge dy + (B - P_z)dx \wedge dz + (C - P_u)dx \wedge du \\ &+ \Big(\int B_y dx - \int A_z dx\Big)dy \wedge dz + \Big(\int C_y dx - \int A_u dx\Big)dy \wedge du \\ &+ \Big(\int C_z dx - \int B_u dx\Big)dz \wedge du. \end{aligned}$$

Thus we have

$$\begin{aligned} \gamma = d\varpi + d(x \cdot dP) + \Big(\alpha u \int kdx\Big)dy \wedge dz + \Big(-(\alpha z + u)\int kdx\Big)dy \wedge du \\ + \Big(F + \int B_u dx - \int C_z dx\Big)dz \wedge du + \phi_1 dy \wedge dz + \phi_2 dy \wedge du. \end{aligned}$$

The first five terms of γ belong to $\tilde{B}^2 + I^2$. It will be denoted by BI. Then $\gamma = BI + \phi_1 dy \wedge dz + \phi_2 dy \wedge du$. By Proposition 5.2, we can consider $H^2_{NP}(\eta_1)$ as $\{\phi_1(y,z,u)dy \wedge dz + \phi_2(y,z,u)dy \wedge du | \phi_1, \phi_2 \in \tilde{\mathcal{G}}\}$ modulo $\tilde{B}^2 + I^2$. Let us define some subspaces of the space of 2-forms on $\mathbb{R}^3(y,z,u)$ as follows.

$$\begin{aligned} U_2 &= \{\phi_1 dy \wedge dz + \phi_2 dy \wedge du \ \mid\ \phi_1, \phi_2 \in \tilde{\mathcal{G}}\}, \\ V_2 &= \{\phi_1 dy \wedge dz + \phi_2 dy \wedge du \in U_2 \ \mid\ (\phi_1)_u = (\phi_2)_z\}, \\ W_2 &= \{\alpha u \tilde{g} dy \wedge dz - (\alpha z + u)\tilde{g} dy \wedge du \ \mid\ \tilde{g} \in \tilde{\mathcal{G}}\}. \end{aligned}$$

Moreover put

$$U_3 = \{\tilde{h} dy \wedge dz \wedge du \ \mid\ \tilde{h} \in \tilde{\mathcal{G}}\}.$$

Since $dU_2 = U_3$, we know that $U_2/V_2 \cong U_3 \cong \tilde{\mathcal{G}}$. Note that $V_2 \subset \tilde{B}^2$ and $W_2 \subset I^2$. Then we have

$$\begin{aligned} H^2_{NP}(\eta_1) &\cong U_2/(V_2 + W_2) \\ &\cong (U_2/V_2)/((V_2 + W_2)/V_2) \\ &\cong (U_2/V_2)/(W_2/V_2 \cap W_2). \end{aligned}$$

Let $\alpha u \tilde{g} dy \wedge dz - (\alpha z + u)\tilde{g} dy \wedge du$ be any element of $V_2 \cap W_2$. Then $\tilde{g}$ must satisfy the equation $(\alpha z + u)\tilde{g}_z + \alpha u \tilde{g}_u = -2\alpha \tilde{g}$. Any solution of this equation has the form $\tilde{g}(y,z,u) = u^{-2}\psi(\frac{\alpha z - u \log u}{u}, y)$, where ψ is any function of 2-variables.

Hence C^∞-solution is only $\tilde{g} = 0$. This means that $V_2 \cap W_2 = 0$. We define a subspace $\tilde{\mathcal{G}}_1$ of $\tilde{\mathcal{G}}$ by

$$\tilde{\mathcal{G}}_1 = \{(\alpha z + u)\tilde{g}_z + \alpha u \tilde{g}_u + 2\alpha\tilde{g} \mid \tilde{g} \in \tilde{\mathcal{G}}\}.$$

Then it is clear that $W_2/V_2 \cap W_2 = W_2 \cong \tilde{\mathcal{G}}_1$. Let $\mathcal{I}_{\mathbb{R}^3}$ be the space of flat functions at the origin defined on $\mathbb{R}^3(y, z, u)$. By the analogous consideration as the case of $H^1_{NP}(\eta_1)$, we obtain

$$H^2_{NP}(\eta_1) \cong \tilde{\mathcal{G}}/\tilde{\mathcal{G}}_1 \cong \mathcal{I}_{\mathbb{R}^3}/\tilde{\mathcal{G}}_1 \cap \mathcal{I}_{\mathbb{R}^3}.$$

For the computation of $H^3_{NP}(\eta_1)$, let $\delta = Adx \wedge dy \wedge dz + Bdx \wedge dy \wedge du + Cdx \wedge dz \wedge du + Ddy \wedge dz \wedge du$ be any element of $\tilde{Z}^3 = \Omega^3$. For this δ, put $\rho = -(\int Ady)dx \wedge dz - (\int Bdy)dx \wedge du$ and put $\lambda = (C - \int B_z dy + \int A_u dy)dx \wedge dz \wedge du + Ddy \wedge dz \wedge du$. Then by Lemma 5.1(a), we have $\delta = d\rho + \lambda \in \tilde{B}^3 + I^3$. This implies $H^3_{NP}(\eta_1) = 0$. □

Remark 5.1. In computing $H^1_{NP}(\eta_1)$ and $H^2_{NP}(\eta_1)$, we mentioned the last isomorphisms by using $\mathcal{I}_{\mathbb{R}^2}$ and $\mathcal{I}_{\mathbb{R}^3}$. The same facts also hold for η_2, η_3 and η_4.

For other Nambu-Poisson tensors η_i, $2 \le i \le 7$, we can compute the corresponding Nambu-Poisson cohomologies by using the analogous methods as in the case of η_1 except for the slight modification. So we state only the results of computations by emphasizing the differences between the cases of η_i, $2 \le i \le 7$ and that of η_1.

The results including Theorem 5.5 are summarized in the following table. Each H^*_{NP} is described under "isomorphism". For example, in η_1-case, we should read that H^1_{NP} is "isomorphic" to $\tilde{\mathcal{F}}/\tilde{\mathcal{F}}_1$.

Table 3. Nambu-Poisson Cohomology of Type (II)

cohomology	H^0_{NP}	H^1_{NP}	H^2_{NP}	$H^k_{NP},\ k \ge 3$
η_1	$\mathbb{R}$	$\tilde{\mathcal{F}}/\tilde{\mathcal{F}}_1$	$\tilde{\mathcal{G}}/\tilde{\mathcal{G}}_1$	0
η_2	$U \subset C^\infty(\mathbb{R})$	$\tilde{\mathcal{F}}/\tilde{\mathcal{F}}_2$	$\tilde{\mathcal{G}}/\tilde{\mathcal{G}}_2$	0
η_3	$\mathbb{R}$	$\tilde{\mathcal{F}}/\tilde{\mathcal{F}}_3$	$\tilde{\mathcal{G}}/\tilde{\mathcal{G}}_3$	0
η_4	$\mathbb{R}$	$\tilde{\mathcal{F}}/\tilde{\mathcal{F}}_4$	$\tilde{\mathcal{G}}/\tilde{\mathcal{G}}_4$	0
η_5	$C^\infty(\mathbb{R}^+)$	$C^\infty(\mathbb{R}^+)$	$C^\infty(\mathbb{R}^2_+)$	0
η_6	$C^\infty(\mathbb{R})$	0	0	0
η_7	$C^\infty(\mathbb{R})$	0	0	0

In the above Table 3, we used the following notations:

- U is a subspace of $C^\infty(\mathbb{R})$;
- $\tilde{\mathcal{F}}_1 = \{(\alpha z + u)\tilde{h}_z + \alpha u \tilde{h}_u + 2\alpha \tilde{h} \mid \tilde{h} \in \tilde{\mathcal{F}}\}$;
- $\tilde{\mathcal{G}}_1 = \{(\alpha z + u)\tilde{g}_z + \alpha u \tilde{g}_u + 2\alpha \tilde{g} \mid \tilde{g} \in \tilde{\mathcal{G}}\}$;
- $\tilde{\mathcal{F}}_2 = \{(\alpha + \beta)\tilde{h} + \alpha z \tilde{h}_z + \beta u \tilde{h}_u \mid \tilde{h} \in \tilde{\mathcal{F}}\}$;
- $\tilde{\mathcal{G}}_2 = \{\alpha z \tilde{g}_z + \beta u \tilde{g}_u + (\alpha + \beta)\tilde{g} \mid \tilde{g} \in \tilde{\mathcal{G}}\}$;
- $\tilde{\mathcal{F}}_3 = \{(\alpha z - \beta u)\tilde{h}_z + (\beta z + \alpha u)\tilde{h}_u + 2\alpha \tilde{h} \mid \tilde{h} \in \tilde{\mathcal{F}}\}$;
- $\tilde{\mathcal{G}}_3 = \{(\alpha z - \beta u)\tilde{g}_z + (\beta z + \alpha u)\tilde{g}_u + 2\alpha \tilde{g} \mid \tilde{g} \in \tilde{\mathcal{G}}\}$;
- $\tilde{\mathcal{F}}_4 = \{z\tilde{h}_z + u\tilde{h}_u + 2\tilde{h} \mid \tilde{h} \in \tilde{\mathcal{F}}\}$;
- $\tilde{\mathcal{G}}_4 = \{z\tilde{g}_z + u\tilde{g}_u + 2\tilde{g} \mid \tilde{g} \in \tilde{\mathcal{G}}\}$;
- $C^\infty(\mathbb{R}^+)$ is a subspace of $C^\infty(\mathbb{R})$ consisting of functions which are defined on $\mathbb{R}^+$;
- $C^\infty(\mathbb{R}^2_+)$ is a subspace of $C^\infty(\mathbb{R}^2)$ consisting of functions whose second variable is defined only on $\mathbb{R}^+$.

Remark 5.2. If we compute H^*_{NP} in the category of formal functions (in short, in the formal category), we have the following results.

(1) In cases of η_1, η_3 and η_4, then we have $H^1_{NP} = H^2_{NP} = 0$.

(2) In case of η_2, put $U = H^0_{NP}$. If $\alpha\beta > 0$, then $U \cong \mathbb{R}$. On the contrary, if α and β are integers which satisfy $\alpha\beta < 0$, then $U \cong C^\infty(\mathbb{R})$. Let $\alpha = q/p$ and $\beta = s/r$ be two irreducible rational numbers with $\alpha\beta < 0$, and put $d =$L.C.M of $\{p, r\}$. Then U is a subspace of $C^\infty(\mathbb{R})$ generated by $\phi(t) = t^{kd}, \ \ k = 0, 1, 2, ...$

If β/α is a negative rational number, then H^1_{NP} and H^2_{NP} are infinite dimensional in the formal category. Hence they are also infinite dimensional in the C^∞-category. If β/α is a positive rational number or an irrational number, then $H^1_{NP} = H^2_{NP} = 0$ in the formal category.

§5.4. Computation of Nambu-Poisson cohomology of type (III)

By an easy consideration, we know that $\sharp_2(\Omega^2) = \mathfrak{g}_\phi$ is spanned by $\langle \phi\frac{\partial}{\partial x}, \phi\frac{\partial}{\partial y}, \phi\frac{\partial}{\partial z}\rangle$ over $\mathcal{F}$. Moreover we know that each $I^i, \ 1 \leq i \leq 4$ coincides with (f) of Lemma 5.1. Hence each Nambu-Poisson cohomology of $H^k_{NP}(\eta_\phi)$ of Type (III) is completely isomorphic to that of $H^k_{NP}(\eta_6)$. Thus we have

Proposition 5.6. *Let $\eta_\phi = \phi\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y} \wedge \frac{\partial}{\partial z}$, where ϕ is a linear function on $\mathbb{R}^4$. Then we have*

$$H^0_{NP}(\eta_\phi) \cong C^\infty(\mathbb{R}),$$
$$H^k_{NP}(\eta_\phi) = 0, \ \ k \geq 1.$$

§5.5. Computation of Nambu-Poisson cohomology of type (VI)

We will only treat here the generic case. Namely we suppose that there exists non-zero constant k such that $b_3 = ka_3$, $b_4 = ka_4$. Then we have $p = -k(q+1) = -k(-kr+2)$ and $q - 1 = -kr$. Now a Nambu-Poisson tensor η_ψ can be written as

$$\eta_\psi = \{(-kr+2)x + ry + a_3z + a_4u\}\Big(k\frac{\partial}{\partial y} \wedge \frac{\partial}{\partial z} \wedge \frac{\partial}{\partial u} + \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial z} \wedge \frac{\partial}{\partial u}\Big).$$

Then the Lie algebra $\mathfrak{g}$ corresponding to η_ψ is as follows.

$$\mathfrak{g} = \Big\langle x\frac{\partial}{\partial x} + kx\frac{\partial}{\partial y}, y\frac{\partial}{\partial x} + ky\frac{\partial}{\partial y}, z\frac{\partial}{\partial x} + kz\frac{\partial}{\partial y}, u\frac{\partial}{\partial x} + ku\frac{\partial}{\partial y},$$
$$x\frac{\partial}{\partial z}, y\frac{\partial}{\partial z}, z\frac{\partial}{\partial z}, u\frac{\partial}{\partial z}, x\frac{\partial}{\partial u}, y\frac{\partial}{\partial u}, z\frac{\partial}{\partial u}, u\frac{\partial}{\partial u}\Big\rangle.$$

Recall that I^k is a subspace of Ω^k whose element $c \in I^k$ satisfies $c(\mathfrak{g}, \dots, \mathfrak{g}) = 0$.

Lemma 5.7. *Let $A, B, C, D, E \in \mathcal{F}$. Then we have*

$$\begin{aligned}
I^1 =& \{Adx + Bdy \mid A + kB = 0\},\\
I^2 =& \{Adx \wedge dy + Bdx \wedge dz + Cdx \wedge du + Ddy \wedge dz\\
& \quad + Edy \wedge du \mid B + kD = 0, \ C + kE = 0\},\\
I^3 =& \{Adx \wedge dy \wedge dz + Bdx \wedge dy \wedge du + Cdx \wedge dz \wedge du\\
& \quad + Ddy \wedge dz \wedge du \mid C + kD = 0\},\\
I^4 =& \Omega^4.
\end{aligned}$$

Proof. Straightforward calculation. □

Theorem 5.8.

$$\begin{aligned}
H^0_{NP}(\eta_\psi) &\cong C^\infty(\mathbb{R}),\\
H^1_{NP}(\eta_\psi) &\cong C^\infty(\mathbb{R}^2)/C^\infty(\mathbb{R}),\\
H^2_{NP}(\eta_\psi) &\cong C^\infty(\mathbb{R}^3)/C^\infty(\mathbb{R}^2),\\
H^3_{NP}(\eta_\psi) &\cong \mathcal{F}/C^\infty(\mathbb{R}^3),\\
H^k_{NP}(\eta_\psi) &= 0, \ \ k \geq 4,
\end{aligned}$$

where $\mathcal{F} = C^\infty(\mathbb{R}^4)$.

Proof. To compute $H^*_{NP}(\eta_\psi)$, we will use Proposition 5.2 again. The space $H^0_{NP}(\eta_\psi)$ is consisting of functions $f \in \mathcal{F}$ which are $\mathfrak{g}$-invariant. Hence each $f \in H^0_{NP}(\eta_\psi)$ must satisfy $f = f(x,y)$ and $r \cdot f_x + k \cdot r \cdot f_y = 0$ for any linear function r on $\mathbb{R}^4$. These conditions are easily lead us to the fact that $f = \phi(kx - y)$, where ϕ is any C^∞-function of one variable. Hence $H^0_{NP}(\eta_\psi) \cong C^\infty(\mathbb{R})$.

Next let us compute $H^1_{NP}(\eta_\psi)$. Put $c = Adx + Bdy + Cdz + Ddu$. Then $c \in \tilde{Z}^1$ if and only if

$$\begin{cases} D_z = C_u, \\ C_x - A_z + k(C_y - B_z) = 0, \\ D_x - A_u + k(D_y - B_u) = 0. \end{cases}$$

By the first equation, there exists a function $h \in \mathcal{F}$ such that $C = h_z,\ \ D = h_u$. Substituting these equations into second and third equations, we have

$$\frac{\partial}{\partial z}(h_x - A + k(h_y - B)) = 0,$$
$$\frac{\partial}{\partial u}(h_x - A + k(h_y - B)) = 0.$$

Hence we know that there exists a function $S(x,y)$ such that

$$A = h_x + kh_y - kB - S(x,y).$$

Then $c \in \tilde{Z}^1$ can be rewritten as follows.

$$\begin{aligned} c =& (h_x + kh_y - kB - S(x,y))dx + Bdy + h_z dz + h_u du \\ =& dh + kh_y dx - h_y dy - kBdx + Bdy - S(x,y)dx. \end{aligned}$$

Since $dh + kh_y dx - h_y dy - kBdx + Bdy$ is an element of $\tilde{B}^1 + I^1$ by Lemma 5.7, we have $c \equiv -S(x,y)dx \pmod{\tilde{B}^1 + I^1}$. Moreover $S(x,y)dx \in \tilde{B}^1$ if and only if $S(x,y) = S(x)$. Hence we finally obtain that $H^1_{NP}(\eta_\psi) \cong C^\infty(\mathbb{R}^2)/C^\infty(\mathbb{R})$ by Proposition 5.2.

Next let us compute $H^2_{NP}(\eta_\psi)$. By Proposition 5.2, $c = Adx \wedge dy + Bdx \wedge dz + Cdx \wedge du + Ddy \wedge dz + Edy \wedge du + Fdz \wedge du$ is contained in $\tilde{Z}^2$ if and only if

$$B_u - C_z + F_x + k(D_u - E_z + F_y) = 0.$$

This equation is equivalent to

$$B + kD = \int (C + kE)_z du - \int (F_x + kF_y) du + \phi(x,y,z),$$

for some C^∞-function $\phi(x,y,z)$. Since $Adx\wedge dy - kDdx\wedge dz - kEdx\wedge du + Ddy\wedge dz + Edy\wedge du$ is an element of I^2 by Lemma 5.7, we have

$$c \equiv (B+kD)dx\wedge dz + (C+kE)dx\wedge du + Fdz\wedge du \pmod{I^2}.$$

Thus $c\in\tilde{Z}^2$ can be rewritten as follows.

$$\begin{aligned} c \equiv &\Big(\int (C+kE)_z du - \int (F_x+kF_y)du + \phi(x,y,z)\Big)dx\wedge dz \\ &+ (C+kE)dx\wedge du + Fdz\wedge du \pmod{I^2}. \end{aligned}$$

Put $\rho = -(\int(C+kE)du)dx$, and $\delta = -(\int Fdu)dz$. Then

$$\begin{aligned} \tilde{B}^2 \ni d\rho = &(C+kE)dx\wedge du + \Big(\int (C+kE)_y du\Big)dx\wedge dy \\ &+ \Big(\int (C+kE)_z du\Big)dx\wedge dz, \end{aligned}$$

and

$$\begin{aligned} \tilde{B}^2 \ni d\delta = &-\Big(\int F_x du\Big)dx\wedge dz - \Big(\int F_y du\Big)dy\wedge dz \\ &+ Fdz\wedge du. \end{aligned}$$

Hence

$$\begin{aligned} c \equiv &-\Big(\int (C+kE)_y du\Big)dx\wedge dy - k\Big(\int F_y du\Big)dx\wedge dz \\ &+ \Big(\int F_y du\Big)dy\wedge dz + \phi(x,y,z)dx\wedge dz \pmod{\tilde{B}^2 + I^2}. \end{aligned}$$

Recall that $-k(\int F_y du)dx\wedge dz + (\int F_y du)dy\wedge dz$, and $-(\int(C+kE)_y du)dx\wedge dy$ are elements of I^2. Thus $c\equiv \phi(x,y,z)dx\wedge dz \pmod{\tilde{B}^2+I^2}$. $\phi(x,y,z)dx\wedge dz$ is an exact 2-form if and only if $\phi(x,y,z)=\phi(x,z)$. Thus we have $H^2_{NP}(\eta_\psi)\cong C^\infty(\mathbb{R}^3)/C^\infty(\mathbb{R}^2)$.

To compute $H^3_{NP}(\eta_\psi)$, put $c = Adx\wedge dy\wedge dz + Bdx\wedge dy\wedge du + Cdx\wedge dz\wedge du + Ddy\wedge dz\wedge du \in \tilde{Z}^3 = \Omega^3$. Since $Adx\wedge dy\wedge dz + Bdx\wedge dy\wedge du - kDdx\wedge dz\wedge du + Ddy\wedge dz\wedge du$ is contained in I^3 by Lemma 5.7, we have $c\equiv (C+kD)dx\wedge dz\wedge du \pmod{I^3}$. Note that 3-form $(C+kD)dx\wedge dz\wedge du$ is contained in $\tilde{B}^3$ if and only if $\frac{\partial}{\partial y}(C+kD)=0$. Then using Proposition 5.2, we have $H^3_{NP}(\eta_\psi)\cong \mathcal{F}/C^\infty(\mathbb{R}^3)$.

For $k\geq 4$, it is clear that $H^k_{NP}(\eta_\psi)=0$, since $\Lambda^k\mathfrak{g}=0$. $\square$

§6. Computation of Nambu-Poisson Cohomology: Quadratic Case

§6.1. Notation and general remarks

In this section we compute Nambu-Poisson cohomology in the case of quadratic Nambu-Poisson tensor. Let us consider $\eta = (x^2+y^2+z^2+u^2)\frac{\partial}{\partial x}\wedge\frac{\partial}{\partial y}\wedge\frac{\partial}{\partial z}$, which is a Nambu-Poisson tensor of order 3 on $\mathbb{R}^4(x, y, z, u)$. As usual, we denote the Nambu-Poisson cohomology of $(\mathbb{R}^4, \eta)$ by $H^*_{NP}(\mathbb{R}^4, \eta)$. To compute $H^*_{NP}(\mathbb{R}^4, \eta)$, we will essentially use the result of computations of $H^*_{NP}(\mathbb{R}^3, \eta')$, where $\eta' = (x^2 + y^2 + z^2)\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y} \wedge \frac{\partial}{\partial z}$.

First of all we review an equivalent cohomology to Nambu-Poisson cohomology, which is due to P. Monnier [9]. Let M be an m-dimensional C^∞-manifold with a volume form Ω. For $h \in C^\infty(M)$, we define the operator

$$\begin{aligned} d_h : \Omega^k(M) &\longrightarrow \Omega^{k+1}(M) \\ \alpha &\mapsto hd\alpha - kdh \wedge \alpha. \end{aligned}$$

It is easy to prove that $d_h \circ d_h = 0$. We denote by $H^*_h(M)$ the cohomology of this complex. Let η be an element of $\Gamma(\Lambda^m(TM))$. Recall that such η becomes always a Nambu-Poisson tensor [10]. Then P. Monnier proved the following [9].

Proposition 6.1. *If we put $h = i_\eta\Omega$, then $H^*_{NP}(M, \eta)$ is isomorphic to $H^*_h(M)$.*

Remark 6.1. It is easy to see that if g is a function on M which does not vanish on M, then the cohomologies $H^*_h(M)$ and $H^*_{hg}(M)$ are isomorphic.

Throughout this section, we will use the following notations:

- $\mathcal{F}$ is the algebra of real-valued C^∞ functions on $\mathbb{R}^4(x, y, z, u)$;
- $\mathcal{F}'$ is the algebra of real-valued C^∞ functions on $\mathbb{R}^3(x, y, z)$;
- $\chi(\mathbb{R}^4)$ is the $\mathcal{F}$-module of vector fields on $\mathbb{R}^4$;
- $\chi'(\mathbb{R}^4) = \{A\frac{\partial}{\partial x} + B\frac{\partial}{\partial y} + C\frac{\partial}{\partial z} | A, B, C \in \mathcal{F}\}$;
- $f = x^2 + y^2 + z^2 + u^2$;
- $f' = x^2 + y^2 + z^2$;
- Ω^k =the space of k-forms on $\mathbb{R}^4$;
- $\Omega'_1 = \{Adx + Bdy + Cdz | A, B, C \in \mathcal{F}\}$;
- $\Omega'_2 = \{Ady \wedge dz + Bdz \wedge dx + Cdx \wedge dy | A, B, C \in \mathcal{F}\}$;
- $\Omega'_3 = \{Adx \wedge dy \wedge dz | A \in \mathcal{F}\}$.

If we choose $\Omega = dx \wedge dy \wedge dz$ as the volume form on $\mathbb{R}^3$, then we have $f' = i_{\eta'}\Omega$. First we compute $H^*_{NP}(\mathbb{R}^3, \eta')$, which is isomorphic to $H^*_{f'}(\mathbb{R}^3)$ by Proposition 6.1. In the formal category (i.e. all coefficients of differential forms are formal power series), the following results were obtained by P. Monnier [9].

Proposition 6.2. *In the formal case,* $H^0_{f'} \cong \mathbb{R}$, $H^1_{f'} \cong \mathbb{R}$, $H^2_{f'} = 0$ *and* $H^3_{f'} \cong \mathbb{R}$.

We want to compute $H^*_{f'}$ in the C^∞-category, and we will show that Proposition 6.2 still holds even in the C^∞-category. First it is clear that $H^0_{f'} \cong \mathbb{R}$. R. Ibáñez *et al.* [7] proved independently of P. Monnier [9] that $H^1_{f'} \cong \mathbb{R}$. Hence it only remains to compute $H^2_{f'}$ and $H^3_{f'}$. To compute them, we use Proposition 6.2.

Let β be a 2-cocycle. Then by definition, β satisfies $f'd\beta = 2df' \wedge \beta$. Denote by $[\beta]$ the formal Taylor expansion of β at the origin. Then by Proposition 6.2, there exists a formal 1-form $[\alpha]$ such that $[\beta] = f'd[\alpha] - df' \wedge [\alpha]$. Hence we can find a 1-form α, whose formal Taylor expansion at the origin is $[\alpha]$. Put $\beta' = \beta - (f'd\alpha - df' \wedge \alpha)$. Then β' is flat (i.e. $[\beta'] = 0$) and satisfies $f'd\beta' = 2df' \wedge \beta'$. $\frac{\beta'}{f'^2}$ is also flat and $d(\frac{\beta'}{f'^2}) = \frac{1}{f'^3}(f'd\beta' - 2df' \wedge \beta') = 0$. Hence there exists a flat 1-form $\tilde{\alpha}$ such that $\frac{\beta'}{f'^2} = d\tilde{\alpha}$. Put $\tilde{\alpha} = \frac{\alpha'}{f'}$. Then α' is a flat 1-form, and we get $\beta' = {f'}^2 d\tilde{\alpha} = f'd\alpha' - df' \wedge \alpha'$. Finally we have

$$\beta = f'd(\alpha + \alpha') - df' \wedge (\alpha + \alpha').$$

This means $H^2_{f'} = 0$.

Next let us compute $H^3_{f'}$. The space of 3-cocycles $Z^3_{f'}$ is clearly isomorphic to $\mathcal{F}'$. And the space of 3-coboundaries $B^3_{f'}$ is isomorphic to the following space $\mathcal{F}_1$.

$$\mathcal{F}_1 = \left\{ f' \left(\frac{\partial A}{\partial x} + \frac{\partial B}{\partial y} + \frac{\partial C}{\partial z} \right) - 4(xA + yB + zC); A, B, C \in \mathcal{F}' \right\}.$$

Lemma 6.3. *Let $\mathcal{I}$ be the subspace of $\mathcal{F}'$ consisting of functions which are flat at the origin. Then $\mathcal{I} \subset \mathcal{F}_1$.*

Proof. For $q \in \mathcal{I}$, put

$$A = (f')^2 \int \frac{q}{(f')^3} dx, \ \ B = 0, \ \ C = 0.$$

Then $f'(\frac{\partial A}{\partial x} + \frac{\partial B}{\partial y} + \frac{\partial C}{\partial z}) - 4(xA + yB + zC) = q$. Hence we have that $q \in \mathcal{F}_1$. $\square$

Denote by F' (resp. F_1) the formal algebra corresponding to $\mathcal{F}'$ (resp. $\mathcal{F}_1$). Let T be a mapping from $\mathcal{F}'$ to F', where $T(h)$ is the formal Taylor expansion of h at the origin. Let $\pi : F' \to F'/F_1$ be the canonical projection, and put $\tilde{T} = \pi \circ T$. Then $\tilde{T}$ is a surjective linear mapping and it is clear that $\ker \tilde{T} = \mathcal{F}_1$ by Lemma 6.3. Since $F'/F_1 \cong \mathbb{R}$ by Proposition 6.2, we get that

$$H^3_{f'} \cong \mathcal{F}'/\mathcal{F}_1 \cong F'/F_1 \cong \mathbb{R}.$$

Thus we obtained the following proposition.

Proposition 6.4. *In C^∞-case, it still holds that $H^0_{f'} \cong \mathbb{R}$, $H^1_{f'} \cong \mathbb{R}$, $H^2_{f'} = 0$ and $H^3_{f'} \cong \mathbb{R}$.*

For the Nambu-Poisson tensor $\eta = f\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y} \wedge \frac{\partial}{\partial z}$ defined on $\mathbb{R}^4$, we know that

$$\sharp_2(\Omega^2) = \{fX | X \in \chi'(\mathbb{R}^4)\}.$$

$\sharp_2(\Omega^2)$ is denoted by $\mathfrak{g}$, which is isomorphic to $\Omega^2/\ker \sharp_2$. Note also that $\Omega^2/\ker \sharp_2$ is isomorphic to Ω'_2. $\mathfrak{g}$ is, of course, a Lie subalgebra of $\chi(\mathbb{R}^4)$.

Since $H^0_{NP}(\mathbb{R}^4, \eta) = \{g \in \mathcal{F} | \ Xg = 0 \text{ for all } X \in \mathfrak{g}\}$, it is clear that $H^0_{NP}(\mathbb{R}^4, \eta) \cong C^\infty(\mathbb{R})$.

In computing Nambu-Poisson cohomology, we use Proposition 6.4. To do this, we need the formal Taylor expansion of a function $A \in \mathcal{F}$ with respect to the variable u, which is denoted by $\bar{A}$. In other words, three variables x, y and z are regarded as parameters. And we say that $\bar{A}$ is the *u-formal Taylor expansion* of A. This terminology will be also used for differential forms and vector fields. Thus we can express $\bar{A}$ (similarly $\bar{B}$ and $\bar{C}$) as follows.

$$\begin{cases} \bar{A} = a_0 + ua_1 + u^2a_2 + \cdots, \\ \bar{B} = b_0 + ub_1 + u^2b_2 + \cdots, \\ \bar{C} = c_0 + uc_1 + u^2c_2 + \cdots, \end{cases} \tag{3}$$

where $a_k, b_k, c_k \in \mathcal{F}'$.

To compute $H^k_{NP}(\mathbb{R}^4, \eta)$, $\ k \geq 1$, let us define a linear mapping $d' : \mathcal{F} \to \Omega'_1$ by

$$d'g = \frac{\partial g}{\partial x}dx + \frac{\partial g}{\partial y}dy + \frac{\partial g}{\partial z}dz.$$

This operator d' is naturally extended to a linear mapping from Ω'_k to Ω'_{k+1}. Moreover we define $d'_f : \Omega'_k \to \Omega'_{k+1}$ by

$$d'_f(\alpha) = fd'\alpha - kd'f \wedge \alpha, \ \ \alpha \in \Omega'_k.$$

Then $d'_f \circ d'_f = 0$, and we denote by $H^*_{d'_f}$ the cohomology space with respect to d'_f.

If we define $b : \chi'(\mathbb{R}^4) \to \Omega'_2$ by $b(X) = i(X)dx \wedge dy \wedge dz$, then we obtain that $\sharp_2(b(X)) = fX$ and that $\sharp_2(\{b(X), b(Y)\}) = [\sharp_2(b(X)), \sharp_2(b(Y))] = [fX, fY]$.

Following the similar method of P. Monnier [9], if $\phi : C^k(\Omega'_2, \mathcal{F}) \to \Omega'_k$ is defined by

$$\phi(c^k)(X_1, \dots, X_k) = c^k(b(X_1), \dots, b(X_k)), \quad X_1, \dots, X_k \in \chi'(\mathbb{R}^4),$$

then ϕ is a linear isomorphism and we can prove the following.

Proposition 6.5. *The following diagram is commutative.*

$$\begin{array}{ccc} C^k(\Omega'_2, \mathcal{F}) & \xrightarrow{\phi} & \Omega'_k \\ \partial \big\downarrow & & \big\downarrow d'_f \\ C^{k+1}(\Omega'_2, \mathcal{F}) & \xrightarrow[\phi]{} & \Omega'_{k+1} \end{array}$$

Hence $H^*_{NP}(\mathbb{R}^4, \eta) \cong H^*_{d'_f}$.

Proof. We prove only for the case $k = 1$. For $c \in C^1(\Omega'_2, \mathcal{F})$, put $\phi(c) = \alpha$. For any $X, Y \in \chi'(\mathbb{R}^4)$, we can directly get

$$\{b(X), b(Y)\} = f \cdot b([X, Y]) - (Xf) \cdot b(Y) + (Yf) \cdot b(X),$$

from the definition of the bracket $\{,\}$ on Ω'_2. Using this equation, we have

$$\begin{aligned} \phi(\partial c)(X, Y) &= (\partial c)(b(X), b(Y)) \\ &= fX \cdot c(b(Y)) - fY \cdot c(b(X)) - c(\{b(X), b(Y)\}) \\ &= fX \cdot \alpha(Y) - fY \cdot \alpha(X) - c(f \cdot b([X, Y]) \\ &\quad + (Xf) \cdot b(Y) - (Yf) \cdot b(X)) \\ &= fX \cdot \alpha(Y) - fY \cdot \alpha(X) - f\alpha([X, Y]) \\ &\quad - (Xf) \cdot \alpha(Y) + (Yf) \cdot \alpha(X) \\ &= f \cdot d'\alpha(X, Y) - (d'f \wedge \alpha)(X, Y) \\ &= (d'_f\alpha)(X, Y) = (d'_f \circ \phi(c))(X, Y). \end{aligned}$$

Thus $\phi \circ \partial = d'_f \circ \phi$. ☐

§6.2. Computation of $H^1_{NP}(\mathbb{R}^4, \eta)$

In this subsection, we compute $H^1_{NP}(\mathbb{R}^4, \eta)$. In order to do this, we have only to compute $H^1_{d'_f}$ by Proposition 6.5. The space of 1-coboundaries, which is denoted by B'_1, is the set of 1-forms $fd'g$, $g \in \mathcal{F}$. Let Z'_1 be the space of 1-cocycles. Then for $\alpha = Adx + Bdy + Cdz \in \Omega'_1$, α is an element of Z'_1 if and only if $fd'\alpha = d'f \wedge \alpha$. This equation is equivalent to the following three equations.

$$
(4) \qquad \begin{cases} f \cdot \Big(\dfrac{\partial B}{\partial x} - \dfrac{\partial A}{\partial y}\Big) = 2xB - 2yA, \\ f \cdot \Big(\dfrac{\partial C}{\partial y} - \dfrac{\partial B}{\partial z}\Big) = 2yC - 2zB, \\ f \cdot \Big(\dfrac{\partial A}{\partial z} - \dfrac{\partial C}{\partial x}\Big) = 2zA - 2xC. \end{cases}
$$

Note that the u-formal Taylor expansion of α is written as $\bar{\alpha} = \alpha_0 + u\alpha_1 + u^2\alpha_2 + \cdots$, where $\alpha_p = a_p dx + b_p dy + c_p dz, a_p, b_p, c_p \in \mathcal{F}'$. And three equations (4) induce the u-formal Taylor expansions. Comparing constant terms with respect to u in them, we have

$$
(5) \qquad \begin{cases} f' \cdot \Big(\dfrac{\partial b_0}{\partial x} - \dfrac{\partial a_0}{\partial y}\Big) = 2xb_0 - 2ya_0, \\ f' \cdot \Big(\dfrac{\partial c_0}{\partial y} - \dfrac{\partial b_0}{\partial z}\Big) = 2yc_0 - 2zb_0, \\ f' \cdot \Big(\dfrac{\partial a_0}{\partial z} - \dfrac{\partial c_0}{\partial x}\Big) = 2za_0 - 2xc_0. \end{cases}
$$

These three equations (5) essentially appeared in computing $H^1_{NP}(\mathbb{R}^3, \eta' = f'\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y} \wedge \frac{\partial}{\partial z})$. By Proposition 6.4, $H^1_{NP}(\mathbb{R}^3, \eta')$ is isomorphic to $\mathbb{R}$. The generator of $H^1_{NP}(\mathbb{R}^3, \eta')$ is df' and this means that there exist a real number k_0 and a function $g_0 \in \mathcal{F}'$ such that

$$
(6) \qquad \begin{cases} a_0 = k_0 \cdot 2x + f' \cdot \dfrac{\partial g_0}{\partial x}, \\ b_0 = k_0 \cdot 2y + f' \cdot \dfrac{\partial g_0}{\partial y}, \\ c_0 = k_0 \cdot 2z + f' \cdot \dfrac{\partial g_0}{\partial z}. \end{cases}
$$

Since $\alpha_0 = a_0dx + b_0dy + c_0dz$, we obtain that $\alpha_0 = k_0df' + f'dg_0$. Similarly if we compare the coefficients of u in the u-formal Taylor expansions, we can get $\alpha_1 = k_1 df' + f'dg_1$, where $k_1 \in \mathbb{R}$ and $g_1 \in \mathcal{F}'$. But if we compare the coefficients of u^2, the situation is slightly different. In fact, we have

$$\begin{cases} f' \cdot \Big(\dfrac{\partial b_2}{\partial x} - \dfrac{\partial a_2}{\partial y}\Big) + \Big(\dfrac{\partial b_0}{\partial x} - \dfrac{\partial a_0}{\partial y}\Big) = 2xb_2 - 2ya_2, \\ f' \cdot \Big(\dfrac{\partial c_2}{\partial y} - \dfrac{\partial b_2}{\partial z}\Big) + \Big(\dfrac{\partial c_0}{\partial y} - \dfrac{\partial b_0}{\partial z}\Big) = 2yc_2 - 2zb_2, \\ f' \cdot \Big(\dfrac{\partial a_2}{\partial z} - \dfrac{\partial c_2}{\partial x}\Big) + \Big(\dfrac{\partial a_0}{\partial z} - \dfrac{\partial c_0}{\partial x}\Big) = 2za_2 - 2xc_2. \end{cases} \tag{7}$$

These equations (7) can be rewritten as follows.

$$\begin{cases} f'\Big(\dfrac{\partial(b_2 - \frac{\partial g_0}{\partial y})}{\partial x} - \dfrac{\partial(a_2 - \frac{\partial g_0}{\partial x})}{\partial y}\Big) = 2x\Big(b_2 - \dfrac{\partial g_0}{\partial y}\Big) - 2y\Big(a_2 - \dfrac{\partial g_0}{\partial x}\Big), \\ f'\Big(\dfrac{\partial(c_2 - \frac{\partial g_0}{\partial z})}{\partial y} - \dfrac{\partial(b_2 - \frac{\partial g_0}{\partial y})}{\partial z}\Big) = 2y\Big(c_2 - \dfrac{\partial g_0}{\partial z}\Big) - 2z\Big(b_2 - \dfrac{\partial g_0}{\partial y}\Big), \\ f'\Big(\dfrac{\partial(a_2 - \frac{\partial g_0}{\partial x})}{\partial z} - \dfrac{\partial(c_2 - \frac{\partial g_0}{\partial z})}{\partial x}\Big) = 2z\Big(a_2 - \dfrac{\partial g_0}{\partial x}\Big) - 2x\Big(c_2 - \dfrac{\partial g_0}{\partial z}\Big). \end{cases} \tag{8}$$

Thus we can apply Proposition 6.4 to (8), and we have that there exist a real number k_2 and $g_2 \in \mathcal{F}'$ such that

$$\begin{cases} a_2 - \dfrac{\partial g_0}{\partial x} = k_2 \cdot 2x + f'\dfrac{\partial g_2}{\partial x}, \\ b_2 - \dfrac{\partial g_0}{\partial y} = k_2 \cdot 2y + f'\dfrac{\partial g_2}{\partial y}, \\ c_2 - \dfrac{\partial g_0}{\partial z} = k_2 \cdot 2z + f'\dfrac{\partial g_2}{\partial z}. \end{cases} \tag{9}$$

Hence $\alpha_2 = k_2 df' + f'dg_2 + dg_0$. By the same methods, we know that each α_p, $(p \geq 3)$ has the form $\alpha_p = k_p df' + f'dg_p + dg_{p-2}$, where $k_p \in \mathbb{R}$ and $g_{p-2}, g_p \in \mathcal{F}'$. These mean that $\bar{\alpha}$ has the following expression. Note that $df' = d'f$ and that $f' + u^2 = f$.

$$\bar{\alpha} = (k_0 + k_1 u + k_2 u^2 + \cdots)d'f + f \cdot d'(g_0 + ug_1 + u^2 g_2 + \cdots).$$

To obtain the final result, we need the following lemma, which is a generalization of E. Borel theorem. This will be proved in the analogous way as K. Abe and K. Fukui, Lemma 4.4 [1]. (See also R. Narasimhan [12], §1.5.2 and §1.5.3.) We put $\vec{r} = (x, y, z, u)$ and $|\vec{r}| = \sqrt{x^2 + y^2 + z^2 + u^2}$. Then a function $F(\vec{r}) \in C^\infty(\mathbb{R}^4)$ is said to be *m-flat* as a function of u at $(x, y, z, 0)$ if $\frac{\partial^\alpha}{\partial u^\alpha} F(x, y, z, 0) = 0$ for $\alpha \leq m$.

Lemma 6.6. *For each integer $p \geq 0$, let $c_p(x,y,z) \in C^\infty(\mathbb{R}^3)$. Then there exists $G(\vec{r}) \in C^\infty(\mathbb{R}^4)$ such that the partial derivatives with respect to the last variable of G at any point $(x,y,z,0) \in \mathbb{R}^4$ are*

$$\frac{\partial^p G}{\partial u^p}(x,y,z,0) = p! c_p(x,y,z) \quad p \geq 0.$$

Proof. Let $T_m(\vec{r}) = \sum_{p=0}^{m} c_p(x,y,z)u^p$ for $\vec{r} \in \mathbb{R}^4$. Let $H(\vec{r}) \in C^\infty(\mathbb{R}^4)$ such that $H(\vec{r}) = 0$ for $|\vec{r}| \leq 1/2$, $H(\vec{r}) = 1$ for $|\vec{r}| \geq 1$ and $H(\vec{r}) \geq 0$ for any $\vec{r} \in \mathbb{R}^4$. For a positive number δ, put

$$g_\delta(\vec{r}) = H\Big(\frac{\vec{r}}{\delta}\Big)(T_{m+1}(\vec{r}) - T_m(\vec{r})).$$

Clearly $g_\delta \in C^\infty(\mathbb{R}^4)$ and vanishes near 0. Moreover $T_{m+1} - T_m$ is m-flat as a function of u at any point $(x,y,z,0)$. Hence as in the proof of Lemma 1.5.2 [12], there exists a positive number δ_m such that

$$\sum_{p=0}^{m} \frac{1}{p!}\Big|\frac{\partial^p}{\partial u^p}(g_{\delta_m} - (T_{m+1} - T_m))(\vec{r})\Big| < 2^{-m}.$$

Put $g_m = g_{\delta_m}$. If we define

$$G = T_0 + \sum_{m=0}^{\infty}(T_{m+1} - T_m - g_m),$$

then as in the proof of Lemma 1.5.3 [12], we get that the function G is the desired function. □

By Lemma 6.6, we obtain that there exist a C^∞-function $k(u)$ and a C^∞-function $g(x,y,z,u)$ such that $\overline{k(u)} = k_0 + k_1 u + k_2 u^2 + \cdots$, and $\overline{g(x,y,z,u)} = g_0 + ug_1 + u^2 g_2 + \cdots$. Put $\alpha' = k(u)d'f + fd'g$, and put $\alpha - \alpha' = \alpha_f$. Then α_f is a 1-cocycle and it satisfies $\overline{\alpha_f} = 0$ (u-flat 1-form). Let $k_1(u)$ be a flat function of one variable u. Then $(\alpha_f - k_1(u)d'f)/f$ is a well-defined 1-form on $\mathbb{R}^4$, and it satisfies

$$d'\Big(\frac{\alpha_f - k_1(u)d'f}{f}\Big) = \frac{1}{f^2}(fd'\alpha_f - d'f \wedge (\alpha_f - k_1(u)d'f)) = 0.$$

Hence, as is easily seen, there exists a flat function $\tilde{g}(x,y,z,u)$ such that $(\alpha_f - k_1(u)d'f)/f = d'\tilde{g}$. And we obtain that $\alpha \in Z_1'$ has the following form:

$$\alpha = \alpha_f + \alpha' = (k(u) + k_1(u))d'f + fd'(g + \tilde{g}).$$

α is, by definition, cohomologous to $(k(u)+k_1(u))d'f$. Moreover $l(u)d'f$ is contained in B'_1 if and only if $l(u)$ is a flat function at $u=0$. In fact, note that in this case $l(u)\log f$ is a C^∞-function and it holds that $l(u)d'f = fd'(l(u)\log f) \in B'_1$. Thus we obtain that $H^1_{NP}(\mathbb{R}^4,\eta)$ is isomorphic to $\mathbb{R}[[u]]$, which is the space of formal power series of one variable u.

§6.3. Computation of $H^2_{NP}(\mathbb{R}^4,\eta)$

We will compute $H^2_{NP}(\mathbb{R}^4,\eta)$. By Proposition 6.5, we will compute $H^2_{d'_f}$. Every computation proceeds in the analogous way as the case of $H^1_{d'_f}$. The space of 2-coboundaries B'_2 is, by definition, the set of 2-forms $d'_f\gamma = fd'\gamma - d'f\wedge\gamma,\ \ \gamma\in\Omega'_1$. Let Z'_2 be the space of 2-cocycles. Then for $\beta = Ady\wedge dz + Bdz\wedge dx + Cdx\wedge dy \in \Omega'_2$, β is an element of Z'_2 if and only if $fd'\beta = 2d'f\wedge\beta$. This is equivalent to

$$f\cdot\Big(\frac{\partial A}{\partial x}+\frac{\partial B}{\partial y}+\frac{\partial C}{\partial z}\Big) = 4(xA+yB+zC). \tag{10}$$

The u-formal Taylor expansion (with respect to u) of β is written as $\bar\beta = \beta_0+u\beta_1+u^2\beta_2+\cdots$, where $\beta_p = a_p dy\wedge dz + b_p dz\wedge dx + c_p dx\wedge dy,\ \ a_p, b_p, c_p\in\mathcal{F}'$. Then the equation (10) has the u-formal Taylor expansion.

Comparing constant terms in it, we have

$$f'\cdot\Big(\frac{\partial a_0}{\partial x}+\frac{\partial b_0}{\partial y}+\frac{\partial c_0}{\partial z}\Big) = 4(xa_0+yb_0+zc_0). \tag{11}$$

This is equivalent to $d_{f'}\beta_0 = 0$ for $\beta_0 = a_0 dy\wedge dz + b_0 dz\wedge dx + c_0 dx\wedge dy$. Recall that $H^2_{NP}(\mathbb{R}^3,\eta') = 0$ by Proposition 6.4. In other words, if $d_{f'}\beta_0 = 0$, then β_0 must be a coboundary. This means that we can find a 1-form α_0 such that $\beta_0 = f'd\alpha_0 - df'\wedge\alpha_0$.

Comparing the coefficients of u, we can also find a 1-form α_1 such that $\beta_1 = f'd\alpha_1 - df'\wedge\alpha_1$. Moreover if $p\ge 2$ we can find p-form α_p such that $\beta_p = f'd\alpha_p - d'f\wedge\alpha_p + d\alpha_{p-2}$. The u-formal Taylor expansion of β is as follows.

$$\begin{aligned}\bar\beta &= \sum_{p=0}^{\infty} u^p\beta_p\\ &= \sum_{p=0}^{\infty} u^p(f'd\alpha_p - d'f\wedge\alpha_p) + \sum_{p=0}^{\infty} u^{p+2}d\alpha_p\end{aligned}$$

$$= \sum_{p=0}^{\infty} u^p (f' d\alpha_p - df' \wedge \alpha_p + u^2 d\alpha_p)$$
$$= \sum_{p=0}^{\infty} u^p (f d\alpha_p - d' f \wedge \alpha_p)$$
$$= f d' \Big(\sum_{p=0}^{\infty} u^p \alpha_p \Big) - d' f \wedge \Big(\sum_{p=0}^{\infty} u^p \alpha_p \Big).$$

Put $\hat{\alpha} = \sum_{p=0}^{\infty} u^p \alpha_p$. Then $\bar{\beta} = f d' \hat{\alpha} - d' f \wedge \hat{\alpha}$. By Lemma 6.6, there exists a 1-form $\alpha' \in \Omega_1'$ such that $\bar{\alpha'} = \hat{\alpha}$. Put $\beta' = f d' \alpha' - d' f \wedge \alpha'$. Then $\bar{\beta} = \bar{\beta}'$ and hence if we put $\tilde{\beta} = \beta - \beta'$, then $\tilde{\beta}$ is a flat 2-form of Ω_2'. Moreover it is easy to see that $f d' \tilde{\beta} = 2 d' f \wedge \tilde{\beta}$, which means $\tilde{\beta} \in Z_2'$. Then by the same method as the proof of $H^2_{f'} = 0$ (C^∞-case), we can prove that there exists a flat 1-form α_2 such that $\tilde{\beta} = f d' \alpha_2 - d' f \wedge \alpha_2$. Hence β has the following form:

$$\beta = \beta' + \tilde{\beta} = f d' (\alpha' + \alpha_2) - d' f \wedge (\alpha' + \alpha_2),$$

and thus $\beta \in B_2'$. Hence we get $H^2_{NP}(\mathbb{R}^4, \eta) = 0$.

§6.4. Computation of $H^3_{NP}(\mathbb{R}^4, \eta)$

Let Z_3' be the space of 3-cocycles. Since $\Omega_4' = 0$, it holds that $Z_3' = \Omega_3'$. Hence Z_3' is isomorphic to $\mathcal{F}$. Let B_3' be the space of 3-coboundaries. Then every element of B_3' is written as

$$\begin{aligned} d'_f \beta &= f d' \beta - 2 d' f \wedge \beta \\ &= \{ f \Big(\frac{\partial A}{\partial x} + \frac{\partial B}{\partial y} + \frac{\partial C}{\partial z} \Big) - 4(xA + yB + zC) \} dx \wedge dy \wedge dz, \end{aligned}$$

where $\beta = A dy \wedge dz + B dz \wedge dx + C dx \wedge dy$ is an arbitrary element of Ω_2'.

Put $\mathcal{B} = \{ f(\frac{\partial A}{\partial x} + \frac{\partial B}{\partial y} + \frac{\partial C}{\partial z}) - 4(xA + yB + zC) | A, B, C \in \mathcal{F} \}$. Then, by Proposition 6.5, $H^3_{NP}(\mathbb{R}^4, \eta)$ is isomorphic to $\mathcal{F}/\mathcal{B}$.

Lemma 6.7. *Put $\mathcal{I} = \{ h \in \mathcal{F} | \frac{\partial^p h}{\partial u^p}(x, y, z, 0) = 0,\ p \geq 0 \}$. i.e., each element h of $\mathcal{I}$ is u-flat. Then $\mathcal{I} \subset \mathcal{B}$.*

Proof. For $h \in \mathcal{I}$, it is clear that h/f^3 is an element of $\mathcal{F}$. Put $A = f^2 \int \frac{h}{f^3} dx, B = 0$ and $C = 0$. Then we have

$$f \Big(\frac{\partial A}{\partial x} + \frac{\partial B}{\partial y} + \frac{\partial C}{\partial z} \Big) - 4(xA + yB + zC) = h.$$

Hence $h \in \mathcal{B}$. □

Put $\hat{F} = \{\bar{A} | A \in \mathcal{F}\}$ and $\hat{B} = \{\bar{A} | A \in \mathcal{B}\}$. We also denote by $\mathcal{F}'_0$ the subspace of functions $g(x, y, z) \in \mathcal{F}'$ with $g(0, 0, 0) = 0$.

Proposition 6.8. $\hat{F}/\hat{B} \cong \mathbb{R}[[u]]$.

Proof. For any element $g = f(\frac{\partial A}{\partial x} + \frac{\partial B}{\partial y} + \frac{\partial C}{\partial z}) - 4(xA + yB + zC) \in \mathcal{B}$, its u-formal Taylor expansion is

$$\begin{aligned}\hat{B} \ni \bar{g} &= f\Big(\frac{\partial \bar{A}}{\partial x} + \frac{\partial \bar{B}}{\partial y} + \frac{\partial \bar{C}}{\partial z}\Big) - 4(x\bar{A} + y\bar{B} + z\bar{C}) \\ &= \sum_{p=0}^{\infty} \Big[u^p \Big\{f'\Big(\frac{\partial a_p}{\partial x} + \frac{\partial b_p}{\partial y} + \frac{\partial c_p}{\partial z}\Big) - 4(xa_p + yb_p + zc_p)\Big\} \\ &\quad + u^{p+2}\Big(\frac{\partial a_p}{\partial x} + \frac{\partial b_p}{\partial y} + \frac{\partial c_p}{\partial z}\Big)\Big].\end{aligned}$$

Put $g_p = f'\Big(\frac{\partial a_p}{\partial x} + \frac{\partial b_p}{\partial y} + \frac{\partial c_p}{\partial z}\Big) - 4(xa_p + yb_p + zc_p)$ and $h_p = \frac{\partial a_p}{\partial x} + \frac{\partial b_p}{\partial y} + \frac{\partial c_p}{\partial z}$ for non-negative integer p. Then every $\bar{g} \in \hat{B}$ has the following expression.

$$\bar{g} = (g_0 + u^2 h_0) + u(g_1 + u^2 h_1) + \cdots + u^p(g_p + u^2 h_p) + \cdots .$$

First recall that $H^3_{NP}(\mathbb{R}^3, \eta') \cong \mathbb{R}$ by Proposition 6.4. Hence for any non-negative integer p, it holds that

$$\{g_p \mid a_p, b_p, c_p \in \mathcal{F}'\} = \mathcal{F}'_0.$$

If we put $W_p = \{g_p + u^2 h_p \mid a_p, b_p, c_p \in \mathcal{F}'\}$, then $\bar{g}$ is contained in $W_0 + uW_1 + \cdots + u^p W_p + \cdots$. Note that h_p is *not* completely determined by g_p. To show this precisely, let us consider the following linear partial differential equation with three unknown functions $a, b, c \in \mathcal{F}'$:

$$(*) \qquad f'\Big(\frac{\partial a}{\partial x} + \frac{\partial b}{\partial y} + \frac{\partial c}{\partial z}\Big) - 4(xa + yb + zc) = 0.$$

We define a subspace $\mathcal{F}''_0$ of $\mathcal{F}'$ by

$$\mathcal{F}''_0 = \Big\{\frac{\partial a}{\partial x} + \frac{\partial b}{\partial y} + \frac{\partial c}{\partial z} \mid \text{a triplet } (a, b, c) \text{ is a solution of } (*)\Big\}.$$

Since (a, b, c) is a solution of the differential equation $(*)$, there exist three functions $A, B, C \in \mathcal{F}'$ such that

$$(12) \qquad \begin{cases} a = f'(C_y - B_z) + 2(zB - yC), \\ b = f'(A_z - C_x) + 2(xC - zA), \\ c = f'(B_x - A_y) + 2(yA - xB). \end{cases}$$

Recall that this fact is equivalent to $H^2_{f'} = 0$. Put $h = \frac{\partial a}{\partial x} + \frac{\partial b}{\partial y} + \frac{\partial c}{\partial z}$. If h is an element of $\mathcal{F}''_0$, then it is clear that h vanishes at the origin and hence $h \in \mathcal{F}'_0$. Thus $\mathcal{F}''_0$ becomes a subspace of $\mathcal{F}'_0$.

Let g_p have the following two expressions:

$$\begin{aligned} g_p &= f'\Big(\frac{\partial a_p}{\partial x} + \frac{\partial b_p}{\partial y} + \frac{\partial c_p}{\partial z}\Big) - 4(xa_p + yb_p + zc_p) \\ &= f'\Big(\frac{\partial a'_p}{\partial x} + \frac{\partial b'_p}{\partial y} + \frac{\partial c'_p}{\partial z}\Big) - 4(xa'_p + yb'_p + zc'_p) \end{aligned}$$

for two triplets (a_p, b_p, c_p) and (a'_p, b'_p, c'_p). Then we have

$$\begin{aligned} &f'\Big(\frac{\partial(a_p - a'_p)}{\partial x} + \frac{\partial(b_p - b'_p)}{\partial y} + \frac{\partial(c_p - c'_p)}{\partial z}\Big) \\ &\quad -4\{x(a_p - a'_p) + y(b_p - b'_p) + z(c_p - c'_p)\} = 0. \end{aligned}$$

Hence

$$h_p - h'_p = \frac{\partial(a_p - a'_p)}{\partial x} + \frac{\partial(b_p - b'_p)}{\partial y} + \frac{\partial(c_p - c'_p)}{\partial z}$$

is an element of $\mathcal{F}''_0$, where $h'_p = \frac{\partial a'_p}{\partial x} + \frac{\partial b'_p}{\partial y} + \frac{\partial c'_p}{\partial z}$. Then it is easy to see that $h_p + \mathcal{F}''_0$, which denotes a coset of h_p in $\mathcal{F}'/\mathcal{F}''_0$, is uniquely determined by g_p. And each W_p has the following expression:

$$W_p = \{g_p + u^2(h_p + \mathcal{F}''_0) \mid g_p \in \mathcal{F}'_0\}.$$

Let $\phi_p : W_p \longrightarrow \mathcal{F}'_0$ be a surjective linear mapping defined by $\phi_p(g_p + u^2(h_p + \mathcal{F}''_0)) = g_p$. It is clear that ϕ_p is well-defined and that $g_p = 0$ means $h_p \in \mathcal{F}''_0$. Hence $W_p/u^2\mathcal{F}''_0 \cong \mathcal{F}'_0$, and we have $W_p \cong \mathcal{F}'_0 + u^2\mathcal{F}''_0$. Now $\hat{B}$ becomes as follows. (Recall that $\mathcal{F}''_0$ is a subspace of $\mathcal{F}'_0$.)

$$\begin{aligned} \hat{B} &= W_0 + uW_1 + u^2W_2 + \cdots + u^pW_p + \cdots \\ &\cong (\mathcal{F}'_0 + u^2\mathcal{F}''_0) + u(\mathcal{F}'_0 + u^2\mathcal{F}''_0) + u^2(\mathcal{F}'_0 + u^2\mathcal{F}''_0) \\ &\quad + \cdots + u^p(\mathcal{F}'_0 + u^2\mathcal{F}''_0) + \cdots \\ &= \mathcal{F}'_0 + u\mathcal{F}'_0 + u^2\mathcal{F}'_0 + \cdots + u^p\mathcal{F}'_0 + \cdots \\ &= \mathbb{R}[[u]]\mathcal{F}'_0. \end{aligned}$$

Since

$$\begin{aligned} \hat{F} &= \mathcal{F}' + u\mathcal{F}' + u^2\mathcal{F}' + \cdots \\ &= (\mathbb{R} + \mathcal{F}'_0) + u(\mathbb{R} + \mathcal{F}'_0) + u^2(\mathbb{R} + \mathcal{F}'_0) + \cdots \\ &= \mathbb{R}[[u]] \oplus \mathbb{R}[[u]]\mathcal{F}'_0, \end{aligned}$$

we obtain that $\hat{F}/\hat{B} \cong \mathbb{R}[[u]]$. □

Let $T : \mathcal{F} \to \hat{F}$ be a linear mapping defined by $T(A) = \bar{A}$. For any $q \in T^{-1}(\hat{B})$, there exists $Q \in \hat{B}$ such that $T(q) = Q$. On the other hand, since $T(\mathcal{B}) = \hat{B}$, there exists $q_1 \in \mathcal{B}$ such that $T(q_1) = Q$. Hence $q - q_1 \in \mathcal{I}$. By Lemma 6.7, we have $q \in \mathcal{B}$, and hence $T^{-1}(\hat{B}) = \mathcal{B}$. Thus by Proposition 6.8,

$$\mathcal{F}/\mathcal{B} \cong \hat{F}/\hat{B} \cong \mathbb{R}[[u]].$$

Now we summarize the results obtained in this section.

Theorem 6.9. *Let $\eta = (x^2 + y^2 + z^2 + u^2)\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y} \wedge \frac{\partial}{\partial z}$ be a Nambu-Poisson tensor on $\mathbb{R}^4(x, y, z, u)$. Then*

$$\begin{aligned} &H^0_{NP}(\mathbb{R}^4, \eta) \cong C^\infty(\mathbb{R}),\\ &H^1_{NP}(\mathbb{R}^4, \eta) \cong \mathbb{R}[[u]],\\ &H^2_{NP}(\mathbb{R}^4, \eta) = 0,\\ &H^3_{NP}(\mathbb{R}^4, \eta) \cong \mathbb{R}[[u]],\\ &H^k_{NP}(\mathbb{R}^4, \eta) = 0, \quad k \geq 4. \end{aligned}$$

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