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Long Range Scattering for the Maxwell-Schrödinger System with Large Magnetic Field Data and Small Schrödinger Data†

By

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Abstract

We study the theory of scattering for the Maxwell-Schrödinger system in the Coulomb gauge in space dimension 3. We prove in particular the existence of modified wave operators for that system with no size restriction on the magnetic field data in the framework of a direct method which requires smallness of the Schrödinger data, and we determine the asymptotic behaviour in time of solutions in the range of the wave operators.

§1. Introduction

This paper is devoted to the theory of scattering and more precisely to the construction of modified wave operators for the Maxwell-Schrödinger system $(\mathrm{MS})_3$ in $3+1$ dimensional space time. That system describes the evolution of a charged nonrelativistic quantum mechanical particle interacting with the (classical) electromagnetic field it generates. It can be written as follows:

$$\begin{cases} i\partial_t u = -(1/2)\Delta_A u + A_0 u \\ \square A_0 - \partial_t(\partial_t A_0 + \nabla\cdot A) = |u|^2 \\ \square A + \nabla(\partial_t A_0 + \nabla\cdot A) = \mathrm{Im}\,\overline{u}\nabla_A u. \end{cases} \tag{1.1}$$

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Here u and (A, A_0) are respectively a complex valued function and an $I\!R^{3+1}$ valued function defined in space time $I\!R^{3+1}$, $\nabla_A = \nabla - iA$, $\Delta_A = \nabla_A^2$ and $\square = \partial_t^2 - \Delta$ is the d'Alembertian in $I\!R^{3+1}$. We shall consider that system exclusively in the Coulomb gauge $\nabla \cdot A = 0$. In that gauge, one can replace the system (1.1) by a formally equivalent one in the following standard way. The second equation of (1.1) can be solved for A_0 by

$$A_0 = -\Delta^{-1}|u|^2 = (4\pi|x|)^{-1} * |u|^2 \equiv g(|u|^2) \tag{1.2}$$

Substituting (1.2) into the first and last equations of (1.1) yields the new system

$$\begin{cases} i\partial_t u = -(1/2)\Delta_A u + g(|u|^2)u \\ \square A = P \,\mathrm{Im}\, \overline{u} \nabla_A u \end{cases} \tag{1.3}$$

where $P = 1\!\!1 - \nabla\Delta^{-1}\nabla$ is the projector on divergence free vector fields, together with the Coulomb gauge condition $\nabla \cdot A = 0$ which is formally preserved by the evolution. From now on we restrict our attention to the system (1.3).

The $(\mathrm{MS})_3$ system is known to be locally well posed in sufficiently regular spaces [11], [12] and to have global weak solutions in the energy space [9] in various gauges including the Coulomb gauge. However that system is so far not known to be globally well posed in any space.

A large amount of work has been devoted to the theory of scattering for nonlinear equations and systems centering on the Schrödinger equation, in particular for nonlinear Schrödinger (NLS) equations, Hartree equations, Klein-Gordon-Schrödinger (KGS), Wave-Schrödinger (WS) and Maxwell-Schrödinger (MS) systems. As in the case of the linear Schrödinger equation, one must distinguish the short range case from the long range case. In the former case, ordinary wave operators are expected and in a number of cases proved to exist, describing solutions where the Schrödinger function behaves asymptotically like a solution of the free Schrödinger equation. In the latter case, ordinary wave operators do not exist and have to be replaced by modified wave operators including a suitable phase in their definition. In that respect, the $(\mathrm{MS})_3$ system (1.1) belongs to the borderline (Coulomb) long range case, because of the t^{-1} decay in L^∞ norm of solutions of the wave equation. Such is the case also for the Hartree equation with $|x|^{-1}$ potential, for the Wave-Schrödinger system $(\mathrm{WS})_3$ in $I\!R^{3+1}$ and for the Klein-Gordon-Schrödinger system $(\mathrm{KGS})_2$ in $I\!R^{2+1}$.

The construction of modified wave operators for the previous long range equations and systems has been tackled by two methods. The first one was initiated in [13] on the example of the NLS equation in $I\!R^{1+1}$ and subsequently applied to the NLS equation in $I\!R^{2+1}$ and $I\!R^{3+1}$ and to the Hartree equation [1], to the $(\mathrm{KGS})_2$ system [14], [15], [16], [17], to the $(\mathrm{WS})_3$ system [18] and to the $(\mathrm{MS})_3$ system [19], [21]. That method is rather direct, starting from the original equation or system. It will be sketched below. It is restricted to the (Coulomb) limiting long range case, and requires a smallness condition on the asymptotic state of the Schrödinger function. Early applications of the method required in addition a support condition on the Fourier transform of the Schrödinger asymptotic state and a smallness condition of the Klein-Gordon or Maxwell field in the case of the $(\mathrm{KGS})_2$ or $(\mathrm{MS})_3$ system respectively [14], [21]. The support condition was subsequently removed for the $(\mathrm{KGS})_2$ and $(\mathrm{MS})_3$ system and the method was applied to the $(\mathrm{WS})_3$ system without a support condition, at the expense of adding a correction term to the Schrödinger asymptotic function [15], [18], [19]. The smallness condition of the KG field was then removed for the $(\mathrm{KGS})_2$ system, first with and then without a support condition [16], [17]. Finally the smallness condition on the wave field was removed for the $(\mathrm{WS})_3$ system, without a support condition or a correction term to the Schrödinger asymptotic function [8].

In the present paper, we extend the results of our previous paper [8] from the $(\mathrm{WS})_3$ system to the $(\mathrm{MS})_3$ system in the Coulomb gauge (1.3). In particular we prove the existence of modified wave operators without any smallness condition on the magnetic potential A, and without a support condition or a correction term on the asymptotic Schrödinger function. In addition, in the same spirit as in [8], we treat the problem in function spaces that are as large as possible, namely with regularity as low as possible. As a consequence, we require only a much lower regularity of the asymptotic state than in previous works.

For completeness and although we shall not make use of that fact in the present paper, we mention that the same problem for the Hartree equation and for the $(\mathrm{WS})_3$ and $(\mathrm{MS})_3$ system can also be treated by a more complex method where one first applies a phase-amplitude separation to the Schrödinger function. The main interest of that method is to remove the smallness condition on the Schrödinger function, and to go beyond the Coulomb limiting case for the Hartree equation. That method has been applied in particular to the $(\mathrm{WS})_3$ system and to the $(\mathrm{MS})_3$ system in a special case [4], [5], [6].

We now sketch briefly the method of construction of the modified wave operators initiated in [13]. That construction basically consists in solving the Cauchy problem for the system (1.3) with infinite initial time, namely in constructing solutions (u, A) with prescribed asymptotic behaviour at infinity in time. We restrict our attention to time going to $+\infty$. That asymptotic behaviour is imposed in the form of suitable approximate solutions (u_a, A_a) of the system (1.3). The approximate solutions are parametrized by data $(u_+, A_+, \dot{A}_+)$ which play the role of (actually would be in simpler e.g. short range cases) initial data at time zero for a simpler evolution. One then looks for exact solutions (u, A) of the system (1.3), the difference of which with the given asymptotic ones tends to zero at infinity in time in a suitable sense, more precisely, in suitable norms. The wave operator is then defined traditionally as the map $\Omega_+ : (u_+, A_+, \dot{A}_+) \to (u, A, \partial_t A)(0)$. However what really matters is the solution (u, A) in the neighbourhood of infinity in time, namely in some interval $[T, \infty)$, and we shall restrict our attention to the construction of such solutions. Continuing such solutions down to $t = 0$ is a somewhat different question, connected with the global Cauchy problem at finite times, which we shall not touch here, especially since the $(\mathrm{MS})_3$ system is not known to be globally well posed in any function space.

The construction of solutions (u, A) with prescribed asymptotic behaviour (u_a, A_a) is performed in two steps.

Step 1. One looks for (u, A) in the form $(u, A) = (u_a + v, A_a + B)$ with $\nabla \cdot A_a = \nabla \cdot B = 0$. The system satisfied by the new functions (v, B) can be written as

$$\begin{cases} i\partial_t v = -(1/2)\Delta_A v + g(|u|^2)v + G_1 - R_1 \\ \square B = G_2 - R_2 \end{cases} \tag{1.4}$$

where G_1 and G_2 are defined by

$$\begin{cases} G_1 = iB \cdot \nabla_{A_a} u_a + (1/2)B^2 u_a + g\left(|v|^2 + 2\,\mathrm{Re}\,\overline{u}_a v\right) u_a \\ G_2 = P\,\mathrm{Im}\,\left(\overline{v}\nabla_A v + 2\overline{v}\nabla_A u_a\right) - PB|u_a|^2 \end{cases} \tag{1.5}$$

and the remainders are defined by

$$\begin{cases} R_1 = i\partial_t u_a + (1/2)\Delta_{A_a} u_a - g\left(|u_a|^2\right) u_a \\ R_2 = \square A_a - P\,\mathrm{Im}\,\overline{u}_a \nabla_{A_a} u_a. \end{cases} \tag{1.6}$$

It is technically useful to consider also the partly linearized system for functions (v', B')

$$\begin{cases} i\partial_t v' = -(1/2)\Delta_A v' + g(|u|^2)v' + G_1 - R_1 \\ \square B' = G_2 - R_2. \end{cases} \tag{1.7}$$

The first step of the method consists in solving the system (1.4) for (v, B), with (v, B) tending to zero at infinity in time in suitable norms, under assumptions on (u_a, A_a) of a general nature, the most important of which being decay assumptions on the remainders R_1 and R_2. That can be done as follows. One first solves the linearized system (1.7) for (v', B') with given (v, B) and initial data $(v', B')(t_0) = 0$ for some large finite t_0. One then takes the limit $t_0 \to \infty$ of that solution, thereby obtaining a solution (v', B') of (1.7) which tends to zero at infinity in time. That construction defines a map $\phi : (v, B) \to (v', B')$. One then shows by a contraction method that the map ϕ has a fixed point. That first step will be performed in Section 2.

Step 2. The second step of the method consists in constructing approximate asymptotic solutions (u_a, A_a) satisfying the general estimates needed to perform Step 1. With the weak time decay allowed by our treatment of Step 1, one can take the simplest version of the asymptotic form used in previous works [6], [19], [21]. Thus we choose

$$u_a = MD\exp(-i\varphi)w_+ \tag{1.8}$$

where

$$M \equiv M(t) = \exp(ix^2/2t), \tag{1.9}$$

$$D(t) = (it)^{-n/2}D_0(t), \quad (D_0(t)f)(x) = f(x/t), \tag{1.10}$$

φ is a real phase to be chosen below and $w_+ = Fu_+$. We furthermore choose A_a in the form $A_a = A_0 + A_1$ where A_0 is the solution of the free wave equation $\square A_0 = 0$ given by

$$A_0 = \cos\omega t A_+ + \omega^{-1}\sin\omega t \dot{A}_+ \tag{1.11}$$

where $\omega = (-\Delta)^{1/2}$, and where

$$A_1(t) = \int_t^\infty dt'(\omega t')^{-1}\sin(\omega(t'-t))P\,x|u_a(t')|^2. \tag{1.12}$$

Substituting (1.8) into (1.12) yields

$$A_1(t) = t^{-1}D_0(t)\widetilde{A}_1 \tag{1.13}$$

where

$$\widetilde{A}_1 = \int_1^\infty d\nu\,\nu^{-3}\omega^{-1}\sin(\omega(\nu-1))D_0(\nu)P\,x|w_+|^2. \tag{1.14}$$

In particular $\widetilde{A}_1$ is constant in time. We finally choose φ by imposing

$$\varphi(1) = 0, \qquad \partial_t\varphi = t^{-1}\left(g(|w_+|^2) - x\cdot\widetilde{A}_1\right) \tag{1.15}$$

so that

$$\varphi = (\ell n\ t) \left(g(|w_+|^2) - x \cdot \widetilde{A}_1 \right) . \tag{1.16}$$

We shall show in Section 3 that the previous choice fulfils the conditions needed for Step 1, under suitable assumptions on the asymptotic state $(u_+, A_+, \dot{A}_+)$.

In order to state our results we introduce some notation. We denote by F the Fourier transform, by $< \cdot, \cdot >$ the scalar product in L^2 and by $\| \cdot \|_r$ the norm in $L^r \equiv L^r(I\!R^3)$, $1 \le r \le \infty$ and we define $\delta(r) = 3/2 - 3/r$. For any nonnegative integer k and for $1 \le r \le \infty$, we denote by W_r^k the Sobolev spaces

$$W_r^k = \left\{ u : \|u; W_r^k\| = \sum_{\alpha : 0 \le |\alpha| \le k} \|\partial_x^\alpha u\|_r < \infty \right\}$$

where α is a multiindex, so that $H^k = W_2^k$. We shall need the weighted Sobolev spaces $H^{k,s}$ defined for k, $s \in I\!R$ by

$$H^{k,s} = \left\{ u : \|u; H^{k,s}\| = \|(1+x^2)^{s/2}(1-\Delta)^{k/2} u\|_2 < \infty \right\}$$

so that $H^k = H^{k,0}$. For any interval I, for any Banach space X and for any q, $1 \le q \le \infty$, we denote by $L^q(I, X)$ (resp. $L^q_{loc}(I, X)$) the space of L^q integrable (resp. locally L^q integrable) functions from I to X if $q < \infty$ and the space of measurable essentially bounded (resp. locally essentially bounded) functions from I to X if $q = \infty$. For any $h \in \mathcal{C}([1, \infty), I\!R^+)$, non increasing and tending to zero at infinity and for any interval $I \subset [1, \infty)$, we define the space

$$\begin{aligned} X(I) = \Big\{ & (v, B) : v \in \mathcal{C}(I, H^2) \cap \mathcal{C}^1(I, L^2), \\ & \| (v,B); X(I) \| \equiv \mathop{\rm Sup}_{t \in I} h(t)^{-1} \Big(\| v(t); H^2 \| + \| \partial_t v(t) \|_2 + \| v; L^{8/3}(J, W_4^1) \| \\ & + \| B; L^4(J, W_4^1) \| + \| \partial_t B; L^4(J, L^4) \| \Big) < \infty \Big\} \end{aligned} \tag{1.17}$$

where $J = [t, \infty) \cap I$.

We can now state our result.

Proposition 1.1. *Let $h(t) = t^{-1}(2 + \ell n\ t)^2$ and let $X(\cdot)$ be defined by* (1.17). *Let u_a be defined by* (1.8) *with $w_+ = Fu_+$ and with φ defined by* (1.16) (1.2) (1.14). *Let $A_a = A_0 + A_1$ with A_0 defined by* (1.11) *and A_1 by* (1.13) (1.14). *Let $u_+ \in H^{3,1} \cap H^{1,3}$ with $\| xw_+ \|_4$ and $\| w_+ \|_3$ sufficiently small. Let $\nabla^2 A_+$, $\nabla \dot{A}_+$, $\nabla^2(x \cdot A_+)$ and $\nabla(x \cdot \dot{A}_+) \in W_1^1$ with A_+, $x \cdot A_+ \in L^3$ and $\dot{A}_+$, $x \cdot \dot{A}_+ \in L^{3/2}$ and let $\nabla \cdot A_+ = \nabla \cdot \dot{A}_+ = 0$.*

Then there exists T, $1 \le T < \infty$ and there exists a unique solution (u, A) of the system (1.3) *such that $(u - u_a, A - A_a) \in X([T, \infty))$. Furthermore $\nabla(A - A_a)$, $\partial_t(A - A_a) \in \mathcal{C}([T, \infty), L^2)$ and A satisfies the estimate*

$$\| \nabla(A - A_a)(t) \|_2 \vee \| \partial_t(A - A_a)(t) \|_2 \le C t^{-3/2}(2 + \ell n\ t)^2 \tag{1.18}$$

for some constant C depending on $(u_+, A_+, \dot{A}_+)$ and for all $t \ge T$.

Remark 1.1. The only smallness conditions bear on $\| x w_+ \|_4$ and on $\| w_+ \|_3$ and are required by the magnetic interaction and the Hartree interaction (1.2) respectively. In particular there is no smallness condition on $(A_+, \dot{A}_+)$.

Remark 1.2. The assumptions A_+, $x \cdot A_+ \in L^3$ and $\dot{A}_+$, $x \cdot \dot{A}_+ \in L^{3/2}$ serve to exclude the occurrence of constant terms in A_+, $x \cdot A_+$, $\dot{A}_+$, $x \cdot \dot{A}_+$ and of terms linear in x in A_+, $x \cdot A_+$, but are otherwise implied by the W_1^1 assumptions on those quantities through Sobolev inequalities.

Remark 1.3. The assumptions on A_+, $\dot{A}_+$ imply that $\omega^{1/2} A_+$, $\omega^{-1/2} \dot{A}_+ \in H^1$ through Sobolev inequalities. As a consequence the free wave solution A_0 defined by (1.11) belongs to $L^4(I\!R, W_4^1)$ by Strichartz inequalities, with $\partial_t A_0 \in L^4(I\!R, L^4)$ [3]. In particular A_0 satisfies the local in time regularity of B required in the definition of the space $X(\cdot)$. Furthermore ∇A_+, $\dot{A}_+ \in L^2$ and therefore ∇A_0, $\partial_t A_0 \in (\mathcal{C} \cap L^\infty)(I\!R, L^2)$, namely A_0 is a finite energy solution of the wave equation.

§2. The Cauchy Problem at Infinite Initial Time

In this section we perform the first step of the construction of solutions of the system (1.3) as described in the introduction, namely we construct solutions (v, B) of the system (1.4) defined in a neighbourhood of infinity in time and tending to zero at infinity under suitable regularity and decay assumptions on the asymptotic functions (u_a, A_a) and on the remainders R_i. As a preliminary to that study, we need to solve the Cauchy problem with finite initial time for the linearized system (1.7). That system consists of two independent equations. The second one is simply a wave equation with an inhomogeneous term and the Cauchy problem with finite or infinite initial time for it is readily solved under suitable assumptions on the inhomogeneous term, which will be fulfilled in the applications. The first one is a Schrödinger equation with time dependent magnetic and scalar potentials and with time dependent inhomogeneity, which we rewrite in a more concise form and with slightly different notation as

$$i\partial_t v = -(1/2)\Delta_A v + Vv + f. \tag{2.1}$$

We first give some preliminary results on the Cauchy problem with finite initial time for that equation at the level of regularity of H^2. The following proposition is a minor variation of Proposition 3.2 in [7].

Proposition 2.1. *Let I be an interval, let $A \in \mathcal{C}(I, L^4 + L^\infty)$, $\partial_t A \in L^1_{loc}(I, L^4 + L^\infty)$, $V \in \mathcal{C}(I, L^2 + L^\infty)$, $\partial_t V \in L^1_{loc}(I, L^2 + L^\infty)$, $f \in \mathcal{C}(I, L^2)$ and $\partial_t f \in L^1_{loc}(I, L^2)$. Let $t_0 \in I$ and $v_0 \in H^2$. Then*

(1) *There exists a unique solution $v \in \mathcal{C}(I, H^2) \cap \mathcal{C}^1(I, L^2)$ of (2.1) in I with $v(t_0) = v_0$. That solution is actually unique in $\mathcal{C}(I, H^1)$. For all $t \in I$, the following equality holds*:

$$\| v(t) \|_2^2 - \| v_0 \|_2^2 = \int_{t_0}^t dt' \ 2\,\mathrm{Im} < v, f > (t'). \tag{2.2}$$

(2) *Let in addition $A \in L^2_{loc}(I, L^\infty)$, $\nabla A \in L^1_{loc}(I, L^\infty)$ and $V \in L^1_{loc}(I, L^\infty)$. Then for all $t \in I$, the following equality holds*:

(2.3)

$$\| \partial_t v(t) \|_2^2 - \| (-(1/2)\Delta_A v_0 + V v_0 + f)\,(t_0) \|_2^2 = \int_{t_0}^t dt' \ 2\,\mathrm{Im} < \partial_t v, f_1 > (t')$$

where

$$f_1 = i\,(\partial_t A) \cdot \nabla_A v + (\partial_t V)\,v + \partial_t f. \tag{2.4}$$

Furthermore the solution is unique in $\mathcal{C}(I, L^2)$.

We shall make an essential use of the well-known Strichartz inequalities for the Schrödinger equation [2], [10], [22], which we recall for completeness. We define

$$U(t) = \exp(i(t/2)\Delta). \tag{2.5}$$

A pair of Hölder exponents (q, r) will be called admissible if $0 \le 2/q = 3/2 - 3/r \le 1$. For any r, $1 \le r \le \infty$, we define $\bar{r}$ by $1/r + 1/\bar{r} = 1$.

Lemma 2.1. *The following inequalities hold.*
(1) *For any admissible pair (q, r) and for any $u \in L^2$*

$$\| U(t)u; L^q(I\!R, L^r) \| \leq C \| u \|_2 . \tag{2.6}$$

(2) *Let I be an interval and let $t_0 \in I$. Then for any admissible pairs (q_i, r_i), $i = 1, 2$,*

$$\| \int_{t_0}^{t} dt' U(\cdot - t') f(t'); L^{q_1}(I, L^{r_1}) \| \leq C \| f; L^{\overline{q}_2}(I, L^{\overline{r}_2}) \| . \tag{2.7}$$

In addition to the Strichartz inequalities for the Schrödinger equation, we shall need special cases of the Strichartz inequalities for the wave equation [3], [10]. Let I be an interval, let $t_0 \in I$ and let $B(t_0) = \partial_t B(t_0) = 0$. Then

$$\| B; L^4(I, L^4) \| \leq C \| \Box B; L^{4/3}(I, L^{4/3}) \|, \tag{2.8}$$

$$\| \nabla B; L^4(I, L^4) \| \vee \| \partial_t B; L^4(I, L^4) \| \leq C \| \nabla \Box B; L^{4/3}(I, L^{4/3}) \|, \tag{2.9}$$

$$\mathop{\mathrm{Sup}}_{t \in I} (\| \nabla B(t) \|_2 \vee \| \partial_t B(t) \|_2) \leq \| \Box B; L^1(I, L^2) \| . \tag{2.10}$$

We now begin the construction of solutions of the system (1.4). For any T, t_0 with $1 \leq T < t_0 \leq \infty$, we denote by I the interval $I = [T, t_0]$ and for any $t \in I$, we denote by J the interval $J = [t, t_0]$. In all this section, we denote by h a function in $\mathcal{C}([1, \infty), I\!R^+)$ such that for some $\lambda > 0$, the function $\overline{h}(t) \equiv t^\lambda h(t)$ is non increasing and tends to zero as $t \to \infty$, and we denote by j, k nonnegative integers.

We shall make repeated use of the following lemma.

Lemma 2.2. *Let $1 \leq q$, $q_k \leq \infty (1 \leq k \leq n)$ be such that*

$$\mu \equiv 1/q - \sum_k 1/q_k \geq 0.$$

Let $f_k \in L^{q_k}(I)$ satisfy

$$\| f_k; L^{q_k}(J) \| \leq N_k \ h(t) \tag{2.11}$$

for $1 \leq k \leq n$, for some constants N_k and for all $t \in I$.

Let $\rho \geq 0$ such that $n\lambda + \rho > \mu$. Then the following inequality holds for all $t \in I$

$$\| \left(\prod_k f_k \right) t^{-\rho}; L^q(J) \| \leq C \left(\prod_k N_k \right) h(t)^n \ t^{\mu - \rho} \tag{2.12}$$

where

$$C = \left(1 - 2^{-q(n\lambda+\rho-\mu)}\right)^{-1/q} . \tag{2.13}$$

Proof. For $t \in I$, we define $I_j = [t2^j, t2^{j+1}] \cap I$ so that $J = \underset{j\geq 0}{\cup} I_j$. We then rewrite $L^q(J) = \ell_j^q(L^q(I_j))$. We estimate

$$\begin{aligned} \parallel \left(\prod_k f_k\right) t^{-\rho}; L^q(J) \parallel &\leq \parallel \left(\prod_k \parallel f_k; L^{q_k}(I_j) \parallel\right) \parallel t^{-\rho}; L^{1/\mu}(I_j) \parallel ; \ell_j^q \parallel \\ &\leq \left(\prod_k N_k\right) \parallel h(t2^j)^n (t2^j)^{-\rho+\mu}; \ell_j^q \parallel \\ &\leq \left(\prod_k N_k\right) \overline{h}(t)^n t^{-n\lambda-\rho+\mu} \parallel 2^{j(-n\lambda-\rho+\mu)}; \ell_j^q \parallel \end{aligned}$$

from which (2.12) follows. □

Remark 2.1. In some special cases, the dyadic decomposition is not needed for the proof of Lemma 2.2. For instance if all the q_k are infinite, one can estimate

$$\begin{aligned} &\parallel h(t)^n t^{-\rho} \parallel_q \leq \overline{h}(t)^n \parallel t^{-\rho-n\lambda} \parallel_q \\ &\leq C\ \overline{h}(t)^n\ t^{-\rho-n\lambda+1/q} = C\ h(t)^n\ t^{-\rho+\mu} \end{aligned} \tag{2.14}$$

by a direct application of Hölder's inequality in J. The same situation occurs if $\rho > \mu$.

In order to estimate the Hartree interaction term (1.2), we shall use the following Lemma. We recall that $\delta(r) = 3/2 - 3/r$.

Lemma 2.3. *The following estimates hold.*

(1)

$$\parallel g(\overline{v}_1 v_2) v_3 \parallel_{\overline{r}_4} \leq C \prod_{1\leq i\leq 3} \parallel v_i \parallel_{r_i} \tag{2.15}$$

for $0 \leq \delta_i = \delta(r_i) \leq 1$, $1 \leq i \leq 4$, $\sum \delta_i = 1$, $0 < \delta_1 + \delta_2 < 1$.

(2)

$$\parallel g(\overline{v}_1 v_2) \parallel_\infty \leq C \parallel v_2 \parallel_{r_2} \left(\parallel v_1 \parallel_{r_{1+}} \parallel v_1 \parallel_{r_{1-}}\right)^{1/2} \tag{2.16}$$

for $0 < 3/r_1 = 2 - 3/r_2 \leq 2$, $1/r_{1\pm} = (1 \mp \varepsilon)/r_1$, $\varepsilon > 0$.

Proof. Part (1) follows from the Hölder and Hardy-Littlewood-Sobolev inequalities.
Part (2) is proved by separating $|x|^{-1}$ into short and long distance parts, applying the Hölder inequality, and optimizing the result with respect to the point of separation (see [1]). □

We can now state the main result of this section.

Proposition 2.2. *Let h be defined as above with $\lambda = 3/8$ and let $X(\cdot)$ be defined by* (1.17). *Let u_a, A_a, R_1 and R_2 be sufficiently regular* (*for the following estimates to make sense*) *and satisfy the estimates*

$$\| \partial_t^j \nabla^k u_a(t) \|_r \leq c \; t^{-\delta(r)} \quad \text{for } 2 \leq r \leq \infty \tag{2.17}$$

and in particular

$$\| u_a \|_3 \leq c_3 \; t^{-1/2} \quad , \quad \| \nabla u_a \|_4 \leq c_4 \; t^{-3/4}, \tag{2.18}$$

$$\| \nabla^2 u_a(t) \|_4 \vee \| \partial_t \nabla u_a(t) \|_4 \leq c \; t^{-3/4}, \tag{2.19}$$

$$\| \partial_t^j \nabla^k A_a(t) \|_\infty \leq a \; t^{-1}, \tag{2.20}$$

$$\| \partial_t^j \nabla^k R_1; L^1([t, \infty), L^2) \| \leq r_1 \; h(t), \tag{2.21}$$

$$\| R_2; L^{4/3}([t, \infty), W^1_{4/3}) \| \leq r_2 \; h(t), \tag{2.22}$$

for $0 \leq j + k \leq 1$, for some constants c, c_3, c_4, a, r_1 and r_2 with c_3 and c_4 sufficiently small and for all $t \geq T_0 \geq 1$. Then there exists T, $T_0 \leq T < \infty$ and there exists a unique solution (v, B) of the system (1.4) *in $X([T, \infty))$. If in addition*

$$\| R_2; L^1([t, \infty), L^2) \| \leq r_2 \; t^{-1/2} \; h(t) \tag{2.23}$$

for all $t \geq T$, then ∇B, $\partial_t B \in \mathcal{C}([T, \infty), L^2)$ and B satisfies the estimate

$$\| \nabla B(t) \|_2 \vee \| \partial_t B(t) \|_2 \leq C \left(t^{-1/2} + t^{1/4} h(t) \right) h(t) \tag{2.24}$$

for some constant C and for all $t \geq T$.

Proof. We follow the sketch given in the introduction. Let $T_0 \leq T < \infty$ and let $(v, B) \in X([T, \infty))$. In particular (v, B) satisfies

$$\| v(t) \|_2 \leq N_0 \; h(t) \tag{2.25}$$

$$\| v; L^4(J, L^3) \| \vee \| v; L^{8/3}(J, L^4) \| \leq N_1 \; h(t) \tag{2.26}$$

$$\| B; L^4(J, L^4) \| \leq N_2 \ h(t) \tag{2.27}$$

$$\| \partial_t v(t) \|_2 \leq N_3 \ h(t) \tag{2.28}$$

$$\| \nabla v; L^4(J, L^3) \| \vee \| \nabla v; L^{8/3}(J, L^4) \| \leq N_4 \ h(t) \tag{2.29}$$

$$\| \Delta v(t) \|_2 \leq N_5 \ h(t) \tag{2.30}$$

$$\| \nabla B; L^4(J, L^4) \| \vee \| \partial_t B; L^4(J, L^4) \| \leq N_6 \ h(t) \tag{2.31}$$

for some constants N_i, $0 \leq i \leq 6$ and for all $t \geq T$, with $J = [t, \infty)$. Furthermore from (2.25) (2.30) it follows that

$$\| \nabla v(t) \|_2 \leq (N_0 N_5)^{1/2} \ h(t) \equiv N_{1/2} \ h(t) \tag{2.32}$$

for all $t \geq T_0$. We first construct a solution (v', B') of the system (1.7) in $X([T, \infty))$. For that purpose, we take t_0, $T < t_0 < \infty$ and we solve the system (1.7) in $X(I)$ where $I = [T, t_0]$ with initial condition $(v', B')(t_0) = 0$. Let (v'_{t_0}, B'_{t_0}) be the solution thereby obtained. The existence of v'_{t_0} follows from Proposition 2.1 with $V = g(|u|^2)$ and $f = G_1 - R_1$. We want to take the limit of (v'_{t_0}, B'_{t_0}) as $t_0 \to \infty$ and for that purpose we need estimates of (v'_{t_0}, B'_{t_0}) in $X(I)$ that are uniform in t_0. Omitting the subscript t_0 for brevity we define

$$N'_0 = \operatorname*{Sup}_{t \in I} \ h(t)^{-1} \| v'(t) \|_2 \tag{2.33}$$

$$N'_1 = \operatorname*{Sup}_{t \in I} \ h(t)^{-1} \Big(\| v'; L^4(J, L^3) \| \vee \| v'; L^{8/3}(J, L^4) \| \Big) \tag{2.34}$$

$$N'_2 = \operatorname*{Sup}_{t \in I} \ h(t)^{-1} \| B'; L^4(J, L^4) \| \tag{2.35}$$

$$N'_3 = \operatorname*{Sup}_{t \in I} \ h(t)^{-1} \| \partial_t v'(t) \|_2 \tag{2.36}$$

$$N'_4 = \operatorname*{Sup}_{t \in I} \ h(t)^{-1} \Big(\| \nabla v'; L^4(J, L^3) \| \vee \| \nabla v'; L^{8/3}(J, L^4) \| \Big) \tag{2.37}$$

$$N'_5 = \operatorname*{Sup}_{t \in I} \ h(t)^{-1} \| \Delta v'(t) \|_2 \tag{2.38}$$

$$N'_6 = \operatorname*{Sup}_{t \in I} \ h(t)^{-1} \left(\| \nabla B'; L^4(J, L^4) \| \vee \| \partial_t B'; L^4(J, L^4) \| \right) \tag{2.39}$$

where $J = [t, \infty) \cap I$ and we set out to estimate the various N'_i. We also define the auxiliary quantities

$$N'_{1/2} = \operatorname*{Sup}_{t \in I} \ h(t)^{-1} \| \nabla v'(t) \|_2 \tag{2.40}$$

$$\widetilde{N}'_{1/2} = \operatorname*{Sup}_{t \in I} \ h(t)^{-1} \| \nabla_A v'(t) \|_2 \tag{2.41}$$

so that in particular $N'_{1/2} \leq (N'_0 N'_5)^{1/2}$.

We shall use the notation

$$\| f; L^q(J, L^r) \| = \| \| f \|_r; L^q(J) \| = \| \| f \|_r \|_q,$$

namely with the inner norm taken in $L^r(\mathbb{R}^3)$ and the outer norm taken in $L^q(J)$. Furthermore we shall use a shorthand notation for two important cases, namely

$$\| \cdot \,; L^1(J, L^2) \| = \| \cdot \|_+ \quad \text{and} \quad \| \cdot \,; L^{4/3}(J, L^{4/3}) \| = \| \cdot \|_* \,.$$

We first estimate N_0', defined by (2.33). From (2.2) we obtain

$$\| v'(t) \|_2 \leq \| G_1 \|_+ + \| R_1 \|_+$$

with G_1 defined by (1.5). We estimate

$$\| B \cdot \nabla u_a \|_+ \leq \| \| B \|_4 \| \nabla u_a \|_4 \|_1 \leq C c_4 N_2 \; h(t)$$

by Lemma 2.2,

$$\begin{aligned} \| B \cdot A_a u_a \|_+ &\leq \| \| B \|_4 \| A_a \|_\infty \| u_a \|_4 \|_1 \\ &\leq caN_2 h \| t^{-7/4} \|_{4/3} \leq caN_2 t^{-1} \; h(t), \\ \| B^2 u_a \|_+ &\leq \| \| B \|_4^2 \| u_a \|_\infty \|_1 \leq cN_2^2 h^2 \| t^{-3/2} \|_2 \leq cN_2^2 t^{-1} \; h(t)^2, \end{aligned}$$

$$\| g(\overline{u}_a v) u_a \|_+ \leq C \; \| \| v \|_2 \| u_a \|_3^2 \|_1 \leq C c_3^2 N_0 \; h(t)$$

by Lemma 2.3, part (1) and Lemma 2.2,

$$\begin{aligned} \| g(|v|^2) u_a \|_+ &\leq C \; \| \| v \|_3 \| v \|_2 \| u_a \|_3 \|_1 \\ &\leq C c_3 N_0 N_1 t^{1/4} \; h(t)^2 \end{aligned}$$

by Lemma 2.3, part (1) and Lemma 2.2 again.

Collecting the previous estimates yields

$$\begin{aligned} N_0' \leq C_0 \Big(& c_4 N_2 + caN_2 T^{-1} + c_3^2 N_0 + cN_2^2 T^{-1} \; h(T) \\ & + cN_0 N_1 T^{-1/8} \overline{h}(T) + r_1 \Big) \end{aligned} \tag{2.42}$$

which is of the form

$$N_0' \leq C_0 \left(c_4 N_2 + c_3^2 N_0 + r_1 + (o(1); N_0, N_1, N_2) \right) \tag{2.43}$$

where $(o(1); \cdot, \cdots, \cdot)$ denotes a quantity depending on the variables indicated and tending to zero as $T \to \infty$ when those variables are fixed.

We next estimate the Strichartz norms of v', namely N_1' defined by (2.34). By Lemma 2.1, in addition to the contribution of $G_1 - R_1$ estimated above, we need to estimate

$$iA \cdot \nabla v' + (1/2)A^2 v' + g(|u|^2)v'$$

in some $L^{\overline{q}}(J, L^{\overline{r}})$ for admissible (q, r). We estimate

$$\begin{aligned} \| A_a \cdot \nabla v' \|_+ &\leq a N_{1/2}' \| t^{-1} h \|_1 \leq 3a N_{1/2}' \ h(t), \\ \| A_a^2 v' \|_+ &\leq a^2 N_0' \| t^{-2} h \|_1 \leq a^2 N_0' t^{-1} \ h(t), \\ \| B \cdot \nabla v'; L^{8/5}(J, L^{4/3}) \| &\leq \| \| B \|_4 \| \nabla v' \|_2 \|_{8/5} \leq C \ N_2 \ N_{1/2}' h(t) \overline{h}(t), \end{aligned}$$

by Lemma 2.2,

$$\| B^2 v'; L^2(J, L^{6/5}) \| \leq C \, \| \| B \|_4^{3/2} \| \nabla B \|_4^{1/2} \| v' \|_2 \|_2 \leq C N_2^{3/2} N_6^{1/2} N_0' \ h(t)^3$$

by Sobolev inequalities and Lemma 2.2,

$$\| g(|u_a|^2) v' \|_+ \leq \| \| g(|u_a|^2) \|_\infty \| v' \|_2 \|_1 \leq C c^2 N_0' \ h(t)$$

by Lemma 2.3, part (2),

$$\| g(|v|^2) v'; L^{4/3}(J, L^{3/2}) \| \leq C \, \| \| v \|_3 \| v \|_2 \| v' \|_2 \|_{4/3} \leq C N_0 N_1 N_0' t^{1/2} \ h(t)^3$$

by Lemma 2.3, part (1) and Lemma 2.2. The term $g(\overline{u}_a v)v'$ need not be considered because it is controlled by the previous ones.

Collecting the previous estimates yields

(2.44)
$$\begin{aligned} N_1' \leq C_1 \Big\{ & c_4 N_2 + ca N_2 T^{-1} + c_3^2 N_0 + c N_2^2 T^{-1} \ h(T) + c N_0 N_1 T^{-1/8} \overline{h}(T) \\ & + \Big(c^2 + a^2 T^{-1} + N_2^{3/2} N_6^{1/2} \ h(T)^2 + N_0 N_1 T^{-1/4} \overline{h}(T)^2 \Big) N_0' + r_1 \\ & + a N_{1/2}' + N_2 N_{1/2}' \overline{h}(T) \Big\} \end{aligned}$$

which is of the form

$$\begin{aligned} N_1' \leq C_1 \Big(& c_4 N_2 + c_3^2 N_0 + c^2 N_0' + a N_{1/2}' + r_1 \\ & + (o(1); N_0, N_1, N_2, N_6, N_0', N_{1/2}') \Big). \end{aligned} \tag{2.45}$$

We now turn to the estimates of B'. We first estimate B' in $L^4(J, L^4)$, namely we estimate N_2' defined by (2.35), by the use of (1.5) (2.8). For that purpose we estimate G_2 in $L^{4/3}(J, L^{4/3})$. The linear terms in v are estimated by

$$\begin{aligned}\| \overline{v}\nabla u_a \|_* &\leq \|\| v \|_2 \| \nabla u_a \|_4 \|_{4/3} \\ &\leq c_4 N_0 \| t^{-3/4} h \|_{4/3} \leq 2c_4 N_0 \; h(t),\end{aligned}$$

$$\begin{aligned}\| \overline{v} A_a u_a \|_* &\leq \|\| v \|_2 \| A_a \|_\infty \| u_a \|_4 \|_{4/3} \\ &\leq acN_0 \| t^{-7/4} h \|_{4/3} \leq acN_0 t^{-1} \; h(t).\end{aligned}$$

The linear term in B is estimated by

$$\begin{aligned}\| B|u_a|^2 \|_* &\leq \|\| B \|_4 \| u_a \|_4^2 \|_{4/3} \\ &\leq c^2 N_2 h \| t^{-3/2} \|_2 \leq c^2 N_2 t^{-1} \; h(t).\end{aligned}$$

The quadratic terms in v^2 are estimated by

$$\| \overline{v}\nabla v \|_* \leq \|\| v \|_4 \| \nabla v \|_2 \|_{4/3} \leq CN_1 N_{1/2} \; h(t)\overline{h}(t),$$

by Lemma 2.2,

$$\begin{aligned}\| A_a |v|^2 \|_* &\leq \|\| v \|_4 \| v \|_2 \| A_a \|_\infty \|_{4/3} \\ &\leq aN_0 N_1 h \| t^{-1} h \|_{8/3} \leq aN_0 N_1 t^{-5/8} \; h(t)^2.\end{aligned}$$

The quadratic terms in Bv need not be considered because

$$2|\overline{v} B u_a| \leq \left|B|u_a|^2\right| + \left|B|v|^2\right|.$$

The cubic term $B|v|^2$ is estimated by

$$\begin{aligned}\| B|v|^2 \|_* &\leq \| B; L^4(L^4) \| \; \| v; L^3(L^r) \|^{3/2} \| v; L^\infty(L^6) \|^{1/2} \\ &\leq CN_2 N_1^{3/2} N_{1/2}^{1/2} \; h(t)^3\end{aligned}$$

where $3 < r = 18/5 < 4$ so that $(3, r)$ is an admissible pair and that the middle norm is controlled by N_1.

Collecting the previous estimates yields

$$\begin{aligned}N_2' \leq C_2 \Big\{ & c_4 N_0 + acN_0 T^{-1} + c^2 N_2 T^{-1} + r_2 \\ & + N_1 N_{1/2} \overline{h}(T) + aN_0 N_1 T^{-5/8} \; h(T) + N_2 N_1^{3/2} N_{1/2}^{1/2} \; h(T)^2 \Big\}\end{aligned} \tag{2.46}$$

which is of the form

$$N_2' \leq C_2 \left(c_4 N_0 + r_2 + (o(1); N_0, N_1, N_{1/2}, N_2) \right). \tag{2.47}$$

We next complete the estimates of B' by estimating $\nabla B'$ and $\partial_t B'$ in $L^4(J, L^4)$, namely we estimate N'_6 defined by (2.39), through the use of (1.5) (2.9). For that purpose we estimate ∇G_2 in $L^{4/3}(J, L^{4/3})$. Now

$$\begin{aligned}(2.48)\quad \nabla G_2 = 2P\,\mathrm{Im}\left((\nabla \overline{v})\nabla_A v + (\nabla \overline{v})\nabla_A u_a + (\nabla \overline{u}_a)\nabla_A v\right) \\ -P(\nabla A)\left(|v|^2 + 2\,\mathrm{Re}\,\overline{u}_a v\right) - P(\nabla B)|u_a|^2 - 2PB\,\mathrm{Re}\,\overline{u}_a \nabla u_a.\end{aligned}$$

The estimate of ∇G_2 in $L^{4/3}(J, L^{4/3})$ proceeds exactly as that of G_2 in the same space, with one additional gradient acting on each factor in each term, except for two facts. First because of the symmetry of the quadratic form $P\ \mathrm{Im}\ (\overline{v}_1 \nabla_A v_2)$, we can always ensure that no terms occur with two derivatives on v or u_a. Second, the quadratic terms coming from $\overline{v} B u_a$ have to be estimated explicitly because they are no longer estimated by polarization. When hitting v, and additional gradient produces a replacement of N_0 by $N_{1/2}$ and of N_1 by N_4 in the estimates. When hitting B, it produces a replacement of N_2 by N_6. When hitting u_a or A_a, it only requires higher regularity of these functions, but does not change the form of the estimates. With those remarks available, only the terms from $\nabla(\overline{v} B u_a)$ and from $B\nabla|v|^2$ need new estimates.

The linear terms in v are estimated by

$$\begin{aligned}\| (\nabla \overline{v})\nabla u_a \|_* &\le 2c_4 N_{1/2}\ h(t),\\ \| (\nabla \overline{v}) A_a u_a \|_* &\le acN_{1/2} t^{-1}\ h(t),\\ \| \overline{v}(\nabla A_a) u_a + \overline{v} A_a \nabla u_a \|_* &\le 2acN_0 t^{-1}\ h(t).\end{aligned}$$

The linear terms in B are estimated by

$$\begin{aligned}\| (\nabla B)|u_a|^2 \|_* &\le c^2 N_6 t^{-1}\ h(t),\\ \| B\overline{u}_a \nabla u_a \|_* &\le c^2 N_2 t^{-1}\ h(t).\end{aligned}$$

The quadratic terms in v^2 are estimated by

$$\begin{aligned}\| |\nabla v|^2 \|_* &\le \| \| \nabla v \|_4 \| \nabla v \|_2 \|_{4/3} \le CN_4 N_{1/2}\ h(t)\overline{h}(t),\\ \| (\nabla \overline{v}) A_a v \|_* &\le \|\| \nabla v \|_2 \| A_a \|_\infty \| v \|_4 \|_{4/3} \le aN_{1/2} N_1 t^{-5/8}\ h(t)^2,\\ \| (\nabla A_a)|v|^2 \|_* &\le aN_0 N_1 t^{-5/8}\ h(t)^2.\end{aligned}$$

The quadratic terms in Bv are estimated by

$$\begin{aligned}\| (\nabla \overline{v}) B u_a \|_* &\le \| \| \nabla v \|_2 \| B \|_4 \| u_a \|_\infty \|_{4/3}\\ &\le cN_{1/2} N_2 h \| t^{-3/2} h \|_2 \le cN_{1/2} N_2 t^{-1}\ h(t)^2\end{aligned}$$

and similarly

$$\| \overline{v}(\nabla B)u_a \|_* \le cN_0N_6t^{-1}\ h(t)^2,$$
$$\| \overline{v}B\nabla u_a \|_* \le cN_0N_2t^{-1}\ h(t)^2.$$

The cubic terms from $B|v|^2$ are estimated by

$$\| (\nabla B)|v|^2 \|_* \le CN_6N_1^{3/2}N_{1/2}^{1/2}\ h(t)^3,$$
$$\begin{aligned}\| B\overline{v}\nabla v \|_* &\le C \| B; L^4(L^4) \| \| v; L^4(L^3) \|^{1/2} \| \nabla v; L^4(L^3) \|^{3/2} \\ &\le CN_2N_1^{1/2}N_4^{3/2}\ h(t)^3.\end{aligned}$$

Collecting the previous estimates yields

(2.49)
$$\begin{aligned}N_6' \le C_6\Big\{ & c_4N_{1/2} + ac(N_{1/2}+N_0)T^{-1} + c^2(N_2+N_6)T^{-1} + r_2 + N_{1/2}N_4\overline{h}(T) \\ & + a(N_{1/2}+N_0)N_1T^{-5/8}\ h(T) + c\left(N_2N_{1/2}+N_2N_0+N_6N_0\right)T^{-1}\ h(T) \\ & + \left(N_6N_1^{3/2}N_{1/2}^{1/2} + N_2N_1^{1/2}N_4^{3/2}\right)\ h(T)^2\Big\}\end{aligned}$$

which is of the form

$$N_6' \le C_6\left(c_4N_{1/2} + r_2 + (o(1); N_0, N_1, N_{1/2}, N_4, N_2, N_6)\right). \tag{2.50}$$

We now come back to the estimates of v' and we first estimate $\partial_t v'$ in L^2, namely we estimate N_3' defined by (2.36) by using (2.3). Here however we encounter a technical difficulty due to the fact that B a priori does not satisfy the assumption $\nabla B \in L^1_{loc}(I, L^\infty)$ needed in Proposition 2.1, part (2) in order to derive (2.3). We circumvent that difficulty by first regularizing B, introducing the associated solution v' which then satisfies (2.3), deriving the N_3' estimate for the auxiliary solution, and removing the regularization by a limiting procedure, which preserves the estimate. Here in order not to burden the proof with technicalities, we provide only the derivation of the estimates from (2.3) and we refer to the proof of Proposition 3.2, part (1) in [7] for the technical details. From (2.3) (2.4) with $V = g(|u|^2)$ and $f = G_1 - R_1$, we obtain

(2.51)
$$\| \partial_t v' \|_2 \le \| i(\partial_t A)\cdot\nabla_A v' + \left(\partial_t g(|u|^2)\right)v' + \partial_t G_1 - \partial_t R_1 \|_+ + \| (G_1 - R_1)(t_0) \|_2$$

with G_1 defined in (1.5).

We first estimate the terms containing v', starting with $i(\partial_t A)\cdot\nabla_A v'$.

$$\begin{aligned}\| (\partial_t A_a)\cdot\nabla_A v' \|_+ &\le \| \| \partial_t A_a \|_\infty \| \nabla_A v' \|_2 \|_1 \\ &\le a\widetilde{N}_{1/2}' \| t^{-1}h \|_1 \le 3a\widetilde{N}_{1/2}'\ h(t),\end{aligned}$$

$$\begin{aligned}
\| (\partial_t B)\cdot \nabla v' \|_+ &\le \| \| \partial_t B \|_4 \| \nabla v' \|_4 \|_1 \le C N_6 N_4' \ h(t)\overline{h}(t),\\
\| (\partial_t B)\cdot A_a v' \|_+ &\le \| \| \partial_t B \|_4 \| A_a \|_\infty \| v' \|_4 \|_1\\
&\le a N_6 N_1' h^2 \| t^{-1} \|_{8/3} \le a N_6 N_1' t^{-5/8} \ h(t)^2,\\
\| (\partial_t B)\cdot B v' \|_+ &\le C \| \| \partial_t B \|_4 \left(\| B \|_4 \| \nabla B \|_4^3 \right)^{1/4} \| v' \|_4 \|_1\\
&\le C N_6^{7/4} N_2^{1/4} N_1' t^{1/8} \ h(t)^3
\end{aligned}$$

by Lemma 2.2.

We next estimate the terms coming from $(\partial_t g(|u|^2))v'$.

$$\begin{aligned}
\| g\left(\overline{u}_a \partial_t u_a\right) v' \|_+ &\le \| \| g\left(\overline{u}_a \partial_t u_a\right) \|_\infty \| v' \|_2 \|_1 \le C c^2 N_0' \ h(t),\\
\| g\left(\overline{u}_a \partial_t v\right) v' \|_+ &\le \| \| g\left(\overline{u}_a \partial_t v\right) \|_\infty \| v' \|_2 \|_1 \le C c N_3 N_0' \ h(t)^2,\\
\| g\left((\partial_t \overline{u}_a) v\right) v' \|_+ &\le C c N_0 N_0' \ h(t)^2
\end{aligned}$$

by Lemma 2.3, part (2) and Lemma 2.2,

$$\| g\left(\overline{v} \partial_t v\right) v' \|_+ \le \| \| v \|_3 \| \partial_t v \|_2 \| v' \|_3 \|_1 \le C N_1 N_3 N_1' t^{1/2} \ h(t)^3$$

by Lemma 2.3, part (1) and Lemma 2.2.

We next estimate $\partial_t G_1$. The estimates are similar to those performed when estimating v' in L^2, with an additional time derivative acting on each factor in each term. This has the effect of requiring more regularity on (A_a, u_a) when that derivative hits (A_a, u_a), without changing the form the estimate, and of replacing one factor N_2 by N_6 when that derivative hits B and one factor N_0 by N_3 when that derivative hits v. Thus we obtain

$$\begin{aligned}
\| (\partial_t B)\cdot \nabla_{A_a} u_a \|_+ &\le N_6 (C c_4 + a c t^{-1}) \ h(t),\\
\| B \cdot \partial_t \nabla_{A_a} u_a \|_+ &\le c N_2 (C + a t^{-1}) \ h(t),\\
\| (\partial_t B) B u_a \|_+ &\le c N_2 N_6 t^{-1} \ h(t)^2,\\
\| B^2 \partial_t u_a \|_+ &\le c N_2^2 t^{-1} \ h(t)^2.\\
\| \partial_t \left(g(\overline{u}_a v) u_a\right) \|_+ &\le C c_3 \left(c_3 N_3 + c N_0\right) \ h(t)\\
\| \partial_t \left(g(|v|^2) u_a\right) \|_+ &\le C N_1 \left(c_3 N_3 + c N_0\right) t^{1/4} \ h(t)^2.
\end{aligned}$$

We finally estimate $\| \partial_t v'(t_0) \|_2$ and for that purpose we need pointwise (in time) estimates of R_1 and of B. Now from (2.21) it follows that

$$\| R_1(t) \|_2 \le \| \partial_t R_1 \|_+ \le r_1 \ h(t) \tag{2.52}$$

while from (2.27) (2.31)

$$\| B(t) \|_4^4 \le 4 \int_t^\infty dt' \ \| B(t') \|_4^3 \| \partial_t B(t') \|_4 \le 4 N_2^3 N_6 \ h(t)^4$$

and therefore

$$\parallel B(t) \parallel_4 \leq \widetilde{N}_2 \ h(t) \equiv \sqrt{2} \left(N_2^3 N_6 \right)^{1/4} \ h(t).$$

We then estimate

(2.53)
$$\begin{aligned}
\parallel G_1 \parallel_2 \leq \parallel B \parallel_4 (\parallel \nabla u_a \parallel_4 + \parallel A_a \parallel_\infty \parallel u_a \parallel_4) + \parallel B \parallel_4^2 \parallel u_a \parallel_\infty \\
+ \parallel g \left(|v|^2 + 2 \operatorname{Re} \overline{u}_a v \right) \parallel_6 \parallel u_a \parallel_3 \\
\leq c\widetilde{N}_2 (1 + at^{-1}) t^{-3/4} \ h(t) + c\widetilde{N}_2^2 t^{-3/2} \ h(t)^2 \\
+ C \Big(c_3^2 N_0 t^{-1} \ h(t) + c_3 N_0^{3/2} N_{1/2}^{1/2} t^{-1/2} \ h(t)^2 \Big)
\end{aligned}$$

by Lemma 2.3 for the terms containing g and the definitions.

Collecting the previous estimates and in particular (2.52) (2.53) taken at $t_0 \geq t$, we obtain

(2.54)
$$\begin{aligned}
N_3' \leq C_3 \Big\{ & a\widetilde{N}_{1/2}' + c_4 N_6 + cN_2 + ca(N_6 + N_2)T^{-1} + c_3^2 N_3 + c^2 (N_0 + N_0') + r_1 \\
& + N_6 N_4' \overline{h}(T) + aN_6 N_1' T^{-5/8} \ h(T) + cN_2 (N_6 + N_2) T^{-1} \ h(T) \\
& + c \left(N_3 + N_0 \right) N_0' \ h(T) + c \left(N_3 + N_0 \right) N_1 T^{-1/8} \overline{h}(T) \\
& + N_6^{7/4} N_2^{1/4} N_1' T^{-1/4} \ h(T) \overline{h}(T) + N_1 N_3 N_1' T^{-1/4} \overline{h}(T)^2 \\
& + c\widetilde{N}_2 (1 + aT^{-1}) T^{-3/4} + c\widetilde{N}_2^2 T^{-3/2} \ h(T) \\
& + c_3^2 N_0 T^{-1} + c_3 N_0^{3/2} N_{1/2}^{1/2} T^{-1/2} \ h(T) \Big\}
\end{aligned}$$

which is of the form

(2.55)
$$\begin{aligned}
N_3' \leq C_3 \Big(& a\widetilde{N}_{1/2}' + c_4 N_6 + cN_2 + c_3^2 N_3 + c^2 \left(N_0 + N_0' \right) + r_1 \\
& + (o(1); N_0, N_0', N_1, N_1', N_{1/2}, N_2, \widetilde{N_2}, N_3, N_4', N_6) \Big).
\end{aligned}$$

We next estimate $\parallel \Delta_A v' \parallel_2$. From (1.7), we obtain

$$\begin{aligned}
\parallel \Delta_A v' \parallel &\leq 2 \left(\parallel \partial_t v' \parallel_2 + \parallel g(|u|^2) v' \parallel_2 + \parallel G_1 \parallel_2 + \parallel R_1 \parallel_2 \right) \\
&\leq 2 \left(N_3' + r_1 \right) \ h(t) + 2 \parallel g(|u|^2) v' \parallel_2 + 2 \parallel G_1 \parallel_2 .
\end{aligned}$$

Furthermore

(2.56)
$$\begin{aligned}
\parallel g(|u|^2) v' \parallel_2 + \parallel G_1 \parallel_2 \leq \parallel g(|u|^2) \parallel_\infty \parallel v' \parallel_2 + \parallel G_1 \parallel_2 \\
\leq c\widetilde{N}_2 (1 + at^{-1}) t^{-3/4} \ h(t) + c\widetilde{N}_2^2 t^{-3/2} \ h(t)^2 \\
+ C \Big(c^2 N_0' t^{-1} h(t) + N_0 N_{1/2} N_0' \ h(t)^3 + c_3^2 N_0 t^{-1} \ h(t) \\
+ c_3 N_0^{3/2} N_{1/2}^{1/2} t^{-1/2} \ h(t)^2 \Big) \equiv M_1 \ h(t)
\end{aligned}$$

by Lemma 2.3 for the terms containing g and by (2.53), so that

$$\begin{aligned}(2.57)\qquad \| \Delta_A v'(t) \|_2 &\leq 2\,(N'_3 + r_1 + M_1)\ \ h(t)\\ &= 2\left(N'_3 + r_1 + \left(o(1); \widetilde{N}_2, N_0, N'_0, N_{1/2}\right)\right)\ h(t).\end{aligned}$$

As a consequence,

$$\begin{aligned}(2.58)\qquad \widetilde{N}'_{1/2} &\leq (2N'_0(N'_3 + r_1 + M_1(T)))^{1/2}\\ &\leq \left(2N'_0\left(N'_3 + r_1 + \left(o(1); \widetilde{N}_2, N_0, N'_0, N_{1/2}\right)\right)\right)^{1/2}.\end{aligned}$$

We next estimate $\| \Delta v'(t) \|_2$, namely N'_5 defined by (2.38). From

$$\Delta_A v' = \Delta v' - 2iA_a \cdot \nabla_A v' - 2iB \cdot \nabla v' + (A_a^2 - B^2)v'$$

we obtain

$$\begin{aligned}\| \Delta v' \|_2 \leq \| \Delta_A v' \|_2 &+2 \| A_a \|_\infty \| \nabla_A v' \|_2 + \| A_a \|_\infty^2 \| v' \|_2\\ &+2 \| B \|_4 \| \nabla v' \|_4 + \| B \|_4^2 \| v' \|_\infty .\end{aligned}$$

Now

$$\begin{aligned}\| \nabla v' \|_4 &\leq C \| \Delta v' \|_2^{7/8} \| v' \|_2^{1/8},\\ \| v' \|_\infty &\leq C \| \Delta v' \|_2^{3/4} \| v' \|_2^{1/4},\end{aligned}$$

and therefore

$$\begin{aligned}\| \Delta v' \|_2 \leq (1+\varepsilon) \| \Delta_A v' \|_2 &+(1+\varepsilon^{-1}) \| A_a \|_\infty^2 \| v' \|_2\\ &+\varepsilon \| \Delta v' \|_2 + C_\varepsilon \| B \|_4^8 \| v' \|_2 .\end{aligned}$$

Taking $\varepsilon = 1/3$ yields

$$\| \Delta v' \|_2 \leq 2 \| \Delta_A v' \|_2 + C \left(\| A_a \|_\infty^2 + \| B \|_4^8\right) \| v' \|_2$$

and therefore by (2.38) (2.57)

$$(2.59)\qquad N'_5 \leq 4\,(N'_3 + r_1 + M_1) + C\left(a^2 T^{-2} + \widetilde{N}_2^8 h(T)^8\right) N'_0$$

which is of the form

$$(2.60)\qquad N'_5 \leq 4\left(N'_3 + r_1 + (o(1); \widetilde{N}_2, N_0, N'_0, N_{1/2})\right).$$

We finally estimate the Strichartz norms of $\nabla v'$. For that purpose, by Lemma 2.1, we have to estimate the following quantity in the sum of spaces of the type $L^{\overline{q}}(J, L^{\overline{r}})$ for admissible pairs (q, r):

(2.61)
$$\begin{aligned} Q &= \nabla\left(iA\cdot\nabla v' + (1/2)A^2 v' + g(|u|^2)v' + G_1 - R_1\right) \\ &= iA\cdot\nabla^2 v' + i\nabla A\cdot\nabla_A v' + (1/2)A^2\nabla v' + \nabla(g(|u|^2)v') + \nabla G_1 - \nabla R_1. \end{aligned}$$

The estimates are similar to those performed when estimating $\| v' \|_2$ and the Strichartz norms of v' (see the proof of (2.42) (2.44)), with an additional gradient acting on each factor in each term, thereby producing the replacement of N_0 by $N_{1/2}$, of N'_0 by $N'_{1/2}$, of $N'_{1/2}$ by N'_5 and of N_2 by N_6 at suitable places. More precisely, the terms containing v' are estimated by

$$\begin{aligned} \| A_a\cdot\nabla^2 v' \|_+ &\le \| \| A_a \|_\infty \| \Delta v' \|_2 \|_1 \le 3aN'_5\ h(t), \\ \| \nabla A_a\cdot\nabla_A v' \|_+ &\le \| \| \nabla A_a \|_\infty \| \nabla_A v' \|_2 \|_1 \le 3a\widetilde{N}'_{1/2}\ h(t), \\ \| A_a^2\nabla v' \|_+ &\le \| \| A_a \|_\infty^2 \| \nabla v' \|_2 \|_1 \le a^2 N'_{1/2} t^{-1} h(t), \\ \| B\cdot\nabla^2 v'; L^{8/5}(J, L^{4/3}) \| &\le \| \| B \|_4 \| \Delta v' \|_2 \|_{8/5} \le CN_2 N'_5\ h(t)\overline{h}(t), \\ \| \nabla B\cdot\nabla_A v'; L^{8/5}(J, L^{4/3}) \| &\le \| \| \nabla B \|_4 \| \nabla_A v' \|_2 \|_{8/5} \\ &\le CN_6 \widetilde{N}'_{1/2}\ h(t)\overline{h}(t), \\ \| B^2\nabla v'; L^2(J, L^{6/5}) \| &\le C\ \| \| B \|_4^{3/2} \| \nabla B \|_4^{1/2} \| \nabla v' \|_2 \|_2 \\ &\le CN_2^{3/2} N_6^{1/2} N'_{1/2}\ h(t)^3, \\ \| \nabla\left(g(|u_a|^2)v'\right) \|_+ &\le Cc^2\left(N'_0 + N'_{1/2}\right)\ h(t), \\ \| \nabla\left(g(\overline{u}_a v)v'\right); L^{4/3}(J, L^{3/2}) \| &\le Cc\left(N_0 N'_0 + N_0 N'_{1/2} + N_{1/2} N'_0\right) t^{1/4} h(t)^2, \\ \| \nabla\left(g(|v|^2)v'\right); L^{4/3}(J, L^{3/2}) \| &\le CN_1\left(N_0 N'_{1/2} + N_{1/2} N'_0\right) t^{1/2} h(t)^3 \end{aligned}$$

where we have used again Lemmas 2.2 and 2.3 in the estimates of the terms containing g.

The terms from ∇G_1 are estimated by

$$\begin{aligned} \| \nabla B\cdot\nabla_{Aa} u_a \|_+ &\le N_6\left(Cc_4 + cat^{-1}\right) h(t), \\ \| B\cdot\nabla\nabla_{Aa} u_a \|_+ &\le CcN_2\left(1 + at^{-1}\right)\ h(t), \\ \| B^2\nabla u_a \|_+ &\le cN_2^2 t^{-1}\ h(t)^2, \\ \| B(\nabla B)u_a \|_+ &\le cN_2 N_6 t^{-1}\ h(t)^2. \\ \| \nabla\left(g(\overline{u}_a v)u_a\right) \|_+ &\le Cc_3\left(c_3 N_{1/2} + cN_0\right)\ h(t), \\ \| \nabla\left(g(|v|^2)u_a\right) \|_+ &\le CN_1\left(c_3 N_{1/2} + cN_0\right) t^{1/4} h(t)^2. \end{aligned}$$

Collecting the previous estimates yields

(2.62)
$$\begin{aligned} N'_4 \le C_4 \Big\{ & a\left(N'_5 + \widetilde{N}'_{1/2}\right) + a^2 N'_{1/2} T^{-1} + c\,(N_6 + N_2)\left(1 + aT^{-1}\right) \\ & + c^2\left(N_0 + N'_0 + N_{1/2} + N'_{1/2}\right) + r_1 \\ & + \left(N_2 N'_5 + N_6 \widetilde{N}'_{1/2}\right)\overline{h}(T) + cN_2(N_6+N_2)T^{-1}\ h(T) \\ & + c\left(\left(N_{1/2} + N_0\right)(N_1 + N'_0) + N_0 N'_{1/2}\right) T^{-1/8}\overline{h}(T) \\ & + N_2^{3/2} N_6^{1/2} N'_{1/2} h(T)^2 + N_1\left(N_0 N'_{1/2} + N_{1/2} N'_0\right) T^{-1/4}\overline{h}(T)^2 \Big\} \end{aligned}$$

which is of the form

(2.63)
$$\begin{aligned} N'_4 \le C_4 \Big(& a\left(N'_5 + \widetilde{N}'_{1/2}\right) + c(N_6 + N_2) + c^2\left(N_0 + N'_0 + N_{1/2} + N'_{1/2}\right) + r_1 \\ & + (o(1); N_0, N'_0, N_1, N_{1/2}, N'_{1/2}, \widetilde{N}'_{1/2}, N'_5, N_2, N_6)\Big). \end{aligned}$$

From the previous estimates, more precisely from (2.42) (2.44) (2.46) (2.49) (2.54) (2.59) (2.62), it follows that the N'_i, $0 \le i \le 6$ are estimated in terms of the N_i, $0 \le i \le 6$, provided T is sufficiently large. In fact (2.42) (2.46) (2.49) provide estimates of N'_0, N'_2 and N'_6. Denoting by C a general constant depending on T and on the N_i, it follows from (2.44) (2.59) and from the definition of $N'_{1/2}$, $\widetilde{N}'_{1/2}$ that

$$N'_1 \le C\left(1 + N'_{1/2}\right), N'_{1/2} \vee \widetilde{N}'_{1/2} \le C\left(1 + N'_5\right)^{1/2}, N'_5 \le 4N'_3 + C \tag{2.64}$$

so that it remains only to estimate N'_3 and N'_4. Substituting the previous estimates into (2.54) (2.62) yields

$$\begin{cases} N'_3 \le C_3 N_6 N'_4 \overline{h}(T) + \text{terms sublinear in } N'_3 \\ N'_4 \le 4C_4 N'_3\left(a + N_2\overline{h}(T)\right) + \text{terms sublinear in } N'_3 \end{cases} \tag{2.65}$$

which ensure the required estimate of N'_3, N'_4 provided T is sufficiently large so that

$$4C_3C_4\left(a + N_2\overline{h}(T)\right) N_6 \overline{h}(T) < 1, \tag{2.66}$$

which we assume from now on. Note that the terms responsible for that large T condition are the terms $\partial_t B \cdot \nabla v'$ from (2.51) and $A \cdot \nabla^2 v'$ from (2.61). No such condition was required at this stage in the simpler case of the $(\mathrm{WS})_3$ system [8]. The estimates obtained for the N'_i are obviously uniform in t_0.

We now take the limit $t_0 \to \infty$ of (v'_{t_0}, B'_{t_0}), restoring the subscript t_0 for that part of the argument. Let $T < t_0 < t_1 < \infty$ and let (v'_{t_0}, B'_{t_0}) and (v'_{t_1}, B'_{t_1}) be the corresponding solutions of (1.7). From the L^2 norm conservation of the difference $v'_{t_0} - v'_{t_1}$ and from (2.42), it follows that for all $t \in [T, t_0]$

$$\| v'_{t_0}(t) - v'_{t_1}(t) \|_2 = \| v'_{t_1}(t_0) \|_2 \le K_0 h(t_0) \tag{2.67}$$

where K_0 is the RHS of (2.42), while from (1.7) (2.8) (2.9) (2.46) (2.49) and the initial conditions, it follows that

$$\begin{aligned} &\| B'_{t_0} - B'_{t_1}; L^4([T,t_0], W^1_4) \| \vee \| \partial_t (B'_{t_0} - B'_{t_1}); L^4([T,t_0], L^4) \| \\ &\le C \| G_2 - R_2; L^{4/3}([t_0,t_1], W^1_{4/3}) \| \le (K_2 + K_6) h(t_0) \end{aligned} \tag{2.68}$$

where K_2 and K_6 are the RHS of (2.46) and (2.49) respectively.

It follows from (2.67) (2.68) that there exists $(v', B') \in L^\infty_{loc}([T,\infty), L^2) \oplus L^4_{loc}([T,\infty), W^1_4)$ with $\partial_t B' \in L^4_{loc}([T,\infty), L^4)$ such that (v'_{t_0}, B'_{t_0}) converges to (v', B') in that space when $t_0 \to \infty$. From the uniformity in t_0 of the estimates (2.42) (2.46) (2.49), it follows that (v', B') satisfies the same estimates in $[T, \infty)$, namely that (2.43) (2.47) (2.50) hold with N'_i defined by (2.33) (2.35) (2.39) with $I = [T, \infty)$. Furthermore it follows by a standard compactness argument that $(v', B') \in X([T,\infty))$ and that v' satisfies the remaining estimates, namely (2.45) (2.55) (2.60) (2.63) with the remaining N'_i again defined by (2.34) (2.36) (2.37) (2.38) with $I = [T, \infty)$. Clearly (v', B') satisfies the system (1.7).

From now on, I denotes the interval $[T, \infty)$. The previous construction defines a map $\phi : (v, B) \to (v', B')$ from $X(I)$ to itself. The next step consists in proving that the map ϕ is a contraction on a suitable closed bounded set $\mathcal{R}$ of $X(I)$. We define $\mathcal{R}$ by the conditions (2.25)-(2.31) for some constants N_i and for all $t \in I$. We first show that for a suitable choice of N_i and for sufficiently large T, the map ϕ maps $\mathcal{R}$ into $\mathcal{R}$. By (2.43) (2.45) (2.47) (2.50) (2.55) (2.60)

(2.63), it suffices for that purpose that

(2.69)
$$\begin{cases} (N'_0 \le) C_0 \Big(c_4 N_2 + c_3^2 N_0 + r_1 + o(1) \Big) \le N_0 \\ (N'_1 \le) C_1 \Big(c_4 N_2 + c_3^2 N_0 + c^2 N'_0 + a N'_{1/2} + r_1 + o(1) \Big) \le N_1 \\ (N'_2 \le) C_2 \Big(c_4 N_0 + r_2 + o(1) \Big) \le N_2 \\ (N'_6 \le) C_6 \Big(c_4 N_{1/2} + r_2 + o(1) \Big) \le N_6 \\ (N'_3 \le) C_3 \Big(a \widetilde{N}'_{1/2} + c_4 N_6 + c N_2 + c_3^2 N_3 + c^2 (N_0 + N'_1) + r_1 + o(1) \Big) \le N_3 \\ (N'_4 \le) C_4 \Big(a \left(N'_5 + \widetilde{N}'_{1/2} \right) + c(N_6 + N_2) + c^2 (N_0 + N'_0 + N_{1/2} + N'_{1/2}) \\ \qquad\qquad + r_1 + o(1) \Big) \le N_4 \\ (N'_5 \le) 4 \Big(N'_3 + r_1 + o(1) \Big) \le N_5 \end{cases}$$

where we have omitted the dependence of the $o(1)$ terms on N_i and N'_i. We know in addition that

$$N'_{1/2} \vee \widetilde{N}'_{1/2} \le 2 \Big(N'_0 (N'_3 + r_1 + o(1)) \Big)^{1/2}. \tag{2.70}$$

In order to ensure (2.69), we proceed as follows. We first choose N_0 and N_2 by imposing

$$\begin{cases} N_0 = C_0 \left(c_4 N_2 + c_3^2 N_0 + r_1 + 1 \right) \\ N_2 = C_2 \left(c_4 N_0 + r_2 + 1 \right) \end{cases} \tag{2.71}$$

which is possible under the smallness condition on c_3 , c_4

$$C_0 \left(C_2 c_4^2 + c_3^2 \right) < 1, \tag{2.72}$$

and we impose the condition that $o(1) \le 1$ in (2.43) (2.47) by taking T sufficiently large (depending on N_0, N_2 just chosen and on $N_{1/2}$, N_1 to be chosen later). This ensures the $N'_0 \le N_0$ and $N'_2 \le N_2$ parts of (2.69). Furthermore one can replace N'_0 by N_0 in all the remaining estimates. We next impose

(2.73)
$$\begin{cases} N_5 = 4 \left(N_3 + r_1 + 1 \right) \\ N_6 = C_6 \left(c_4 (N_0 N_5)^{1/2} + r_2 + 1 \right) = C_6 \left(2 c_4 \left(N_0 (N_3 + r_1 + 1) \right)^{1/2} + r_2 + 1 \right) \end{cases}$$

and we impose $o(1) \le 1$ in (2.50) and (2.60) by taking T sufficiently large depending on the relevant N_i. This ensures the $N'_6 \le N_6$ part of (2.69) together with the inequality

$$N'_5 \le 4 \left(N'_3 + r_1 + 1 \right) \tag{2.74}$$

which will ensure the $N'_5 \le N_5$ part of (2.69) as soon as the $N'_3 \le N_3$ part holds. Furthermore, under the choices and assumptions made so far, (2.74) implies

$$N'_{1/2} \vee \widetilde{N}'_{1/2} \le 2\left(N_0(N'_3 + r_1 + 1)\right)^{1/2} . \tag{2.75}$$

We now substitute (2.75) into (2.44), we substitute (2.74) (2.75) into (2.62) and we substitute the results and (2.75) again into (2.54), thereby obtaining an estimate of the form

$$\begin{aligned} N'_3 \le f\left(N'_3, \{N_i\}\right) \equiv C_3\Big(& 2a\left(N_0(N'_3 + r_1 + 1)\right)^{1/2} \\ & + c_4 C_6\left(2c_4\left(N_0(N_3 + r_1 + 1)\right)^{1/2} + r_2 + 1\right) \\ & + cN_2 + c_3^2 N_3 + 2c^2 N_0 + r_1 + o(1)\Big) \end{aligned} \tag{2.76}$$

where $f(N'_3, \{N_i\})$ is a positive increasing concave function of N'_3 for fixed T and N_i. It follows therefrom that (2.76) will imply $N'_3 \le N_3$ provided we ensure that

$$N_3 \ge f\left(N_3, \{N_i\}\right) . \tag{2.77}$$

This is obtained by imposing

$$\begin{aligned} N_3 = C_3\Big(& 2\left(a + C_6 c_4^2\right)\left(N_0(N_3 + r_1 + 1)\right)^{1/2} + c_4 C_6 (r_2 + 1) \\ & + cN_2 + c_3^2 N_3 + 2c^2 N_0 + r_1 + 1\Big) \end{aligned} \tag{2.78}$$

which is possible under the smallness condition $C_3 c_3^2 < 1$, and by imposing that $o(1) \le 1$ in (2.76) by taking T sufficiently large depending on the N_i.

It is then a simple matter to choose N_1 and N_4 in order to ensure the $N'_1 \le N_1$ and $N'_4 \le N_4$ parts of (2.69), since all the N'_i in the RHS of (2.44) and (2.62) are now under control. It suffices to choose

$$N_1 = C_1\left(c_4 N_2 + 2c^2 N_0 + a(N_0 N_5)^{1/2} + r_1 + 1\right) \tag{2.79}$$

$$N_4 = C_4\left(aN_5 + (a + 2c^2)(N_0 N_5)^{1/2} + c(N_6 + N_2) + 2c^2 N_0 + r_1 + 1\right) \tag{2.80}$$

and to impose that $o(1) \le 1$ in (2.45) (2.63) by taking T sufficiently large depending on the N_i (with the N'_i in the $o(1)$ terms being estimated by the N_i).

We now show that the map ϕ is a contraction on $\mathcal{R}$ for a suitable norm defined on $X(I)$. Let $(v_i, B_i) \in \mathcal{R}$ and $(v'_i, B'_i) = \phi((v_i, B_i))$, $i = 1, 2$. For any pair of functions f_1, f_2, we define $f_\pm = (1/2)(f_1 \pm f_2)$ so that $f_1 = f_+ + f_-$, $f_2 = f_+ - f_-$ and $(fg)_\pm = f_+ g_\pm + f_- g_\mp$. In particular $u_+ = u_a + v_+$, $u_- = v_-$, $A_+ = A_a + B_+$, $A_- = B_-$, $(\Delta_A)_- = -2iB_- \cdot \nabla_{A_+}$, and similarly for the primed quantities. Since $\mathcal{R}$ is convex and stable under ϕ, (v_+, B_+) and (v'_+, B'_+) belong to $\mathcal{R}$, namely satisfy (2.25)-(2.31). Corresponding to (1.7), (v'_-, B'_-) satisfies the system

$$\begin{cases} i\partial_t v'_- = -(1/2)(\Delta_A)_+ v'_- + g(|u|^2_+)v'_- + iB_- \cdot \nabla_{A_+}(u_a + v'_+) \\ \qquad\qquad +g\left(2\,\mathrm{Re}\left(\overline{u}_a + \overline{v}_+\right)v_-\right)(u_a + v'_+) \\ \Box B'_- = 2P\,\mathrm{Im}\left(\overline{v}_- \nabla_{A_+}(u_a + v_+)\right) - PB_-\left(|u_a + v_+|^2 + |v_-|^2\right). \end{cases} \tag{2.81}$$

Here however, in contrast with the case of the $(\mathrm{WS})_3$ system where the corresponding map ϕ can be shown to be a contraction for the whole norm of $X(I)$, we encounter a difficulty due to the derivative coupling in the covariant Laplacian. In fact if D is a differential operator of order m, a straightforward energy estimate of $\| Dv'_- \|_2$ from (2.81) yields

$$\partial_t \| Dv'_- \|_2 \leq \| B_- \cdot D\nabla_{A_+} v'_+ \|_2 + \text{other terms}$$

and requires therefore a control of v'_+ at order $m+1$, so that one can hope to contract norms of v of degree at most one less than those occurring in the definition of $X(I)$. Fortunately, because of the special algebraic properties of the equations, it turns out that the lowest two semi norms of $X(I)$ for the differences, namely those corresponding to N_0 and N_2, can be decoupled from the higher ones and can be contracted on the bounded sets of $X(I)$. This follows from the fact that the symmetry of the quadratic form $P\,\mathrm{Im}\,(\overline{v}_1 \nabla_A v_2)$ has made it possible to avoid having a gradient acting on v_- in the equation for B'_- in (2.81). Thus we shall show that ϕ is a contraction for the pair of semi norms

$$\begin{cases} N_0 = \sup\limits_{t\in I} h(t)^{-1} \| v(t) \|_2, \\ N_2 = \sup\limits_{t\in I} h(t)^{-1} \| B; L^4([t,\infty), L^4) \| . \end{cases} \tag{2.82}$$

Let (N_{0-}, N_{2-}) and (N'_{0-}, N'_{2-}) be the corresponding semi norms of (v_-, B_-) and (v'_-, B'_-) respectively. We have to estimate (N'_{0-}, N'_{2-}) in terms of (N_{0-}, N_{2-}). We first estimate N'_{0-}. From (2.81) we obtain

$$\| v'_-(t) \|_2 \leq \| B_- \cdot \nabla_{A_+}(u_a + v'_+) \|_+ + \| g\left(2\,\mathrm{Re}\left(\overline{u}_a + \overline{v}_+\right) v_-\right)(u_a + v'_+) \|_+ . \tag{2.83}$$

The terms not containing v'_+ are estimated as in the proof of (2.42), namely

$$\begin{aligned} \| B_- \cdot \nabla u_a \|_+ &\leq C c_4 N_{2-} \ h(t), \\ \| B_- \cdot A_a u_a \|_+ &\leq c a N_{2-} t^{-1} \ h(t), \\ \| B_- B_+ u_a \|_+ &\leq c N_{2-} N_2 t^{-1} \ h(t)^2, \\ \| g\,(\overline{u}_a v_-)\, u_a \|_+ &\leq C c_3^2 N_{0-} \ h(t), \\ \| g\,(\overline{v}_+ v_-)\, u_a \|_+ &\leq C c_3 N_{0-} N_1 t^{1/4} \ h(t)^2. \end{aligned}$$

We next estimate the terms containing v'_+.

$$\| B_- \cdot \nabla v'_+ \|_+ \leq \| \, \| B_- \|_4 \, \| \nabla v'_+ \|_4 \, \|_1 \leq C \ N_{2-} \ N_4 \ h(t) \ \overline{h}(t)$$

by Lemma 2.2,

$$\begin{aligned} \| B_- \cdot A_a v'_+ \|_+ &\leq \| \| B_- \|_4 \| A_a \|_\infty \| v'_+ \|_4 \|_1 \\ &\leq a N_{2-} N_1 h^2 \ \| t^{-1} \|_{8/3} \leq a N_{2-} N_1 t^{-5/8} \ h(t)^2, \\ \| B_- B_+ v'_+ \|_+ &\leq C \ \| \| B_- \|_4 \left(\| B_+ \|_4 \| \nabla B_+ \|_4^3 \right)^{1/4} \| v'_+ \|_4 \|_1 \\ &\leq C \ N_{2-} (N_2 \ N_6^3)^{1/4} \ N_1 \ t^{1/8} \ h(t)^3 \end{aligned}$$

by Lemma 2.2,

$$\begin{aligned} \| g\,(\overline{u}_a v_-)\, v'_+ \|_+ &\leq C \ \| \| u_a \|_3 \| v_- \|_2 \| v'_+ \|_3 \|_1 \leq C c_3 N_{0-} N_1 t^{1/4} \ h(t)^2, \\ \| g\,(\overline{v}_+ v_-)\, v'_+ \|_+ &\leq C \ \| \| v_+ \|_3 \| v_- \|_2 \| v'_+ \|_3 \|_1 \leq C N_{0-} N_1^2 t^{1/2} \ h(t)^3, \end{aligned}$$

by Lemma 2.3, part (1) and Lemma 2.2.

Collecting the previous estimates yields

(2.84)
$$\begin{aligned} N'_{0-} \ \leq C_0 \Big\{ & N_{2-} \Big(c_4 + acT^{-1} + cN_2 T^{-1} \ h(T) + N_4 \overline{h}(T) + aN_1 T^{-5/8} \ h(T) \\ & + (N_2 N_6^3)^{1/4} N_1 T^{-1/4} \ h(T) \overline{h}(T) \Big) + N_{0-} \Big(c_3^2 + cN_1 T^{-1/8} \overline{h}(T) \\ & + N_1^2 T^{-1/4} \overline{h}(T)^2 \Big) \Big\} \end{aligned}$$

which is of the form

$$N'_{0-} \leq C_0 \Big(N_{2-} (c_4 + o(1)) + N_{0-} (c_3^2 + o(1)) \Big). \qquad (2.85)$$

We next estimate N'_{2-}. From (2.8) (2.81) we obtain

(2.86)
$$\| B'_-; L^4(J, L^4) \| \leq C \left(\| \overline{v}_- \nabla_{A_+} (u_a + v_+) \|_* + \| B_- (|u_a|^2 + |v_+|^2) \|_* \right).$$

The linear terms are estimated as in the proof of (2.46), namely

$$\begin{aligned}\| \overline{v}_- \nabla u_a \|_* &\le 2c_4 N_{0-}\ h(t),\\ \| \overline{v}_- A_a u_a \|_* &\le acN_{0-} t^{-1}\ h(t),\\ \| B_- |u_a|^2 \|_* &\le c^2 N_{2-} t^{-1}\ h(t).\end{aligned}$$

The non linear terms are estimated in a slightly different way. The quadratic terms are estimated by

$$\begin{aligned}\| \overline{v}_- \nabla v_+ \|_* &\le \|\| v_- \|_2 \| \nabla v_+ \|_4 \|_{4/3} \le CN_{0-} N_4\ h(t)\overline{h}(t),\\ \| \overline{v}_- A_a v_+ \|_* &\le \|\| A_a \|_\infty \| v_- \|_2 \| v_+ \|_4 \|_{4/3} \le aN_{0-} N_1 t^{-5/8}\ h(t)^2,\\ \| \overline{v}_- B_+ u_a \|_* &\le \|\| v_- \|_2 \| B_+ \|_4 \| u_a \|_\infty \|_{4/3}\\ &\le cN_{0-} N_2 h \| t^{-3/2} h \|_2 \le cN_{0-} N_2 t^{-1}\ h(t)^2.\end{aligned}$$

The cubic terms are estimated by

$$\begin{aligned}\| \overline{v}_- B_+ v_+ \|_* &\le C \|\| v_- \|_2 \left(\| B_+ \|_4 \| \nabla B_+ \|_4^3 \right)^{1/4} \| v_+ \|_4 \|_{4/3}\\ &\le C\ N_{0-} (N_2 N_6^3)^{1/4} N_1 t^{1/8}\ h(t)^3\end{aligned}$$

by Lemma 2.2,

$$\| B_- |v_+|^2 \|_* \le CN_{2-} N_1^{3/2} N_{1/2}^{1/2}\ h(t)^3.$$

Collecting the previous estimates yields

(2.87)

$$\begin{aligned}N'_{2-} \le C_2 \Big\{ & N_{0-} \Big(c_4 + acT^{-1} + N_4 \overline{h}(T) + aN_1 T^{-5/8}\ h(T) + cN_2 T^{-1}\ h(T)\\ & + (N_2 N_6^3)^{1/4} N_1 T^{-1/4}\ h(T)\overline{h}(T) \Big) + N_{2-} \left(c^2 T^{-1} + N_1^{3/2} N_{1/2}^{1/2} h(T)^2 \right) \Big\}\end{aligned}$$

which is of the form

$$N'_{2-} \le C_2 \left(c_4 + o(1) \right) N_{0-} + o(1) N_{2-}. \tag{2.88}$$

It follows from (2.85) (2.88) that the map ϕ is a contraction for the pair of semi norms (N_0, N_2) on the set $\mathcal{R}$ under the smallness condition

$$C_0 \left(C_2 c_4^2 + c_3^2 \right) < 1 \tag{2.89}$$

and for T sufficiently large. Since the set $\mathcal{R}$ is closed for the norm defined by the pair (N_0, N_2), it follows therefrom that the system (1.4) has a solution in

$\mathcal{R}$. This proves the existence part of Proposition 2.2. The uniqueness part follows from (2.85) (2.88) again with $N'_{i-} = N_{i-}$.

We remark at this point that the constants C_0, C_2 appearing in (2.89) can be taken to be the same as in (2.72) so that the two smallness conditions actually coincide. In fact those constants are determined by the linear terms in the estimates, and those terms are the same in both cases. There may occur additional, different constants coming from the non linear terms. They have been omitted in (2.84) (2.87).

It remains to prove the last statement of Proposition 2.2 and for that purpose we need to estimate the energy norm of B'. From (1.7) (2.10) it follows that for all $t \in I$

$$\| \nabla B'(t) \|_2 \vee \| \partial_t B'(t) \|_2 \leq \| G_2 - R_2 \|_+ \tag{2.90}$$

where G_2 is defined by (1.5). We estimate the various terms of G_2 successively. The linear terms in v are estimated by

$$\begin{aligned} \| \overline{v} \nabla u_a \|_+ &\leq \| \| v \|_2 \| \nabla u_a \|_\infty \|_1 \\ &\leq c\, N_0 \| t^{-3/2} h \|_1 \leq 2cN_0 t^{-1/2}\ h(t), \end{aligned}$$

$$\begin{aligned} \| \overline{v} A_a u_a \|_+ &\leq \| \| v \|_2 \| A_a \|_\infty \| u_a \|_\infty \|_1 \\ &\leq acN_0 \| t^{-5/2} h \|_1 \leq acN_0 t^{-3/2} h(t). \end{aligned}$$

The linear term in B is estimated by

$$\begin{aligned} \| B |u_a|^2 \|_+ &\leq \| \| B \|_4 \| u_a \|_4 \| u_a \|_\infty \|_1 \\ &\leq c^2 N_2 h \| t^{-3/4-3/2} \|_{4/3} \leq c^2 N_2 t^{-3/2}\ h(t). \end{aligned}$$

The quadratic terms in v^2 are estimated by

$$\| \overline{v} \nabla v \|_+ \leq \| \| v \|_4 \| \nabla v \|_4 \|_1 \leq C N_1 N_4 t^{1/4}\ h(t)^2$$

by Lemma 2.2,

$$\begin{aligned} \| A_a |v|^2 \|_+ &\leq \| \| A_a \|_\infty \| v \|_4^2 \|_1 \\ &\leq a N_1^2 h^2 \| t^{-1} \|_4 \leq a N_1^2 t^{-3/4}\ h(t)^2. \end{aligned}$$

The quadratic terms in Bv again need not be considered. The cubic term $B|v|^2$ is estimated by

$$\begin{aligned} \| B|v|^2 \|_+ &\leq C \| B; L^4(L^4) \| \| v; L^{8/3}(L^4) \|^{5/4} \| \nabla v; L^{8/3}(L^4) \|^{3/4} \\ &\leq C N_2 N_1^{5/4} N_4^{3/4}\ h(t)^3. \end{aligned}$$

Collecting the previous estimates and using (2.23), we obtain

(2.91)
$$\| \nabla B'(t) \|_2 \vee \| \partial_t B'(t) \|_2 \leq C\Big(cN_0 t^{-1/2} + acN_0 t^{-3/2} + c^2 N_2 t^{-3/2}$$
$$+ N_1 N_4 t^{1/4}\ h(t) + aN_1^2 t^{-3/4}\ h(t) + N_2 N_1^{5/4} N_4^{3/4}\ h(t)^2 + r_2 t^{-1/2}\Big)\ h(t)$$

which proves that the solution of (1.4) constructed previously satisfies (2.24).

□

Remark 2.2. The only smallness condition on u is the condition (2.72), coming from N_0 and from its coupling with N_2. The subsequent condition $C_3 c_3^2 < 1$ needed for the choice of N_3 comes in fact from exactly the same estimate as the c_3^2 contribution to N_0', so that the latter condition is actually the $c_4 = 0$ special case of (2.72) and is therefore weaker than (2.72). That fact is hidden by the use of overall constants C_0 and C_3 in the estimates of N_0' and N_3'.

§3. Remainder Estimates and Completion of the Proof

In this section, we first prove that the choice of asymptotic functions (u_a, A_a) made in the introduction satisfies the assumptions of Proposition 2.2 for the choice of h made in Proposition 1.1, under suitable assumptions on the asymptotic state $(u_+, A_+, \dot{A}_+)$. We then combine those results with Proposition 2.2 to complete the proof of Proposition 1.1.

We first supplement the definition of (u_a, A_a) with some additional properties of a general character. In addition to the representation (1.13) (1.14) of A_1, we need a representation of $\partial_t A_1$. From (1.12) it follows that

$$\partial_t A_1(t) = -\int_t^\infty dt' \cos(\omega(t'-t)) t'^{-1} Px|u_a(t')|^2 \tag{3.1}$$

so that upon substitution of (1.8) we obtain

$$\partial_t A_1(t) = t^{-2} D_0(t) \widetilde{\widetilde{A}}_1 \tag{3.2}$$

where

$$\widetilde{\widetilde{A}}_1 = -\int_t^\infty d\nu \nu^{-3} \cos(\omega(\nu - 1)) D_0(\nu) Px|w_+|^2. \tag{3.3}$$

On the other hand, from (1.13)

$$\nabla A_1(t) = t^{-2} D_0(t) \nabla \widetilde{A}_1. \tag{3.4}$$

We shall need the operator

$$J \equiv J(t) = x + it\nabla. \tag{3.5}$$

The asymptotic form A_a for A has been chosen in order to make R_2 small. In fact R_2 can be rewritten as

$$R_2 = \Box A_a + P\left(t^{-1}\,\mathrm{Re}\,\overline{u}_a J u_a + (A_a - x/t)|u_a|^2\right) \tag{3.6}$$

and A_a has been chosen in such a way that

$$\Box A_a = P(x/t)|u_a|^2 \tag{3.7}$$

so that

$$R_2 = P\left(t^{-1}\,\mathrm{Re}\,\overline{u}_a J u_a + A_a|u_a|^2\right). \tag{3.8}$$

Under general assumptions on (u_a, A_a), of the same type as in Proposition 2.2 (see especially (2.17) (2.20)) but not making use of their special form, we can prove that R_2 satisfies the assumptions needed for that proposition with the choice of h required for Proposition 1.1.

Lemma 3.1. *Let (u_a, A_a) satisfy the estimates*

$$\| u_a(t) \|_r \le ct^{-\delta(r)} \qquad \text{for } 2 \le r \le \infty, \tag{3.9}$$

$$\| \nabla u_a(t) \|_4 \le ct^{-3/4}, \tag{3.10}$$

$$\| Ju_a(t) \|_2 \le c_1(1 + \ell n\ t), \tag{3.11}$$

$$\| A_a(t) \|_\infty \vee \| \nabla A_a(t) \|_\infty \le at^{-1} \tag{3.12}$$

for all $t \ge 1$. Then R_2 satisfies the estimates

$$\begin{aligned} \| R_2; L^{4/3}([t,\infty), L^{4/3}) \| \vee \| \nabla R_2; L^{4/3}([t,\infty), L^{4/3}) \| \\ \le r_2 t^{-1}(1 + \ell n\ t), \end{aligned} \tag{3.13}$$

$$\| R_2; L^1([t,\infty), L^2) \| \le r_2 t^{-3/2}(1 + \ell n\ t) \tag{3.14}$$

for some constant r_2 and for all $t \ge 1$.

Proof. We estimate

$$\begin{aligned} \| R_2(t) \|_{4/3} &\le C \| u_a \|_4 \left(t^{-1} \| Ju_a \|_2 + \| A_a \|_\infty \| u_a \|_2\right) \\ &\le Ct^{-7/4} c\,(c_1(1 + \ell n\ t) + ac) \end{aligned}$$

which implies the first estimate of (3.13) by integration,

$$\begin{aligned}\| R_2(t) \|_2 &\leq \| u_a \|_\infty \left(t^{-1} \| Ju_a \|_2 + \| A_a \|_\infty \| u_a \|_2\right) \\ &\leq t^{-5/2} c\, (c_1(1+\ell n\ t) + ac)\end{aligned}$$

which implies (3.14) by integration.

In order to prove the second estimate of (3.13), we note that the quadratic form

$$P\,\mathrm{Re}\,\left(t^{-1}\overline{v}_1 Jv_2 + A_a \overline{v}_1 v_2\right)$$

is symmetric in v_1, v_2, so that

$$\begin{aligned}\nabla R_2 = 2\ P\,\mathrm{Re}\,&\left(t^{-1}(\nabla\overline{u}_a)\ Ju_a + A_a(\nabla\overline{u}_a)u_a\right) \\ &+ P\,(\nabla(A_a + x/t))\,|u_a|^2\end{aligned}$$

and therefore

$$\begin{aligned}\| \nabla R_2 \|_{4/3} \leq C\Big(&\| \nabla u_a \|_4 \left(t^{-1} \| Ju_a \|_2 + \| A_a \|_\infty \| u_a \|_2\right) \\ &+ \left(\| \nabla A_a \|_\infty + t^{-1}\right) \| u_a \|_4 \| u_a \|_2 \Big) \\ \leq Ct^{-7/4} &\left(c\,(c_1(1+\ell n\ t) + ac) + c^2(a+1)\right)\end{aligned}$$

from which the second estimate of (3.13) follows by integration. □

We now turn to R_1. We first skim R_1 of some harmless terms. Expanding the covariant Laplacian and using again J, we rewrite R_1 as

$$R_1 = R_{1,1} + R_{1,2} \tag{3.15}$$

where

$$R_{1,1} = i\partial_t u_a + (1/2)\Delta u_a + t^{-1}(x \cdot A_1)u_a - g(|u_a|^2)u_a, \tag{3.16}$$

$$R_{1,2} = t^{-1}(x \cdot A_0)u_a - t^{-1}A_a \cdot Ju_a - (1/2)A_a^2 u_a. \tag{3.17}$$

In the same way as for R_2, we can show that $R_{1,2}$ satisfies the assumptions needed for Proposition 2.2 with the choice of h required for Proposition 1.1 under general assumptions on (u_a, A_a) not making use of their special form.

Lemma 3.2. *Let u_a, A_a and A_0 satisfy the estimates*

$$\| \partial_t^j \nabla^k u_a \|_2 \leq c, \tag{3.18}$$

$$\| \partial_t^j \nabla^k Ju_a \|_2 \leq c_1(1+\ell n\ t), \tag{3.19}$$

$$\| \partial_t^j \nabla^k A_a \|_\infty \leq at^{-1}, \tag{3.20 $\equiv$ 2.20}$$

$$\| \partial_t^j \nabla^k (x \cdot A_0) \|_\infty \leq a_0 t^{-1}, \tag{3.21}$$

for $0 \le j + k \le 1$ *and for all* $t \ge 1$. *Then* $R_{1,2}$ *satisfies the estimates*

$$\| \partial_t^j \nabla^k R_{1,2} \|_2 \le r_{1,2} t^{-2} (1 + \ell n\ t), \tag{3.22}$$

for $0 \le j + k \le 1$, *for some constant* $r_{1,2}$ *and for all* $t \ge 1$.

Proof. We estimate

$$\begin{aligned} \| R_{1,2} \|_2 &\le t^{-2} \left((a_0 + (1/2)a^2)c + ac_1(1 + \ell n\ t) \right), \\ \| \nabla R_{1,2} \|_2 &\le t^{-2} \left((2a_0 + (3/2)a^2)c + 2ac_1(1 + \ell n\ t) \right), \\ \| \partial_t R_{1,2} \|_2 &\le \text{idem} + t^{-3} \left((a_0 + ac_1(1 + \ell n\ t) \right). \end{aligned}$$

□

We now turn to $R_{1,1}$. We shall need the commutation relations

$$\nabla MD = MD \left(ix + t^{-1} \nabla \right) \equiv MD \widetilde{\nabla}, \tag{3.23}$$

$$i\partial_t MD = MD \left(i\partial_t + (1/2)x^2 - it^{-1}(x \cdot \nabla + 3/2) \right) \equiv MDi\widetilde{\partial}_t, \tag{3.24}$$

$$JMD = iMD\nabla, \tag{3.25}$$

$$(i\partial_t + (1/2)\Delta)\, MD = MD \left(i\partial_t + (2t^2)^{-1} \Delta \right). \tag{3.26}$$

In particular (3.23) (3.24) are taken as the definitions of $\widetilde{\nabla}$ and $\widetilde{\partial}_t$. From the choice (1.8) of u_a and from (3.26), it follows that

$$R_{1,1} = MD \left(i\partial_t + (2t^2)^{-1} \Delta + t^{-1} x \cdot \widetilde{A}_1 - t^{-1} g(|w_+|^2) \right) \exp(-i\varphi) w_+. \tag{3.27}$$

The choice (1.15) of φ has been taylored to cancel the two long range terms in (3.27), so that

$$R_{1,1} = (2t^2)^{-1} MD\Delta \exp(-i\varphi) w_+. \tag{3.28}$$

We now have to prove that the previous choice of (u_a, A_a) satisfies the remaining assumptions of Proposition 2.2 and of Lemmas 3.1 and 3.2. More precisely we have to prove that (u_a, A_a) satisfies the estimates (2.17) (2.19) (2.20) (3.19) (3.21) and the analogue of (3.22) for $R_{1,1}$. (Note that (3.9) (3.10) (3.18) are special cases of (2.17) and that (3.12) is a special case of (3.20) which is identical with (2.20)).

The contribution of A_0 to A_a and to $R_{1,2}$ will be taken care of by the following general estimates of solutions of the wave equation.

Lemma 3.3. *Let A_0 be defined by* (1.11) *and let $k \geq 0$ be an integer. Let A_+ and $\dot{A}_+$ satisfy the conditions*

$$\nabla^2 A_+, \nabla \dot{A}_+ \in W_1^k, \quad A_+ \in L^3, \dot{A}_+ \in L^{3/2}. \tag{3.29}$$

Then A_0 satisfies the estimates

$$\begin{cases} \| A_0(t); W_\infty^k \| \leq a_0 t^{-1} \\ \| \partial_t A_0(t); W_\infty^{k-1} \| \leq a_0 t^{-1} & \textit{for } k \geq 1. \end{cases} \tag{3.30}$$

A proof can be found in [20]. As mentioned in Remark 1.2, the assumptions $A_+ \in L^3$ and $\dot{A}_+ \in L^{3/2}$ serve to exclude constants in A_+ and $\dot{A}_+$ and linear terms in x in A_+, but are otherwise controlled by the W_1^k assumption through Sobolev inequalities.

We next derive some preliminary estimates of $\widetilde{A}_1$ and $\widetilde{\widetilde{A}}_1$.

Lemma 3.4. *Let $k \geq 0$ be an integer. Then the following estimates hold.*

$$\| \omega^{k+1} \widetilde{A}_1 \|_2 \vee \| \omega^k \widetilde{\widetilde{A}}_1 \|_2 \leq (k + 1/2)^{-1} \| \omega^k x |w_+|^2 \|_2, \tag{3.31}$$

$$\| \omega^{k+1} (x \cdot \widetilde{A}_1) \|_2 \leq (k - 1/2)^{-1} \Big(\| \omega^k x^2 |w_+|^2 \|_2 + 2 \| \omega^k x |w_+|^2 \|_2 \Big) \quad \textit{for } k \geq 1, \tag{3.32}$$

$$\| \nabla^k \widetilde{A}_1 \|_\infty \leq C \| \omega^k x |w_+|^2 ; H^1 \|, \tag{3.33}$$

$$\| \nabla^{k-1} \widetilde{\widetilde{A}}_1 \|_\infty \leq C \| \omega^k x |w_+|^2 ; H^1 \| \quad \textit{for } k \geq 1, \tag{3.34}$$

$$\| \nabla^k (x \cdot \widetilde{A}_1) \|_\infty \leq C \Big(\| \omega^k x^2 |w_+|^2 ; H^1 \| + \| \omega^k x |w_+|^2 ; H^1 \| \Big) \quad \textit{for } k \geq 1. \tag{3.35}$$

Proof. (3.31) follows immediately from (1.14) and (3.3). From (1.14) and from the commutation relation

$$[x; P] = -2\omega^{-2} \nabla$$

it follows that

(3.36)

$$x \cdot \widetilde{A}_1 = \int_1^\infty d\nu \, \nu^{-2} \omega^{-1} \sin(\omega(\nu - 1)) D_0(\nu) \left\{ P \cdot (x \otimes x) |w_+|^2 - 2\omega^{-2} \nabla \cdot x |w_+|^2 \right\}$$

so that

$$\begin{aligned}\| \omega^{k+1} x \cdot \widetilde{A}_1 \|_2 \le & \int_1^\infty d\nu\ \nu^{-2} \Big(\| \omega^k D_0(\nu) x^2 |w_+|^2 \|_2 \\ & + 2(\nu - 1) \| \omega^{k+1} D_0(\nu) \omega^{-1} x |w_+|^2 \|_2 \Big) \\ \le & \int_1^\infty d\nu\ \nu^{-1/2-k} \left(\| \omega^k x^2 |w_+|^2 \|_2 + 2 \| \omega^k x |w_+|^2 \|_2 \right)\end{aligned}$$

which implies (3.32). Finally (3.33)–(3.35) follow from (3.31) (3.32) and from the fact that $\dot{H}^1 \cap \dot{H}^2 \subset L^\infty$. □

As an immediate corollary, we obtain the following estimates of A_1 and $\partial_t A_1$.

Corollary 3.1. *The following estimates hold.*

$$\| A_1(t) \|_\infty \le C \| w_+; H^{2,1} \|^2\ t^{-1}, \tag{3.37}$$

$$\| \partial_t A_1(t) \|_\infty \vee \| \nabla A_1(t) \|_\infty \le C \| w_+; H^{2,1} \|^2\ t^{-2}. \tag{3.38}$$

Proof. The result follows from (1.13) (3.2) (3.4) and from (3.33) (3.34). □

We next derive the remaining estimates of u_a and of $R_{1,1}$. The following proposition is slightly stronger than needed.

Proposition 3.1. *Let u_a be defined by* (1.8) *with $w_+ = Fu_+$ and with φ defined by* (1.16) (1.2) (1.14) *and let $R_{1,1}$ be given by* (3.28). *Let $u_+ \in H^{3,1} \cap H^{1,3}$. Then the following estimates hold for some constants c, c_1 and $r_{1,1}$, for $0 \le j + k \le 1$ and for all $t \ge 1$:*

$$\| \partial_t^j \nabla^k u_a(t) \|_r \le c\ t^{-\delta(r)} \qquad \text{for } 2 \le r \le \infty. \tag{3.39 $\equiv$ 2.17}$$

In particular

$$\| u_a(t) \|_3 \le \| w_+ \|_3\ t^{-1/2}, \tag{3.40}$$

$$\| \nabla u_a(t) \|_4 \le \left(\| x w_+ \|_4 + O(t^{-1} \ell n\ t) \right) t^{-3/4}. \tag{3.41}$$

$$\| \partial_t^j \nabla^{k+1} u_a(t) \|_r \le c t^{-\delta(r)} \qquad \text{for } 2 \le r \le 6, \tag{3.42}$$

$$\| \partial_t^j \nabla^k J u_a(t) \|_r \le c_1 (1 + \ell n\ t) t^{-\delta(r)} \qquad \text{for } 2 \le r \le 6, \tag{3.43}$$

$$\| \partial_t^j \nabla^k R_{1,1}(t) \|_2 \le r_{1,1} t^{-2} (1 + \ell n\ t)^2. \tag{3.44}$$

Proof. From the commutation relations (3.23) (3.24), it follows that for any differential operator Z

$$\| \partial_t^j \nabla^k MDZ \exp(-i\varphi)w_+ \|_r = t^{-\delta(r)} \| \widetilde{\partial}_t^j \widetilde{\nabla}^k Z \exp(-i\varphi)w_+ \|_r \ . \tag{3.45}$$

From (1.8) (3.28) and from the commutation relation (3.25), it follows that in order to derive (3.39)–(3.44) we have to estimate norms of the type

$$\| \widetilde{\partial}_t^j \widetilde{\nabla}^k Z \exp(-i\varphi)w_+ \|_r$$

for $0 \leq j + k \leq 1$, for suitable choices of Z and r, and with suitable r-independent time behaviour. The relevant choices are

$$\begin{array}{lll} Z = 1, & 2 \leq r \leq \infty & \text{for } (3.39) \\ Z = \widetilde{\nabla} \text{or} \nabla, & 2 \leq r \leq 6 & \text{for } (3.42)\ (3.43), \\ Z = t^{-2}\Delta, & r = 2 & \text{for } (3.44). \end{array}$$

Expanding $\widetilde{\partial}_t^j \ \widetilde{\nabla}^k$ according to the definitions (3.23) (3.24) and omitting the commutators of derivatives with powers of x and t which generate terms of lower order, we are led to estimate norms of the type $\| Z \exp(-i\varphi)w_+ \|_r$ for the following choices of Z, r:

$$\begin{array}{ll} Z = 1, x, t^{-1}\nabla, x^2, \partial_t, t^{-1}x \cdot \nabla & \text{with } 2 \leq r \leq \infty \text{ for } (3.39), \\ Z = x, x^2, t^{-1}x\nabla, x^3, x\partial_t, t^{-1}x^2\nabla, t^{-1}\nabla, t^{-2}\nabla^2, t^{-1}\partial_t\nabla, t^{-2}x\nabla^2 & \\ & \text{with } 2 \leq r \leq 6 \text{ for } (3.42), \\ Z = \nabla, x\nabla, t^{-1}\nabla^2, x^2\nabla, \partial_t\nabla, t^{-1}x\nabla^2 & \text{with } 2 \leq r \leq 6 \text{ for } (3.43), \\ Z = \Delta, x\Delta, t^{-1}\nabla\Delta, x^2\Delta, \partial_t\Delta, t^{-1}x\nabla\Delta & \text{with } r = 2 \text{ for } (3.44), \end{array}$$

where we have omitted an overall t^{-2} factor in the last case.

We expand the derivatives acting on $\exp(-i\varphi)w_+$ by the Leibnitz rule and we estimate the expressions thereby obtained by the Hölder inequality. For that purpose we need some control of φ. From Lemma 2.3 it follows easily that for $w_+ \in H^3$, $\nabla g(|w_+|^2) \in H^4$ and in particular $\nabla^k g(|w_+|^2) \in L^\infty$ for $0 \leq k \leq 3$. Together with Lemma 3.4, this provides an estimate of $\| \partial_t^j \nabla^k \varphi \|_r$ for $j = 0, 1$, for $k = 1, 2$ and $r = \infty$ and for $k = 3$ and $r = 6$. With that information available, we apply the Hölder inequality according to the following rules:

(i) all the explicit powers of x are attached to w_+. In addition, whenever there appears a factor $\partial_t\varphi$ (with no space derivative), one power x is extracted from the $\widetilde{A}_1$ part of $\partial_t\varphi$ and attached to w_+ (since $\widetilde{A}_1$ belongs to L^∞ but a priori $\partial_t\varphi$ does not).

(ii) The x amputated contribution of $\partial_t \varphi$ generated by rule (i) and all the factors $\partial_t^j \nabla^k \varphi$ with $k = 1, 2$ are estimated in L^∞. The factors $\nabla^3 \varphi$ are estimated in L^6 (in fact in H^1). Such factors occur only from the $t^{-1} x^s \nabla \Delta$ terms in the proof of (3.44).

(iii) The previous rules generate norms of the type $\| x^s \nabla^k w_+ \|_r$ for w_+. Those norms are estimated by H^1 norms of the same quantities for $2 < r \leq 6$ and by H^2 norms for $6 < r \leq \infty$.

(iv) The time dependence of the various terms follows from the explicit t dependence of the operators Z of the previous list, together with the fact that $\| \partial_t^j \nabla^k \varphi \|_r$ generates a factor t^{-1} for $j = 1$ and a factor $\ell n\ t$ for $j = 0$.

With the previous rules available, the proof reduces to an elementary book keeping exercise, which will be omitted. We simply remark that the dominant terms as regards w_+ have $x^3 \nabla$, $x^2 \nabla^2$ and $x \nabla^3$ which are exactly controlled by the assumption $w_+ \in H^{1,3} \cap H^{3,1}$, equivalent to the assumption $u_+ \in H^{3,1} \cap H^{1,3}$. As regards the time dependence, the dominant terms come from $x^s \nabla \varphi$ in the proof of (3.43) thereby generating a factor $\ell n\ t$, and from $x^s \Delta \exp(-i\varphi)$ in the proof of (3.44), generating $x^s |\nabla \varphi|^2$ and therefore a factor $(\ell n\ t)^2$.

Finally (3.40) is the special case $j = k = 0$, $Z = 1$, $r = 3$ of (3.45), while (3.41) follows from the estimate

$$\| \nabla u_a(t) \|_4 \leq \left(\| x w_+ \|_4 + t^{-1} \left(\| \nabla w_+ \|_4 + \| \nabla \varphi \|_\infty \| w_+ \|_4 \right) \right) t^{-3/4}. \tag{3.46}$$

□

We can now complete the proof of Proposition 1.1.

Proof of Proposition 1.1. It suffices to show that the assumptions of Proposition 2.2 are satisfied for the choice $h(t) = t^{-1}(2 + \ell n\, t)^2$ made in Proposition 1.1. Now the assumptions (2.17) (2.19) follow from (3.39) (3.42) of Proposition 3.1, the assumption (2.20) follows from Lemma 3.3 and Corollary 3.1. The assumption (2.21) follows from Lemma 3.2 as regards $R_{1,2}$ and from (3.44) of Proposition 3.1 as regards $R_{1,1}$. The assumptions (3.11) of Lemma 3.1 and (3.19) of Lemma 3.2 are special cases of (3.43). The assumption (3.21) of Lemma 3.2 follows from Lemma 3.3 and from the fact that if A_0 is a solution of the free wave equation $\square A_0$ in the Coulomb gauge $\nabla \cdot A_0 = 0$, with initial data $(A_+, \dot{A}_+)$, then also $x \cdot A_0$ is a solution of the free wave equation, namely $\square(x \cdot A_0) = 0$, with initial data $(x \cdot A_+, x \cdot \dot{A}_+)$. Finally the assumptions (2.22) (2.23) follow from Lemma 3.1.

The smallness conditions needed for Proposition 2.2 bear on c_3 and c_4. Now from (3.40), $c_3 = \| w_+ \|_3$ while from (3.41) or (3.46)

$$c_4 = \| xw_+ \|_4 + O\left(T_0^{-1} \ell n\ T_0\right).$$

Since the estimates are used only for $t \geq T$, one can replace T_0 by T in that expression, and the last term can be made arbitrarily small by taking T sufficiently large, so that the smallness condition of c_4 reduces to the smallness of $\| xw_+ \|_4$.

□

Remark 3.1. The regularity assumptions on u_+ or w_+ could be somewhat weakened. The strongest assumptions come from the Δw_+ term in $R_{1,1}$ and from the x^3, $x\nabla^2$ and $x^2\nabla$ operators Z in the estimate of $\partial_t \nabla u_a$. On the one hand the Δw_+ term in $R_{1,1}$ could be eliminated by the choice

$$w(t) = U(1/t)^* w_+$$

at the expense of generating either a more complicated and less explicit φ or additional terms in R_2. On the other hand, we have obtained on L^6 estimate of $\partial_t \nabla u_a$ whereas an L^4 estimate was sufficient. Only a minor weakening of the assumptions on u_+ could be achieved along those lines, and we shall not press that point any further.

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