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Decay of Solutions of Wave-type Pseudo-differential Equations over p–adic Fields

By

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Abstract

We show that the solutions of p–adic pseudo-differential equations of wave type have a decay similar to the solutions of classical generalized wave equations.

§1. Introduction

During the eighties several physical models using p–adic numbers were proposed. Particularly various models of p–adic quantum mechanics [11], [13], [21], [22]. As a consequence of this fact several new mathematical problems emerged, among them, the study of p–adic pseudo-differential equations [8], [22]. In this paper we initiate the study of the decay of the solutions of wave-type pseudo-differential equations over p–adic fields; these equations were introduced by Kochubei [9] in connection with the problem of characterizing the p–adic wave functions using pseudo-differential operators. We show that the solutions of p–adic wave-type equations have a decay similar to the solutions of classical generalized wave equations.

Let K be a p–adic field, i.e. a finite extension of $\mathbb{Q}_p$. Let R_K be the valuation ring of K, P_K the maximal ideal of R_K, and $\overline{K} = R_K/P_K$ the residue field of K. Let π denote a fixed local parameter of R_K. The cardinality

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of $\overline{K}$ is denoted by q. For $z \in K$, $v(z) \in \mathbb{Z} \cup \{+\infty\}$ denotes the valuation of z, and $|z|_K = q^{-v(z)}$. Let $\mathbb{S}(K^n)$ denote the $\mathbb{C}$-vector space of Schwartz-Bruhat functions over K^n, the dual space $\mathbb{S}'(K^n)$ is the space of distributions over K^n. Let $\mathcal{F}$ denote the Fourier transform over $\mathbb{S}(K^{n+1})$. The reader can consult any of the references [6], [22], [23] for an exposition of the theory of distributions over p–adic fields.

This article aims to study the following initial value problem:

$$(1) \qquad \begin{cases} (Hu)(x,t) = 0, \; x \in K^n, \;\; t \in K \\ \\ u(x,0) = f_0(x), \end{cases}$$

where $n \geq 1$, $f_0(x) \in \mathbb{S}(K^n)$, and

$$\begin{array}{rcl} H : \mathbb{S}(K^{n+1}) & \longrightarrow & \mathbb{S}(K^{n+1}) \\ \Phi & \longrightarrow & \mathcal{F}^{-1}_{(\tau,\xi)\longrightarrow(x,t)} \left(|\tau - \phi(\xi)|_K \, \mathcal{F}_{(x,t)\longrightarrow(\tau,\xi)} \Phi \right), \end{array}$$

is a pseudo-differential operator with symbol $|\tau - \phi(\xi)|_K$, where $\phi(\xi)$ is a polynomial in $K[\xi_1, \ldots, \xi_n]$ satisfying $\phi(0) = 0$. In the case in which $\phi(\xi) = a_1\xi_1^2 + \cdots + a_n\xi_n^2$, H is called a *Schrödinger-type pseudo-differential operator*; this operator was introduced by Kochubei in [9]. For $n = 1$ the solution of (1) appears in the formalism of p–adic quantum mechanics as the wave function for the free particle [21]. The problem of characterizing the p–adic wave functions as solutions of some pseudo-differential equation remains open.

Let $\Psi(\cdot)$ denote an additive character of K trivial on R_K but no on P_K^{-1}. By passing to the Fourier transform in (1) one gets that

$$|\tau - \phi(\xi)|_K \, \mathcal{F}_{(x,t)\longrightarrow(\tau,\xi)} u = 0.$$

Then any distribution of the form $\mathcal{F}^{-1}g$ with g a distribution supported on $\tau - \phi(\xi) = 0$ is a solution. By taking

$$g(\xi, \tau) = (\mathcal{F}_{x\to\xi} f_0)\, \delta(\tau - \phi(\xi)),$$

where δ is the Dirac distribution, one gets

$$(2) \qquad u(x,t) = \int_{K^n} \Psi\left(t\phi(\xi) + \sum_{i=1}^{n} x_i \xi_i \right) (\mathcal{F}_{x\to\xi} f_0)(\xi)\, |d\xi|,$$

here $|d\xi|$ is the Haar measure of K^n normalized so that $vol(R_K^n) = 1$.

In this paper we show that the decay of $u(x,t)$ is completely similar to the decay of the solution of the following initial value problem:

$$(3)\qquad \begin{cases} \frac{\partial u^{\text{arch}}(x,t)}{\partial t} = i\phi\left(D\right)u^{\text{arch}}\left(x,t\right), x \in \mathbb{R}^n, \;\; t\in\mathbb{R} \\ \\ u^{\text{arch}}\left(x,0\right) = f_0\left(x\right), \end{cases}$$

here $\phi\left(D\right)$ is a pseudo-differential operator having symbol $\phi\left(\xi\right)$. In this case

$$(4)\qquad u^{\text{arch}}\left(x,t\right) = \int_{\mathbb{R}^n} \exp 2\pi i\left(t\phi\left(\xi\right) + \sum_{i=1}^{n} x_i\xi_i\right)\left(\mathcal{F}_{x\to\xi}f_0\right)\left(\xi\right)d\xi$$

is the solution of the initial value problem (3). If $\phi\left(\xi\right) = \xi_1^2 + \cdots + \xi_n^2$, i.e. $\phi\left(D\right)$ is the Laplacian, $u^{\text{arch}}\left(x,t\right)$ satisfies

$$(5)\qquad \left\|u^{\text{arch}}\left(x,t\right)\right\|_{L^{\frac{2(n+2)}{n}}} \leq c\left\|f_0\right\|_{L^2},$$

(see [19]). If $n=1$ and $\phi\left(\xi\right) = \xi^3$, $u^{\text{arch}}\left(x,t\right)$ satisfies

$$(6)\qquad \left\|u^{\text{arch}}\left(x,t\right)\right\|_{L^8} \leq c\left\|f_0\right\|_{L^2},$$

(see [10]). We show that $u\left(x,t\right)$ satisfies (5), if $\phi\left(\xi\right) = \xi_1^2 + \cdots + \xi_n^2$ (see Corollary 2), and that $u\left(x,t\right)$ satisfies (6), if $\phi\left(\xi\right) = \xi^3$ (see Corollary 3). These two results are particular cases of our main result which describes the decay of $u\left(x,t\right)$ in $L^{\sigma}\left(K^{n+1}\right)$ when $\phi\left(\xi\right)$ is a non-degenerate polynomial with respect to its Newton polyhedron (see Theorem 6). The proof is achieved by adapting standard techniques in PDEs and by using number-theoretic techniques for estimating exponential sums modulo π^m. Indeed, like in the classical case the estimation of the decay rate can be reduced to the problem of estimating of the restriction of Fourier transforms to non-degenerate hypersurfaces [17]; we solve this problem (see Theorems 4, 5) by reducing it to the estimation of exponential sums modulo π^m (see Theorems 2, 3). These exponential sums are related to the Igusa zeta function for non-degenerate polynomials [3], [7], [25], [26]. More precisely, by using Igusa's method, the estimation of these exponential sums can be reduced to the description of the poles of twisted local zeta functions [3], [25], [26]. It is important to mention that all the results of this paper are valid in positive characteristic, i.e. if $K = \mathbb{F}_q((T))$, $q = p^n$, and $p > c$. Here c is a constant that depends on the Newton polyhedron of the polynomial ϕ.

The restriction of Fourier transforms in $\mathbb{R}^n$ (see e.g. [17, Chapter VIII]) was first posed and partially solved by Stein [5]. This problem have been intensively studied during the last thirty years [1], [17], [19], [24]. Recently Mockenhaupt and Tao have studied the restriction problem in $\mathbb{F}_q^n$ [12]. In this paper we initiate the study of the restriction problem in the non-archimedean field setting.

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§2. The Non-archimedean Principle of the Stationary Phase

Given $f(x) \in K[x]$, $x = (x_1, \ldots, x_m)$, we denote by

$$C_f(K) = \left\{ z \in K^m \mid \frac{\partial f}{\partial x_1}(z) = \cdots = \frac{\partial f}{\partial x_m}(z) = 0 \right\}$$

the critical set of the mapping $f : K^m \to K$. If $f(x) \in R_K[x]$, we denote by $\overline{f}(x)$ its reduction modulo π, i.e. the polynomial obtained by reducing the coefficients of $f(x)$ modulo π.

Give a compact open set $A \subset K^m$, we set

$$E_A(z,f) = \int_A \Psi(zf(x)) \, |dx| ,$$

for $z \in K$, where $|dx|$ is the normalized Haar measure of K^m. If $A = R_K^m$ we use the simplified notation $E(z,f)$ instead of $E_A(z,f)$. If $f(x) \in R_K[x]$, then

$$E(z,f) = q^{-nm} \sum_{x \bmod \pi^n} \Psi(zf(x)) ;$$

thus $E(z,f)$ is a generalized Gaussian sum.

Lemma 1. *Let $f(x) \in R_K[x]$, $x = (x_1, \ldots, x_m)$, be a non-constant polynomial. Let A be the preimage of $\overline{A} \subseteq \mathbb{F}_q^m$ under the canonical homomorphism $R_K^m \to (R_K/P_K)^m$. If $C_f(K) \cap A = \emptyset$, then there exists a constant $I(f,A)$ such that*

$$E(z,f) = 0, \quad \textit{for} \quad |z|_K > q^{2I(f,A)+1}.$$

Proof. We define

$$I(f,a) = \min_{1 \le i \le m} \left\{ v\left(\frac{\partial f}{\partial x_i}(a) \right) \right\},$$

for any $a \in A$, and

$$I(f,A) = \sup_{a \in A} \{ I(f,a) \} .$$

Since A is compact and $C_f(K) \cap A = \emptyset$, $I(f,A) < \infty$.

We denote by a^* an equivalence class of R_K^m modulo $\left(P_K^{I(f,A)+1}\right)^m$, and by $a \in R_K^m$ a fixed representative of a^*. By decomposing A into equivalence classes modulo $\left(P_K^{I(f,A)+1}\right)^m$, one gets

$$E(z,f) = \sum_{a^* \subseteq A} q^{-m(I(f,A)+1)} \int_{R_K^m} \Psi\left(zf\left(a+\pi^{I(f,A)+1}x\right)\right)|dx|.$$

Thus, it is sufficient to show that $\int_{R_K^m} \Psi\left(zf\left(a+\pi^{I(f,A)+1}x\right)\right)|dx| = 0$ for $|z|_K > q^{2I(f,A)+1}$.

On the other hand, if $a = (a_1, \ldots, a_m)$, then

$$\frac{f\left(a+\pi^{I(f,A)+1}x\right) - f(a)}{\pi^{I(f,A)+1+\alpha_0}}$$

equals

$$\sum_{i=1}^{m} \pi^{-\alpha_0} \frac{\partial f}{\partial x_i}(a)(x-a_i) + \pi^{I(f,A)+1-\alpha_0}\,(\text{higher order terms}),$$

where

$$\alpha_0 = \min_i \left\{ v\left(\frac{\partial f}{\partial x_i}(a)\right)\right\}.$$

Therefore

$$f\left(a+\pi^{I(f,A)+1}x\right) - f(a) = \pi^{I(f,A)+1+\alpha_0}\widetilde{f}(x) \tag{7}$$

with $\widetilde{f}(x) \in R_K[x]$, and since $C_f(K) \cap A = \emptyset$, there exists an $i_0 \in \{1, \ldots, m\}$ such that

$$\overline{\frac{\partial \widetilde{f}}{\partial x_{i_0}}}(\overline{a}) \neq 0. \tag{8}$$

We put $y = \Phi(x) = (\Phi_1(x), \ldots, \Phi_m(x))$ where

$$\Phi_i(x) = \begin{cases} \widetilde{f}(x) & i = i_0 \\ x_i & i \neq i_0. \end{cases}$$

Since $\Phi_1(x), \ldots, \Phi_m(x)$ are restricted power series and

$$\overline{J\left(\frac{(y_1,\ldots,y_m)}{(x_1,\ldots,x_m)}\right)} = \overline{\frac{\partial \widetilde{f}}{\partial x_{i_0}}}(\overline{a}) \neq 0,$$

the non-archimedean implicit function theorem implies that $y = \Phi(x)$ gives a measure-preserving map from R_K^m to R_K^m (see [7, Lemma 7.43]). Therefore

$$\int_{R_K^m} \Psi\left(zf\left(a+\pi^{I(f,A)+1}x\right)\right)|dx|$$
$$= \Psi(zf(a)) \int_{R_K} \Psi\left(z\pi^{I(f,A)+1+\alpha_0} y_{i_0}\right)|dy_{i_0}| = 0,$$

for $v(z) < -(I(f,A)+1+\alpha_0)$, i.e. for $|z|_K > q^{I(f,A)+1+\alpha_0}$, and a fortiori

$$\int_{R_K^m} \Psi\left(zf\left(a+\pi^{I(f,A)+1}x\right)\right)|dx| = 0,$$

for $|z|_K > q^{2I(f,A)+1}$ and any a. □

Theorem 1. *Let $f(x) \in K[x]$, $x = (x_1, \dots, x_m)$, be a non-constant polynomial. Let $B \subset K^m$ be a compact open set. If $C_f(K) \cap B = \emptyset$, then there exist a constant $c(f,B)$ such that*

$$E_B(z,f) = 0, for\,|z|_K \geq c(f,B).$$

Proof. By taking a covering $\cup_i (y_i + (\pi^\alpha R_K)^m)$ of B, $E_B(z,f)$ can be expressed as linear combination of integrals of the form $E(z,f_i)$ with $f_i(x) \in K[x]$. After changing z by $z\pi^\beta$, we may suppose that $f_i(x) \in R_K[x]$. By applying Lemma 1 we get that $E(z,f_i) = 0$, for $|z|_K > c_i$. Therefore

$$E_B(z,f) = 0, \quad \text{for } |z|_K > \max_i c_i. \tag{9}$$

□

We note that the previous result implies that

$$E_B(z,f) = O(|z|_K^{-M}),$$

for any $M \geq 0$. This is the standard form of the principle of the stationary phase.

§3. Local Zeta Functions and Exponential Sums

In this section we review some results about exponential sums and Newton polyhedra that will be used in the next section. For $x \in K$ we denote by $ac(x) = x\pi^{-v(x)}$ its angular component. Let $f(x) \in R_K[x]$, $x = (x_1, \dots, x_m)$

be a non-constant polynomial, and $\chi : R_K^\times \to \mathbb{C}^\times$ a character of $R_K^\times$, the group of units of R_K. We formally put $\chi(0) = 0$. To these data one associates the Igusa local zeta function,

$$Z(s, f, \chi) = \int_{R_K^n} \chi(acf(x))|f(x)|_K^s \mid dx \mid, \ \ s \in \mathbb{C},$$

for $Re(s) > 0$, where $\mid dx \mid$ denotes the normalized Haar measure of K^n. The Igusa local zeta function admits a meromorphic continuation to the complex plane as a rational function of q^{-s}. Furthermore, it is related to the number of solutions of polynomial congruences modulo π^m and exponential sums modulo π^m [2], [7].

§3.1. Exponential sums associated with non-degenerate polynomials

We set $\mathbb{R}_+ = \{x \in \mathbb{R} \mid x \geqq 0\}$. Let $f(x) = \sum_l a_l x^l \in K[x]$, $x = (x_1, \dots, x_m)$ be a non-constant polynomial satisfying $f(0) = 0$. The set $\text{supp}(f) = \{l \in \mathbb{N}^m \mid a_l \neq 0\}$ is called the support of f. The Newton polyhedron $\Gamma(f)$ of f is defined as the convex hull in $\mathbb{R}_+^m$ of the set

$$\bigcup_{l \in \text{supp}(f)} \left(l + \mathbb{R}_+^m\right).$$

We denote by $\langle \cdot, \cdot \rangle$ the usual inner product of $\mathbb{R}^m$, and identify $\mathbb{R}^m$ with its dual by means of it. We set

$$\langle a_\gamma, x \rangle = m(a_\gamma),$$

for the equation of the supporting hyperplane of a facet γ (i.e. a face of codimension 1 of $\Gamma(f)$) with perpendicular vector $a_\gamma = (a_1, \dots, a_n) \in \mathbb{N}^n \smallsetminus \{0\}$, and $\sigma(a_\gamma) := \sum_i a_i$.

Definition 1. A polynomial $f(x) \in K[x]$ is called non-degenerate with respect to its Newton polyhedron $\Gamma(f)$, if it satisfies the following two properties: (i) $C_f(K) = \{0\} \subset K^n$; (ii) for every proper face $\gamma \subset \Gamma(f)$, the critical set $C_{f_\gamma}(K)$ of $f_\gamma(x) := \sum_{i \in \gamma} a_i x^i$ satisfies $C_{f_\gamma}(K) \cap (K \smallsetminus \{0\})^m = \emptyset$.

We note that the above definition is not standard because it requires that the origin be an isolated critical point (see e.g. [3], [4], [26]). The condition (ii) can be replaced by

$$\{x \in K^m \mid f_\gamma(x) = 0\} \cap C_{f_\gamma}(K) \cap (K \smallsetminus \{0\})^m = \emptyset. \tag{10}$$

If K has characteristic $p > 0$, by using Euler's identity, it can be verified that condition (ii) in the above definition is equivalent to (10), if p does not divide the $m(a_\gamma) \neq 0$, for any facet γ.

In [26] the author showed that if f is non-degenerate with respect $\Gamma(f)$, then the poles of $\left(1 - q^{-1-s}\right) Z(s, f, \chi_{\text{triv}})$ and $Z(s, f, \chi)$, $\chi \neq \chi_{\text{triv}}$, have the form

$$s = -\frac{\sigma\left(a_\gamma\right)}{m(a_\gamma)} + \frac{2\pi i}{\log\, q} \frac{k}{m(a_\gamma)},\ k \in \mathbb{Z},$$

for some facet γ of $\Gamma(f)$ with perpendicular a_γ, and $m(a_\gamma) \neq 0$ (see [26, Theorem A, and Lemma 4.4]). Furthermore, if $\chi \neq \chi_{\text{triv}}$ and the order of χ does not divide any $m(a_\gamma) \neq 0$, where γ is a facet of $\Gamma(f)$, then $Z(s, f, \chi)$ is a polynomial in q^{-s}, and its degree is bounded by a constant independent of χ (see [26, Theorem B]). These two results imply that for $\mid z \mid_K$ big enough $E(z, f)$ is a finite $\mathbb{C}$-linear combination of functions of the form

$$\chi(ac(z)) \mid z \mid_K^\lambda \left(\log_q(|z|_K)\right)^\gamma,$$

with coefficients independent of z, and with $\lambda \in \mathbb{C}$ a pole of

$$(1 - q^{-1-s}) Z(s, f, \chi_{\text{triv}}) \text{ or of } Z(s, f, \chi), \chi \neq \chi_{\text{triv}},$$

and $\gamma \in \mathbb{N}$, $\gamma \leqq$ (multiplicity of pole λ) -1 (see [2, Corollary 1.4.5]). Moreover all poles λ appear effectively in this linear combination. Therefore

$$|E(z, f)| \leqq c \mid z \mid_K^{-\beta_f + \epsilon}, \tag{11}$$

with $\epsilon > 0$, and

$$\beta_f := \min_\tau \left\{ \frac{\sigma\left(a_\tau\right)}{m(a_\tau)} \right\},$$

where τ runs through all facets of $\Gamma(f)$ satisfying $m(a_\tau) \neq 0$. The point

$$T_0 = (\beta_f^{-1}, \ldots, \beta_f^{-1}) \in \mathbb{Q}^m$$

is the intersection point of the boundary of the Newton polyhedron $\Gamma(f)$ with the diagonal $\Delta = \{(t, \ldots, t) \mid t \in \mathbb{R}\} \subset \mathbb{R}^m$. By combining estimation (11) and Theorem 1, we obtain the following result.

Theorem 2. *Let $f(x) \in K[x]$ be non-degenerate with respect to its Newton polyhedron $\Gamma(f)$. Let $B \subset K^m$ a compact open subset. Then*

$$|E_B(z, f)| \leqq c \mid z \mid_K^{-\beta_f + \epsilon},$$

for any $\epsilon > 0$.

We have to mention that the previous result is known by the experts, however the author did not find a suitable reference for the purposes of this article. If K has characteristic $p > 0$, the previous result is valid if p does not divide the $m(a_\tau) \neq 0$ [26, Corollary 6.1].

§3.2. Exponential Sums Associated with Quasi-homogeneous Polynomials

Definition 2. Let $f(x) \in K[x]$, $x = (x_1, \dots, x_m)$ be a non-constant polynomial satisfying $f(0) = 0$. The polynomial $f(x)$ is called quasi-homogeneous of degree d with respect $\alpha = (\alpha_1, \dots, \alpha_m) \in (\mathbb{N} \setminus \{0\})^m$, if it satisfies

$$f(\lambda^{\alpha_1} x_1, \dots, \lambda^{\alpha_m} x_m) = \lambda^d f(x), \text{ for every } \lambda \in K.$$

In addition, if $C_f(K)$ is the origin of K^m, then $f(x)$ is called a non-degenerate quasi-homogeneous polynomial.

The non-degenerate quasi-homogeneous polynomials are a subset of the non-degenerate polynomials with respect to the Newton polyhedron. For these type of polynomials the bound (11) can be improved:

$$|E(z, f)| \leq c\, |z|_K^{-\beta_f}, \tag{12}$$

where $\beta_f = \frac{1}{d} \sum_{i=1}^m \alpha_i$. By using the techniques exposed in [25, Theorem 3.5], and [26, Lemma 2.4] follow that the poles of $\left(1 - q^{-1-s}\right) Z(s, f, \chi_{\text{triv}})$ and $Z(s, f, \chi)$, $\chi \neq \chi_{\text{triv}}$, have the form

$$s = -\frac{\sigma(\alpha)}{d} + \frac{2\pi i}{\log q} \frac{k}{d}, \; k \in \mathbb{Z}.$$

Then by using the same reasoning as before, we obtain (12). This estimate and Theorem 1 imply the following result.

Theorem 3. *Let $f(x) \in K[x]$, $x = (x_1, \dots, x_m)$ be a non-degenerate quasi-homogeneous polynomial of degree d with respect to $\alpha = (\alpha_1, \dots, \alpha_m)$. Let $B \subset K^m$ be a compact open set. Then*

$$|E_B(z, f)| \leq c\, |z|_K^{-\beta_f}.$$

If K has characteristic $p > 0$, the above result is valid, if p does not divide $\sigma(\alpha)$.

§4. Fourier Transform of Measures Supported on Hypersurfaces

Let Y be a closed smooth submanifold of K^n of dimension $n-1$. If

$$I = \{i_1, \dots, i_{n-1}\} \text{ with } i_1 < i_2 < \cdots < i_{n-1} \tag{13}$$

is a subset of $\{1, \dots, n\}$ we denote by ω_{Y_I} the differential form induced on Y by $dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_{n-1}}$, and by $d\sigma_{Y_I}$ the corresponding measure on Y. The canonical measure of Y is defined as

$$d\sigma_Y = \sup_I \; \{d\sigma_{Y_I}\}$$

where I runs through all the subsets of form (13). Given S a compact open subset of K^n with characteristic function Θ_S, we define $d\mu_{Y,S} = d\mu_Y = \Theta_S d\sigma_Y$. The canonical measure $d\mu_Y$ was introduced by Serre in [14]. The Fourier transform of $d\mu_Y$ is defined as

$$\widehat{d\mu_Y(\xi)} = \int_Y \Psi\left(-[x,\xi]\right) d\mu_Y(x),$$

where $[x,y] := \sum_{i=1}^n x_i y_i$, with $x, y \in K^n$. The analysis of the decay of $|\widehat{d\mu_Y(\xi)}|$ as $\|\xi\|_K := \max_i \{|\xi_i|_K\}$ approaches infinity plays a central role in this paper. This analysis can be simplified taking into account the following facts. Any compact open set of K^n is a finite union of classes modulo π^e, by taking e big enough, and taking into account that $Y \cap y + (\pi^e R_K)^n$ is a hypersurface of the form

$$\{x \in y + (\pi^e R_K)^n \mid x_n = \phi(x_1, \dots, x_{n-1})\}$$

with ϕ an analytic function satisfying

$$\phi(0) = \frac{\partial \phi}{\partial x_1}(0) = \cdots = \frac{\partial \phi}{\partial x_{n-1}}(0) = 0, \tag{14}$$

(see [14, p. 147]), we may assume that Y is a hypersurface of the form $x_n - \phi(x_1, \dots, x_{n-1}) = 0$, with ϕ satisfying (14). In this case $d\sigma_Y(x) = |dx_1| \dots |dx_{n-1}|$, the normalized Haar measure of K^{n-1}.

Finally we want to mention that if $X = \{x \in K^n \mid f(x) = 0\}$ is a hypersurface then

$$\frac{dx_1 \dots dx_{n-1}}{\left|\frac{\partial f}{\partial x_n}\right|_K}$$

is a measure on a neighborhood of X provided that $\left|\frac{\partial f}{\partial x_n}\right|_K \neq 0$ (see [7, Section 7.6]). This measure is not intrinsic to X, but if S is small enough, it coincides with $d\mu_X = \Theta_S d\sigma_X$ for a polynomial of type $f(x) = x_n - \phi(x_1, \ldots, x_{n-1})$. The Serre measure allow us to define $\widehat{d\mu_Y(\xi)}$ intrinsically for an arbitrary submanifold Y.

The rest of this section is dedicated to describe the asymptotics of $\widehat{d\mu_Y}$ when ϕ is a non-degenerate polynomial with respect $\Gamma(\phi)$.

Theorem 4. *Let $\phi(x) \in R_K[x]$, $x = (x_1, \ldots, x_{n-1})$, be a non-degenerate polynomial with respect to its Newton polyhedron $\Gamma(\phi)$. Let Θ_S be the characteristic function of a compact open set S, let*

$$Y = \{x \in K^n \mid x_n = \phi(x_1, \ldots, x_{n-1})\},$$

and let $d\mu_Y = \Theta_S d\sigma_Y$. Then

$$\left|\widehat{d\mu_Y(\xi)}\right| \leq c\,\|\xi\|_K^{-\beta}, \tag{15}$$

for $0 \leq \beta \leq \beta_\phi - \epsilon$, for $\epsilon > 0$. Furthermore, if ϕ is a non-degenerate quasi-homogeneous polynomial, (15) *is valid for $0 \leq \beta \leq \beta_\phi$.*

Proof. Since S is compact by passing to a sufficiently fine covering

$$\bigcup_i (x_i, \phi(x_i)) + (\pi^{e_0} R_K)^n,$$

with $e_0 > 0$, we may suppose that $S = (x_i, \phi(x_i)) + (\pi^{e_0} R_K)^n$. In the case $x_i = 0$,

$$\widehat{d\mu_Y(\xi)} = \int\limits_{(\pi^{e_0} R_K)^{n-1}} \Psi\left(-\xi_n \phi(x) - [x, \xi']\right) |dx|, \tag{16}$$

where $\xi' = (\xi_1, \ldots, \xi_{n-1})$. If $\xi' = 0$, Theorem 2 implies that

$$\left|\widehat{d\mu_Y(\xi)}\right| \leq c\,|\xi_n|_K^{-\beta} = c\,\|\xi\|_K^{-\beta}, \tag{17}$$

for $0 \leq \beta \leq \beta_\phi - \epsilon$, $\epsilon > 0$. Furthermore, if ϕ is a non-degenerate quasi-homogeneous polynomial then (17) is valid for $0 \leq \beta \leq \beta_\phi$ (cf. Theorem 3).

Since $\widehat{d\mu_Y(\xi)} = \widehat{d\mu_Y(\xi_n, \xi')}$ is a continuous function with respect to ξ', estimation (17) remains valid if

$$\frac{|\xi_i|_K}{|\xi_n|_K} \leq c, \; i = 1, \ldots, n-1, \tag{18}$$

for some small positive constant c. Then we may suppose that

$$\frac{|\xi_i|_K}{|\xi_n|_K} > c, \; i = 1, \ldots, n-1. \tag{19}$$

Since $(\pi^{e_0} R_K)^{n-1}$ is small enough, (19) implies that the system of equations

$$\frac{\partial \phi(x)}{\partial x_j} = \frac{\xi_i}{\xi_n}, \; j = 1, \ldots, n-1,$$

does not have solutions in $(\pi^{e_0} R_K)^{n-1}$, and then the critical set of the polynomial

$$F(x, \xi) = \xi_n \phi(x) + [x, \xi']$$

does not meet $(\pi^{e_0} R_K)^{n-1}$ if $\xi_n \neq 0$. By applying Theorem 1, it follows that

$$\widehat{d\mu_Y(\xi)} = 0, \text{ for } \|\xi\|_K \text{ big enough.} \tag{20}$$

Then for $\|\xi\|_K$ big enough, (17) and (20) imply that

$$\left|\widehat{d\mu_Y(\xi)}\right| \leq A \|\xi\|_K^{-\beta}, \quad \text{for } 0 \leq \beta \leq \beta_\phi. \tag{21}$$

In the case $x_i \neq 0$, by using the fact that the origin is the only critical point of ϕ, a similar reasoning shows that $\widehat{d\mu_Y(\xi)} = 0$, for $\|\xi\|_K$ big enough. Therefore estimation (21) is valid for any compact open set S. □

§4.1. Restriction of the Fourier Transform to Non-degenerate Hypersurfaces

Let X be a submanifold of K^n with $d\sigma_X$ its canonical measure. We set $d\mu_{Y,S} = \Theta_S d\sigma_Y$, where Θ_S is the characteristic function of a compact open set S in K^n. We say that the L^ρ *restriction property* is valid for X if there exists a $\tau(\rho)$ so that

$$\left(\int_X |\mathcal{F}g(\xi)|_K^\tau \, d\mu_{X,S}(\xi) \right)^{\frac{1}{\tau}} \leq c_{\tau,\rho}(S) \|g\|_{L^\rho}$$

holds for each $g \in \mathbb{S}(K^n)$ and any compact open set S of K^n.

The restriction problem in $\mathbb{R}^n$ (see e.g. [17, Chapter VIII]) was first posed and partially solved by Stein [5]. This problem have been intensively studied during the last thirty years [1], [17], [19], [24]. Recently Mockenhaupt and

Tao have studied the restriction problem in $\mathbb{F}_q^n$ [12]. In this paper we study the restriction problem in the non-archimedean field setting. More precisely, in the case in which X is a non-degenerate hypersurface and $\tau = 2$. The proof of the restriction property in the non-archimedean case uses the Lemma of interpolation of operators (see e.g. [17, Chapter IX]) and the estimates for oscillatory integrals obtained in the previous section. The interpolation Lemma given in [17, Chapter IX] is valid in the non-archimedean case. For the sake of completeness we rewrite this lemma here.

Let $\{U^z\}$ be a family of operators on the strip $a \leq \mathrm{Re}(z) \leq b$ defined by

$$(U^z g)(x) = \int_{K^n} \mathfrak{K}_z(x,y)\, g(y)\, |dy|,$$

where the kernels $\mathfrak{K}_z(x,y)$ have a fixed compact support and are uniformly bounded for $(x,y) \in K^n \times K^n$ and $a \leq \mathrm{Re}(z) \leq b$. We also assume that for each (x,y), the function $\mathfrak{K}_z(x,y)$ is analytic in $a < \mathrm{Re}(z) < b$ and is continuous in the closure $a \leq \mathrm{Re}(z) \leq b$, and that

$$\begin{cases} \|U^z g\|_{L^{\tau_0}} \leq M_0 \|g\|_{L^{\rho_0}}, \text{when } \mathrm{Re}(z) = a, \\ \\ \|U^z g\|_{L^{\tau_1}} \leq M_1 \|g\|_{L^{\rho_1}}, \text{when } \mathrm{Re}(z) = b; \end{cases}$$

here (τ_i, ρ_i) are two pairs of given exponents with $1 \leq \tau_i, \rho_i \leq \infty$.

Lemma 2 (Interpolation Lemma [17, Chapter IX]). *Under the above hypotheses,*

$$\left\| U^{a(1-\theta)+b\theta} g \right\|_{L^\tau} \leq M_0^{1-\theta} M_1^{\theta} \|g\|_{L^\rho}$$

where $0 \leq \theta \leq 1$, $\frac{1}{\tau} = \frac{(1-\theta)}{\tau_0} + \frac{\theta}{\tau_1}$, *and* $\frac{1}{\rho} = \frac{(1-\theta)}{\rho_0} + \frac{\theta}{\rho_1}$.

Theorem 5. *Let $\phi(x) \in K[x]$, $x = (x_1, \ldots, x_{n-1})$, be a non-degenerate polynomial with respect to its Newton polyhedron $\Gamma(\phi)$. Let*

$$Y = \{x \in K^n \mid x_n = \phi(x_1, \ldots, x_{n-1})\}$$

with the measure $d\mu_{Y,S} = \Theta_S d\sigma_Y$, where Θ_S is the characteristic function of a compact open subset S of K^n. Then

$$\left(\int_Y |\mathcal{F}g(\xi)|_K^2 \, d\mu_Y(\xi) \right)^{\frac{1}{2}} \leq c(Y) \|g\|_{L^\rho}, \tag{22}$$

holds for each $1 \leq \rho < \frac{2\left(1+\beta_\phi\right)}{2+\beta_\phi}$. *Furthermore, if* ϕ *is a non-degenerate quasi-homogeneous polynomial,* (22) *holds for each* $1 \leq \rho \leq \frac{2(1+\beta_\phi)}{2+\beta_\phi}$.

Proof. We first note that

$$\int_Y |\mathcal{F}g(\xi)|_K^2 \, d\mu_{Y,S}(\xi) = \int_Y \mathcal{F}g(\xi)\,\overline{\mathcal{F}g(\xi)} d\mu_{Y,S}(\xi) = \int_{K^n} (Tg)(x)\,\overline{g(x)}\,|dx| \tag{23}$$

where $(Tg)(x) = (g * \mathfrak{K})(x)$ with

$$\mathfrak{K}(x) = \int_Y \Psi([x,\xi])\, d\mu_{Y,S}(\xi) = \widehat{d\mu_{Y,S}}(-x).$$

The theorem follows from (23) by Hölder's inequality if we show that

$$\|T(g)\|_{L^{\rho_0'}} \leq c\,\|g\|_{L^{\rho_0}}$$

where ρ_0' is the dual exponent of ρ_0. Now we define $\mathfrak{K}_z(x)$ as equal to

$$\gamma(z) \int_{K^n} \Psi([x,\xi]) \left|\xi_n - \phi\left(\xi'\right)\right|_K^{-1+z} \eta\left(\xi_n - \phi\left(\xi'\right)\right) \Theta_S\left(\xi', \phi\left(\xi'\right)\right) |d\xi|,$$

where $\gamma(z) = \left(\frac{1-q^{-z}}{1-q^{-1}}\right)$, $\xi' = \left(\xi_1, \dots, \xi_{n-1}\right)$, $\eta(\xi)$ is the characteristic function of the ball $P_K^{e_0}$, $e_0 \geq 1$, and $\mathrm{Re}(z) > 0$. By making $y = \xi_n - \phi\left(\xi'\right)$ in the above integral we obtain

$$\mathfrak{K}_z(x) = \zeta_z(x_n)\,\mathfrak{K}(x)$$

with

$$\zeta_z(x_n) = \gamma(z) \int_K \Psi(x_n y)\,|y|_K^{-1+z}\,\eta(y)\,|dy|, \quad \mathrm{Re}(z) > 0.$$

On the other hand,

$$\zeta_z(x_n) = \begin{cases} q^{-e_0 z}, & \text{if} \quad |x_n|_K \leq q^{e_0}; \\ \left(\frac{1-q^{z-1}}{1-q^{-1}}\right) |x_n|_K^{-z}, & \text{if} \quad |x_n|_K > q^{e_0}, \end{cases}$$

(for a similar calculation the reader can see [20, page 54]), then $\zeta_z(x_n)$ has an analytic continuation to the complex plane as an entire function; also $\zeta_0(x_n) = 1$, and $|\zeta_z(x_n)| \leq c\,|x_n|_K^{-\mathrm{Re}(z)}$ where $|x_n|_K \geq q^{e_0}$. Therefore $\zeta_z(x_n)$ has an analytic continuation to an entire function satisfying the following properties:

(i) $\mathfrak{K}_0(x) = \mathfrak{K}(x)$,

(ii) $|\mathfrak{K}_{-\beta+i\gamma}(x)| \leq c$, for $x \in K^n$, $\gamma \in \mathbb{R}$, and $0 \leq \beta \leq \beta_\phi - \epsilon$, $\epsilon > 0$,

(iii) $|\mathcal{F}\mathfrak{K}_{1+i\gamma}(\xi)| \leq c$, for $\xi \in K^n$, and $\gamma \in \mathbb{R}$.

In fact (ii) follows from Theorem 4, and (iii) is an immediate consequence of the definition of $\mathfrak{K}_z(x)$.

Now we consider the analytic family of operators $T_z(g) = (g * \mathfrak{K}_z)(x)$. From (ii) one has

$$\|T_{-\beta+i\gamma}(g)\|_{L^\infty} \leq c\,\|g\|_{L^1},$$

for $0 \leq \beta \leq \beta_\phi - \epsilon$, $\epsilon > 0$, and $\gamma \in \mathbb{R}$, and from (iii) and Plancherel's Theorem one gets

$$\|T_{1+i\gamma}(g)\|_{L^2} \leq c\,\|g\|_{L^2},$$

for $\gamma \in \mathbb{R}$. By applying the Interpolation Lemma with

$$\theta = \frac{\beta}{1+\beta},$$

we obtain

$$\|T_0(g)\|_{L^{\rho'}} \leq c\,\|g\|_{L^\rho},$$

with ρ' the dual exponent of $\rho = \frac{2(1+\beta)}{2+\beta}$, and $0 \leq \beta \leq \beta_\phi - \epsilon$, $\epsilon > 0$. Therefore the previous estimate for $\|T_0(g)\|_{L^{\rho'}}$ is valid for $1 \leq \rho \leq \frac{2(1+\beta_\phi-\epsilon)}{2+\beta_\phi-\epsilon}$. In the quasi-homogeneous case the estimate is valid for $1 \leq \rho \leq \frac{2(1+\beta_\phi)}{2+\beta_\phi}$. □

Our proof of Theorem 5 is strongly influenced by Stein's proof for the restriction problem in the case of a smooth hypersurface in $\mathbb{R}^n$ with non-zero Gaussian curvature [16].

§5. Asymptotic Decay of Solutions of Wave-type Equations

Like in the classical case [19], the decay of the solutions of wave-type pseudo-differential equations can be deduced from the restriction theorem proved in the previous section, taking into account that the following two problems are completely equivalent if $\frac{1}{\rho} + \frac{1}{\sigma} = 1$:

Problem 1. *For which values of* ρ, $1 \leq \rho < 2$, *is it true that* $f \in L^{\rho}(K^n)$ *implies that* $\mathcal{F}f$ *has a well-defined restriction to* Y *in* $L^2(d\mu_{Y,S})$ *with*

$$\left(\int_Y |\mathcal{F}f|^2 \, d\mu_{Y,S}\right)^{\frac{1}{2}} \leq c_\rho \|f\|_{L^\rho}?$$

Problem 2. *For which values of* σ, $2 < \sigma \leq \infty$, *is it true that the distribution* $g d\mu_{Y,S}$ *for each* $g \in L^2(d\mu_{Y,S})$ *has Fourier transform in* $L^\sigma(K^n)$ *with*

$$\left\|\mathcal{F}\left(g d\mu_{Y,S}\right)\right\|_{L^\sigma} \leq c_\sigma \left(\int_Y |g|^2 \, d\mu_{Y,S}\right)^{\frac{1}{2}}?$$

§5.1. Wave-type Equations with Non-degenerate Symbols

Theorem 6 (Main Result). *Let* $\phi(\xi) \in K[\xi]$, $\xi = (\xi_1, \ldots, \xi_n)$ *be a non-degenerate polynomial with respect* $\Gamma(\phi)$. *Let*

$$\begin{aligned} H : \mathbb{S}(K^n) &\longrightarrow \mathbb{S}(K^n) \\ \Phi &\longrightarrow \mathcal{F}^{-1}_{(\tau,\xi)\longrightarrow(x,t)}\left(|\tau - \phi(\xi)|_K \, \mathcal{F}_{(x,t)\longrightarrow(\tau,\xi)}\Phi\right), \end{aligned}$$

be a pseudo-differential operator with symbol $|\tau - \phi(\xi)|_K$. *Let* $u(x,t)$ *be the solution of the following initial value problem*:

$$\begin{cases} (Hu)(x,t) = 0, \ x \in K^n, \ t \in K, \\ u(x,0) = f_0(x), \end{cases}$$

where $f_0(x) \in \mathbb{S}(K^n)$, *then*

$$\|u(x,t)\|_{L^\sigma(K^{n+1})} \leq A \|f_0(x)\|_{L^2(K^n)}, \tag{24}$$

for $\frac{2(1+\beta_\phi)}{\beta_\phi} < \sigma \leq \infty$. *Furthermore, if* ϕ *is a quasi-homogeneous polynomial,* (24) *is valid for* $\frac{2(1+\beta_\phi)}{\beta_\phi} \leq \sigma \leq \infty$.

Proof. Since

$$\begin{aligned} u(x,t) &= \int_{K^n} \Psi\left(t\phi(\xi) + [x,\xi]\right) \mathcal{F}f_0(\xi) \, |d\xi| \\ &= \int_Y \Psi\left(\left[\underline{x}, \underline{\xi}\right]\right) \mathcal{F}f_0\left(\underline{\xi}\right) d\mu_{Y,S}\underline{(\xi)}, \end{aligned}$$

where $\underline{\xi}= \left(\xi, \xi_{n+1}\right) \in K^{n+1}$, $\underline{x}= (x,t) \in K^{n+1}$,

$$Y = \left\{ \underline{\xi} \in K^{n+1} \mid \xi_{n+1} = \phi\left(\xi\right) \right\},$$

and $d\mu_{Y,S} = \Theta_S d\sigma_Y$, with Θ_S the characteristic function of a compact open set S containing the support of $\mathcal{F}f_0$. By applying Theorem 5, replacing n with $n+1$, and dualizing, one gets

$$\left\Vert u\left(x,t\right)\right\Vert_{L^{\sigma}(K^{n+1})} \leq A \left\Vert f_0\left(x\right)\right\Vert_{L^2(K^n)}, \tag{25}$$

where $\sigma = \frac{2(1+\beta)}{\beta}$ is the dual exponent of ρ in Theorem 5, and $0 \leq \beta < \beta_\phi$, therefore (25) is valid for $\frac{2\left(1+\beta_\phi\right)}{\beta_\phi} < \sigma \leq \infty$. □

The following corollary follows immediately from the previous theorem by using the fact that $\mathbb{S}(K^n)$ is dense in $L^{\sigma}(K^n)$ for $1 \leq \sigma < \infty$.

Corollary 1. *With the hypothesis of Theorem* 6, *if* $f_0 \in L^2(K^n)$, *then* $u(x,t) \in L^{\sigma}(K^{n+1})$, *for* $\frac{2\left(1+\beta_\phi\right)}{\beta_\phi} < \sigma < \infty$. *Furthermore, if* ϕ *is a quasi-homogeneous polynomial,* $u(x,t) \in L^{\sigma}(K^{n+1})$, *for* $\frac{2\left(1+\beta_\phi\right)}{\beta_\phi} \leq \sigma < \infty$.

§5.2. Wave-type Equations with Quasi-homogeneous Symbols

As a consequence of the previous theorem we obtain the following two corollaries.

Corollary 2. *With the hypothesis of Theorem* 6, *if* $\phi\left(\xi\right) = \xi_1^2 + \cdots + \xi_n^2$, *then*

$$\left\Vert u\left(x,t\right)\right\Vert_{L^{\frac{2(2+n)}{n}}(K^{n+1})} \leq A \left\Vert f_0\left(x\right)\right\Vert_{L^2(K^n)}.$$

Corollary 3. *With the hypothesis of Theorem* 6, *if* $\phi\left(\xi\right) = \xi^d$, *then*

$$\left\Vert u\left(x,t\right)\right\Vert_{L^{2(d+1)}(K^2)} \leq A \left\Vert f_0\left(x\right)\right\Vert_{L^2(K)}.$$

In particular if $d = 3$, *then*

$$\left\Vert u\left(x,t\right)\right\Vert_{L^{8}(K^2)} \leq A \left\Vert f_0\left(x\right)\right\Vert_{L^2(K)}.$$

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