

Integrality of Gopakumar–Vafa Invariants of Toric Calabi–Yau Threefolds

By

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Abstract

The Gopakumar–Vafa invariants are numbers defined as certain linear combinations of the Gromov–Witten invariants. We prove that the GV invariants of a toric Calabi–Yau threefold are integers and that the invariants for high genera vanish. The proof of the integrality is based on elementary number theory and that of the vanishing uses the operator formalism and the exponential formula.

§1. Introduction

A toric Calabi–Yau (TCY) threefold is a three-dimensional smooth toric variety of finite type, whose canonical bundle is trivial. For example, the total space of the rank two vector bundle over \mathbb{P}^1 , $\mathcal{O}(a_1) \oplus \mathcal{O}(a_2) \rightarrow \mathbb{P}^1$, such that $a_1 + a_2 = -2$ and the total space of the canonical bundle of a smooth toric surface are TCY threefolds.

Thanks to the duality of open and closed strings, a procedure to write down the partition function of the 0-pointed Gromov–Witten (GW) invariants of any TCY threefold X became available [AKMV]. By the partition function, we mean the exponential of the the generating function. One only has to draw a labeled planar graph from the fan of X and combine a certain quantity according to the shape and the labels of the graph. See [Z1][LLZ1][LLZ2][LLLZ] for the mathematical formulation and the proof. In this article, we call the graph the toric graph of X and refer to the quantity as the three point function.

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One open problem concerning the Calabi–Yau threefold is the Gopakumar–Vafa (GV) conjecture [GV]. We define the Gopakumar–Vafa invariants as certain linear combinations of the GW invariants in the manner of [BP]. One statement of the conjecture is that the GV invariants are integers and that only finite number of them are nonzero (in a given homology class). This is remarkable given that the GW invariants themselves are, in general, not integers but rational numbers. Other statement is that the GV invariants are equal to “the number of BPS states” in the M-theory compactified on the TCY threefold. A mathematical formulation in this direction was proposed in [HST]. Recently, the studies using the relation to the instanton counting appeared [LiLZ][AK].

The first statement of the GV conjecture was proved by Peng [P] in the case of the canonical bundles of Fano toric surfaces. The aim of this article is to prove it for general TCY threefolds. We put the problem in a combinatorial setting and prove the combinatorial version of the statement. The proof consists of two parts corresponding to the integrality and the vanishing for high genera. The proof of the former is based on elementary number theory and basically the same as that of [P]. The proof of the latter uses the operator formalism and the exponential formula. It is the generalization of the results of [K].

The organization of the paper is as follows. In Section 2, we define a generalization of the toric graph, the partition function and the free energy. In Section 3, we state main results. In Section 4, we explain that the first statement of the GV conjecture follows from these results. In Sections 5 and 6, we give proofs of the integrality and the vanishing, respectively. Appendix contains a proof of a lemma.

§2. Partition Function

In this section, we first define the notion of the generalized toric (GT) graph. Then we introduce the three point function and define the partition function and the free energy of the GT graph.

§2.1. Generalized toric graph

Throughout this article, we assume that a graph has the finite edge set and vertex set and has no self-loop.

A flag f is a pair of a vertex v and an edge e such that e is incident on v . The flag whose edge is the same as f and vertex is the other endpoint of the

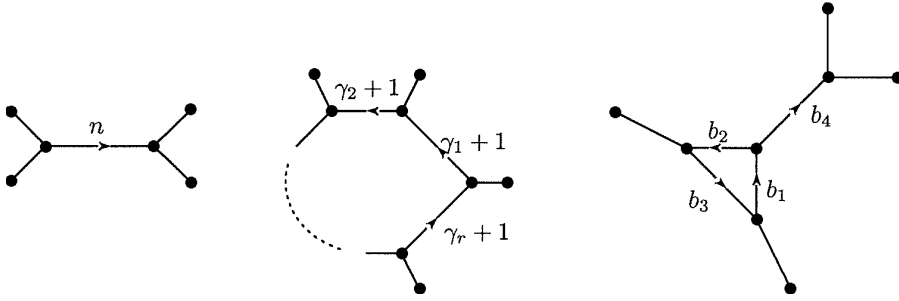


Figure 1. Examples of the GT graph $(n, \gamma_1, \dots, \gamma_r, b_1, \dots, b_4 \in \mathbb{Z})$.

edge is denoted by $-f$.

$$\begin{array}{ccc}
 v & \xrightarrow{e} & v' \\
 \bullet & \text{---} & \bullet
 \end{array}
 \quad
 \begin{array}{l}
 f = (v, e) \\
 -f = (v', e)
 \end{array}$$

A connected planar graph Γ is a *trivalent planar graph* if all vertices are either trivalent or univalent. The set of trivalent vertices is denoted by $V_3(\Gamma)$. The set of edges whose two endpoints are both trivalent is denoted by $E_3(\Gamma)$. The set of flags whose edges are in $E_3(\Gamma)$ is denoted by $F_3(\Gamma)$.

Definition 2.1. A trivalent planar graph with a label $n_f \in \mathbb{Z}$ on every flag $f \in F_3(\Gamma)$ together with a drawing into \mathbb{R}^2 is a *generalized toric graph* (GT graph) if it satisfies the following conditions.

1. $n_f = -n_{-f}$.
2. The drawing has no crossing.

n_f is called the *framing* of the flag f .

Since $n_f = -n_{-f}$, assigning framings is the same as assigning each edge an integer and a direction. Therefore, we add an auxiliary direction to every edge $e \in E_3(\Gamma)$ and redraw the graph as follows.



The direction of the edge is taken arbitrarily. The label on an edge e is denoted by n_e .

Examples of the GT graphs are shown in Figure 1.

§2.2. Partition and notations

We summarize notations (mainly) on partitions ([M]).

A *partition* is a non-increasing sequence $\lambda = (\lambda_1, \lambda_2, \dots)$ of nonnegative integers containing only finitely many nonzero terms. The nonzero λ_i 's are called the parts. The number of parts is the *length* of λ , denoted by $l(\lambda)$. The sum of the parts is the *weight* of λ , denoted by $|\lambda|$: $|\lambda| = \sum_i \lambda_i$. If $|\lambda| = d$, λ is a partition of d . The set of all partitions of d is denoted by \mathcal{P}_d and the set of all partitions by \mathcal{P} . Let $m_k(\lambda) = \#\{\lambda_i : \lambda_i = k\}$ be the *multiplicity* of k where $\#$ denotes the number of elements of a finite set. Let $\text{aut}(\lambda)$ be the symmetric group acting as the permutations of the equal parts of λ : $\text{aut}(\lambda) \cong \prod_{k \geq 1} \mathfrak{S}_{m_k(\lambda)}$. Then $\#\text{aut}(\lambda) = \prod_{k \geq 1} m_k(\lambda)!$. We define

$$z_\lambda = \prod_{i=1}^{l(\lambda)} \lambda_i \cdot \#\text{aut}(\lambda),$$

which is the number of the centralizers of the conjugacy class associated to λ .

A partition $\lambda = (\lambda_1, \lambda_2, \dots)$ is identified as the Young diagram with λ_i boxes in the i -th row ($1 \leq i \leq l(\lambda)$). The Young diagram with λ_i boxes in the i -th column is its *transposed* Young diagram. The corresponding partition is called the *conjugate partition* and denoted by λ^t . Note that $\lambda_i^t = \sum_{k \geq i} m_k(\lambda)$.

We define

$$\kappa(\lambda) = \sum_{i=1}^{l(\lambda)} \lambda_i(\lambda_i - 2i + 1).$$

This is equal to twice the sum of contents $\sum_{x \in \lambda} c(x)$ where $c(x) = j - i$ for the box x at the (i, j) -th place in the Young diagram λ . Thus, $\kappa(\lambda)$ is always even and satisfies $\kappa(\lambda^t) = -\kappa(\lambda)$.

$\mu \cup \nu$ denotes the partition whose parts are $\mu_1, \dots, \mu_{l(\mu)}, \nu_1, \dots, \nu_{l(\nu)}$ and $k\mu$ the partition $(k\mu_1, k\mu_2, \dots)$ for $k \in \mathbb{N}$.

For a finite set of integers $s = (s_1, s_2, \dots, s_l)$, we use the following notations.

$$|s| = \sum_i s_i.$$

When s has at least one nonzero element, we define

$$\text{gcd}(s) = \text{the greatest common divisor of } \{|s_i|, s_i \neq 0\}$$

where $|s_i|$ is the absolute value of s_i .

Throughout this paper, we use the letter q for a variable. We define

$$[k] = q^{\frac{k}{2}} - q^{-\frac{k}{2}} \quad (k \in \mathbb{Q}),$$

which is called the q -number. For a partition λ and a finite set s as above, we use the shorthand notations

$$[\lambda] = \prod_{i=1}^{l(\lambda)} [\lambda_i], \quad [s] = [s_1] \dots [s_l].$$

§2.3. Three point function

Let q^ρ and $q^{\lambda+\rho}$ be the following (infinite) sequences:

$$q^\rho = (q^{-i+\frac{1}{2}})_{i \geq 1}, \quad q^{\lambda+\rho} = (q^{\lambda_i-i+\frac{1}{2}})_{i \geq 1}.$$

The Schur function and skew-Schur function are denoted by s_λ and $s_{\lambda/\mu}$.

Definition 2.2. Let $(\lambda^1, \lambda^2, \lambda^3)$ be a triple of partitions. The *three point function* is

$$C_{\lambda^1, \lambda^2, \lambda^3}(q) = q^{\frac{\kappa(\lambda^3)}{2}} s_{\lambda^2}(q^\rho) \sum_{\eta \in \mathcal{P}} s_{\lambda^1/\eta}(q^{\lambda^{2t}+\rho}) s_{\lambda^3t/\eta}(q^{\lambda^2+\rho}).$$

This is a rational function in $q^{\frac{1}{2}}$. An important property of the three point function is the cyclic symmetry:

$$C_{\lambda^1, \lambda^2, \lambda^3}(q) = C_{\lambda^2, \lambda^3, \lambda^1}(q) = C_{\lambda^3, \lambda^1, \lambda^2}(q).$$

See [ORV] for a proof. Various identities can be found in [Z2].

Since the variables q^ρ and $q^{\lambda+\rho}$ are infinite sequences, let us explain how to compute the (skew-) Schur function. For a sequence of variables $x = (x_1, x_2, \dots)$, the elementary symmetric function $e_i(x)$ ($i \geq 0$) and the completely symmetric function $h_i(x)$ ($i \geq 0$) are obtained from the generating functions:

$$\sum_{i=0}^{\infty} e_i(x) z^i = \prod_{i=0}^{\infty} (1 + x_i z), \quad \sum_{i=0}^{\infty} h_i(x) z^i = \prod_{i=0}^{\infty} (1 - x_i z)^{-1}.$$

The skew-Schur function $s_{\mu/\nu}(x)$ is written in terms of $e_i(x)$ or $h_i(x)$:

$$(1) \quad s_{\mu/\nu}(x) = \det (e_{\mu_i^t - \nu_j^t - i + j}(x))_{1 \leq i, j \leq l(\mu^t)} = \det (h_{\mu_i - \nu_j - i + j}(x))_{1 \leq i, j \leq l(\mu)}.$$

In the determinants, $h_{-i}(x)$ and $e_{-i}(x)$ ($i > 0$) are assumed to be zero. For variables q^ρ , we can compute the elementary and the completely symmetric functions by using the identities [M]:

$$\prod_{i=0}^{\infty} (1 + q^i z) = \sum_{i=0}^{\infty} \frac{q^{\frac{i(i-1)}{2}}}{(1-q) \dots (1-q^i)} z^i,$$

$$\prod_{i=0}^{\infty} (1 - q^i z)^{-1} = \sum_{i=0}^{\infty} \frac{1}{(1-q) \dots (1-q^i)} z^i.$$

Therefore, for the variable q^ρ ,

$$(2) \quad e_i(q^\rho) = \frac{q^{-\frac{i(i-1)}{4}}}{[1] \dots [i]}, \quad h_i(q^\rho) = \frac{q^{\frac{i(i-1)}{4}}}{[1] \dots [i]}.$$

For the variable $q^{\lambda+\rho}$, $e_i(q^{\lambda+\rho})$ and $h_i(q^{\lambda+\rho})$ are computed from the generating functions:

$$(3) \quad \sum_{i=0}^{\infty} e_i(q^{\lambda+\rho}) z^i = \prod_{i=1}^{l(\lambda)} \frac{1 + q^{\lambda_i - i + \frac{1}{2}} z}{1 + q^{-i + \frac{1}{2}} z} \cdot \left(\sum_{k=0}^{\infty} e_k(q^\rho) z^k \right),$$

$$\sum_{i=0}^{\infty} h_i(q^{\lambda+\rho}) z^i = \prod_{i=1}^{l(\lambda)} \frac{1 - q^{-i + \frac{1}{2}} z}{1 - q^{\lambda_i - i + \frac{1}{2}} z} \cdot \left(\sum_{k=0}^{\infty} h_k(q^\rho) z^k \right).$$

In this way, we can explicitly compute the skew-Schur functions and the three point functions. Here are some examples of three point functions.

$$C_{(1),\emptyset,\emptyset}(q) = \frac{1}{[1]}, C_{(2),\emptyset,\emptyset}(q) = \frac{q^2}{(q-1)(q^2-1)}, C_{(1,1),\emptyset,\emptyset}(q) = \frac{q}{(q-1)(q^2-1)},$$

$$C_{(1),(1),\emptyset}(q) = \frac{q^2 - q + 1}{(1-q)^2}, C_{(1),(1),(1)}(q) = \frac{q^4 - q^3 + q^2 - q + 1}{q^{\frac{1}{2}}(q-1)^3}.$$

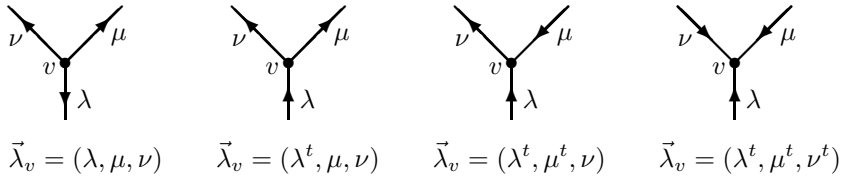
More examples can be found in [AKMV], Section 8.

§2.4. Partition function

First we set some notations. Consider a GT graph Γ .

- We associate one formal variable to every edge $e \in E_3(\Gamma)$. The variable associated to e is denoted by Q_e . $\vec{Q} = (Q_e)_{e \in E_3(\Gamma)}$.
- A *degree* is a set $\vec{d} = (d_e)_{e \in E_3(\Gamma)}$ of nonnegative integers which is not $\vec{0}$.

- A set $\vec{\lambda} = (\lambda_e)_{e \in E_3(\Gamma)}$ of partitions is called a Γ -partition. $\vec{\lambda}$ is of degree \vec{d} if $(|\lambda_e|)_{e \in E_3(\Gamma)} = \vec{d}$. Note that picking one Γ -partition is the same as assigning a partition to every edge of $E_3(\Gamma)$.
- Given a Γ -partition $\vec{\lambda}$, we define $\vec{\lambda}_v$ for a vertex $v \in V_3(\Gamma)$ as follows.



It depends on the directions of three incident edges and their partitions. If an incident edge is not in $E_3(\Gamma)$, then we assume that the empty partition \emptyset is assigned to it. (Although such edge is not directed, it is irrelevant since $\emptyset^t = \emptyset$.)

- For a Γ -partition $\vec{\lambda}$, we set

$$(4) \quad Y_{\vec{\lambda}}(q) = \prod_{e \in E_3(\Gamma)} (-1)^{d_e(n_e+1)} q^{\frac{n_e \kappa(\lambda_e)}{2}} \prod_{v \in V_3(\Gamma)} C_{\vec{\lambda}_v}(q).$$

Definition 2.3. The partition function of a GT graph Γ is

$$\mathcal{Z}^\Gamma(q; \vec{Q}) = 1 + \sum_{\vec{d}; \text{degree}} \mathcal{Z}_{\vec{d}}^\Gamma(q) \vec{Q}^{\vec{d}},$$

where $\vec{Q}^{\vec{d}} = \prod_{e \in E_3(\Gamma)} Q_e^{d_e}$ and

$$\mathcal{Z}_{\vec{d}}^\Gamma(q) = \sum_{\substack{\vec{\lambda}; \Gamma\text{-partition} \\ \text{of degree } \vec{d}}} Y_{\vec{\lambda}}(q).$$

Definition 2.4. The free energy of Γ is defined as

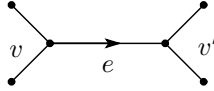
$$\mathcal{F}^\Gamma(q; \vec{Q}) = \log \mathcal{Z}^\Gamma(q; \vec{Q}).$$

The coefficient of $\vec{Q}^{\vec{d}}$ is denoted by $\mathcal{F}_{\vec{d}}^\Gamma(q)$.

§2.5. Examples of partition function

We calculate the partition function for the GT graphs in Figure 1.

2.5.1. First, we compute the partition function for the left GT graph. Let us name trivalent vertices and the middle edge as follows.



A Γ -partition consists of only one partition associated to the edge e : $\vec{\lambda} = (\lambda)$. For this Γ -partition,

$$\begin{aligned} Y_{\vec{\lambda}}(q) &= (-1)^{(n+1)|\lambda|} q^{n\frac{\kappa(\lambda)}{2}} C_{\lambda, \emptyset, \emptyset}(q) C_{\lambda^t, \emptyset, \emptyset}(q) \\ &= (-1)^{(n+1)|\lambda|} q^{n\frac{\kappa(\lambda)}{2}} s_{\lambda}(q^{\rho}) s_{\lambda^t}(q^{\rho}) \\ &= (-1)^{(n+1)|\lambda|} q^{(n-1)\frac{\kappa(\lambda)}{2}} s_{\lambda}(q^{\rho})^2. \end{aligned}$$

In the last line, we have used the identity $s_{\lambda^t}(q^{\rho}) = q^{-\frac{\kappa(\lambda)}{2}} s_{\lambda}(q^{\rho})$ [Z2].

Since a degree \vec{d} consists of only one component d associated to the edge e , we write d instead of \vec{d} . We also write Q_e as Q for simplicity. The partition function is

$$\begin{aligned} Z^{\Gamma}(q; Q) &= 1 + \sum_{d=1}^{\infty} Z_d^{\Gamma}(q) Q^d, \\ Z_d^{\Gamma}(q) &= (-1)^{(n+1)d} \sum_{\lambda \in \mathcal{P}_d} q^{(n-1)\frac{\kappa(\lambda)}{2}} s_{\lambda}(q^{\rho})^2. \end{aligned}$$

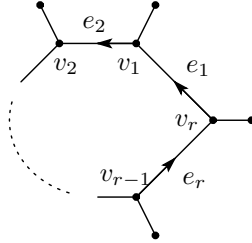
This GT graph represents the total space of $\mathcal{O}(n-1) \oplus \mathcal{O}(-n-1) \rightarrow \mathbb{P}^1$ and the free energy $\mathcal{F}^{\Gamma}(q; Q)$ is nothing but the generating function of the GW invariants.

2.5.2. Next, we compute the partition function for the middle GT graph. We introduce the *two-point function*

$$W_{\mu, \nu}(q) = (-1)^{|\mu|+|\nu|} q^{\frac{\kappa(\mu)+\kappa(\nu)}{2}} \sum_{\eta \in \mathcal{P}} s_{\mu/\eta}(q^{-\rho}) s_{\nu/\eta}(q^{-\rho}) \quad (\mu, \nu \in \mathcal{P}),$$

where $q^{-\rho} = (q^{i-\frac{1}{2}})_{i \geq 1}$. It is a rational function in $q^{\frac{1}{2}}$ and satisfies $q^{\frac{\kappa(\mu)}{2}} W_{\mu, \nu}(q) = C_{\mu^t, \emptyset, \nu}(q)$ (Proposition 4.5, [Z2]).

Let us name the edge with the framing $\gamma_i + 1$ as e_i ($1 \leq i \leq r$) and the trivalent vertex incident on e_i and e_{i+1} as v_i .



Let $\lambda = (\lambda^1, \dots, \lambda^r)$ be a Γ -partition where λ^i is a partition assigned to edge e_i ($1 \leq i \leq r$). For v_i , $\vec{\lambda}_{v_i} = (\lambda^{it}, \emptyset, \lambda^{i+1})$. Therefore

$$\begin{aligned} Y_{\vec{\lambda}}(q) &= \prod_{i=1}^r (-1)^{\gamma_i |\lambda^i|} q^{(\gamma_i+1)\frac{\kappa(\lambda^i)}{2}} C_{\lambda^{it}, \emptyset, \lambda^{i+1}}(q) \\ &= \prod_{i=1}^r (-1)^{\gamma_i |\lambda^i|} q^{\frac{\gamma_i \kappa(\lambda^i)}{2}} W_{\lambda^i, \lambda^{i+1}}(q). \end{aligned}$$

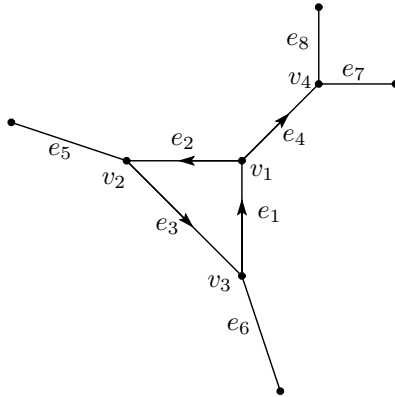
Here $\lambda^{r+1} = \lambda^1$ is assumed.

We associate formal variables Q_1, \dots, Q_r to e_1, \dots, e_r , respectively (in the previous notation, $Q_i = Q_{e_i}$). Then the partition function is

$$\begin{aligned} Z^{\Gamma}(q; Q_1, \dots, Q_r) &= 1 + \sum_{\substack{\vec{d}=(d_1, \dots, d_r); \\ \vec{d} \neq \vec{0}, \\ d_i \geq 0}} \prod_{i=1}^r (-1)^{\gamma_i d_i} Q_i^{d_i} \sum_{\substack{(\lambda^1, \dots, \lambda^r) \\ \lambda^i \in \mathcal{P}_{d_i}}} \prod_{i=1}^r q^{\frac{\gamma_i \kappa(\lambda^i)}{2}} W_{\lambda^i, \lambda^{i+1}}(q). \end{aligned}$$

The GT graph represents a complete smooth toric surface S if $(\gamma_1, \dots, \gamma_r)$ is equal to the set of self-intersection numbers of the toric invariant curves in S . In such a case, the free energy $\mathcal{F}^{\Gamma}(q, \vec{Q})$ is equal to the generating function of the GW invariants of the canonical bundle of S .

2.5.3. Finally, we compute the partition function of the right GT graph. Let us name trivalent vertices and edge as follows.



$E_3(\Gamma) = \{e_1, e_2, e_3, e_4\}$ and $V_3(\Gamma) = \{v_1, v_2, v_3, v_4\}$. Let $\vec{\lambda} = (\lambda^1, \dots, \lambda^4)$ be a Γ -partition where λ^i is a partition assigned to the edge e_i . For each trivalent vertex,

$$(5) \quad \begin{aligned} \vec{\lambda}_{v_1} &= (\lambda^{1t}, \lambda^4, \lambda^2), & \vec{\lambda}_{v_2} &= (\lambda^{2t}, \emptyset, \lambda^3), \\ \vec{\lambda}_{v_3} &= (\lambda^{3t}, \emptyset, \lambda^1), & \vec{\lambda}_{v_4} &= (\emptyset, \lambda^{4t}, \emptyset). \end{aligned}$$

Therefore

$$\begin{aligned} Y_{\vec{\lambda}}(q) &= (-1)^{\sum_{i=1}^4 (b_i+1)|\lambda^i|} q^{\sum_{i=1}^4 b_i \frac{\kappa(\lambda^i)}{2}} C_{\lambda^{1t}, \lambda^4, \lambda^{2t}}(q) C_{\lambda^{2t}, \emptyset, \lambda^{3t}}(q) \\ &\quad \times C_{\lambda^{3t}, \emptyset, \lambda^{1t}}(q) C_{\emptyset, \lambda^{4t}, \emptyset} \end{aligned}$$

and the partition function is

$$\begin{aligned} \mathcal{Z}^\Gamma(q; \vec{Q}) &= 1 + \sum_{\substack{\vec{d}=(d_1, d_2, d_3, d_4); \\ \vec{d} \neq \vec{0}, \\ d_i \geq 0}} \mathcal{Z}_{\vec{d}}^\Gamma(q) Q_1^{d_1} \dots Q_4^{d_4}, \\ \mathcal{Z}_{\vec{d}}^\Gamma(q) &= \sum_{\substack{\vec{\lambda}=(\lambda^1, \lambda^2, \lambda^3, \lambda^4); \\ \lambda^i \in \mathcal{P}_{d_i}}} Y_{\vec{\lambda}}(q). \end{aligned}$$

When $b_1 = b_2 = b_3 = 2$ and $b_4 = 0$, the GT graph represents the flop of the total space of the canonical bundle of the Hirzebruch surface \mathbb{F}_1 and the free energy is equal to the generating function of the GW invariants.

§3. Main Results

In this section, we state main results of this article. Let us define

Definition 3.1.

$$G_{\vec{d}}^\Gamma(q) = \sum_{k; k|d_0} \frac{\mu(k)}{k} \mathcal{F}_{\vec{d}/k}^\Gamma(q^k) \quad (d_0 = \gcd(\vec{d})),$$

where $\mu(k)$ is the Möbius function.

We set $t = [1]^2$ and define $\mathcal{L}[t]$ by

$$\mathcal{L}[t] = \left\{ \frac{f_2(t)}{f_1(t)} \mid f_1(t), f_2(t) \in \mathbb{Z}[t], f_1(t) : \text{monic} \right\}.$$

$\mathcal{L}[t]$ is a subring of the ring of rational functions $\mathbb{Q}(t)$ [P].

The main results of the paper are

Proposition 3.2.

$$G_{\vec{d}}^\Gamma(q) \in \mathcal{L}[t].$$

Proposition 3.3.

$$t \cdot G_{\vec{d}}^\Gamma(q) \in \mathbb{Q}[t].$$

We will prove Propositions 3.2 and 3.3 in Sections 5 and 6, respectively.

Propositions 3.2 and 3.3 imply that the numerator of $t \cdot G_{\vec{d}}^\Gamma(q)$ is divisible by the denominator. Since the denominator is monic, the quotient is a polynomial in t with integer coefficients. Thus

Corollary 3.4. $t \cdot G_{\vec{d}}^\Gamma(q) \in \mathbb{Z}[t].$

What does this corollary mean? By the formula of the Möbius function

$$(6) \quad \sum_{k': k'|k} \mu(k') = \begin{cases} 1 & (k = 1) \\ 0 & (k > 1, k \in \mathbb{N}), \end{cases}$$

the free energy in degree \vec{d} is written as

$$\mathcal{F}_{\vec{d}}^\Gamma(q) = \sum_{k; k|d_0} \frac{1}{k} G_{\vec{d}/k}^\Gamma(q^k).$$

In fact, Definition 3.1 was obtained by inverting this relation [BP]. Let us write the corollary as follows.

$$G_{\vec{d}}^\Gamma(q) = \sum_{g \geq 0} n_{\vec{d}}^g(\Gamma) (-t)^{g-1}$$

where $\{n_d^g(\Gamma)\}_{g \geq 0}$ is a sequence of integers only finite number of which is nonzero. Note that Proposition 3.2 implies the integrality of $n_d^g(\Gamma)$. Proposition 3.3 implies the vanishing of n_d^g at large g (and also at $g < 0$). We find that the free energy is written in terms of these integers as

$$(7) \quad \mathcal{F}_d^\Gamma(q) = \sum_{g \geq 0} \sum_{k; k|d_0} n_{d/k}^g(\Gamma) \frac{(-t_k)^{g-1}}{k},$$

where $t_k = [k]^2$.

Before moving to the proof of the propositions, we explain the geometric meaning of these results.

§4. Toric Calabi–Yau Threefold and Gopakumar–Vafa Conjecture

Given a toric Calabi–Yau threefold (TCY threefold) X , a planar graph is determined canonically from the fan of X . It is called the *toric graph* of X and it is a GT graph or the graph union of GT graphs. In this section, we first describe how to draw the toric graph. Then we explain the relation between the free energy of the toric graph and the generating function of the GW invariants of X . Finally, we see that (7) implies the integrality and the vanishing for high genera of the Gopakumar–Vafa invariants.

§4.1. TCY threefold

A *Calabi–Yau toric threefold* is a three-dimensional, smooth toric variety X of finite type, whose canonical bundle K_X is a trivial line bundle. The last condition is called the *Calabi–Yau condition*. For simplicity, we impose one more condition, which implies that the fundamental group $\pi_1(X)$ is trivial and that $H^2(X) \cong \text{Pic}(X)$.

A toric variety X is constructed from a fan Σ , which is a collection of cones. The fan of X is unique up to $SL(3, \mathbb{Z})$ since the action of $SL(3, \mathbb{Z})$ on a fan is offset by the change of the coordinate functions.

The conditions on X is rephrased in terms of those on the fan Σ as follows.

Finite type: X is of finite type if its fan Σ is a finite set.

Smoothness: X is smooth if and only if the minimal set of generators of every cone forms a part of a \mathbb{Z} -basis of \mathbb{R}^3 . (Here the generators of a cone mean the shortest integral vectors that generate the cone.)

Calabi–Yau: The canonical bundle of X is trivial if and only if there exists a vector $u \in (\mathbb{R}^3)^*$ satisfying

$$\langle \omega_i, u \rangle = 1$$

for all generators ω_i of the fan. Using the action of $SL(3, \mathbb{Z})$, we take

$$u = (0, 0, 1).$$

Therefore every generators of a fan of a toric Calabi–Yau threefolds is of the form $(*, *, 1)$. Note that such fan can not be complete. Equivalently, the toric variety X is noncompact.

Other assumption: We assume that there exists at least one 3-cone and that every 1 or 2-cone of the fan Σ is a face of some 3-cone. This implies that

$$\pi_1(X) = \{id\}, \quad H^2(X) \cong \text{Pic}(X).$$

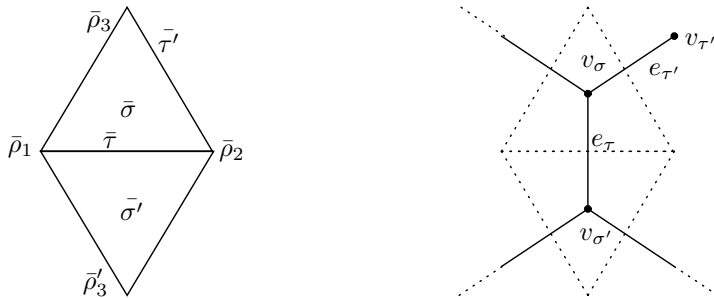
See [F] for a proof.

§4.2. Toric graph

Since all the generators are of the form $(*, *, 1)$, it is sufficient to see the section $\bar{\Sigma}$ of the fan Σ at the height 1. We will write the section of a cone σ as $\bar{\sigma}$.

From $\bar{\Sigma}$, we draw a labeled graph as follows.

1. Draw a vertex v_σ inside every 2-simplex $\bar{\sigma}$.
2. Draw an edge e_τ transversally to every 1-simplex $\bar{\tau}$ as follows.
 - (a) If $\bar{\tau}$ is the boundary of two 2-simplices $\bar{\sigma}, \bar{\sigma}'$, let e_τ join v_σ and $v_{\sigma'}$.
 - (b) If $\bar{\tau}$ is the boundary of only one 2-simplex $\bar{\sigma}$, let e_τ be incident to v_σ ; add one vertex v_τ to other endpoint.



- To every flag f whose edge is of type 2a, we assign an integer label n_f as follows. For (v, σ) and (v, σ') in the above figure, the labels are

$$\frac{a_1 - a_2}{2} \text{ for } (v_\sigma, e_\tau), \quad \frac{-a_1 + a_2}{2} \text{ for } (v_{\sigma'}, e_\tau).$$

Here a_1, a_2 are integers defined by

$$\omega'_3 = -a_1\omega_1 - a_2\omega_2 - \omega_3$$

where $\omega_1, \omega_2, \omega_3$ and ω'_3 are generators of the 1-cones ρ_1, ρ_2, ρ_3 and ρ'_3 , respectively. Since $a_1 + a_2 = -2$ by the Calabi–Yau condition, these are integers. The label is called the *framing* of the flag. For reference, we computed the framings for flags in Figure 2.

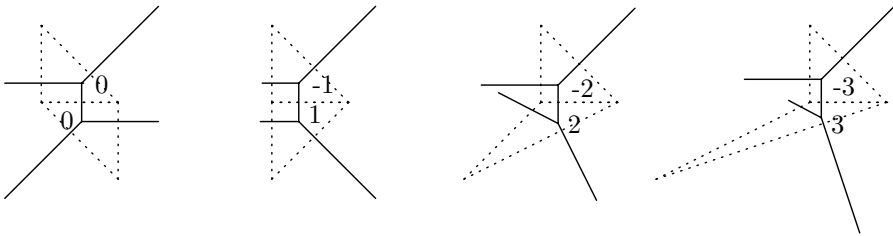


Figure 2. Examples of framings.

The resulting graph is the *toric graph* of the TCY threefold X . Note that the toric graph is unique although the fan is unique only up to the action of $SL(3, \mathbb{Z})$.

Examples of the toric graphs are shown in Figures 3 and 4. See also Figure 1.

It is clear that each connected component of a toric graph is a GT graph. Therefore we define the partition function of the toric graph by the product of the partition functions of its connected components.

Let us summarize the information on X read from the toric graph Γ :

- $v \in V_3(\Gamma)$ represents a torus fixed point p_v .
- $e \in E_3(\Gamma)$ represents a curve $C_e \cong \mathbb{P}^1$. If the two endpoints of $e \in E_3(\Gamma)$ is v, v' , then $p_v, p_{v'}$ are two torus fixed points in C_e . The framing n_f of $f = (v, e)$ represents the degrees of the normal bundle: $NC_e \cong \mathcal{O}_{\mathbb{P}^1}(n_f - 1) \oplus \mathcal{O}_{\mathbb{P}^1}(-n_f - 1)$.

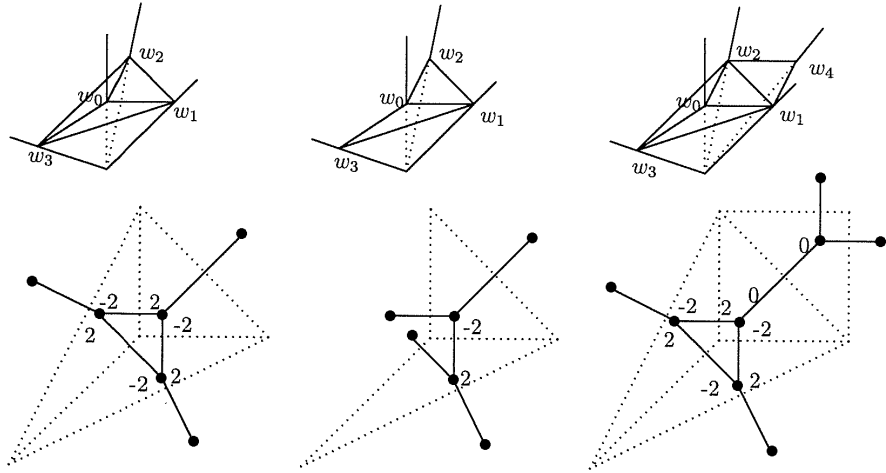


Figure 3. Examples of fans (upstairs) and toric graphs (downstairs). The left is the canonical bundle of \mathbb{P}^2 , the middle is the total space of the vector bundle $\mathcal{O}(1) \oplus \mathcal{O}(-3) \rightarrow \mathbb{P}^1$ and the left is the flop of the canonical bundle of the Hirzebruch surface \mathbb{F}_1 . w_i ($0 \leq i \leq 4$) are the generators: $w_0 = (0, 0, 1)$, $w_1 = (1, 0, 1)$, $w_2 = (0, 1, 1)$, $w_3 = (-1, -1, 1)$ and $w_4 = (1, 1, 1)$.

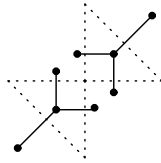


Figure 4. An example of the toric graph with more than one connected components. This corresponds to the canonical bundle of the noncomplete toric surface $\mathbb{P}^1 \times \mathbb{P}^1 \setminus \{(0, 0), (\infty, \infty)\}$.

§4.3. Geometric meaning of free energy

Let X be a TCY threefold. Roughly speaking, the (0-pointed) Gromov–Witten invariant $N_\beta^g(X)$ is the number obtained by the integration of 1 over the moduli of the 0-pointed stable maps from a curve of genus g whose image belong to the homology class $\beta \in H_2^{cpt}(X, \mathbb{Z})$. (See [LLLZ] for the precise definition.) We define the generating function with a fixed homology class β :

$$\mathcal{F}_\beta(X) = \sum_{g \geq 0} g_s^{2g-2} N_\beta^g(X).$$

In this article, we use the symbol g_s as the genus expansion parameter.

Let Γ be the toric graph of X and $\mathcal{F}_d^\Gamma(q)$ be the free energy in a degree \vec{d} . Note that any degree \vec{d} determines a homology class with the compact support,

$$[\vec{d} \cdot \vec{C}] = \sum_{e \in E_3(\Gamma)} d_e [C_e].$$

The proposal of [AKMV] (proposition 7.4 [LLLZ]) is that the generating function $\mathcal{F}_\beta(X)$ is equal to the sum of the free energy in degrees \vec{d} such that $\vec{d} \cdot \vec{C} = \beta$, under the identification $q = e^{\sqrt{-1}g_s}$:

$$(8) \quad \mathcal{F}_\beta(X) = \sum_{[\vec{d} \cdot \vec{C}] = \beta} \mathcal{F}_d^\Gamma(e^{\sqrt{-1}g_s}).$$

Actually, each $\mathcal{F}_d^\Gamma(q)$ has the meaning in the localization calculation: it is the contribution from the fixed point loci in the moduli stacks of stable maps whose image curves are $\vec{d} \cdot \vec{C}$.

§4.4. Gopakumar–Vafa conjecture

Let us define the numbers $\{n_\beta^g(X)\}_{g \geq 0, \beta \in H_2(X; \mathbb{Z})}$ by rewriting $\{\mathcal{F}_\beta(X)\}_{\beta \in H_2(X; \mathbb{Z})}$ in the form below.

$$(9) \quad \mathcal{F}_\beta(X) = \sum_{g \geq 0} \sum_{k; k|\beta} \frac{n_{\beta/k}^g(X)}{k} \left(2 \sin \frac{kg_s}{2}\right)^{2g-2}.$$

$n_\beta^g(X)$ is called the *Gopakumar–Vafa invariant*. The Gopakumar–Vafa conjecture states the followings [GV].

1. $n_\beta^g(X) \in \mathbb{Z}$ and $n_\beta^g(X) = 0$ for every fixed β and $g \gg 1$.
2. Moreover, $n_\beta^g(X)$ is equal to the number of certain BPS states in M-theory (see [HST] for a mathematical formulation).

The first part of the conjecture follows from corollary 3.4 since the GV invariant is written as

$$n_\beta^g(X) = \sum_{\vec{d}; [\vec{d} \cdot \vec{C}] = \beta} n_d^g(\Gamma),$$

by (7), (8) and (9).

§5. Proof of Proposition 3.2

In this section, we give a proof of Proposition 3.2.

§5.1. Outline of proof

The proof proceeds as follows. Firstly, we take the logarithm of the partition function using the Taylor expansion. For a degree \vec{d} , we define

$$D(\vec{d}) = \{\vec{\delta} : \text{degree}|\delta_e \leq d_e \text{ for all } e \in E_3(\Gamma)\}.$$

This is just the set of degrees smaller than or equal to \vec{d} . Consider assigning a nonnegative integer to each element of $D(\vec{d})$; in other words, consider a set of nonnegative integers

$$n = \{n_{\vec{\delta}} \in \mathbb{Z}_{\geq 0} \mid \vec{\delta} \in D(\vec{d})\}.$$

We call such a set a *multiplicity* in \vec{d} if it satisfies

$$n \cdot \vec{d} := \sum_{\vec{\delta} \in D(\vec{d})} n_{\vec{\delta}} \vec{\delta} = \vec{d}.$$

With these notations, the free energy in degree \vec{d} is written as

$$\mathcal{F}_{\vec{d}}^{\Gamma}(q) = \sum_{\substack{n; \\ \text{multiplicity} \\ \text{in } \vec{d}}} \frac{|n|!}{\prod_{\vec{\delta} \in D(\vec{d})} n_{\vec{\delta}}!} \frac{(-1)^{|n|-1}}{|n|} \prod_{\vec{\delta} \in D(\vec{d})} (\mathcal{Z}_{\vec{\delta}}^{\Gamma}(q))^{n_{\vec{\delta}}}.$$

We further rewrite it. Let $d_0 = \text{gcd}(\vec{d})$.

$$\mathcal{F}_{\vec{d}}^{\Gamma}(q) = \sum_{k; k|d_0} \sum_{\substack{n; \text{multiplicity} \\ \text{in } \vec{d}/k, \\ \text{gcd}(n)=1}} \frac{(k|n|)!}{\prod_{\vec{\delta} \in D(\vec{d}/k)} (kn_{\vec{\delta}})!} \frac{(-1)^{k|n|-1}}{k|n|} \left(\prod_{\vec{\delta} \in D(\vec{d}/k)} (\mathcal{Z}_{\vec{\delta}}^{\Gamma}(q))^{n_{\vec{\delta}}} \right)^k.$$

Then

$$G_{\vec{d}}^{\Gamma}(q) = \sum_{k; k|d_0} \sum_{\substack{n; \text{multiplicity} \\ \text{in } \vec{d}/k, \\ \text{gcd}(n)=1}} \frac{1}{k|n|} \times \left[\sum_{k'; k'|k} \mu\left(\frac{k}{k'}\right) \frac{(k'|n|)!}{\prod_{\vec{\delta} \in D(\vec{d}/k)} (k'n_{\vec{\delta}})!} \frac{(-1)^{k'|n|-1}}{k'|n|} \left(\prod_{\vec{\delta} \in D(\vec{d}/k)} (\mathcal{Z}_{\vec{\delta}}^{\Gamma}(q^{k/k'}))^{n_{\vec{\delta}}} \right)^{k'} \right].$$

Each summand turns out to be an element of $\mathcal{L}[t]$ by the next lemmas.

Lemma 5.1. For any degree \vec{d} ,

$$\mathcal{Z}_{\vec{d}}^{\Gamma}(q) \in \mathcal{L}[t].$$

Lemma 5.2. Let $n = (n_1, \dots, n_l)$ be the set of nonnegative integers such that $\gcd(n) = 1$. For $R(t) \in \mathcal{L}[t]$, $k \in \mathbb{N}$ and n ,

$$\frac{1}{k|n|} \sum_{k'; k'|k} \mu\left(\frac{k}{k'}\right) \frac{(k'|n|)!}{(k'n_1)! \cdots (k'n_l)!} \frac{(-1)^{k'|n|}}{k'|n|} R(t_{k/k'})^{k'} \in \mathcal{L}[t].$$

The proofs of Lemmas 5.1 and 5.2 are given in Subsection 5.2 and Appendix A, respectively.

Thus $G_{\vec{d}}^{\Gamma}(q) \in \mathcal{L}[t]$ and Proposition 3.2 is proved.

§5.2. Proof of Lemma 5.1

In this subsection, we give a proof of Lemma 5.1. The main point is in showing that $\mathcal{Z}_{\vec{d}}^{\Gamma}(q)$, which is a priori a function in $q^{\frac{1}{2}}$, is actually a function in t . We use two key facts here. Let $\mathbb{Z}_0[t]$ be the ring of monic polynomials and let $\mathbb{Z}^+[q, q^{-1}]$ be the subring of the ring of Laurent polynomials in q whose elements are symmetric with respect to q, q^{-1} . The one fact is that [BP]

$$t_k := [k]^2 \in \mathbb{Z}_0[t] \quad (k \in \mathbb{N}).$$

The other is that (see [K], Lemma 6.2)

$$\mathbb{Z}[t] \cong \mathbb{Z}^+[q, q^{-1}].$$

We first state preliminary lemmas.

Lemma 5.3.

(i) $h_i(q^\rho)$ is written in the form

$$h_i(q^\rho) = q^{i/2} \frac{f_2(q)}{f_1(t)}$$

with $f_2(q) \in \mathbb{Z}[q, q^{-1}]$ and $f_1(q) \in \mathbb{Z}_0[t]$.

(ii) $e_i(q^\rho) = (-1)^i h_i(q^\rho)|_{q \rightarrow q^{-1}}$.

(iii) $h_i(q^{\lambda+\rho})$ is written in the form

$$h_i(q^{\lambda+\rho}) = q^{i/2} \frac{f_2^\lambda(q)}{f_1^\lambda(t)}$$

with $f_2^\lambda(q) \in \mathbb{Z}[q, q^{-1}]$ and $f_1^\lambda(t) \in \mathbb{Z}_0[t]$.

(iv) $e_i(q^{\lambda+\rho}) = (-1)^i h_i(q^{\lambda^t+\rho})|_{q \rightarrow q^{-1}}$.

(v) $s_{\mu/\nu}(q^{\lambda+\rho})$ is written in the following form:

$$s_{\mu/\nu}(q^{\lambda+\rho}) = q^{\frac{|\mu|-|\nu|}{2}} \frac{s_2^{\lambda,\mu,\nu}(q)}{s_1^{\lambda,\mu,\nu}(t)}$$

with $s_2^{\lambda,\mu,\nu}(q) \in \mathbb{Z}^+[q, q^{-1}]$ and $s_1^{\lambda,\mu,\nu}(t) \in \mathbb{Z}_0[t]$.

(vi) $s_{\mu^t/\nu^t}(q^{\lambda+\rho}) = (-1)^{|\mu|-|\nu|} s_{\mu/\nu}(q^{\lambda+\rho})|_{q \rightarrow q^{-1}}$.

(vii) The three point function is written in the following form:

$$C_{\lambda^1, \lambda^2, \lambda^3}(q) = q^{\frac{|\lambda^1|+|\lambda^2|+|\lambda^3|}{2}} \frac{c_2^{\lambda^1, \lambda^2, \lambda^3}(q)}{c_1^{\lambda^1, \lambda^2, \lambda^3}(t)}$$

where $c_2^{\lambda^1, \lambda^2, \lambda^3}(q) \in \mathbb{Z}[q, q^{-1}]$ and $c_1^{\lambda^1, \lambda^2, \lambda^3}(t) \in \mathbb{Z}_0[t]$.

(viii)

$$C_{\lambda^{1t}, \lambda^{2t}, \lambda^{3t}}(q) = (-1)^{|\lambda^1|+|\lambda^2|+|\lambda^3|} C_{\lambda^1, \lambda^2, \lambda^3}(q^{-1}).$$

Proof. (i). Recall the expression (2). If we multiply both the denominator and the numerator by $[1] \dots [i]$, we obtain

$$h_i(q^\rho) = q^{\frac{i}{2}} \frac{q^{\frac{i(i-3)}{4}} [1] \dots [i]}{t_1 \dots t_i}.$$

This proves (i).

(ii) follows from (2).

(iii) follows from (i) and the generating function (3).

(iv) follows from (3) and the identity:

$$\prod_{i=1}^{l(\lambda)} \frac{1 + q^{\lambda_i - i + \frac{1}{2}} z}{1 + q^{-i + \frac{1}{2}} z} = \prod_{j=1}^{l(\lambda^t)} \frac{1 + q^{j - \frac{1}{2}} z}{1 + q^{-\lambda_j^t + j - \frac{1}{2}} z}.$$

(This identity can be proved by showing that the LHS is equal to $\prod_{i=1}^{r(\lambda)} (1 + q^{\lambda_i - i + \frac{1}{2}} z) / (1 + q^{-(\lambda_i^t - i + \frac{1}{2})} z)$ where $r(\lambda)$ denotes the number of diagonal boxes in the Young diagram of λ .)

(v) follows from (iii) and (1):

$$\begin{aligned} s_{\mu/\nu}(q^{\lambda+\rho}) &= \det (h_{\mu_i - \nu_j - i + j}(q^{\lambda+\rho}))_{i,j} \\ &= q^{\frac{|\mu|-|\nu|}{2}} \det (q^{\frac{-\mu_i + \nu_j + i - j}{2}} h_{\mu_i - \nu_j - i + j}(q^{\lambda+\rho}))_{i,j}. \end{aligned}$$

(vi) follows from (iv) and (1)

$$\begin{aligned} s_{\mu^t/\nu^t}(q^{\lambda+\rho}) &= \det \left(e_{\mu_i-\nu_j-i+j}(q^{\lambda+\rho}) \right)_{i,j} \\ &= (-1)^{|\mu|-|\nu|} \det \left(h_{\mu_i-\nu_j-i+j}(q^{\lambda^t+\rho}) \right)_{i,j} \Big|_{q \rightarrow q^{-1}} \quad (\because \text{(iv)}) \\ &= (-1)^{|\mu|-|\nu|} s_{\mu/\nu}(q^{\lambda^t+\rho}) \Big|_{q \rightarrow q^{-1}}. \end{aligned}$$

(vii) and (viii) follow from (v) and (vi), respectively. □

Now we prove Lemma 5.1. Let $\vec{\lambda} = (\lambda_e)_{e \in E_3(\Gamma)}$ be a Γ -partition. By (vii), $Y_{\vec{\lambda}}(q)$ (defined in (4)) is written in the form

$$Y_{\vec{\lambda}}(q) = \frac{Y_2^{\vec{\lambda}}(q)}{Y_1^{\vec{\lambda}}(t)}$$

with $Y_2^{\vec{\lambda}}(q) \in \mathbb{Z}[q, q^{-1}]$ and $Y_1^{\vec{\lambda}}(t) \in \mathbb{Z}_0[t]$. Moreover, by (viii), it holds that

$$Y_{\vec{\lambda}^t}(q) = \frac{Y_2^{\vec{\lambda}}(q^{-1})}{Y_1^{\vec{\lambda}}(t)} = Y_{\vec{\lambda}}(q^{-1}),$$

where $\vec{\lambda}^t = (\lambda_e^t)_{e \in E_3(\Gamma)}$. Therefore

$$Y_{\vec{\lambda}}(q) + Y_{\vec{\lambda}^t}(q) \in \mathcal{L}[t] \quad (\vec{\lambda} \neq \vec{\lambda}^t), \quad Y_{\vec{\lambda}}(q) \in \mathcal{L}[t] \quad (\vec{\lambda} = \vec{\lambda}^t).$$

Thus

$$\mathcal{Z}_{\vec{d}}^{\Gamma}(q) = \frac{1}{2} \sum_{\substack{\vec{\lambda}; \Gamma\text{-partition} \\ \text{of degree } \vec{d}, \\ \vec{\lambda} \neq \vec{\lambda}^t}} \left(Y_{\vec{\lambda}}(q) + Y_{\vec{\lambda}^t}(q) \right) + \sum_{\substack{\vec{\lambda}; \Gamma\text{-partition} \\ \text{of degree } \vec{d}, \\ \vec{\lambda} = \vec{\lambda}^t}} Y_{\vec{\lambda}}(q) \in \mathcal{L}[t].$$

Note that the prefactor 1/2 does not matter because the same term appears twice if $\vec{\lambda} \neq \vec{\lambda}^t$. The proof of Lemma 5.1 is finished.

§6. Proof of Proposition 3.3

In this section, we give a proof of Proposition 3.3. We first rewrite the three point function and the partition function in the operator formalism (Subsections 6.2 and 6.4). Then we express the partition function as the sum of certain quantities - *combined amplitude* - of not necessarily connected graphs (Subsection 6.5). By using the exponential formula, we obtain the free energy as the sum over connected graphs (Subsection 6.6). Then we show that the

proposition follows from the property of the combined amplitudes of the connected graphs (Subsection 6.7). This proof is almost the same as [K], where the same proposition was proved for the middle graph in Figure 1.

There are, however, two technical difficulties in generalization. They occur when writing the partition function in the operator formalism due to the existence of trivalent vertices whose three incident edges are in $E_3(\Gamma)$. The one difficulty is how to incorporate the variables such as $q^{\lambda+\rho}$. It is overcome by the fact that the i -th power sum of $q^{\lambda+\rho}$ is equal to the matrix element of $\mathcal{E}_0(i)$ with respect to the state $|v_\lambda\rangle$ (Subsection 6.1.1). The other is how to perform the summation when the states $|v_\lambda\rangle$ and $|v_{\lambda^t}\rangle$ appear simultaneously. It is solved by introducing the operator R that transforms $|v_\lambda\rangle$ to $|v_{\lambda^t}\rangle$ (Subsection 6.1.2).

The expression of the three point function $C_{\lambda^1, \lambda^2, \lambda^3}$ thus obtained does not possess the cyclic symmetry with respect to three partitions $\lambda^1, \lambda^2, \lambda^3$. Therefore it becomes necessary to specify the order of three flags around every trivalent vertex. For this purpose we will introduce the notion of the *flag-order* (Subsection 6.3).

We omit the explanation of the operator formalism. Please see [K], Section 2.1.

§6.1. Technical preliminary

This subsection is devoted to the solution to the two technical problems mentioned previously.

6.1.1. Power sum

We express the power sum functions of the variables $q^{\lambda+\rho}$ as a matrix element.

For a sequence of variables $x = (x_1, x_2, \dots)$, the i -th power sum function is defined by $p_i(x) = \sum_{j \geq 1} x_j^i$. The power sum function associated to a partition ν is defined by $p_\nu(x) = \prod_{i=1}^{l(\nu)} p_{\nu_i}(x)$.

Consider the variable $q^{\lambda+\rho} = (q^{\lambda_i - i + \frac{1}{2}})_{i \geq 1}$ associated to a partition λ . The i -th power sum function is equal to

$$(10) \quad p_i(q^{\lambda+\rho}) = \underbrace{\sum_{j=1}^{l(\lambda)} (q^{i(\lambda_j - j + \frac{1}{2})} - q^{i(-j + \frac{1}{2})})}_{\star} + \frac{1}{[i]} \quad (i \in \mathbb{Z} \setminus \{0\}).$$

It turns out to be written as the matrix element of the operator $\mathcal{E}_0(i)$,

$$\mathcal{E}_0(i) = \sum_{k \in \mathbb{Z} + \frac{1}{2}} q^{ik} E_{k,k} + \frac{1}{[i]} \quad (i \in \mathbb{Z} \setminus \{0\}).$$

Lemma 6.1.

$$p_i(q^{\lambda+\rho}) = \langle v_\lambda | \mathcal{E}_0(i) | v_\lambda \rangle = -\langle v_{\lambda^t} | \mathcal{E}_0(-i) | v_{\lambda^t} \rangle.$$

The lemma implies that

$$p_\nu(q^{\lambda+\rho}) = \langle v_\lambda | \mathcal{E}_0(\nu) | v_\lambda \rangle, \quad p_\nu(q^{\lambda^t+\rho}) = (-1)^{l(\nu)} \langle v_\lambda | \mathcal{E}_0(-\nu) | v_\lambda \rangle,$$

where

$$(11) \quad \mathcal{E}_0(\pm\nu) := \mathcal{E}_0(\pm\nu_1) \cdots \mathcal{E}_0(\pm\nu_{l(\nu)}).$$

Proof. Note that the state $|v_\lambda\rangle$ is written as

$$|v_\lambda\rangle = \psi_{\lambda_1 - \frac{1}{2}} \cdots \psi_{\lambda_{l(\lambda)} - l(\lambda) + \frac{1}{2}} \psi_{-l(\lambda) + \frac{1}{2}}^* \cdots \psi_{-\frac{1}{2}}^* |0\rangle.$$

Therefore the action of $\mathcal{E}_0(i)$ on $|v_\lambda\rangle$ follows from the commutation relations:

$$[\mathcal{E}_0(i), \psi_k] = q^{ik} \psi_k, \quad [\mathcal{E}_0(i), \psi_k^*] = -q^{ik} \psi_k^*.$$

Thus we have

$$\mathcal{E}_0(i) |v_\lambda\rangle = p_i(q^{\lambda+\rho}) |v_\lambda\rangle.$$

So if we take the pairing with $\langle v_\lambda |$, we obtain the first line.

To show the second, we need the next identities. The term (\star) in (10) is written in three ways:

$$\begin{aligned} (\star) &= \sum_{j=1}^{l(\lambda)} (q^{i(\lambda_j - j + \frac{1}{2})} - q^{i(-j + \frac{1}{2})}) \\ &= \sum_{j=1}^{r(\lambda)} (q^{i(\lambda_j - j + \frac{1}{2})} - q^{-i(\lambda_j^t - j + \frac{1}{2})}) \\ &= - \sum_{j=1}^{l(\lambda^t)} (q^{-i(\lambda_j^t - j + \frac{1}{2})} - q^{-i(-j + \frac{1}{2})}). \end{aligned}$$

In the second line, $r(\lambda)$ denotes the number of diagonal boxes in the Young diagram of λ . This implies

$$p_i(q^{\lambda+\rho}) = -p_i(q^{\lambda^t+\rho})|_{q \rightarrow q^{-1}} = -\langle v_{\lambda^t} | \mathcal{E}_0(-i) | v_{\lambda^t} \rangle.$$

□

6.1.2. Transposition operator

We define the *transposition operator* R as follows. The actions of R on the charge zero subspace $\Lambda_0^{\frac{\infty}{2}} V$ and its dual space are defined by

$$R|v_{\lambda}\rangle = R|v_{\lambda^t}\rangle, \quad \langle v_{\lambda}|R = \langle v_{\lambda^t}|.$$

The actions of R on the fermions are determined by compatibility¹:

$$R\psi_k R^{-1} = (-1)^{k-\frac{1}{2}} \psi_{-k}^*, \quad R\psi_k^* R^{-1} = (-1)^{-k+\frac{1}{2}} \psi_{-k} \quad \left(k \in \mathbb{Z} + \frac{1}{2}\right).$$

Therefore the actions on other operators are as follows:

$$\begin{aligned} RE_{i,j}R^{-1} &= (-1)^{i-j+1} E_{-j,-i}, \\ R\mathcal{E}_c(n)R^{-1} &= (-1)^{c+1} \mathcal{E}_c(-n) \quad (c, n) \neq (0, 0), \\ R\alpha_m R^{-1} &= (-1)^{m+1} \alpha_m \quad (m \in \mathbb{Z}, m \neq 0), \\ R\mathcal{F}_2 R^{-1} &= -\mathcal{F}_2, \quad RHR^{-1} = H. \end{aligned} \tag{12}$$

The action on a bosonic state $|\mu\rangle$ is obtained from the third of (12):

$$R|\mu\rangle = (-1)^{l(\mu)+|\mu|} |\mu\rangle. \tag{13}$$

§6.2. Three point function

The three point function $C_{\lambda^1, \lambda^2, \lambda^3}(q)$ is written in the operator formalism as follows.

¹One could check the compatibility by using the following expressions:

$$|v_{\lambda}\rangle = \prod_{i=1}^{r(\lambda)} ((-1)^{b_i-\frac{1}{2}} \psi_{a_i} \psi_{-b_i}^*) |0\rangle, \quad \langle v_{\lambda}| = \langle 0| \prod_{i=1}^{r(\lambda)} ((-1)^{b_i-\frac{1}{2}} \psi_{b_i} \psi_{a_i}^*)$$

where $r(\lambda) = \#$ (diagonal boxes in λ), $a_i = \lambda_i - i + \frac{1}{2}$, $b_i = \lambda_i^t - i + \frac{1}{2}$.

Lemma 6.2.

$$C_{\lambda^1, \lambda^2, \lambda^3}(q) = \sum_{\substack{\mu \in \mathcal{P} \\ |\mu| \leq |\lambda^1|, |\lambda^3|}} \sum_{\substack{\nu^1, \nu^2, \nu^3 \in \mathcal{P}; \\ |\nu^2| = |\lambda^2|, \\ |\nu^1| = |\lambda^1| - |\mu|, \\ |\nu^3| = |\lambda^3| - |\mu|}} \frac{(-1)^{l(\nu^1)}}{z_\mu z_{\nu^1} z_{\nu^2} z_{\nu^3} [\nu^2]} \\ \times \langle v_{\lambda^1} | \nu^1 \cup \mu \rangle \langle v_{\lambda^{3t}} | q^{-\mathcal{F}_2} | \nu^3 \cup \mu \rangle \langle v_{\lambda^2} | \mathcal{E}_0(-\nu^1) \mathcal{E}_0(\nu^3) | \nu^2 \rangle.$$

Note that the cyclic symmetry with respect to $\lambda^1, \lambda^2, \lambda^3$ is not manifest in this expression.

Proof. The skew-Schur function in the variables $x = (x_1, x_2, \dots)$ is written as

$$s_{\mu/\eta}(x) = \sum_{\substack{\mu'; |\mu'| = |\mu| - |\eta|, \\ \eta'; |\eta'| = |\eta|}} \frac{p_{\mu'}(x)}{z_{\mu'} z_{\eta'}} \langle v_\mu | \mu' \cup \eta' \rangle \langle \eta' | v_\eta \rangle.$$

Therefore,

$$s_{\lambda^2}(q^\rho) = \sum_{\nu^2 \in \mathcal{P}; |\nu^2| = |\lambda^2|} \frac{1}{z_{\nu^2}} \langle v_{\lambda^2} | \nu^2 \rangle.$$

And

$$\begin{aligned} & \sum_{\eta} s_{\lambda^1/\eta}(q^{\lambda^{2t} + \rho}) s_{\lambda^{3t}/\eta}(q^{\lambda^2 + \rho}) \\ &= \sum_{d=0}^{\min\{|\lambda^1|, |\lambda^3|\}} \sum_{\eta \in \mathcal{P}_d} s_{\lambda^1/\eta}(q^{\lambda^{2t} + \rho}) s_{\lambda^{3t}/\eta}(q^{\lambda^2 + \rho}) \\ &= \sum_{d=0}^{\min\{|\lambda^1|, |\lambda^3|\}} \sum_{\substack{\nu^1; |\nu^1| = |\lambda^1| - d, \\ \nu^3; |\nu^3| = |\lambda^3| - d, \\ \mu, \mu' \in \mathcal{P}_d}} \frac{p_{\nu^1}(q^{\lambda^{2t} + \rho}) p_{\nu^3}(q^{\lambda^2 + \rho})}{z_{\nu^1} z_{\nu^3} z_\mu z_{\mu'}} \langle v_{\lambda^1} | \nu^1 \cup \mu \rangle \langle v_{\lambda^{3t}} | \nu^3 \cup \mu' \rangle \\ & \times \underbrace{\sum_{\eta \in \mathcal{P}_d} \langle \mu | v_\eta \rangle \langle v_\eta | \mu' \rangle}_{= \langle \mu | \mu' \rangle = z_\mu \delta_{\mu, \mu'}}. \end{aligned}$$

By Lemma 6.1, this is equal to

$$\sum_{\nu^1, \nu^3, \mu} \frac{(-1)^{l(\nu^1)}}{z_{\nu^1} z_{\nu^3} z_\mu} \langle v_{\lambda^1} | \nu^1 \cup \mu \rangle \langle v_{\lambda^{3t}} | \nu^3 \cup \mu \rangle \langle v_{\lambda^2} | \mathcal{E}_0(-\nu^1) \mathcal{E}_0(\nu^3) | v_{\lambda^2} \rangle.$$

The factor $q^{\frac{\kappa(\lambda_3)}{2}}$ is written as follows:

$$q^{\frac{\kappa(\lambda_3)}{2}} = \langle v_{\lambda_{3t}} | q^{-\mathcal{F}_2} | v_{\lambda_{3t}} \rangle.$$

Combining the above expressions, we obtain the lemma. □

§6.3. Flag-order

Since the expression in Lemma 6.2 is not cyclic symmetric, we have to specify a counterclockwise order of three flags for every trivalent vertex v . For this reason, we introduce the notion of *flag order* of a GT graph Γ .

Let $F(v)$ be the set of three flags incident on a trivalent vertex $v \in V_3(\Gamma)$ and $F'_3(\Gamma)$ be the set of flags whose vertices are trivalent:

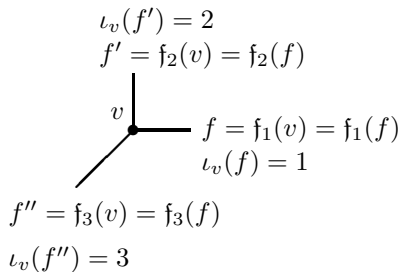
$$F'_3(\Gamma) = \bigcup_{v \in V_3(\Gamma)} F(v).$$

(Note that $F_3(\Gamma) \subset F'_3(\Gamma)$.) A *flag-order* of a GT graph Γ is a map ι from $F'_3(\Gamma)$ to $\{1, 2, 3\}$ satisfying the following conditions: for every $v \in V_3(\Gamma)$,

1. $\iota_v : F(v) \rightarrow \{1, 2, 3\}$ is one-to-one;
2. the disposition of three flags $\iota_v^{-1}(1), \iota_v^{-1}(2), \iota_v^{-1}(3)$ is counterclockwise,

where ι_v is the restriction of ι to $F(v)$.

For convenience of writing, we set $f_i(v) := \iota_v^{-1}(i)$ ($i = 1, 2, 3$) for $v \in V_3(\Gamma)$. We also set $f_i(f) := f_i(v)$ for $f \in F_3(\Gamma)$ where v is the vertex of f .



A flag-order of a GT graph Γ is not unique; there are $3^{\#V_3(\Gamma)}$ flag-orders. In the rest of Section 6, we fix one GT graph Γ and one flag-order ι .

$(\iota(f), \iota(f'))$	N_e	$L_e(\vec{\mu}, \vec{\nu})$	$K_e(\vec{\mu}, \vec{\nu})$
(1, 1)	$-n_e$	$l(\mu) + l(\nu')$	$\langle \mu \cup \nu q^{N_e \mathcal{F}_2} \mu' \cup \nu' \rangle$
(1, 2)	$-n_e$	$l(\mu)$	$\langle \mu \cup \nu \mathcal{E}_0(-\nu^1) \mathcal{E}_0(\nu^3) q^{N_e \mathcal{F}_2} \nu' \rangle$
(1, 3)	$n_e - 1$	$l(\nu)$	$\langle \mu \cup \nu q^{N_e \mathcal{F}_2} \mu' \cup \nu' \rangle$
(2, 1)	$-n_e$	$l(\nu) + l(\nu') + l(\nu^1) + l(\nu^3)$	$\langle \nu \mathcal{E}_0(-\nu^3) \mathcal{E}_0(\nu^1) q^{N_e \mathcal{F}_2} \mu' \cup \nu' \rangle$
(2, 2)	$-n_e$	$l(\nu) + l(\nu^1) + l(\nu^3)$	$\langle \nu \mathcal{E}_0(-\nu^3 \cup \nu^1) \mathcal{E}_0(\nu^1 \cup \nu^3) q^{N_e \mathcal{F}_2} \nu' \rangle$
(2, 3)	$n_e - 1$	0	$\langle \nu \mathcal{E}_0(-\nu^1) \mathcal{E}_0(\nu^3) q^{N_e \mathcal{F}_2} \mu' \cup \nu' \rangle$
(3, 1)	$-n_e - 1$	$l(\nu')$	$\langle \mu \cup \nu q^{N_e \mathcal{F}_2} \mu' \cup \nu' \rangle$
(3, 2)	$-n_e - 1$	0	$\langle \mu \cup \nu \mathcal{E}_0(-\nu^1) \mathcal{E}_0(\nu^3) q^{N_e \mathcal{F}_2} \nu' \rangle$
(3, 3)	n_e	$l(\mu) + l(\nu)$	$\langle \mu \cup \nu q^{N_e \mathcal{F}_2} \mu' \cup \nu' \rangle$

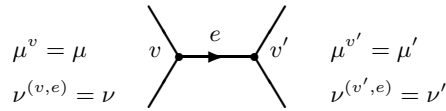


Table 1. $N_e, L_e(\vec{\mu}, \vec{\nu})$ and $K_e(\vec{\mu}, \vec{\nu})$. μ, μ', ν, ν' are as above. When $\iota((v, e)) = 2$, $\nu^1 = \nu^{f_1(v)}$ and when $\iota((v', e)) = 2$, $\nu'^1 = \nu'^{f_1(v')}$, $\nu'^3 = \nu'^{f_3(v')}$.

§6.4. Partition function

The goal of this subsection is to rewrite the partition function $\mathcal{Z}_{\vec{d}}^\Gamma(q)$ in the operator formalism. To state the result, we introduce the following notations.

- We use the symbol $\vec{\mu}$ and $\vec{\nu}$ for tuples of partitions $\vec{\mu} = (\mu^v)_{v \in V_3(\Gamma)}$ and $\vec{\nu} = (\nu^f)_{f \in F_3(\Gamma)}$. A Γ -set of degree \vec{d} is a pair $(\vec{\mu}, \vec{\nu})$ satisfying the following conditions:
 1. for $f = (v, e) \in F_3(\Gamma)$, $|\mu^v| + |\nu^f| = d_e$ if $\iota(f) = 1, 3$;
 2. for $f = (v, e) \in F_3(\Gamma)$, $|\nu^f| = d_e$ if $\iota(f) = 2$;
 3. for $v \in V_3(\Gamma)$, $|\mu^v| = 0$ if $f_1(v) \notin F_3(\Gamma)$ or $f_3(v) \notin F_3(\Gamma)$.
- Integers $N_e, L_e(\vec{\mu}, \vec{\nu})$ and matrix elements $K_e(\vec{\mu}, \vec{\nu})$ ($e \in E_3(\Gamma)$) are defined as in Table 1.

With these notations, $\mathcal{Z}_{\vec{d}}^\Gamma(q)$ is written as follows:

Lemma 6.3.

$$\begin{aligned}
 (14) \quad \mathcal{Z}_{\vec{d}}^\Gamma(q) &= \sum_{\substack{(\vec{\mu}, \vec{\nu}); \Gamma\text{-set} \\ \text{of degree } \vec{d}}} \frac{(-1)^{L_1(\vec{\mu}, \vec{\nu}) + L_2(\vec{\mu}, \vec{\nu})}}{z_{\vec{\mu}} z_{\vec{\nu}}} \prod_{\substack{f \in F_3(\Gamma), \\ \iota(f)=2}} \frac{1}{[\nu^f]} \prod_{\substack{f \in F_3(\Gamma); \\ \iota(f)=1,3, \\ f_2(f) \notin F_3(\Gamma)}} \frac{1}{[\nu^f]} \\
 &\quad \times \prod_{e \in E_3(\Gamma)} K_e(\vec{\mu}, \vec{\nu}).
 \end{aligned}$$

Here $L_1(\vec{\mu}, \vec{\nu})$ and $L_2(\vec{\mu}, \vec{\nu})$ are

$$L_1(\vec{\mu}, \vec{\nu}) = \sum_{e \in E_3(\Gamma)} L_e(\vec{\mu}, \vec{\nu}) + \sum_{\substack{f \in F_3(\Gamma); \\ \iota(f)=1, \\ f_2(f) \notin F_3(\Gamma)}} l(\nu^f), \quad L_2(\vec{\mu}, \vec{\nu}) = \sum_{e \in E_3(\Gamma)} N_e d_e,$$

and $z_{\vec{\mu}} = \prod_{v \in V_3(\Gamma)} z_{\mu^v}$, $z_{\vec{\nu}} = \prod_{f \in F_3(\Gamma)} z_{\nu^f}$.

The details of the RHS of (14) depends on the choice of the flag order ι (cf. Example 6.4-(i)). However, the rest of the proof of Proposition 3.3 proceeds completely in the same way.

Since (14) is quite involved, we compute it for the examples of Subsection 2.5. Then we describe a proof in the case of Example 2.5-3. The proof for a general GT graph is the same and left to the reader.

Example 6.4. (i) Let Γ be the GT graph of Example 2.5-1.

We take the flag order $\iota((v, e)) = 1, \iota((v', e)) = 3$. The value of ι for other flags are determined by the counterclockwise condition. The partition function with respect to this flag order is:

$$\mathcal{Z}_d^\Gamma(q) = \sum_{\nu, \nu' \in \mathcal{P}_d} \frac{(-1)^{d(n+1)}}{z_\nu z_{\nu'} [\nu][\nu']} \langle \nu | q^{(n-1)\mathcal{F}_2} | \nu' \rangle.$$

If we take another flag order $\iota((v, e)) = \iota((v', e)) = 1$, then we obtain another expression:

$$\mathcal{Z}_d^\Gamma(q) = \sum_{\nu, \nu' \in \mathcal{P}_d} \frac{(-1)^{dn+l(\nu)}}{z_\nu z_{\nu'} [\nu][\nu']} \langle \nu | q^{-n\mathcal{F}_2} | \nu' \rangle.$$

In this exmample, one could see that the operator formalism expression depends on the choice of the flag order.

(ii) Let Γ be the GT graph of Example 2.5-2. We take the flag-order

$$\iota(v_{i-1}, e_i) = 3, \quad \iota(v_i, e_i) = 1 \quad (1 \leq i \leq r).$$

We write $\vec{\mu} = (\mu^1, \dots, \mu^r)$ where μ^i is a partition associated to the vertex v^i . We also write $\vec{\nu} = (\vec{\alpha}, \vec{\beta})$, $\vec{\alpha} = (\alpha_1, \dots, \alpha^r)$, $\vec{\beta} = (\beta^1, \dots, \beta^r)$ where α^i and β^i are partitions associated to flags (v_{i-1}, e_i) and (v_i, e_i) (here $v^{-1} = v^r$ is assumed). The condition that a Γ -set $(\vec{\mu}, \vec{\nu})$ is of degree $\vec{d} = (d_1, \dots, d_r)$ is

$$|\mu^{i-1}| + |\alpha^i| = |\mu^i| + |\beta^i| = d_i \quad (1 \leq i \leq r),$$

where $\mu^{-1} = \mu^r$. Then

$$\mathcal{Z}_{\vec{d}}^\Gamma(q) = (-1)^{\sum_{i=1}^r \gamma_i d_i} \sum_{(\mu^i, \alpha^i, \beta^i)} \frac{1}{z_{\vec{\mu}} z_{\vec{\alpha}} z_{\vec{\beta}} [\vec{\alpha}] [\vec{\beta}]} \prod_{i=1}^r \langle \mu^{i-1} \cup \alpha^i | q^{-(\gamma_i+2)\mathcal{F}_2} | \mu^i \cup \beta^i \rangle.$$

(For any tuple of partitions $\vec{\lambda} = (\lambda^1, \dots, \lambda^l)$, $z_{\vec{\lambda}} = z_{\lambda^1} \dots z_{\lambda^l}$ and $[\lambda] = [\lambda^1] \dots [\lambda^l]$.)

(iii) Let Γ be the GT graph in Example 2.5-3. We take the following flag order ι :

$$\iota(v_1, e_1) = \iota(v_2, e_2) = \iota(v_3, e_3) = 1, \quad \iota(v_4, e_4) = 2.$$

We write $\vec{\mu} = (\mu^1, \mu^2, \mu^3, \mu^4)$ where μ^i is a partition associated to v_i . We also write $\vec{\nu} = (\nu^{1,1}, \nu^{1,2}, \nu^{1,4}, \nu^{2,2}, \nu^{2,3}, \nu^{3,3}, \nu^{3,1}, \nu^{4,4})$ where $\nu^{i,j}$ is a partition associated to the flag (v_i, e_j) . The conditions for a Γ -set $(\vec{\mu}, \vec{\nu})$ to be of degree $\vec{d} = (d_1, d_2, d_3, d_4)$ are

$$\begin{aligned} |\mu^1| + |\nu^{1,1}| &= |\mu^3| + |\nu^{3,1}| = d_1, & |\mu^2| + |\nu^{2,2}| &= |\mu^1| + |\nu^{1,2}| = d_2, \\ |\mu^3| + |\nu^{3,3}| &= |\mu^2| + |\nu^{2,3}| = d_3, & |\nu^{1,4}| &= |\nu^{4,4}| = d_4, & |\mu^4| &= 0. \end{aligned}$$

The partition function $\mathcal{Z}_{\vec{d}}^\Gamma(q)$ is written as:

$$\begin{aligned} (15) \quad \mathcal{Z}_{\vec{d}}^\Gamma(q) &= (-1)^{\sum_{i=1}^3 (b_i+1)d_i + b_4 d_4} \sum_{\substack{(\vec{\mu}, \vec{\nu}); \Gamma\text{-set} \\ \text{of degree } \vec{d}}} \frac{(-1)^{l(\nu^{1,2}) + l(\nu^{1,4})}}{z_{\vec{\mu}} z_{\vec{\nu}} [\nu^{1,4}] [\nu^{2,2}] [\nu^{2,3}] [\nu^{3,1}] [\nu^{3,3}] [\nu^{4,4}]} \\ &\times \langle \mu^1 \cup \nu^{1,1} | q^{-(b_1+1)\mathcal{F}_2} | \mu^3 \cup \nu^{3,1} \rangle \langle \mu^2 \cup \nu^{2,2} | q^{-(b_2+1)\mathcal{F}_2} | \mu^1 \cup \nu^{1,2} \rangle \\ &\times \langle \mu^3 \cup \nu^{3,3} | q^{-(b_3+1)\mathcal{F}_2} | \mu^1 \cup \nu^{2,3} \rangle \langle \nu^{1,4} | \mathcal{E}_0(-\nu^{1,1}) \mathcal{E}_0(\nu^{1,2}) q^{-b_4 \mathcal{F}_2} | \nu^{4,4} \rangle. \end{aligned}$$

Now we describe a proof of Lemma 6.3 in the last example.

We apply Lemma 6.2 to $C_{\vec{\lambda}_{v_i}}(q)$ ($1 \leq i \leq 4$):

$$\begin{aligned} C_{\vec{\lambda}_{v_1}}(q) &= \sum_{\mu^1} \sum_{\nu^{1,1}, \nu^{1,4}, \nu^{1,2}} \frac{(-1)^{l(\nu^{1,1})}}{z_{\mu^1} z_{\nu^{1,1}} z_{\nu^{1,4}} z_{\nu^{1,2}} [\nu^{1,4}]} \\ &\times \langle v_{\lambda^{1t}} | \nu^{1,1} \cup \mu^1 \rangle \langle v_{\lambda^{2t}} | q^{-\mathcal{F}_2} | \nu^{1,2} \cup \mu^1 \rangle \langle v_{\lambda^4} | \mathcal{E}_0(-\nu^{1,1}) \mathcal{E}_0(\nu^{1,2}) | \nu^{1,4} \rangle. \end{aligned}$$

$$\begin{aligned}
 C_{\bar{\lambda}_{v_2}}(q) &= \sum_{\mu^2} \sum_{\nu^{2,2}\nu^{2,3}} \frac{(-1)^{l(\nu^{2,2})}}{z_{\mu^2} z_{\nu^{2,2}} z_{\nu^{2,3}}} \langle v_{\lambda^{2t}} | \nu^{2,2} \cup \mu^2 \rangle \langle v_{\lambda^{3t}} | q^{-\mathcal{F}_2} | \nu^{2,3} \cup \mu^2 \rangle \\
 &\quad \times \langle 0 | \mathcal{E}_0(-\nu^{2,2}) \mathcal{E}_0(\nu^{2,3}) | 0 \rangle \\
 &= \sum_{\mu^2} \sum_{\nu^{2,2}\nu^{2,3}} \frac{1}{z_{\mu^2} z_{\nu^{2,2}} z_{\nu^{2,3}} [\nu^{2,2}] [\nu^{2,3}]} \langle v_{\lambda^{2t}} | \nu^{2,2} \cup \mu^2 \rangle \\
 &\quad \times \langle v_{\lambda^{3t}} | q^{-\mathcal{F}_2} | \nu^{2,3} \cup \mu^2 \rangle. \\
 C_{\bar{\lambda}_{v_3}}(q) &= \sum_{\mu^3} \sum_{\nu^{3,3}\nu^{3,1}} \frac{1}{z_{\mu^3} z_{\nu^{3,3}} z_{\nu^{3,1}} [\nu^{3,3}] [\nu^{3,1}]} \langle v_{\lambda^{3t}} | \nu^{3,3} \cup \mu^3 \rangle \\
 &\quad \times \langle v_{\lambda^{1t}} | q^{-\mathcal{F}_2} | \nu^{3,1} \cup \mu^3 \rangle. \\
 C_{\bar{\lambda}_{v_4}}(q) &= \sum_{\nu^{4,4}} \frac{1}{z_{\nu^{4,4}} [\nu^{4,4}]} \langle v_{\lambda^{4t}} | \nu^{4,4} \rangle.
 \end{aligned}$$

The factor $q^{b_i \frac{\kappa(\lambda^i)}{2}}$ ($1 \leq i \leq 4$) is equal to

$$q^{b_i \frac{\kappa(\lambda^i)}{2}} = \langle v_{\lambda^i} | q^{b_i \mathcal{F}_2} | v_{\lambda^i} \rangle = \langle v_{\lambda^{it}} | q^{-b_i \mathcal{F}_2} | v_{\lambda^{it}} \rangle.$$

Next we perform the summation over λ^2 :

$$\begin{aligned}
 &\sum_{\lambda^2} \langle v_{\lambda^{2t}} | q^{-b_2 \mathcal{F}_2} | v_{\lambda^{2t}} \rangle \langle v_{\lambda^{2t}} | q^{-\mathcal{F}_2} | \nu^{1,2} \cup \mu^1 \rangle \langle v_{\lambda^{2t}} | \nu^{2,2} \cup \mu^2 \rangle \\
 &= \langle \mu^2 \cup \nu^{2,2} | q^{-(b_2+1)\mathcal{F}_2} | \nu^{1,2} \cup \mu^1 \rangle
 \end{aligned}$$

The summations over λ^1 and λ^3 are similar. The summation over λ^4 is:

$$\begin{aligned}
 &\sum_{\lambda^4} \langle v_{\lambda^4} | q^{b_4 \mathcal{F}_2} | v_{\lambda^4} \rangle \langle v_{\lambda^4} | \mathcal{E}_0(-\nu^{1,1}) \mathcal{E}_0(\nu^{1,2}) | \nu^{1,4} \rangle \langle v_{\lambda^{4t}} | \nu^{4,4} \rangle \\
 &= \langle \nu^{1,4} | \mathcal{E}_0(-\nu^{1,1}) \mathcal{E}_0(\nu^{1,2}) q^{b_4 \mathcal{F}_2} R | \nu^{4,4} \rangle \\
 &= (-1)^{d_4 + l(\nu^{1,4}) + l(\nu^{1,1}) + l(\nu^{1,2})} \langle \nu^{1,4} | \mathcal{E}_0(-\nu^{1,2}) \mathcal{E}_0(\nu^{1,1}) q^{-b_4 \mathcal{F}_2} | \nu^{4,4} \rangle.
 \end{aligned}$$

In the middle line, R is the transposition operator introduced in Subsection 6.1.2. In passing to the last line, we have moved R to the left using (12) (13) and then exchanged $\mathcal{E}_0(\nu^{1,1})$ and $\mathcal{E}_0(\nu^{1,2})$ using the fact that $\mathcal{E}_0(i)$ and $\mathcal{E}_0(j)$ ($i, j \in \mathbb{Z} \setminus \{0\}$) commute with each other. Combining the above expressions, we obtain (15).

Lemma 6.3 is proved for a general GT graph completely in the same manner.

§6.5. Graph expression

In this subsection, we will express $\mathcal{Z}_d^\Gamma(q)$ as the sum over a certain set of labeled graphs.

Before proceeding, we explain briefly the graph expression introduced in [K], Section 3.2.

6.5.1. Graph expression of VEV

Let \vec{c} and \vec{n} be sequences of integers of the same length l :

$$\vec{c} = (c_1, \dots, c_l), \quad \vec{n} = (n_1, \dots, n_l)$$

such that $(c_i, n_i) \neq (0, 0)$ for $(1 \leq i \leq l)$ and $|\vec{c}| := \sum_{i=1}^l c_i = 0$.

The vacuum expectation value

$$(16) \quad \langle \mathcal{E}_{c_1}(n_1) \mathcal{E}_{c_2}(n_2) \cdots \mathcal{E}_{c_l}(n_l) \rangle$$

is computed by applying the commutation relation:

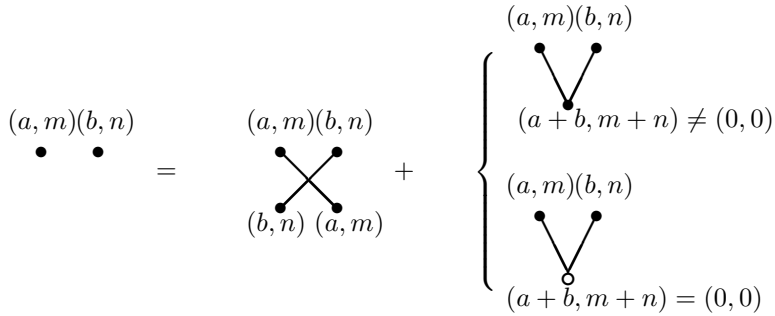
$$\mathcal{E}_a(m) \mathcal{E}_b(n) = \mathcal{E}_b(n) \mathcal{E}_a(m) + \begin{cases} [an - bm] \mathcal{E}_{a+b}(m+n) & (a+b, m+n) \neq (0, 0) \\ a & (a+b, m+n) = (0, 0) \end{cases}$$

For the algorithm to be well-defined, we set the rule that the commutation relation is applied to the rightmost neighboring pair $(\mathcal{E}_a(m), \mathcal{E}_b(n))$ such that $a \geq 0$ and $b < 0$. We also use the relations:

$$\langle \cdots \mathcal{E}_a(m) \rangle = \begin{cases} 0 & (a > 0) \\ \langle \cdots \rangle \frac{1}{|m|} & (a = 0) \end{cases}$$

$$\langle \mathcal{E}_b(n) \cdots \rangle = 0 \quad (b < 0), \quad \text{and } \langle 1 \rangle = 1.$$

We associate to the commutation relation the drawing:

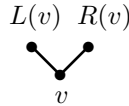


Then graphs are generated over the course of the calculation.

Definition 6.5. $\text{Graph}^\bullet(\vec{c}, \vec{n})$ is the set of graphs generated by this procedure.

Every graph $F \in \text{Graph}_a^\bullet(\vec{c}, \vec{n})$ is the graph union of (binary rooted) trees. We call F a *VEV forest* and a connected component of F a *VEV tree*. The two component label of every vertex v is denoted by (c_v, n_v) and called the *vertex label* of v . We add another label called the *leaf index* to every leaf of F : leaves of F correspond one-to-one to components (c_i, n_i) of (\vec{c}, \vec{n}) ; so the leaf index of a leaf corresponding to (c_i, n_i) is defined to be i .

By construction, every graph $F \in \text{Graph}^\bullet(\vec{c}, \vec{n})$ represents one term in the final result of the computation of the VEV (16). The corresponding term can be recovered from F as follows. For each VEV tree T in F , let $V_2(T)$ be the set of vertices which have two adjacent vertices at the upper level. The upper left and right vertices adjacent to $v \in V_2(T)$ are denoted by $L(v)$ and $R(v)$, respectively.



For a vertex $v \in V_2(T)$, define

$$\xi_v = c_{L(v)}n_{R(v)} - c_{R(v)}n_{L(v)}.$$

We define the *amplitude* $\mathcal{A}(F)$ by

$$\mathcal{A}(F) = \prod_{T: \text{VEV tree in } F} \mathcal{A}(T),$$

$$\mathcal{A}(T) = \begin{cases} \prod_{v \in V_2(T)} [\xi_v] / [n_{\text{root}}] & (n_{\text{root}} \neq 0) \\ c_{L(\text{root})} \prod_{\substack{v \in V_2(T), \\ v \neq \text{root}}} [\xi_v] & (n_{\text{root}} = 0) \end{cases}$$

The amplitude $\mathcal{A}(F)$ is exactly the term corresponding to F . Thus we have

Proposition 6.6.

$$\langle \mathcal{E}_{c_1}(n_1) \mathcal{E}_{c_2}(n_2) \cdots \mathcal{E}_{c_l}(n_l) \rangle = \sum_{F \in \text{Graph}^\bullet(\vec{c}, \vec{n})} \mathcal{A}(F).$$

Proof. Clear. □

The amplitude of a VEV tree T admits an important pole structure. Let us define $m_v := \text{gcd}(c_v, n_v)$ for a leaf v of a VEV tree T and

$$(17) \quad m(T) := \text{gcd}(\{m_v\}_{v: \text{leaf of } T}), \quad \mathcal{B}(T) := \frac{\mathcal{A}(T)}{\prod_{v: \text{leaf}} [m_v]}.$$

Proposition 6.7. *Let T be a VEV tree.*

1. *There exists $g_T \in \mathbb{Z}$ and $f_T(t) \in \mathbb{Z}[t]$ such that*

$$\mathcal{B}(T) = \begin{cases} \frac{g_T}{t_{m(T)}} + f_T(t_{m(T)}) & (m(T) \text{ odd or } n_{\text{root}}/m(T) \text{ even}) \\ \frac{g_T}{t_{m(T)}} \left(1 + \frac{t_{m(T)}/2}{2}\right) + f_T(t_{m(T)}) & (m(T) \text{ even and } n_{\text{root}}/m(T) \text{ odd}) \end{cases}$$

2. *Moreover,*

$$g_T = g_{T_{(0)}} \cdot m(T)^{\#\text{leaves}-1}.$$

Here $T_{(0)}$ is the VEV tree which is the same as T except all the vertex-labels are multiplied by $1/m(T)$.

See Proposition 6.1 and Section 6.3.2 [K] for a proof.

6.5.2. Graph expression

Now we will express $\mathcal{Z}_d^\Gamma(q)$ as the sum over a certain set of labeled graphs. This takes two steps.

The first step is to write the matrix elements $K_e(\vec{\mu}, \vec{\nu})$ as the sum over a set of labeled graphs using Proposition 6.6.

For partitions η, ξ, ξ', η' such that $|\eta| = |\eta'|$, we define $\text{Graph}_a^\bullet(\eta, -\xi, \xi', \eta')$ to be the set $\text{Graph}^\bullet(\vec{c}, \vec{n})$ with

$$\begin{aligned} \vec{c} &= (\eta_{l(\eta)}, \dots, \eta_1, \underbrace{0, \dots, 0}_{l(\xi)+l(\xi')}, -\eta'_1, \dots, -\eta'_{l(\eta')}), \\ \vec{n} &= (\underbrace{0, \dots, 0}_{l(\eta)}, -\xi_1, \dots, -\xi_{l(\xi)}, \xi'_1, \dots, \xi'_{l(\xi')}, a\eta'_1, \dots, a\eta'_{l(\eta')}). \end{aligned}$$

Definition 6.8. We set

$$\text{Graph}_e^\bullet(\vec{\mu}, \vec{\nu}) := \text{Graph}_{N_e}^\bullet(\eta_e, -\xi_e, \xi'_e, \eta'_e) \quad (e \in E_3(\Gamma)).$$

Here $\eta_e, \xi_e, \xi'_e, \eta'_e$ are partitions listed in Table 2.

Then the matrix element $K_e(\vec{\mu}, \vec{\nu})$ is written as follows:

Lemma 6.9.

$$K_e(\vec{\mu}, \vec{\nu}) = \sum_{F \in \text{Graph}_e^\bullet(\vec{\mu}, \vec{\nu})} \mathcal{A}(F).$$

$(\iota(f), \iota(f'))$	η_e	ξ_e	ξ'_e	η'_e
(1, 1)	$\mu \cup \nu$	\emptyset	\emptyset	$\mu' \cup \nu'$
(1, 2)	$\mu \cup \nu$	ν'^1	ν'^3	ν'
(1, 3)	$\mu \cup \nu$	\emptyset	\emptyset	$\mu' \cup \nu'$
(2, 1)	ν	ν^3	ν^1	$\mu' \cup \nu'$
(2, 2)	ν	$\nu^3 \cup \nu'^1$	$\nu^1 \cup \nu'^3$	ν'
(2, 3)	ν	ν^1	ν^3	$\mu' \cup \nu'$
(3, 1)	$\mu \cup \nu$	\emptyset	\emptyset	$\mu' \cup \nu'$
(3, 2)	$\mu \cup \nu$	ν'^1	ν'^3	ν'
(3, 3)	$\mu \cup \nu$	\emptyset	\emptyset	$\mu' \cup \nu'$

Table 2. $\eta_e, \xi_e, \xi'_e, \eta'_e$ for $e \in E_3(\Gamma)$. The notation is the same as in Table 1.

Proof. Since ([K], eq (3))

$$q^{a\mathcal{F}_2} \alpha_{-i} q^{-a\mathcal{F}_2} = \mathcal{E}_{-i}(ai),$$

a matrix element of the form

$$\langle \eta | \mathcal{E}_0(-\xi) \mathcal{E}_0(\xi') q^{a\mathcal{F}_2} | \eta' \rangle \quad (\eta, \xi, \xi', \eta' : \text{partitions}, |\eta| = |\eta'|)$$

is equal to the VEV

$$\langle \mathcal{E}_{n(\eta)}(0) \dots \mathcal{E}_{n_1}(0) \mathcal{E}_0(\xi) \mathcal{E}_0(-\xi) \mathcal{E}_{-\eta'_1}(a\eta'_1) \dots \mathcal{E}_{-\eta'_l(\eta')}(a\eta'_l(\eta')) \rangle.$$

Therefore the lemma follows from Proposition 6.6. □

The second step is to construct a new type of labeled graphs so that $\mathcal{Z}_d^\Gamma(q)$ becomes the sum of quantities over these graphs.

Let us rewrite $\mathcal{Z}_d^\Gamma(q)$ using $\mathcal{B}(F)$ introduced in (17): $\mathcal{B}(F) = \mathcal{A}(F) / [\eta][\xi][\xi'][\eta']$ for $F \in \text{Graph}_d^\bullet(\eta, -\xi, \xi', \eta')$. By Lemmas 6.3 and 6.9,

$$\mathcal{Z}_d^\Gamma(q) = \sum_{\substack{(\vec{\mu}, \vec{\nu}); \Gamma\text{-set} \\ \text{of degree } \vec{d}}} \frac{(-1)^{L_1(\vec{\mu}, \vec{\nu}) + L_2(\vec{\mu}, \vec{\nu})}}{z_{\vec{\mu}} z_{\vec{\nu}}} \underbrace{\prod_{v \in V_3(\Gamma)} [\mu^v]^2 \prod_{f \in \hat{F}_3(\Gamma)} [\nu^f]^2}_{*} \sum_{(F_e)} \prod_{e \in E_3(\Gamma)} \mathcal{B}(F_e).$$

Here

$$\hat{F}_3(\Gamma) := \{f \in F_3(\Gamma) | \iota(f) = 1, 3, f_2(f) \in F_3(\Gamma)\},$$

and the second summation is over the set

$$\prod_{e \in E_3(\Gamma)} \text{Graph}_e^\bullet(\vec{\mu}, \vec{\nu}).$$

So we need to incorporate the factors in (*). Recall that, by construction, leaves in a VEV forest $F_e \in \text{Graph}_e^\bullet(\vec{\mu}, \vec{\nu})$ ($e \in E_3(\Gamma)$) correspond one-to-one to parts of the partitions $\eta_e, \xi_e, \xi'_e, \eta'_e$. Therefore, every leaf corresponds to a part of μ^v ($v \in V_3(\Gamma)$) or ν^f ($f \in F_3(\Gamma)$)[†] (cf. Remark 6.12). For μ^v , there are two leaves associated to every part μ_i^v , the one in F_e such that $(v, e) = f_1(v)$ and the other in $F_{e'}$ such that $(v, e') = f_3(v)$. Similarly, for ν^f with $f \in \hat{F}_3(\Gamma)$, there are two leaves associated to every part ν_i^f of ν^f with $f \in \hat{F}_3(\Gamma)$, the one in F_e such that $f = (v, e)$ and the other in $F_{e'}$ such that $f_2(f) = (v, e')$.

We construct a new graph from

$$(F_e) \in \prod_{e \in E_3(\Gamma)} \text{Graph}_e^\bullet(\vec{\mu}, \vec{\nu})$$

as follows.

1. Assign the label $e \in E_3(\Gamma)$ to each F_e and make the graph union.
2. Join the two leaves associated to μ_i^v ($v \in V_3(\Gamma), 1 \leq i \leq l(\mu^v)$) and attach the label μ_i^v to the new edge. Also join the two leaves associated to ν_i^f ($f \in \hat{F}_3(\Gamma), 1 \leq i \leq l(\nu^f)$) and attach the label ν_i^f to the new edge.

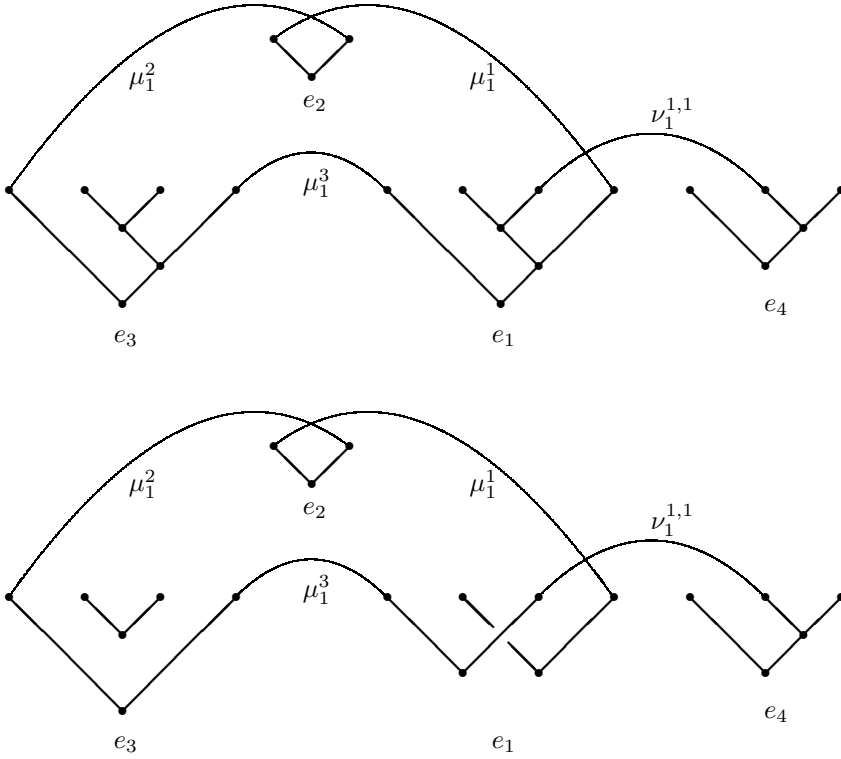
The resulting graph W is a set of VEV forests marked by $e \in E_3(\Gamma)$ and joined through leaves. We call W a *combined forest*. The new edges are called the *bridges*. The label of a bridge b is denoted by $h(b)$.

Definition 6.10. The set of combined forests constructed by the above procedure is denoted by $\text{Comb}_\Gamma^\bullet(\vec{\mu}, \vec{\nu})$. The subset consisting of connected combined forests is denoted by $\text{Comb}_\Gamma^\circ(\vec{\mu}, \vec{\nu})$.

Examples of combined forests are shown in Figure 5. For a combined forest $W \in \text{Comb}_\Gamma^\bullet(\vec{\mu}, \vec{\nu})$, we define

$$\mathcal{H}(W) = (-1)^{L_1(W)+L_2(W)} \prod_{e \in E_3(\Gamma)} \mathcal{B}(F_e) \prod_{b; \text{bridge}} [h(b)]^2,$$

where $L_1(W) = L_1(\vec{\mu}, \vec{\nu})$ and $L_2(W) = L_2(\vec{\mu}, \vec{\nu})$. $\mathcal{H}(W)$ is called the *combined amplitude*.



Γ : the GT graph in Example 2.5-3

$$b_1, b_2, b_3 \neq -1 \text{ and } b_4 \neq 0.$$

$$\mu^1 = \mu^2 = \mu^3 = (1), \mu^4 = \emptyset,$$

$$\nu^{1,1} = \nu^{1,4} = \nu^{2,3} = \nu^{3,3} = \nu^{3,1} = \nu^{4,4} = (1), \nu^{1,2} = \nu^{2,2} = \emptyset.$$

Figure 5. Example of combined forests. (Vertex labels and leaf indices are omitted.)

Proposition 6.11.

$$\mathcal{Z}_d^\Gamma(q) = \sum_{\substack{(\vec{\mu}, \vec{\nu}); \\ \Gamma\text{-set of} \\ \text{degree } \vec{d}}} \frac{1}{z_{\vec{\mu}} z_{\vec{\nu}}} \sum_{W \in \text{Comb}_\Gamma^\bullet(\vec{\mu}, \vec{\nu})} \mathcal{H}(W).$$

Proof. The proposition follows from the definitions of the combined forest and the combined amplitude. □

Remark 6.12. Precisely speaking, the statement (†) is not correct. As an example, consider a VEV forest $F_e \in \text{Graph}_e^\bullet(\vec{\mu}, \vec{\nu})$ where $e \in E_3(\Gamma)$ such that $\eta_e = \mu^v \cup \nu^f$ with $f = (v, e)$. If μ^v and ν^f have equal parts, say $\mu_i^v = \dots = \mu_{i+l-1}^v = \nu_j^f = \dots = \nu_{j+m-1}^f = k \in \mathbb{N}$, then F_e has $l + m$ leaves with the vertex label $(k, 0)$. But there is no preferred way to determine which leaves correspond to which parts. To solve this problem, we promise that these leaves correspond, from left to right, to $\mu_{i+l-1}^v, \dots, \mu_i^v, \nu_{j+m-1}^f, \dots, \nu_j^f$ (i.e., the outer l leaves correspond to the parts of μ^v , and the inner m leaves to those of ν^f). Similarly, if $\eta'_e = \mu^v \cup \nu^f$ and μ^v and ν^f have equal parts, we promise that outer leaves correspond to the parts of μ^v and inner leaves to the parts of ν^f . If $\xi_e = \nu^{f_3(v)} \cup \nu^{f_1(v')}$ (resp. $\xi'_e = \nu^{f_1(v)} \cup \nu^{f_3(v')}$) and if $\nu^{f_3(v)}$ and $\nu^{f_1(v')}$ (resp. $\nu^{f_1(v)}$ and $\nu^{f_3(v')}$) have equal parts, we promise that left leaves correspond to the parts of $\nu^{f_3(v)}$ (resp. $\nu^{f_1(v)}$) and right leaves to the parts of $\nu^{f_1(v')}$ (resp. $\nu^{f_3(v')}$).

§6.6. Free energy

We take the logarithm of the partition function by using the exponential formula. As a result, we obtain the free energy as the sum over connected combined graphs.

Proposition 6.13.

$$\mathcal{F}_d^\Gamma(q) = \sum_{\substack{(\vec{\mu}, \vec{\nu}); \\ \Gamma\text{-set of} \\ \text{degree } \vec{d}}} \frac{1}{z_{\vec{\mu}} z_{\vec{\nu}}} \sum_{W \in \text{Comb}_\Gamma^{\bullet}(\vec{\mu}, \vec{\nu})} \mathcal{H}(W).$$

Proof. We use the formulation in [K], Appendix A.

For a combined forest W , we define \overline{W} to be the graph obtained by forgetting all leaf indices. Two combined forests W and W' are *equivalent* if \overline{W} and \overline{W}' are isomorphic as labeled graphs. The set of equivalence classes in $\text{Comb}_\Gamma^\bullet(\vec{\mu}, \vec{\nu})$ is denoted by $\overline{\text{Comb}}_\Gamma^\bullet(\vec{\mu}, \vec{\nu})$. We define

$$\begin{aligned} \overline{\text{Comb}}_\Gamma^\bullet(d) &= \prod_{|\vec{d}|=d} \prod_{\substack{(\vec{\mu}, \vec{\nu}) \\ \Gamma\text{-set of} \\ \text{degree } \vec{d}}} \overline{\text{Comb}}_\Gamma^\bullet(\vec{\mu}, \vec{\nu}) \quad (d \geq 1), \\ \overline{\text{Comb}}_\Gamma^\bullet &= \prod_{d \geq 1} \overline{\text{Comb}}_\Gamma^\bullet(d). \end{aligned}$$

The set $\overline{\text{Comb}}_\Gamma^\bullet$ is a G-set.

Next we define a map $\Psi : \text{Comb}_\Gamma^\bullet \rightarrow \mathbb{Q}(t)[[\vec{Q}]]$, so that it satisfies

$$(18) \quad \frac{1}{\#\text{aut}(G)} \Psi(\overline{G}) = \sum_{\substack{W \in \text{Comb}_\Gamma^\bullet(\vec{\mu}, \vec{\nu}) \\ \overline{W} = G}} \frac{1}{z_{\vec{\mu}} z_{\vec{\nu}}} \mathcal{H}(W) \vec{Q}^{\vec{d}} \quad (G \in \overline{\text{Comb}}_\Gamma^\bullet(\vec{\mu}, \vec{\nu}))$$

where \vec{d} is the degree of the Γ -set $(\vec{\mu}, \vec{\nu})$. It is easy to see that Ψ is a grade preserving map. Showing that Ψ is multiplicative with respect to the graph union is straightforward. See Appendix B.

$(\overline{\text{Comb}}_\Gamma^\bullet, \mathbb{Q}(t)[[\vec{Q}]], \Psi)$ satisfies the conditions for the GA-triple. Applying the exponential formula to this, we obtain

$$\log \left[1 + \sum_{G \in \overline{\text{Comb}}_\Gamma^\bullet} \frac{1}{\#\text{aut}(G)} \Psi(G) \right] = \sum_{G \in \overline{\text{Comb}}_\Gamma^\circ} \frac{1}{\#\text{aut}(G)} \Psi(G).$$

Here $\overline{\text{Comb}}_\Gamma^\circ$ stands for the subset of $\overline{\text{Comb}}_\Gamma^\bullet$ consisting of all connected graphs. Since the LHS is equal to $\log \mathcal{Z}^\Gamma(q, \vec{Q})$, the RHS is equal to the free energy $\mathcal{F}^\Gamma(q, \vec{Q})$. Substituting (18) back, we obtain Proposition 6.13. \square

§6.7. Proof of Proposition 3.3

Finally we will give a proof of Proposition 3.3.

Given a combined forest W and a positive integer k , there exists a unique combined forest which is the same as W except that all the vertex-labels are multiplied by k . It is denoted by $W_{(k)}$.

We first rewrite the free energy as follows.

$$\mathcal{F}_d^\Gamma(q) = \sum_{k; k|d_0} \sum_{\substack{(\vec{\mu}, \vec{\nu}); \\ \Gamma\text{-set of} \\ \text{degree } \vec{d}/k; \\ \text{gcd}(\vec{\mu}, \vec{\nu})=1}} \frac{1}{z_k \vec{\mu} z_k \vec{\nu}} \sum_{W \in \text{Comb}_\Gamma^\circ(\vec{\mu}, \vec{\nu})} \mathcal{H}(W_{(k)})$$

where $d_0 = \text{gcd}(\vec{d})$. Then, $G_{\vec{d}}^\Gamma(q)$ is equal to

$$\begin{aligned} G_{\vec{d}}^\Gamma(q) &= \sum_{k; k|d_0} \sum_{\substack{(\vec{\mu}, \vec{\nu}); \\ \Gamma\text{-set of} \\ \text{degree } \vec{d}/k; \\ \text{gcd}(\vec{\mu}, \vec{\nu})=1}} \sum_{k'; k'|k} \frac{k'}{k} \mu\left(\frac{k}{k'}\right) \frac{1}{z_{k'} \vec{\mu} z_{k'} \vec{\nu}} \sum_{W \in \text{Comb}_\Gamma^\circ(\vec{\mu}, \vec{\nu})} \mathcal{H}(W_{(k')})|_{q \rightarrow q^{k/k'}} \\ &= \sum_{k; k|d_0} \sum_{\substack{(\vec{\mu}, \vec{\nu}); \\ \Gamma\text{-set of} \\ \text{degree } \vec{d}/k; \\ \text{gcd}(\vec{\mu}, \vec{\nu})=1}} \frac{1}{k \#\text{aut}(\vec{\mu}) \#\text{aut}(\vec{\nu})} \sum_{W \in \text{Comb}_\Gamma^\circ(\vec{\mu}, \vec{\nu})} g_k^\Gamma(W), \end{aligned}$$

where

$$\mathcal{G}_k^\Gamma(W) = \sum_{k':k'|k} \mu\left(\frac{k}{k'}\right) k'^{-l(\vec{\mu})-l(\vec{\nu})+1} \mathcal{H}(W_{(k')}) \Big|_{q \rightarrow q^{k/k'}}.$$

Proposition 3.3 follows from:

Proposition 6.14. *Let $(\vec{\mu}, \vec{\nu})$ be a Γ -set such that $\gcd(\vec{\mu}, \vec{\nu}) = 1$. Let W be a connected combined forest in $\text{Comb}_\Gamma^\circ(\vec{\mu}, \vec{\nu})$ and let k be a positive integer. Then*

$$t \cdot \mathcal{G}_k^\Gamma(W) \in \mathbb{Q}[t].$$

Proof. The proposition is a consequence of the next lemma. □

Lemma 6.15. *Let W be a connected combined forest in $\text{Comb}_\Gamma^\circ(\vec{\mu}, \vec{\nu})$.*

1. *If $\#E(W) - \#V(W) + 1 > 0$, then $\mathcal{H}(W) \in \mathbb{Z}[t]$.*
2. *If $\#E(W) - \#V(W) + 1 = 0$, then $t_m \mathcal{H}(W) \in \mathbb{Z}[t]$ where $m = \gcd(\vec{\mu}, \vec{\nu})$. Moreover, the followings hold:*
3. *Assume that $\gcd(\vec{\mu}, \vec{\nu}) = 1$ and that $\#E(W) - \#V(W) + 1 = 0$. Then if k is odd,*

$$\mathcal{H}(W_{(k)}) - k^{l(\vec{\mu})+l(\vec{\nu})-1} \mathcal{H}(W) \Big|_{t \rightarrow t_k} \in \mathbb{Z}[t];$$

if k is even,

$$\mathcal{H}(W_{(k)}) - k^{l(\vec{\mu})+l(\vec{\nu})-1} \mathcal{H}(W) \Big|_{t \rightarrow t_k} \prod_{T \in VT_I(W)} \left(1 + \frac{t_{m(T)k/2}}{2}\right) \in \mathbb{Z}[t].$$

Here $VT_I(W)$ is the set of VEV trees in W such that $m(T)$ is odd and $n_{\text{root}}/m(T)$ is odd.

Proof. This lemma follows from Proposition 6.7. The proof is completely the same as those of Propositions 6.7 and 6.8 in [K]. □

This completes the proof of Proposition 3.3.

Appendix A. Proof of Lemma 5.2

In this appendix, we give a proof of Lemma 5.2.

A.1. Sublemmas

Let p be a prime integer, $i, b \in \mathbb{N}$.

Lemma A.1.

$$(x_1 + \cdots + x_l)^{p^i b} = (x_1^p + \cdots + x_l^p)^{p^{i-1} b} + \text{term divisible by } p^i \quad (i \geq 1)$$

where x_1, \dots, x_l are indeterminate variables.

Proof. The proof is by induction on i .

$i = 1$:

$$\begin{aligned} (x_1 + \cdots + x_l)^{pb} &= ((x_1 + \cdots + x_l)^p)^b \\ &= (x_1^p + \cdots + x_l^p + \text{term divisible by } p)^b \\ &= (x_1^p + \cdots + x_l^p)^b + \text{term divisible by } p. \end{aligned}$$

$i > 1$: assume that this is true for $i - 1$.

$$\begin{aligned} (x_1 + \cdots + x_l)^{p^{i+1} b} &= ((x_1 + \cdots + x_l)^{p^i b})^p \\ &= ((x_1^p + \cdots + x_l^p)^{p^{i-1} b} + \text{term divisible by } p^i)^p \\ &= (x_1^p + \cdots + x_l^p)^{p^i b} + \text{term divisible by } p^{i+1}. \end{aligned}$$

So the lemma is proved. □

Lemma A.2. Let $n = (n_1, \dots, n_l)$ be the set of nonnegative integers such that $\gcd(n) = 1$.

1. For $k \in \mathbb{N}$,

$$\frac{(k|n)!}{(kn_1)! \cdots (kn_l)!} \equiv 0 \pmod{|n|}.$$

2. For a positive integer $k \in \mathbb{N}$ prime to p ,

$$\frac{(p^i k|n)!}{(p^i kn_1)! \cdots (p^i kn_l)!} \equiv \frac{(p^{i-1} k|n)!}{(p^{i-1} kn_1)! \cdots (p^{i-1} kn_l)!} \pmod{p^i |n|}.$$

Proof. 1. The main idea of the proof is the following. If a rational number r satisfies

$$h_1 \cdot r \in \mathbb{Z} \quad \text{and} \quad h_2 \cdot r \in \mathbb{Z}$$

where h_1, h_2 are natural numbers, then clearly it holds that

$$\gcd(h_1, h_2) \cdot r \in \mathbb{Z}.$$

Let us define

$$C(k, n) := \frac{1}{|n|} \frac{(k|n|)!}{(kn_1)! \dots (kn_l)!} \in \mathbb{Z}.$$

Then

$$n_1 C(k, n) = \frac{(k|n| - 1)!}{(kn_1 - 1)! \dots (kn_l)!} \in \mathbb{Z}.$$

The same holds for every n_i ($1 \leq i \leq l$). Therefore

$$\gcd(n) \cdot C(k, n) \in \mathbb{Z}.$$

Since we assumed $\gcd(n) = 1$, $C(k, n)$ is an integer.

2. Let us write $|n| = p^\alpha n'$ where n' is an integer prime to p . By Lemma (A.1), we have

$$(x_1 + \dots + x_l)^{p^i k|n|} = (x_1^p + \dots + x_l^p)^{p^i k|n|} + \text{term divisible by } p^{i+\alpha} \quad (i \geq 1)$$

Then comparing the coefficient of $x_1^{p^i kn_1} \dots x_l^{p^i kn_l}$, we obtain

$$\frac{(p^i k|n|)!}{(p^i kn_1)! \dots (p^i kn_l)!} - \frac{(p^{i-1} k|n|)!}{(p^{i-1} kn_1)! \dots (p^{i-1} kn_l)!} \equiv 0 \pmod{p^{i+\alpha}}.$$

Moreover, the LHS is divisible by $|n|$ by the result of 1. Therefore the LHS is divisible by $\text{lcm}(|n|, p^{i+\alpha}) = p^i |n|$. □

Lemma A.3. For $R(t) \in \mathcal{L}[t]$, there exists $h(t) \in \mathcal{L}[t]$ such that

$$R(t)^{p^i} = R(t_p)^{p^{i-1}} + p^i h(t).$$

Proof. It was shown in [BP] that

$$t^{p^i b} = (t_p)^{p^{i-1} b} + p^{i\exists} g_1(t) \quad (g(t) \in \mathbb{Z}[t]).$$

Therefore, for $r(t) \in \mathbb{Z}[t]$,

$$(19) \quad r(t_p)^{p^{i-1}} - r(t)^{p^{i-1}} = p^{i\exists} g_2(t), \quad (g_2(t) \in \mathbb{Z}[t])$$

The case $i = 1$ is the above result of [BP] and the proof is by induction on i . On the other hand, it was shown in [P], Lemma 5.1 that, for $r(t) \in \mathbb{Z}[t]$,

$$r(t)^{p^i} - r(t_p)^{p^{i-1}} = p^{i\exists} g_3(t) \quad (g_3(t) \in \mathbb{Z}[t]).$$

Hence, for $r(t) \in \mathbb{Z}[t]$,

$$(20) \quad r(t)^{p^i} - r(t_p)^{p^{i-1}} = p^{i\exists} g_4(t) \quad (g_4(t) \in \mathbb{Z}[t]).$$

Let us write $R(t) = r_1(t)/r_2(t)$ where $r_1(t), r_2(t) \in \mathbb{Z}[t]$ and $r_2(t)$ is monic. Then

$$\begin{aligned} R(t)^{p^i} - R(t_p)^{p^{i-1}} &= \frac{r_1(t)^{p^i}}{r_2(t)^{p^i}} - \frac{r_1(t_p)^{p^{i-1}}}{r_2(t_p)^{p^{i-1}}} \\ &= \frac{1}{r_2(t)^{p^i} r_2(t_p)^{p^{i-1}}} (r_1(t)^{p^i} (r_2(t)^{p^{i-1}} - r_2(t_p)^{p^{i-1}}) + r_2(t)^{p^i} (r_1(t)^{p^i} - r_1(t)^{p^{i-1}})). \end{aligned}$$

By (20), the numerator is written in the form $p^i g_5(t)$ with some $g_5(t) \in \mathbb{Z}[t]$.

The lemma is proved. □

A.2. Proof of Lemma 5.2

Now we give a proof of Lemma 5.2.

If $k = 1$, the statement is trivial, so we prove the $k > 1$ case. Let

$$k = k_1^{a_1} \cdots k_s^{a_s}$$

be the prime decomposition of k . It is sufficient to prove the following for every i ($1 \leq i \leq s$).

$$(21) \quad \sum_{k': k'|k} \mu\left(\frac{k}{k'}\right) \frac{|k'n|! (-1)^{k'|n|}}{(k'n_1)! \cdots (k'n_l)!} R(t_{k/k'})^{k'} = k_i^{a_i} |n|^{\exists} f_i(t), \quad f_i(t) \in \mathcal{L}[t].$$

Since the proof of (21) is the same for any i ($1 \leq i \leq s$), we only show the case $i = 1$. Let $j = k/k_1^{a_1}$. Note that for any divisor k' of k ,

$$\mu\left(\frac{k}{k'}\right) = \begin{cases} \mu(j/j') & k' = k_1^{a_1} j' \\ -\mu(j/j') & k' = k_1^{a_1-1} j' \\ 0 & \text{otherwise.} \end{cases}$$

Therefore

$$\begin{aligned}
 & \text{LHS of (21)} \\
 &= \sum_{j':j'|j} \mu\left(\frac{j}{j'}\right) \left[\frac{|k_1^{a_1} j' n|! (-1)^{k_1^{a_1} j' |n|}}{(k_1^{a_1} j' n_1)! \cdots (k_1^{a_1} j' n_l)!} R(t_{j/j'})^{k_1^{a_1} j'} \right. \\
 & \quad \left. - \frac{|k_1^{a_1-1} j' n|! (-1)^{k_1^{a_1-1} j' |n|}}{(k_1^{a_1-1} j' n_1)! \cdots (k_1^{a_1-1} j' n_l)!} R(t_{k_1 j/j'})^{k_1^{a_1-1} j'} \right] \\
 &= \sum_{j':j'|j} \mu\left(\frac{j}{j'}\right) (-1)^{k_1^{a_1} j' |n|} R(t_{j/j'})^{k_1^{a_1} j'} \\
 & \quad \times \underbrace{\left[\frac{|k_1^{a_1} j' n|!}{(k_1^{a_1} j' n_1)! \cdots (k_1^{a_1} j' n_l)!} - \frac{|k_1^{a_1-1} j' n|!}{(k_1^{a_1-1} j' n_1)! \cdots (k_1^{a_1-1} j' n_l)!} \right]}_{*} \\
 & \quad + \sum_{j':j'|j} \mu\left(\frac{j}{j'}\right) \underbrace{\frac{|k_1^{a_1-1} j' n|!}{(k_1^{a_1-1} j' n_1)! \cdots (k_1^{a_1-1} j' n_l)!}}_{*} \\
 & \quad \times \underbrace{\left[\left((-1)^{j' |n|} R(t_{j/j'})^{j'} \right)^{k_1^{a_1}} - \left((-1)^{j' |n|} R(t_{k_1 j/j'})^{j'} \right)^{k_1^{a_1-1}} \right]}_{\dagger}.
 \end{aligned}$$

By Lemma A.2, (*) is divisible by $k_1^{a_1} |n|$ and (*) is divisible by $|n|$. By Lemma A.3, there exists $h(t_{j/j'}) \in \mathcal{L}[t_{j/j'}]$ such that (†) is written as $k_1^{a_1} h(t_{j/j'})$. Since $\mathcal{L}[t_{j/j'}] \subset \mathcal{L}[t]$, $h(t_{j/j'}) \in \mathcal{L}[t]$. Therefore (21) is proved.

This completes the proof of Lemma 5.2.

Appendix B. Multiplicativity of Ψ

In this section, we show that the map Ψ , introduced in the proof of Proposition 6.13, is multiplicative with respect to the graph union.

Since the combined amplitudes of two equivalent VEV forests are the same, (18) is equivalent to:

$$(22) \quad \Psi(G) = \#\text{aut}(G) \cdot N(W) \frac{1}{z_{\vec{\mu}} z_{\vec{\nu}}} \mathcal{H}(W) \vec{Q}^{\vec{d}} \quad (W \text{ such that } \overline{W} = G),$$

where $N(W)$ is the number of combined forests equivalent to W .

$N(W)$ is described as follows. It is equal to the number of ways to assign leaf indices to leaves of $G \in \overline{\text{Comb}}_{\Gamma}(\vec{\mu}, \vec{\nu})$. Therefore

$$(23) \quad N(W) = N_0(G) \prod_{T: \text{VEV tree in } W} N'(T),$$

where $N'(T)$ is the number of VEV trees equivalent to T and $N_0(G)$ is the number of ways to distribute the leaf indices to VEV trees in G . The latter is

$$(24) \quad N_0(G) = \frac{\#\text{aut}(\vec{\mu}) \#\text{aut}(\vec{\nu})}{\#\text{aut}(G)} \prod_{T: \text{VEV tree in } G} \left(\prod_{f \in F_3(\Gamma) \setminus \hat{F}_3(\Gamma)} \#\text{aut}(\nu^f(T)) \right)^{-1} \\ \times \prod_{\substack{\{T, T'\}; \\ T, T': \text{VEV trees in } G, \\ T \neq T'}} \left(\prod_{v \in V_3(\Gamma)} \#\text{aut}(\mu^v(T, T')) \prod_{f \in \hat{F}_3(\Gamma)} \#\text{aut}(\nu^f(T, T')) \right)^{-1},$$

where

$$\nu^f(T) = \{\nu_i^f \mid \nu_i^f \text{ is associated to a leaf of } T\} \quad (f \in F_3(\Gamma) \setminus \hat{F}_3(\Gamma)), \\ \mu^v(T, T') = \{\mu_i^v \mid \mu_i^v \text{ is associated to a bridge joining } T \text{ and } T'\} \quad (v \in V_3(\Gamma)), \\ \nu^f(T, T') = \{\nu_i^f \mid \nu_i^f \text{ is associated to a bridge joining } T \text{ and } T'\} \quad (f \in \hat{F}_3(\Gamma)).$$

Substituting (23)(24) into (22), we find out that Ψ is multiplicative.

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