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A Similarity Degree Characterization of Nuclear C^*-algebras

By

Gilles Pisier*

Abstract

We show that a C^*-algebra A is nuclear iff there is a number $\alpha < 3$ and a constant K such that, for any bounded homomorphism $u\colon\ A \to B(H)$, there is an isomorphism $\xi\colon\ H \to H$ satisfying $\|\xi^{-1}\|\|\xi\| \le K\|u\|^\alpha$ and such that $\xi^{-1}u(.)\xi$ is a $*$-homomorphism. In other words, an infinite dimensional A is nuclear iff its length (in the sense of our previous work on the Kadison similarity problem) is equal to 2.

In 1955, Kadison [14] formulated the following conjecture: any bounded homomorphism $u\colon\ A \to B(H)$, from a C^*-algebra into the algebra $B(H)$ of all bounded operators on a Hilbert space H, is similar to a $*$-homomorphism, i.e. there is an invertible operator $\xi\colon\ H \to H$ such that $x \to \xi u(x)\xi^{-1}$ satisfies $\xi u(x^*)\xi^{-1} = (\xi u(x)\xi^{-1})^*$ for all x in A. This conjecture remains unproved, although many partial results are known (see [4], [10]). In particular, by [10], we know that u is similar to a $*$-homomorphism iff it is completely bounded (c.b. in short) in the sense of e.g. [17] or [20] (to which we refer for background on c.b. maps). Moreover, we have

$$\|u\|_{cb} = \inf\{\|\xi\|\|\xi^{-1}\|\}$$

where the infimum runs over all invertible ξ such that $\xi u(\cdot)\xi^{-1}$ is a $*$-homomorphism. Recall that, by definition, $\|u\|_{cb} = \sup_{n\ge 1}\|u_n\|$ where $u_n\colon\ M_n(A) \to M_n(B(H))$ is the mapping taking $[a_{ij}]$ to $[u(a_{ij})]$. Thus Kadison's conjecture

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*Partially supported by NSF grant No. 0200690.
Department of Mathematics, Texas A&M University, College Station, TX 77843-3368, U. S. A. and Université Paris VI, Equipe d'Analyse, Case 186, 75252, Paris Cedex 05, France.
e-mail: pisier@math.tamu.edu

is equivalent to the validity of the implication $\|u\| < \infty \Rightarrow \|u\|_{cb} < \infty$. In [18], the author proved that if a C^*-algebra A verifies Kadison's conjecture, then there is a number α for which there exists a constant K so that any bounded homomorphism $u\colon\ A \to B(H)$ satisfies $\|u\|_{cb} \leq K\|u\|^{\alpha}$. Moreover, the smallest number α with this property is an integer denoted by $d(A)$ (and $\alpha = d(A)$ itself satisfies the property).

An analogous parameter can be defined for a discrete group G and it is proved in [18] that G is amenable iff $d(G) \leq 2$. The main result of this note is the analogous equivalence for C^*-algebras: a C^*-algebra A is nuclear (or equivalently amenable, see below) iff $d(A) \leq 2$. In [18], we could only prove a partial result in this direction.

Let A, B be C^*-algebras. Let $\|\ \ \|_{\alpha}$ be a C^*-norm on their algebraic tensor product, denoted by $A \otimes B$; as usual, $A \otimes_{\alpha} B$ then denotes the C^*-algebra obtained by completing $A \otimes B$ with respect to $\|\ \ \|_{\alpha}$. By classical results (see [24]) the set of C^*-norms admits a minimal and a maximal element denoted respectively by $\|\cdot\|_{\min}$ and $\|\cdot\|_{\max}$. Then A is called nuclear if for any B we have $A \otimes_{\min} B = A \otimes_{\max} B$, or equivalently $\|x\|_{\min} = \|x\|_{\max}$ for any x in $A \otimes B$. We refer the reader to [24], [15] for more information on nuclear C^*-algebras. We note in particular that by results due to Connes and Haagerup ([7], [8]), a C^*-algebra is nuclear iff it is amenable as a Banach algebra (in B.E. Johnson's sense).

The main result of this note is as follows.

Theorem 1. *The following properties of C^*-algebra A are equivalent*:

(i) *A is nuclear.*

(ii) *There are $\alpha < 3$ and a constant K such that any bounded homomorphism $u\colon\ A \to B(H)$ satisfies $\|u\|_{cb} \leq K\|u\|^{\alpha}$.*

(iii) *Same as* (ii) *with $K = 1$ and $\alpha = 2$.*

The implication (i) $\Rightarrow$ (iii) is well known (see [2], [4]).

In the terminology of [18], the similarity degree $d(A)$ is the smallest α for which the property considered in (ii) above is satisfied. It is proved in [18] that $d(A)$ is always an integer identified as the smallest length of a specific kind of factorization for matrices with entries in A.

With this terminology, the preceding theorem means that A is nuclear iff $d(A) \leq 2$. In the infinite dimensional case, $d(A) > 1$ hence A is nuclear iff $d(A) = 2$.

In his work on derivations (see [4] and [5]) Erik Christensen isolated the following property D_k for a C^*-algebra. Here k is any number $\geq 1/2$. A C^*-algebra A has property D_k if for any H, any representation $\pi\colon\ A \to B(H)$, and any T in $B(H)$ the derivation $\delta_T\colon\ A \to B(H)$ defined by $\delta_T(a) = \pi(a)T - T\pi(a)$ satisfies

$$\|\delta_T\|_{cb} \le 2k\|\delta_T\|.$$

With this terminology, Theorem 1 implies the following:

Corollary 2. *Let A be a C^*-algebra. The following assertions are equivalent.*

(i) *A is nuclear.*

(ii) *A satisfies property D_k for some $k < 3/2$.*

(iii) *A satisfies property D_1.*

Proof. Here again the fact that (i) $\Rightarrow$ (iii) is well known (see [2], [4]). The equivalence between the similarity problem and the derivation problem was established by Kirchberg in [16]. Refining Kirchberg's estimates, the author proved in [18] (see also [20, p. 139]) that property D_k implies that the similarity degree $d(A)$ is at most $2k$. Thus (ii) $\Rightarrow$ (i) follows from the corresponding implication in Theorem 1. □

The main point in Theorem 1 is (ii) $\Rightarrow$ (i). In our previous work, we could only prove that (ii) implies that A is "semi-nuclear," i.e. that whenever a representation $\pi\colon\ A \to B(H)$ generates a semifinite von Neumann algebra, the latter is injective. In this note, we show that the semifiniteness assumption is not needed. We use the same starting point as in [18], but we feel the idea of the present proof is more transparent than the one in [18]. In particular, we will use the following result which is part of Th.2.9 in [19] (obtained independently in [6]), but the latter is inspired by and closely related to Haagerup's Th. 2.1 in [9].

Theorem 3. *Let $N \subset B(H)$ be a von Neumann algebra. Then N is injective iff there is a constant C such that, for all n, if elements x_i in N ($i = 1, \ldots, n$) admit a decomposition $x_i = \alpha_i + \beta_i$ with $\alpha_i, \beta_i \in B(H)$ such that $\|\sum \alpha_i^*\alpha_i\| \le 1$ and $\|\sum \beta_i\beta_i^*\| \le 1$, then there is a decomposition $x_i = a_i + b_i$ with $a_i, b_i \in N$ such that $\|\sum a_i^*a_i\| \le C^2$ and $\|\sum b_ib_i^*\| \le C^2$.*

The preceding statement can be viewed as the analogue for von Neumann algebras of the characterization of amenable discrete groups obtained in [27] (see also [1]).

Our main (somewhat) new ingredient is as follows.

Theorem 4. *Let $M \subset B(H)$ be a von Neumann algebra with a cyclic vector. Let $y_1, \dots, y_n$ in M' be such that for any $x_1, \dots, x_n$ in M we have*

$$\left\|\sum x_i y_i\right\| \le \max\left\{\left\|\sum x_i^* x_i\right\|^{1/2}, \left\|\sum x_i x_i^*\right\|^{1/2}\right\}. \tag{1}$$

Then there is a decomposition

$$y_i = a_i + b_i$$

with a_i, b_i in M' such that

$$\left\|\sum a_i {a_i}^*\right\| \le 1 \quad \textit{and} \quad \left\|\sum {b_i}^* b_i\right\| \le 1.$$

Proof. We follow a well known kind of argument with roots in [9]; see also [23] and the proof of a theorem due to Kirchberg as presented in [20, §14] that we will follow closely below.

Recall that the "row and column" operator spaces $R_n \subset M_n$ and $C_n \subset M_n$ are defined by:

$$R_n = \mathrm{span}[e_{1i} \mid 1 \le i \le n] \quad C_n = \mathrm{span}[e_{i1} \mid 1 \le i \le n].$$

Let $\Delta_n \subset C_n \oplus R_n$ be the operator space spanned by $\delta_i = e_{i1} \oplus e_{1i}$ $(i = 1, 2, \dots, n)$. Our assumption means that the linear map

$$v\colon\ \Delta_n \otimes_{\min} M \to B(H)$$

defined by

$$v\left(\sum \delta_i \otimes x_i\right) = \sum x_i y_i$$

satisfies $\|v\| \le 1$. (Indeed, it is easy to check that the majorant in (1) is equal to $\|\sum \delta_i \otimes x_i\|_{\min}$.)

Since v is clearly a two-sided M-module map and M has a cyclic vector, it follows by [22] (and unpublished work of Haagerup) that $\|v\|_{cb} = \|v\| \le 1$.

Therefore, by a result due to Wittstock [26] (see also [23]), v can be extended to a two-sided M-module map $\tilde{v} : [C_n \oplus R_n] \otimes_{\min} M \to B(H)$ with $\|\tilde{v}\|_{cb} \le 1$. Let $a_i = \tilde{v}([e_{i1} \oplus 0] \otimes 1)$ and $b_i = \tilde{v}([0 \oplus e_{1i}] \otimes 1)$. Then we have clearly $\|\sum a_i {a_i}^*\|^{1/2} \le \|\tilde{v}\|_{cb} \le 1$ and similarly $\|\sum {b_i}^* b_i\|^{1/2} \le 1$. Moreover, since $\tilde{v}$ is an M-module map, for any m in M and any z in $C_n \oplus R_n$, we must have $m.\tilde{v}(z \otimes 1) = \tilde{v}(z \otimes m) = \tilde{v}(z \otimes 1).m$ and hence $\tilde{v}(z \otimes 1) \in M'$. Thus a_i and b_i are in M'. □

Remark 1. It is easy to see that the preceding result fails without the cyclicity assumption: Just consider the case $M = \mathbb{C}$ and $M' = B(H)$ with $\dim(H) = \infty$.

Remark 2. The same proof gives a criterion for a map $u\colon\ E \to M'$ defined on a subspace $E \subset A$ of a general C^*-algebra A to admit an extension $\tilde{u}\colon\ A \to M'$ with $\|\tilde{u}\|_{dec} \leq 1$. This is essentially the same as Kirchberg's [20, Th. 14.6].

Remark 3. The above Theorem 4 may be viewed as an analogue for the operator space $R_n + C_n$ of Haagerup's [9, Lemma 3.5] devoted to the operator space ℓ_1^n equipped with its maximal structure, in the Blecher-Paulsen sense (see e.g. [20, §3]). Note that while he decomposes into products, we decompose into sums.

Remark 4. Let (E_0, E_1) be a compatible pair of operator spaces in the sense of [20, §2.7]. Then Remark 2 gives a sufficient criterion for a map $u\colon\ E_0 + E_1 \to M'$ to admit a decomposition $u = u_0 + u_1$ with $u_0\colon\ E_0 \to M'$ and $u_1\colon\ E_1 \to M'$ satisfying $\|u_0\|_{cb} \leq 1$ and $\|u_1\|_{cb} \leq 1$. Assume that $E_0 \subset A_0$ and $E_1 \subset A_1$, where A_0, A_1 are C^*-algebras, then this criterion actually ensures that there are extensions

$$\tilde{u}_0\colon\ A_0 \to M' \quad \text{and} \quad \tilde{u}_1\colon\ A_1 \to M'$$

with $\|\tilde{u}_0\|_{dec} \leq 1$ and $\|\tilde{u}_1\|_{dec} \leq 1$. In that formulation, the converse also holds up to a numerical factor 2. Note that, in the special case of interest to us, when $E_0 = C$ and $E_1 = R$, then we can take A_0, A_1 equal to $K(\ell_2)$ (hence nuclear) so that the min and max norms are identical on $(A_0 \oplus A_1) \otimes M$.

Notation. Let $A \subset B(H)$ be any C^*-subalgebra. For any $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ in A, we denote

$$\|(x_j)\|_{R\cap C} = \max\left\{\left\|\sum x_j^* x_j\right\|^{1/2}, \left\|\sum x_j x_j^*\right\|^{1/2}\right\} \tag{2}$$

$$\|(y_j)\|_{R+C} = \inf\left\{\left\|\sum \alpha_j^* \alpha_j\right\|^{1/2} + \left\|\sum \beta_j \beta_j^*\right\|^{1/2}\right\}, \tag{3}$$

where the infimum runs over all α_j, β_j in $B(H)$ such that $y_j = \alpha_j + \beta_j$. Note that, by the injectivity of $B(H)$, the definition of $\|(y_j)\|_{R+C}$ does not really depend on the choice of H or of the embedding $A \subset B(H)$. The corresponding fact for $\|(x_j)\|_{R\cap C}$ is obvious.

Corollary 5. *Let $M \subset B(H)$ be a von Neumann algebra. Then M is injective iff there is a constant C such that, for all n, all $x_1, \dots, x_n$ in M and $y_1, \dots, y_n$ in M', we have*

$$\left\|\sum x_i y_i\right\| \le C\|(x_i)\|_{R\cap C}\|(y_i)\|_{R+C}. \tag{4}$$

Proof. If M has a cyclic vector, then this follows immediately from Theorems 3 and 4 and the well known fact that M' is injective iff M is injective (see [25, p. 174]). Now assume that M has a finite cyclic set, i.e. there are $\xi_1, \dots, \xi_N$ in H such that $M\xi_1 + \cdots + M\xi_N$ is dense in H. Then the vector $(\xi_1, \dots, \xi_N)$ in H^N is cyclic for $M_N(M) \subset M_N(B(H))$. Moreover, it is easy to check that (4) remains true for $M_N(M)$ but with C replaced by a constant $C(N)$ (possibly unbounded when N grows). Nevertheless, by the first part of the proof it follows that $M_N(M)$ is injective and hence, a fortiori, M is injective.

In the general case, let $\{\xi_i \mid i \in I\}$ be a dense subset of H. For any finite subset $J \subset I$, let H_J be the closure of

$$\left\{\sum\nolimits_{j\in J} a_j(\xi_j) \mid a_j \in M\right\}.$$

Note that H_J is an invariant subspace for M, so that (since M is self-adjoint) the orthogonal projection $P_J\colon\ H \to H_J$ belongs to M'. Let $\pi_J(a)$ denote the restriction of a to H_J. Then $\pi_J\colon\ M \to B(H_J)$ is a normal representation, $\pi_J(M)$ admits a finite cyclic set (namely $\{\xi_i \mid i \in J\}$), and it is easy to check that our assumption (4) is still verified by $\pi_J(M)$ on H_J.

Thus, by the first part of the proof, $\pi_J(M)$ is injective. This clearly implies that the von Neumann algebra $M_J \subset B(H)$ generated by $P_J M$ and $I - P_J$ also is injective. Finally, since M is the weak-$*$ closure of the directed union of the M_J's, we conclude that M itself is injective.

Conversely, if M injective then, by Remark 5 below, (4) holds with $C = 1$. □

Remark 5. Let $M \subset B(H)$ be an injective von Neumann algebra, so that there is a projection $P\colon\ B(H) \to M'$ with $\|P\|_{cb} = 1$. Then M satisfies (4) with $C = 1$. To see this, assume $y_i \in M'$ and $\|(y_i)\|_{R+C} < 1$, so that $y_i = \alpha_i + \beta_i$ with $\|\Sigma\alpha_i^*\alpha_i\|^{1/2} + \|\Sigma\beta_i\beta_i^*\|^{1/2} < 1$. Then $y_i = a_i + b_i$ with $a_i, b_i \in M'$ satisfying

$$\left\|\sum a_i^* a_i\right\|^{1/2} + \left\|\sum b_i b_i^*\right\|^{1/2} \le \|P\|_{cb} = 1.$$

Indeed, $a_i = P\alpha_i$ and $b_i = P\beta_i$ clearly verify this.

Then for any $x_1, \dots, x_n$ in M we have by Cauchy-Schwarz

$$\left\|\sum x_i a_i\right\| \le \left\|\sum x_i x_i^*\right\|^{1/2} \left\|\sum a_i^* a_i\right\|^{1/2}$$

and

$$\left\|\sum b_i x_i\right\| \le \left\|\sum x_i^* x_i\right\|^{1/2} \left\|\sum b_i b_i^*\right\|^{1/2},$$

therefore, since

$$\left\|\sum x_i y_i\right\| \le \left\|\sum x_i a_i\right\| + \left\|\sum b_i x_i\right\|,$$

we obtain finally

$$\left\|\sum x_i y_i\right\| \le \|(x_i)\|_{R\cap C}\|(y_i)\|_{R+C}.$$

□

We will also use:

Theorem 6 ([18]). *A unital operator algebra A satisfies property* (ii) *in Theorem* 1 *iff we have*: (iv) *There is a constant K' satisfying the following*: *for any linear map $u\colon\ A \to B(H)$ for which there are a Hilbert space K, bounded linear maps v_1, w_1 from A to $B(K,H)$ and v_2, w_2 from A to $B(H,K)$ such that*

$$\forall a, b \in A \qquad u(ab) = v_1(a)v_2(b) + w_1(a)w_2(b) \tag{5}$$

we have

$$\|u\|_{cb} \le K'(\|v_1\|\|v_2\| + \|w_1\|\|w_2\|).$$

Remark. Note that (5) implies that the bilinear map $(a,b) \to u(ab)$ is c.b. on $\max(A) \otimes_h \max(A)$ with c.b. norm $\le K'(\|v_1\|\|v_2\| + \|w_1\|\|w_2\|)$. Thus, Theorem 6 follows from the case $d = 2$ of [18, Th. 4.2].

Another ingredient is the following Lemma which can be derived from [13] or from the more recent paper [21].

Lemma 7. *Let E be a finite dimensional operator space and let A be a C^*-algebra. Assume that E is a "maximal" operator space* (*equivalently that E^* is a minimal one*). *Then for any c.b. map $u\colon\ A \to E$ we have*

$$\forall n\ \forall a_1, \dots, a_n \in A \qquad \forall \xi_i \in E^*$$

$$\left|\sum\langle u(a_j),\xi_j\rangle\right| \le C\|u\|_{cb}\left(\left\|\sum a_j^*a_j\right\|^{1/2}+\left\|\sum a_ja_j^*\right\|^{1/2}\right) \cdot \sup_{x\in E}\left(\sum|\xi_j(x)|^2\right)^{1/2} \tag{6}$$

where C is a numerical constant.

Proof. We may apply [13, Th. 1.4], arguing as in [18, Lemma 6.3] (using [19, Th. 17.13] to remove the exactness assumption) this yields (6) with $C = 2$. Or we may invoke [21, Th. 0.3] taking into account [21, Lemma 2.3] (to remove the exactness assumption) and then we again obtain (6) with $C = 2$. □

For the convenience of the reader, we reproduce here the elementary Lemma 8 from [18].

Lemma 8. *Let (e_i) be the canonical basis of the operator space* $\max(\ell_2)$. *Let H be any Hilbert space and let X be either $B(\mathbb{C},H)$ or $B(H^*,\mathbb{C})$, or equivalently let X be either the column Hilbert space or the row Hilbert space. Then for all $x_1,\dots,x_n$ in X we have*

$$\left\|\sum_1^n x_i\otimes e_i\right\|_{X\otimes_{\min}\max(\ell_2)} \le \left(\sum\|x_i\|^2\right)^{1/2}.$$

Proof. Assume $X = B(\mathbb{C},H)$ or $B(H^*,\mathbb{C})$. We identify X with H as a vector space. Let (δ_m) be an orthonormal basis in H. Observe that for any finite sequence a_m in $B(\ell_2)$ we have in both cases

$$\left\|\sum\delta_m\otimes a_m\right\|_{\min} \le \left(\sum\|a_m\|^2\right)^{1/2}.$$

whence we have, for any $x_1,\dots,x_n$ in X,

$$\begin{aligned}\left\|\sum x_i\otimes e_i\right\| &= \left\|\sum_m \delta_m\otimes\sum_i\langle x_i,\delta_m\rangle e_i\right\|\\ &\le \left(\sum_m\left\|\sum_i\langle x_i,\delta_m\rangle e_i\right\|^2\right)^{1/2}\\ &= \left(\sum_{m,i}|\langle x_i,\delta_m\rangle|^2\right)^{1/2} = \left(\sum_i\|x_i\|^2\right)^{1/2}.\end{aligned}$$

□

Proof of Theorem 1. As we already observed, it suffices to show that (ii) implies that A is nuclear. Let $\pi\colon\ A \to B(H)$ be a representation and let $M = \pi(A)''$. Using Theorem 6 and Corollary 5, we will show that (ii) implies that M is injective. By the well known results of Choi–Effros and Connes (see [3]), this implies that A is nuclear. Since $\pi(A) \simeq A/\ker(\pi)$ is a quotient of A, it obviously inherits the property (ii). Thus we may as well replace $\pi(A)$ by A: we assume $A \subset B(H)$ and let $M = A''$. It suffices to show that M is injective.

Claim. We claim that for any $x_1, \dots, x_n$ in M and $y_1, \dots, y_n$ in M' we have

$$\left\|\sum x_j y_j\right\| \le 4K'C\|(x_j)\|_{R\cap C}\|(y_j)\|_{R+C}. \tag{7}$$

Note: It may be worthwhile for the reader to note that $\|(y_j)\|_{R+C}$ is (up to a factor 2) in operator space duality with $\|(x_j)\|_{R\cap C}$, namely if we set

$$|||(y_j)||| = \sup\left\{\left\|\sum x_j \otimes y_j\right\|_{\min}\right\}$$

where the sup runs over all (x_j) in $B(\ell_2)$ such that $\|(x_j)\|_{R\cap C} \le 1$, then we have (see e.g. [12])

$$|||(y_j)||| \le \|(y_j)\|_{R+C} \le 2|||(y_j)|||.$$

To prove (7) we introduce the operator space $E = \max(\ell_2^n)$, that is n-dimensional Hilbert space equipped with its "maximal" operator space structure in the Blecher-Paulsen sense (see [20, §3]). Let us now fix an n-tuple (y_j) in M' such that $\|(y_j)\|_{R+C} < 1$. In addition, we fix ξ, η in the unit sphere of H. Then we define a linear map $u\colon\ M \to E$ as follows:

$$u(x) = \sum\nolimits_j \langle x y_j \xi, \eta\rangle e_j$$

where e_j is the canonical basis of ℓ_2^n. We will assume that $E \subset B(K)$ completely isometrically. The reader may prefer to consider instead of u, the bilinear form $(x, \xi) \to \langle u(x), \xi\rangle$ defined on $M \times E^*$ where E^* is now ℓ_2^n equipped with its "minimal" (or commutative) operator space structure obtained by embedding it isometrically into a *commutative* C^*-algebra. We will now apply Theorem 6 to u.

Since we assume $\|(y_j)\|_{R+C} < 1$, we can write

$$y_j = \alpha_j + \beta_j$$

with $\|\Sigma\alpha_j^*\alpha_j\| < 1$ and $\|\Sigma\beta_j\beta_j^*\| < 1$. Note that, since $y_j \in M'$, for all a, b in M we have

$$aby_j = ay_jb$$

and hence

$$u(ab) = V(a,b) + W(a,b)$$

where

$$V(a,b) = \sum \langle a\alpha_j b\xi, \eta\rangle e_j$$
$$W(a,b) = \sum \langle a\beta_j b\xi, \eta\rangle e_j.$$

We now claim that we can write for all a, b in M

$$V(a,b) = v_1(a)v_2(b) \quad \text{and} \quad W(a,b) = w_1(a)w_2(b) \tag{8}$$

where

$$v_1\colon\ M \to B(H\otimes K, K), \quad w_1\colon\ M \to B(H\otimes K, K)$$
$$v_2\colon\ M \to B(K, H\otimes K), \quad w_2\colon\ M \to B(K, H\otimes K)$$

are linear maps all with norm ≤ 1.

Indeed, let us set for $h \in H$, $k \in K$

$$v_1(a)(h\otimes k) = \sum\nolimits_j \langle a\beta_j h, \eta\rangle e_j k$$
$$w_1(a)(h\otimes k) = \langle ah, \eta\rangle k$$
$$v_2(b)(k) = b\xi\otimes k$$
$$w_2(b)(k) = \sum\nolimits_j \alpha_j b\xi\otimes e_j k.$$

Then, it is easy to check (8). Also, we have trivially

$$\|w_1(a)\| = \|a^*\eta\| \le \|a\|$$
$$\|v_2(b)\| = \|b\xi\| \le \|b\|.$$

Moreover, by Lemma 8 we have

$$\|v_1(a)\|^2 \le \sum\nolimits_j \|\beta_j^* a^*\eta\|^2 = \left\langle \sum \beta_j\beta_j^* a^*\eta, a^*\eta\right\rangle$$
$$\le \|a^*\eta\|^2 \le \|a\|^2$$
$$\|w_2(b)\|^2 \le \sum\nolimits_j \|\alpha_j b\xi\|^2 = \left\langle \sum \alpha_j^*\alpha_j b\xi, b\xi\right\rangle$$
$$\le \|b\xi\|^2 \le \|b\|^2.$$

By Theorem 6, it follows that

$$\|u_{|A}\|_{cb} \le 2K'.$$

Since $u\colon\ M \to B(K)$ is clearly normal (i.e. $\sigma(M, M_*)$ continuous) and since A is $\sigma(M, M_*)$ dense in M, we clearly have (by the Kaplansky density theorem)

$$\|u\|_{cb} = \|u_{|A}\|_{cb} \le 2K'.$$

Then by Lemma 7, applied with ξ_j biorthogonal to e_j, we have

$$\forall n\ \forall x_1, \dots, x_n \in M \qquad \left|\left\langle \sum x_j y_j \xi, \eta \right\rangle\right| \le 4K'C\|(x_j)\|_{R\cap C}.$$

Hence, taking the supremum over ξ, η and using homogeneity, we obtain the claimed inequality (7). Then, by Corollary 5, M is injective. □

Remark. Since Lemma 4 actually holds whenever A is an exact operator space (with C replaced by twice the exactness constant [13], [21]), the proof of Theorem 1 shows that any unital, exact (non selfadjoint) operator algebra $A \subset B(H)$ with $d(A) \le 2$ in the sense of [18] satisfies (4) for some C.

The preceding arguments establish the following result of independent interest.

Theorem 9. *A C^*-algebra A is nuclear iff for any C^*-algebra B there is a constant C such that, for all n, all $x_1, \dots, x_n$ in A and all $y_1, \dots, y_n$ in B we have*

$$\left\|\sum x_i \otimes y_i\right\|_{\max} \le C\|(x_i)\|_{R\cap C}\|(y_i)\|_{R+C}. \tag{9}$$

Proof. Let $\pi\colon\ A \to B(H)$ be a representation. Taking $B = \pi(A)'$ (and using the fact that the set of n-tuples (x_i) in A^{**} with $\|(x_i)\|_{R\cap C} \le 1$ is the weak-$*$ closure of its intersection with A^n, see e.g. [20, p. 303]) we see that (9) implies (4) for $M = \pi(A)''$. Since this holds for any π, we may argue as in the preceding proof (replacing π by π_J) to conclude that $\pi(A)''$ is injective, and hence that A is nuclear. Conversely, if A is nuclear it is easy to show (see Remark 5) that (9) holds with $C = 1$. □

Theorem 10. *A C^*-algebra A is nuclear iff for any C^*-algebra B there is a constant C such that for all n, all $x_1, \dots, x_n$ in A and all $y_1, \dots, y_n$ in B we have*

$$\left\|\sum x_i \otimes y_i\right\|_{\max} \le C \left\|\sum x_i \otimes \bar{x}_i\right\|_{\min}^{1/2} \left\|\sum y_i \otimes \bar{y}_i\right\|_{\min}^{1/2},$$

where the min norms are relative to $A \otimes \overline{A}$ and $B \otimes \overline{B}$.

Proof. It is known (see [19, (2.12)]) that $\|\Sigma x_i \otimes \bar{x}_i\|_{\min}^{1/2} \le \|(x_i)\|_{R\cap C}$. Thus, arguing as above, we find that for any representation $\pi\colon\ A \to B(H)$ the von Neumann algebra $M = \pi(A)''$ satisfies the following: if $y_1, \dots, y_n$ in M' are such that $\|\Sigma y_i \otimes \bar{y}_i\|_{\min} < 1$, then there are a_i, b_i in M' with $y_i = a_i + b_i$ such that $\|\Sigma a_i^* a_i\|^{1/2} < C$ and $\|\Sigma b_i b_i^*\|^{1/2} < C$. By [19, Th. 2.9], this ensures that M' is injective, and hence A is nuclear. □

Remark 6. Note however that by [11] the inequality

$$\left\|\sum x_i \otimes \bar{x}_i\right\|_{\max}^{1/2} \le C \left\|\sum x_i \otimes \bar{x}_i\right\|_{\min}^{1/2}$$

characterizes the weak expectation property, which is strictly more general than nuclearity.

Remark. It would be nice to know exactly which families of pairs of operator spaces in duality (F_n, F_n^*) can be used instead of $F_n = R_n \cap C_n$ or $F_n = OH_n$ to characterize nuclearity (or injectivity) analogously to the above Theorems 9 and 10 (note that $F_n = R_n$ or $F_n = C_n$ obviously do not work).

We will say that a function $f\colon\ \mathbb{N} \to \mathbb{R}_+$ is "slowly growing" if, for any $\varepsilon > 0$, there is a constant C_ε such that $f(n) \le C_\varepsilon n^\varepsilon$ for all $n \ge 1$.

The rest of the paper is devoted to a technical refinement, based on the following observation: assume that in Theorem 3 the constant C depends on n, i.e. $C = C(n)$ but that it is "slowly growing". Then N is injective.

Indeed, as for Theorem 3, this observation follows from the same argument as for [19, Th. 2.9], itself based on [9]. Recall Haagerup's characterization of finite injective von Neumann algebras ([9, Lemma 2.2]): N is finite and injective iff for any n-tuple (u_i) of unitaries and any central projection p in N we have

$$n = \|\sum pu_i \otimes \overline{pu_i}\|. \tag{10}$$

Actually, for this to hold it suffices that there exists a slowly growing function $C(n)$ such that for any n-tuple (u_i) of unitaries and any central projection p in N we have

$$n \le C(n)\|\sum pu_i \otimes \overline{pu_i}\|. \tag{11}$$

Indeed, if we set $t = \sum pu_i \otimes \overline{pu_i}$ and if we apply the preceding inequality to $(t^*t)^m$, take the m-th root and let m go to infinity, then we find that (11) implies (10) (a similar trick appears in [9, Lemma 2.2]). Given that this is true, the above observation can be deduced, first in the case when N is semifinite, and then in the general case, from the finite case by the same basic reasoning as in [9].

The following theorems are then easy to obtain in the same way as above.

Theorem 11. *The following properties of a C^*-algebra are equivalent.*

(i) *A is nuclear.*

(ii) *There is a slowly growing function $C\colon \mathbb{N} \to \mathbb{R}_+$ such that for any n and any C^*-algebra B we have*:

(12)
$$\forall (x_i) \in A^n \quad \forall (y_i) \in B^n \qquad \left\|\sum x_i \otimes y_i\right\|_{\max} \le C(n)\|(x_i)\|_{R\cap C}\|(y_i)\|_{R+C}.$$

(iii) *There is a slowly growing function $C\colon \mathbb{N} \to \mathbb{R}_+$ such that for any n and any C^*-algebra B, we have*:

(13)
$$\forall (x_i) \in A^n \qquad \forall (y_i) \in B^n$$
$$\left\|\sum x_i \otimes y_i\right\|_{\max} \le C(n) \left\|\sum x_i \otimes \bar{x}_i\right\|_{\min}^{1/2} \left\|\sum y_i \otimes \bar{y}_i\right\|_{\min}^{1/2}.$$

Corollary 12. *A von Neumann algebra M is injective iff there is a slowly growing function $C\colon \mathbb{N} \to \mathbb{R}_+$ such that, for any n, any mapping $u\colon \Delta_n \to M$ admits an extension $\tilde{u}\colon M_n \oplus M_n \to M$ such that*

$$\|\tilde{u}\|_{cb} \le C(n)\|u\|_{cb}.$$

Remark. Consider a map $u\colon E \to F$ between operator spaces. Let $\gamma(u) = \inf\{\|v\|_{cb}\|w\|_{cb}\}$ where the infimum runs over all Hilbert spaces H and all possible factorizations $u = vw$ of u through $B(H)$ (here $v : B(H) \to F$ and $w : E \to B(H)$). Let M be a von Neumann algebra. Assume that there is a constant C so that, for any n, any $u : R_n \cap C_n \to M$ satisfies $\gamma(u) \le C\|u\|_{cb}$. Then, by the preceding Corollary, M is injective. Actually, even if $C = C(n)$ depends on n, but grows slowly when $n \to \infty$, we conclude that M is injective, and hence, a posteriori, we can factor through $B(H)$ *any* u that takes values in M, regardless of its domain. It seems interesting to investigate which (sequences of) operator spaces have the property that they "force" injectivity like $\{R_n \cap C_n\}$. One can show that $\{OH_n\}$ has that property too, but obviously not $\{R_n\}$ or $\{C_n\}$, since these are themselves injective !

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