

Partial $*$ -algebras of Distributions

By

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Abstract

The problem of multiplying elements of the conjugate dual of certain kind of commutative generalized Hilbert algebras, which are dense in the set of C^∞ -vectors of a self-adjoint operator, is considered in the framework of the so-called duality method. The multiplication is defined by identifying each distribution with a multiplication operator acting on the natural rigged Hilbert space. Certain spaces, that are an abstract version of the Bessel potential spaces, are used to factorize the product.

§1. Introduction

Distributions are, as is well-known, typical objects that can be multiplied only if some very particular circumstances occur. Nevertheless, products of distributions, sometimes understood only in formal sense, frequently appear in physical applications (for instance in quantum field theory) and play a relevant role in the theory of partial differential equations. For these reasons many possibilities of defining a (partial) multiplication have been suggested in the literature (see [1] for an overview) dating back to the famous Schwartz paper devoted to the impossibility of multiplying two Dirac delta measures massed at the same point [2].

Reconsidering an idea developed in [3], we study in this paper the possibility of making of the space $\mathcal{S}'(\mathbb{R}^n)$ a partial $*$ -algebra in the sense of [4]. As is clear, the multiplication of a test function times a tempered distribution, makes of $(\mathcal{S}'(\mathbb{R}^n), \mathcal{S}(\mathbb{R}^n))$ a quasi $*$ -algebra in the sense of Lassner [5, 6] but, in this

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set-up, the corresponding *lattice of multipliers* is rather trivial. For this reason, moving within the framework of the so-called *duality method* [1, Sect. II.5], Russo and one of us proposed a way of *refining* the multiplication in $\mathcal{S}'(\mathbb{R}^n)$. This basically consists in considering distributions as multiplication operators acting on a space \mathcal{D} of test functions and then in discussing the possibility of multiplying these operators. To be more definite let us introduce some notation and basic definitions.

Let \mathcal{D} be a dense subspace of Hilbert space \mathcal{H} . Let us endow \mathcal{D} with a locally convex topology t , stronger than the one induced on \mathcal{D} by the Hilbert norm and let $\mathcal{D}'[t']$ be its topological conjugate dual endowed with the strong dual topology t' defined by the set of seminorms

$$(1) \quad F \mapsto \|F\|_{\mathcal{M}} := \sup_{\phi \in \mathcal{M}} |\langle F, \phi \rangle|$$

where \mathcal{M} runs in the family of bounded subsets of $\mathcal{D}[t]$. In this way we get the familiar triplet

$$\mathcal{D} \subset \mathcal{H} \subset \mathcal{D}'$$

called a *rigged Hilbert space*.

Given a rigged Hilbert space, $\mathcal{D} \subset \mathcal{H} \subset \mathcal{D}'$, we denote with $\mathcal{L}(\mathcal{D}, \mathcal{D}')$ the set of all continuous linear maps from $\mathcal{D}[t]$ into $\mathcal{D}'[t']$. The space $\mathcal{L}(\mathcal{D}, \mathcal{D}')$ carries a natural involution $A \rightarrow A^\dagger$ defined by

$$\langle A^\dagger f, g \rangle = \overline{\langle Ag, f \rangle} \quad \forall f, g \in \mathcal{D}.$$

Furthermore, we denote by $\mathcal{L}^+(\mathcal{D})$ the *-algebra of all closable operators in \mathcal{H} with the properties $\mathcal{D}(A) = \mathcal{D}, \mathcal{D}(A^*) \supseteq \mathcal{D}$ and both A and A^* leave \mathcal{D} invariant (* denotes here the usual Hilbert adjoint). The involution in $\mathcal{L}^+(\mathcal{D})$ is defined by $A \rightarrow A^\dagger = A^* \upharpoonright_{\mathcal{D}}$. The space $\mathcal{L}^+(\mathcal{D})$ is not, in general a subset of $\mathcal{L}(\mathcal{D}, \mathcal{D}')$ but, for instance, when t is the so called *graph-topology* [7] defined by $\mathcal{L}^+(\mathcal{D})$ on \mathcal{D} then $\mathcal{L}^+(\mathcal{D}) \subset \mathcal{L}(\mathcal{D}, \mathcal{D}')$ and $(\mathcal{L}(\mathcal{D}, \mathcal{D}'), \mathcal{L}^+(\mathcal{D}))$ is a quasi *-algebra. In this case, $A^\dagger = A^+$, $\forall A \in \mathcal{L}^+(\mathcal{D})$ (for this reason we denote with \dagger both involutions).

If $U \in \mathcal{S}'(\mathbb{R}^n)$, then the map $L_U : \phi \in \mathcal{S}(\mathbb{R}^n) \rightarrow U\phi \in \mathcal{S}'(\mathbb{R}^n)$ is continuous. Hence, the problem of multiplying two distributions $U, V \in \mathcal{S}'(\mathbb{R}^n)$ can be viewed in terms of multiplication of the corresponding operators L_U, L_V in $\mathcal{L}(\mathcal{S}(\mathbb{R}^n), \mathcal{S}'(\mathbb{R}^n))$ and then investigating conditions under which $L_U \cdot L_V = L_W$ for some $W \in \mathcal{S}'(\mathbb{R}^n)$. This was partially done in [3] also to a certain degree of abstractness: therein, in fact, tempered distributions were considered as a special case of the so called *A-distributions* defined as elements of the (conjugate)

dual of $\mathcal{D}^\infty(A)$ where A is a self-adjoint operator in $L^2(\Omega)$ and Ω is an open subset of \mathbb{R}^n .

The basic idea for solving the general problem of multiplying operators of $\mathcal{L}(\mathcal{D}, \mathcal{D}')$, with a suitable choice of \mathcal{D} , consists [8] in *factorizing* the operators through some intermediate spaces between \mathcal{D} and \mathcal{D}' that we call *interspaces*. In this way, under certain conditions that make of a family of interspaces a *multiplication framework* [11], $\mathcal{L}(\mathcal{D}, \mathcal{D}')$ becomes a *partial *-algebra* [4, 9, 10].

A *partial *-algebra* is a vector space \mathcal{A} with involution $a \rightarrow a^*$ [i.e. $(a + \lambda b)^* = a^* + \bar{\lambda}b^*$; $a = a^{**}$] and a subset $\Gamma \subset \mathcal{A} \times \mathcal{A}$ such that (i) $(a, b) \in \Gamma$ implies $(b^*, a^*) \in \Gamma$; (ii) (a, b) and $(a, c) \in \Gamma$ and $\lambda \in \mathbb{C}$ imply $(a, b + \lambda c) \in \Gamma$; and (iii) if $(a, b) \in \Gamma$, then there exists an element $ab \in \mathcal{A}$ and for this multiplication the distributive property holds in the following sense: if $(a, b) \in \Gamma$ and $(a, c) \in \Gamma$ then, by (ii), $(a, b + c) \in \Gamma$ and

$$ab + ac = a(b + c).$$

Furthermore $(ab)^* = b^*a^*$. The product is not required to be associative. The partial *-algebra \mathcal{A} is said to have a unit if there exists an element e (necessarily unique) such that $e^* = e$, $(e, a) \in \Gamma$, $ea = ae = e$, $\forall a \in \mathcal{A}$.

If $(a, b) \in \Gamma$ then we say that a is a left multiplier of b [and write $a \in L(b)$] or b is a right multiplier of a [$b \in R(a)$]. For $\mathcal{S} \subset \mathcal{A}$ we put $L\mathcal{S} = \bigcap \{L(a) : a \in \mathcal{S}\}$; the set $R\mathcal{S}$ is defined in analogous way. The set $M\mathcal{S} = L\mathcal{S} \cap R\mathcal{S}$ is called the set of universal multipliers of \mathcal{S} .

The paper is organized as follows. In Section 2 we discuss (mostly summarizing or reformulating known results) the general problem of the multiplication in $\mathcal{L}(\mathcal{D}, \mathcal{D}')$.

In Section 3 we consider the particular case where \mathcal{D} is a dense (in the graph topology) subspace of $\mathcal{D}^\infty(A)$, with A a self-adjoint operator in Hilbert space \mathcal{H} and we assume that \mathcal{D} is, at once, a (sort of) generalized commutative Hilbert algebra. The structure of partial *-algebra of the corresponding conjugate dual space \mathcal{D}' is investigated, by associating to each *distribution* $F \in \mathcal{D}'$ a multiplication operator $L_F \in \mathcal{L}(\mathcal{D}, \mathcal{D}')$. The problem is first considered from an abstract point of view. Then, for a fixed family of Banach spaces $\{E_\alpha\}$ and a domain \mathcal{D} , which is a core for all powers of A , we consider the multiplication framework generated by the spaces

$$L_A^{s,\alpha} = \{F \in \mathcal{D}' : (1 + A^2)^{\frac{s}{2}}F \in E_\alpha\}.$$

This abstract family of spaces is interesting in its own since it reduces, when $A = -i \frac{d}{dx}$ and $\{E_\alpha\} = \{L^p(\mathbb{R}^n)\}$, to the so called *Bessel potential spaces*.

In Section 4, the ideas developed in the previous sections are applied to the rigged Hilbert space generated by the tempered distributions.

§2. $\mathcal{L}(\mathcal{D}, \mathcal{D}')$ as Partial *-Algebra

The problem of multiplying operators of $\mathcal{L}(\mathcal{D}, \mathcal{D}')$ has been first considered by Kürsten [8]. Other studies have been carried out in [12] and, more recently, in [13]. In order to keep the paper sufficiently self-contained, we summarize in this Section the basic definitions and main results.

Let

$$\mathcal{D} \subset \mathcal{H} \subset \mathcal{D}'$$

be a rigged Hilbert space with $\mathcal{D}[t]$ a semireflexive space.

Let $\mathcal{E}[t_{\mathcal{E}}]$ be a locally convex space satisfying

$$(2) \quad \mathcal{D} \hookrightarrow \mathcal{E} \hookrightarrow \mathcal{D}'$$

where \hookrightarrow denotes continuous embeddings with dense range. Let \mathcal{E}' denote the conjugate dual of $\mathcal{E}[t_{\mathcal{E}}]$ endowed with its own strong dual topology $t_{\mathcal{E}'}$. Then by duality, \mathcal{E}' is continuously embedded in \mathcal{D}' and the embedding has dense range. Also \mathcal{D} is continuously embedded in \mathcal{E} but in this case the image of \mathcal{D} is not necessarily dense in \mathcal{E}' , unless \mathcal{E} is semi-reflexive. In order to avoid this difficulty, we endow \mathcal{E} with the Mackey topology $\tau(\mathcal{E}, \mathcal{E}') := \tau_{\mathcal{E}}$ and \mathcal{E}' with $\tau(\mathcal{E}', \mathcal{E}) := \tau_{\mathcal{E}'}$. The same can be done, of course, with the spaces \mathcal{D} and \mathcal{D}' themselves. If (2) holds for the initial topologies, then it holds also when each space is endowed with the Mackey topology.

Following [11], a subspace \mathcal{E} of \mathcal{D}' satisfying (2) and endowed with $\tau_{\mathcal{E}}$ will be called an *interspace*. Clearly, if \mathcal{E} is an interspace, then $\mathcal{E}'[\tau_{\mathcal{E}'}]$ is an interspace too.

Let \mathcal{E}, \mathcal{F} be interspaces and $\mathcal{L}(\mathcal{E}, \mathcal{F})$ the linear space of all continuous linear maps from \mathcal{E} into \mathcal{F} . Following [8], we define

$$\mathcal{C}(\mathcal{E}, \mathcal{F}) = \left\{ A \in \mathcal{L}(\mathcal{D}, \mathcal{D}') : A = \tilde{A} \upharpoonright \mathcal{D} \text{ for some } \tilde{A} \in \mathcal{L}(\mathcal{E}, \mathcal{F}) \right\}.$$

It is not difficult to prove that $\mathcal{L}^{\dagger}(\mathcal{D}) = \mathcal{C}(\mathcal{D}, \mathcal{D}) \cap \mathcal{C}(\mathcal{D}', \mathcal{D}')$.

Let now $A, B \in \mathcal{L}(\mathcal{D}, \mathcal{D}')$ and assume that there exists an interspace \mathcal{E} such that $B \in \mathcal{L}(\mathcal{D}, \mathcal{E})$ and $A \in \mathcal{C}(\mathcal{E}, \mathcal{D}')$; it would then be natural to define

$$A \cdot Bf = \tilde{A}(Bf), \quad f \in \mathcal{D}.$$

This product is not, however, well defined, because it may depend on the choice of the interspace \mathcal{E} . As shown by Kürsten [8], this pathology is due to the fact

that \mathcal{D} is not necessarily dense in the intersection $\mathcal{E} \cap \mathcal{F}$ of two interspaces \mathcal{E}, \mathcal{F} , endowed with the projective topology $\tau_{\mathcal{E}} \wedge \tau_{\mathcal{F}}$. For this reason, we give the following definition.

Definition 2.1. A family \mathfrak{L}_0 of interspaces in the rigged Hilbert space $(\mathcal{D}[t], \mathcal{H}, \mathcal{D}'[t'])$ is said to be tight (around \mathcal{D}) if $\mathcal{E} \cap \mathcal{F}$ is an interspace for any pair of interspaces $\mathcal{E}, \mathcal{F} \in \mathfrak{L}_0$.

Definition 2.2. Let \mathfrak{L}_0 be a tight family of interspaces in the rigged Hilbert space $(\mathcal{D}[t], \mathcal{H}, \mathcal{D}'[t'])$. The product $A \cdot B$ of two elements of $\mathcal{L}(\mathcal{D}, \mathcal{D}')$ is defined, with respect to \mathfrak{L}_0 , if there exist three interspaces $\mathcal{E}, \mathcal{F}, \mathcal{G} \in \mathfrak{L}_0$ such that $A \in \mathcal{C}(\mathcal{F}, \mathcal{G})$ and $B \in \mathcal{C}(\mathcal{E}, \mathcal{F})$. In this case the multiplication $A \cdot B$ is defined by:

$$A \cdot B = (\tilde{A}\tilde{B}) \upharpoonright \mathcal{D}$$

or, equivalently, by:

$$A \cdot Bf = \tilde{A}Bf, \quad f \in \mathcal{D}.$$

where \tilde{A} (resp., \tilde{B}) denote the extension of A (resp., B) to \mathcal{E} (resp., \mathcal{F}).

This definition does not depend on the particular choice of the interspaces $\mathcal{E}, \mathcal{F}, \mathcal{G} \in \mathfrak{L}_0$ [8, Proposition 3.2] but may depend on \mathfrak{L}_0 . Of course, we may also suppose that $\mathcal{E} = \mathcal{D}$ and $\mathcal{G} = \mathcal{D}'$. With this choice, for the product $A \cdot B$ to make sense, we need only to require the existence of one interspace \mathcal{F} such that $A \in \mathcal{C}(\mathcal{F}, \mathcal{D}')$ and $B \in \mathcal{C}(\mathcal{D}, \mathcal{F})$. The price is, of course, a loss of information on the range of $A \cdot B$. Nevertheless, given a tight family of interspaces around \mathcal{D} , $\mathcal{L}(\mathcal{D}, \mathcal{D}')$ is not, in general, a partial *-algebra with respect to the multiplication defined above. This is due to the fact that a family of interspaces around \mathcal{D} is not necessarily closed under the operation of taking duals and under finite intersections. We give the following definition [11]:

Definition 2.3. A family \mathfrak{L} of interspaces in the rigged Hilbert space $(\mathcal{D}[t], \mathcal{H}, \mathcal{D}'[t'])$ is called a *multiplication framework* if

- (i) $\mathcal{D} \in \mathfrak{L}$
- (ii) $\forall \mathcal{E} \in \mathfrak{L}$, its dual \mathcal{E}' also belongs to \mathfrak{L}
- (iii) $\forall \mathcal{E}, \mathcal{F} \in \mathfrak{L}$, $\mathcal{E} \cap \mathcal{F} \in \mathfrak{L}$.

In many instances, however, instead of the notion of multiplication framework, a lighter condition can be of some usefulness: we call *generating* a tight

family of interspaces \mathfrak{L}_0 closed under duality and enjoying the following property:

- \mathcal{D} is dense in $\mathcal{E}_1 \cap \dots \cap \mathcal{E}_n$, endowed with its own projective topology, for any finite set $\{\mathcal{E}_1, \dots, \mathcal{E}_n\}$ of elements of \mathfrak{L}_0 .

Proposition 2.4. *Let \mathfrak{L} be a multiplication framework in the rigged Hilbert space $(\mathcal{D}[t], \mathcal{H}, \mathcal{D}'[t'])$. Then $\mathcal{L}(\mathcal{D}, \mathcal{D}')$ with the multiplication defined above is a (non-associative) partial $*$ -algebra.*

The same statement holds true if we replace the multiplication framework \mathfrak{L} with a generating family of interspaces.

In [13], a particular generating family has been constructed in the case where the rigged Hilbert space is that generated by a single self-adjoint operator A with domain $D(A)$ in \mathcal{H} . As usual we put

$$\mathcal{D}^\infty(A) = \bigcap_{n=1}^\infty \mathcal{D}(A^n).$$

Endowed with the topology t_A generated by the set of seminorms

$$f \rightarrow \|A^n f\|, \quad n \in \mathbb{N},$$

$\mathcal{D}^\infty(A)$ is a Fréchet and reflexive domain; let us denote with $\mathcal{D}_{-\infty}(A)$ its conjugate dual with respect to the scalar product of \mathcal{H} and endow it with the strong dual topology t'_A .

Let $\{E_\alpha\}_{\alpha \in I}$ be a family of interspaces, such that each E_α is a Banach space with norm $\|\cdot\|_\alpha$. If the family is closed under duality between $\mathcal{D}_{-\infty}(A)$ and $\mathcal{D}^\infty(A)$, then we can define an involution $\alpha \rightarrow \bar{\alpha}$ in the set of indices, such that $E_{\bar{\alpha}} \simeq (E_\alpha)'$. We assume, in particular, that the sesquilinear form which puts E_α and $E_{\bar{\alpha}}$ in conjugate duality extends the inner product of \mathcal{D} . Thus:

$$|\langle F, G \rangle| \leq \|F\|_\alpha \|G\|_{\bar{\alpha}} \quad \forall F \in E_\alpha \quad \forall G \in E_{\bar{\alpha}}.$$

Let $U(t)$ be a one-parameter group of unitaries generated by A and $\widehat{U}(t)$ its continuous extension to $\mathcal{D}_{-\infty}(A)$. Then, the family $\{E_\alpha\}_{\alpha \in I}$ is compatible with A if the following conditions are satisfied:

- 1) $\widehat{U}(t)E_\alpha = E_\alpha, \quad \forall \alpha \in I$
- 2) $\lim_{t \rightarrow 0} \|\widehat{U}(t)F - F\|_\alpha = 0, \quad \forall F \in E_\alpha, \quad \forall \alpha \in I.$

We define for all $s \in \mathbb{R}$ and $\alpha \in I$ the set $\mathcal{L}_A^{s,\alpha}$:

$$(3) \quad \mathcal{L}_A^{s,\alpha} := \{F \in \mathcal{D}_{-\infty}(A) : J^s F \in E_\alpha\}$$

where $J = (1 + A^2)^{1/2}$.

Lemma 2.5. *If, for every $s < 0$, J^s maps E_α continuously into itself, then if $s_1 < s_2$, $\mathcal{L}_A^{s_2,\alpha} \subset \mathcal{L}_A^{s_1,\alpha}$ and the embedding is continuous.*

Proof. Indeed, one has, for some $C > 0$:

$$\|J^{s_1} F\|_\alpha = \|J^{s_1-s_2} J^{s_2} F\|_\alpha \leq C \|J^{s_2} F\|_\alpha.$$

□

Proposition 2.6. *If $\{E_\alpha\}_{\alpha \in I}$ is a family of Banach spaces compatible with A , then the following statements hold:*

- 1) *For each $s \in \mathbb{R}$ and $\alpha \in I$, $\mathcal{L}_A^{s,\alpha}[\|\cdot\|_{s,\alpha}]$, endowed with the norm $\|F\|_{s,\alpha} = \|(1 + A^2)^{\frac{s}{2}} F\|_\alpha$, is a Banach space, and $(1 + A^2)^{\frac{s}{2}}$ is an isometry of $\mathcal{L}_A^{s,\alpha}$ in E_α .*
- 2) $\mathcal{D}^\infty(A) \hookrightarrow \mathcal{L}_A^{s,\alpha}$, $s \in \mathbb{R}, \alpha \in I$ (\hookrightarrow denotes a continuous embedding).
- 3) $(\mathcal{L}_A^{s,\alpha})' \simeq \mathcal{L}_A^{-s,\bar{\alpha}}$, $s \in \mathbb{R}, \alpha \in I$.
- 4) *For any $s \in \mathbb{R}$, $\mathcal{D}^\infty(A)$ is dense in any finite intersection of the spaces $\mathcal{L}_A^{s,\alpha}$ endowed with the projective norm.*

The notion of compatibility is crucial in the proof of 4). Indeed, in this case, if $F \in \bigcap_{i=1}^n \mathcal{L}_A^{s_i,\alpha_i}$, as shown in [13], the net $\{j_\epsilon^A * F\}$ converges to F with respect to the projective topology. The net $\{j_\epsilon^A * F\}$ is defined by

$$j_\epsilon^A * F = \int_{\mathbb{R}} j_\epsilon(t) \widehat{U}(t) F dt$$

where j_ϵ is the approximate identity constructed from a regularizing function $j \in C_0^\infty(\mathbb{R})$ and $\widehat{U}(t)$ the continuous extension of $U(t)$ to E_α .

This fact and Proposition 2.6 imply that the spaces $\mathcal{L}_A^{s,\alpha}$ constitute a generating family of interspaces, in the sense explained above.

More in general, we may consider a subspace $\mathcal{D} \subset \mathcal{D}^\infty(A)$ endowed with a locally convex topology t which makes of \mathcal{D} a semireflexive space and satisfying the following properties:

- d1) the topology t of \mathcal{D} is finer than the topology induced from $\mathcal{D}^\infty(A)$;
- d2) both A and J map $\mathcal{D}[t]$ continuously into itself;
- d3) for all $n \in \mathbb{N}$, \mathcal{D} is a *core* for J^n , that is: $\overline{J^n \upharpoonright_{\mathcal{D}}} = J^n$.

If \mathcal{D}' denotes the conjugate dual of $\mathcal{D}[t]$, we have the following situation

$$\mathcal{D} \hookrightarrow \mathcal{D}^\infty(A) \hookrightarrow \mathcal{H} \hookrightarrow \mathcal{D}_{-\infty}(A) \hookrightarrow \mathcal{D}'$$

and if $\{G_\alpha\}$ is a multiplication framework between $\mathcal{D}^\infty(A)$ and $\mathcal{D}_{-\infty}(A)$, so it is between \mathcal{D} and \mathcal{D}' also. In analogy with (3), we define

$$L_A^{s,\alpha} = \{F \in \mathcal{D}' : (1 + A^2)^{\frac{s}{2}} F \in E_\alpha\}.$$

The following Lemma, proved in [13] will be needed in what follows.

Lemma 2.7. *For each $s \in \mathbb{R}$ and $\alpha \in I$, the space $L_A^{s,\alpha}[\|\cdot\|_{s,\alpha}]$ is a Banach space. Moreover, the map*

$$F \in L_A^{s,\alpha} \rightarrow F \upharpoonright_{\mathcal{D}_{-\infty}(A)} \in \mathcal{L}_A^{s,\alpha}$$

is an isometric isomorphism of Banach spaces.

Clearly, for the spaces $L_A^{s,\alpha}$ a statement completely analogous to Proposition 2.6 holds.

§3. Multiplication of Distributions

In this Section we will discuss the problem of the multiplication of distributions identifying them with certain multiplication operators acting in the rigged Hilbert space of distributions itself and applying the methods of Section 2. We will try to maintain the situation as abstract as we can; thus, instead of considering specific test function spaces, such as $\mathcal{D}(\mathbb{R}^n)$ or $\mathcal{S}(\mathbb{R}^n)$, we start with a dense domain in Hilbert space which is at once a commutative *-algebra satisfying additional topological requirements.

Let \mathcal{D} be a domain satisfying the conditions d1)-d3) above with respect to a fixed self-adjoint operator A . We assume, in addition, that \mathcal{D} is a *generalized* commutative Hilbert *-algebra in the sense that \mathcal{D} is a commutative *-algebra with respect to the involution $\phi \mapsto \phi^*$ and the multiplication $(\phi, \psi) \mapsto \phi\psi (= \psi\phi)$ and the following conditions hold:

- (h1) $\langle \phi, \psi \rangle = \langle \psi^*, \phi^* \rangle, \quad \forall \phi, \psi \in \mathcal{D};$
- (h2) $\langle \phi\psi, \chi \rangle = \langle \psi, \phi^*\chi \rangle, \quad \forall \phi, \psi, \chi \in \mathcal{D};$
- (h3) the multiplication $(\phi, \psi) \mapsto \phi\psi (= \psi\phi)$ is *jointly* continuous with respect to the topology t of \mathcal{D} ;
- (h4) the involution $\phi \mapsto \phi^*$ is continuous in $\mathcal{D}[t]$;
- (h5) $\mathcal{D} \cdot \mathcal{D}$ is dense in $\mathcal{D}[t]$.

Proposition 3.1. *Let \mathcal{D}' be the (conjugate) dual of \mathcal{D} . If we define the multiplication of an element $F \in \mathcal{D}'$ and an element $\phi \in \mathcal{D}$ by*

$$\langle F\phi, \psi \rangle = \langle \phi F, \psi \rangle := \langle F, \phi^*\psi \rangle, \quad \forall \psi \in \mathcal{D},$$

then $(\mathcal{D}'[t'], \mathcal{D})$ is a topological quasi-algebra.*

Proof. First, we prove that $F\phi \in \mathcal{D}', \forall \phi \in \mathcal{D}$. Indeed, let $\{p_\gamma^{\mathcal{D}}\}_{\gamma \in K}$ denote a directed family of seminorms defining the topology t ; then using the continuity of F and (h3), we can find a bounded subset \mathcal{M} of \mathcal{D} , $\gamma, \delta \in K$ and a positive constant C such that

$$|\langle F\phi, \psi \rangle| = |\langle F, \phi^*\psi \rangle| \leq \|F\|_{\mathcal{M}} p_\gamma^{\mathcal{D}}(\phi^*\psi) \leq C p_\delta^{\mathcal{D}}(\phi) p_\delta^{\mathcal{D}}(\psi), \quad \forall \phi, \psi \in \mathcal{D}.$$

Therefore, $F\phi$ is a continuous linear functional on \mathcal{D} . Now, we define an involution $*$ in \mathcal{D}' which extends the involution of \mathcal{D} . This can be done by setting

$$F \in \mathcal{D}' \mapsto F^* \in \mathcal{D}'$$

where

$$\langle F^*, \phi \rangle := \overline{\langle F, \phi^* \rangle}, \quad \forall \phi \in \mathcal{D}.$$

The involution defined in this way satisfies the equality

$$(F\psi)^* = \psi^* F^*, \quad \forall F \in \mathcal{D}', \forall \psi \in \mathcal{D}.$$

Furthermore, it is continuous, since, if \mathcal{M} is a bounded subset of \mathcal{D} , we have:

$$\|F^*\|_{\mathcal{M}} = \sup_{\phi \in \mathcal{M}} |\langle F^*, \phi \rangle| = \sup_{\phi \in \mathcal{M}} |\langle F, \phi^* \rangle| = \sup_{\phi \in \mathcal{M}^*} |\langle F, \phi \rangle| = \|F\|_{\mathcal{M}^*}$$

where $\mathcal{M}^* := \{\phi^*, \phi \in \mathcal{M}\}$ is bounded as continuous image of a bounded set. For each fixed ϕ in \mathcal{D} , the map

$$F \in \mathcal{D}' \mapsto F\phi \in \mathcal{D}'$$

is continuous. Indeed, let \mathcal{M} be a bounded subset of \mathcal{D} ; we have:

$$\|F\phi\|_{\mathcal{M}} = \|F\|_{\phi\mathcal{M}}.$$

The set $\phi\mathcal{M}$ is still bounded in \mathcal{D} since it is the continuous image of a bounded set. Finally, it is clear, by the semireflexivity, that \mathcal{D} is dense in \mathcal{D}' . \square

From these facts it follows that to each element F of \mathcal{D}' we can associate an operator L_F of multiplication on \mathcal{D} defined by

$$L_F : \phi \in \mathcal{D} \rightarrow F\phi \in \mathcal{D}'.$$

This is a continuous linear map of \mathcal{D} into \mathcal{D}' . Indeed, if \mathcal{M} is bounded in $\mathcal{D}[t]$, by the continuity of F , there exists $C > 0$ and $\gamma \in K$ such that:

$$\sup_{\chi \in \mathcal{M}} |\langle F\phi, \chi \rangle| = \sup_{\chi \in \mathcal{M}} |\langle F, \phi^* \chi \rangle| \leq C \sup_{\chi \in \mathcal{M}} p_\gamma^{\mathcal{D}}(\phi^* \chi).$$

Then, by the continuity of the multiplication and of the involution, we can find a new constant $C_1 > 0$ and $\delta \in K$ such that

$$\sup_{\chi \in \mathcal{M}} |\langle F\phi, \chi \rangle| \leq C_1 p_\delta^{\mathcal{D}}(\phi).$$

Moreover, the map $j : F \in \mathcal{D}' \rightarrow L_F \in \mathcal{L}(\mathcal{D}, \mathcal{D}')$ is injective; indeed, $L_F = 0$ implies $L_F\phi = 0, \forall \phi \in \mathcal{D}$ and so

$$\langle F\phi, \psi \rangle = \langle F, \phi^* \psi \rangle = 0, \quad \forall \phi \in \mathcal{D}, \psi \in \mathcal{D}.$$

The density of $\mathcal{D} \cdot \mathcal{D}$ in \mathcal{D} implies that $F = 0$. Clearly, for $F, G \in \mathcal{D}'$ and $\lambda \in \mathbb{C}$, we have

$$L_{F+G} = L_F + L_G, \quad L_{\lambda F} = \lambda L_F, \quad (L_F)^* = L_{F^*}.$$

Therefore, the problem of multiplying two distributions F, G can be formulated in terms of the multiplication of the corresponding operators L_F and L_G . The multiplication of operators of this kind can then be studied in the terms proposed in [13] and summarized in the previous section. Nevertheless, even though the product $L_F \cdot L_G$ exists in $\mathcal{L}(\mathcal{D}, \mathcal{D}')$, it is not necessarily an operator of multiplication by some distribution. For this to happen, additional conditions must be added. A net (η_ϵ) of elements of \mathcal{D} is called an approximate identity of \mathcal{D} if

$$t - \lim_{\epsilon} \eta_\epsilon \phi = \phi, \quad \forall \phi \in \mathcal{D}.$$

Proposition 3.2. *Assume that \mathcal{D} has an approximate identity and let $X \in \mathcal{L}(\mathcal{D}, \mathcal{D}')$. Then $X = L_V$ for some $V \in \mathcal{D}'$ if, and only if, the following two conditions are fulfilled*

- (i) $X(\phi\psi) = \phi X\psi, \quad \forall \phi, \psi \in \mathcal{D};$
- (ii) *there exists $C > 0$ and $\gamma \in K$ such that*

$$|\langle X\phi, \psi \rangle| \leq Cp_\gamma^{\mathcal{D}}(\phi^*\psi), \quad \forall \phi, \psi \in \mathcal{D}.$$

Proof. The necessity is obvious. As for the sufficiency, let (η_ϵ) be an approximate identity of \mathcal{D} . We define a conjugate linear functional V_η on \mathcal{D} by

$$V_\eta(\phi) = \lim_\epsilon \langle X\eta_\epsilon, \phi \rangle, \quad \phi \in \mathcal{D}.$$

We have

$$|V_\eta(\phi)| = \lim_\epsilon |\langle X\eta_\epsilon, \phi \rangle| \leq C \lim_\epsilon p_\delta^{\mathcal{D}}(\eta_\epsilon\phi) = Cp_\delta^{\mathcal{D}}(\phi), \quad \forall \phi \in \mathcal{D}.$$

Therefore, $V_\eta \in \mathcal{D}'$.

Now, making use of (i), we get, for $\phi, \psi \in \mathcal{D}$,

$$\langle L_{V_\eta}\phi, \psi \rangle = \langle V_\eta, \phi^*\psi \rangle = \lim_\epsilon \langle X\eta_\epsilon, \phi^*\psi \rangle = \langle X\phi, \psi \rangle.$$

Therefore the definition of V_η is independent of (η_ϵ) and X is a multiplication operator. □

Remark 3.3. We notice that the assumption that \mathcal{D} has an approximate identity is used only for the sufficiency. The previous Proposition can be seen as an abstract version of [3, Proposition 3.10] where it was proved for $\mathcal{D} = \mathcal{S}(\mathbb{R}^n)$.

In order to simplify notations, from now on we will not distinguish graphically elements of \mathcal{D}' and \mathcal{D} . This means that we consider \mathcal{D} as a true subspace of \mathcal{D}' .

Lemma 3.4. *Let $u \in \mathcal{D}'$ and let \mathcal{F} and \mathcal{G} be two interspaces. Assume that L_u has a continuous extension $\tilde{L}_u : \mathcal{F} \rightarrow \mathcal{G}$, i.e., $L_u \in \mathcal{C}(\mathcal{F}, \mathcal{G})$. Then also L_{u^*} has a continuous extension from \mathcal{G}' to \mathcal{F}' , and $(\tilde{L}_u)^\dagger = \tilde{L}_{u^*} = (\tilde{L}_u^\dagger)$.*

Proof. Since $\tilde{L}_u \in \mathcal{L}(\mathcal{F}, \mathcal{G})$, then it has an adjoint

$$(\tilde{L}_u)^\dagger : \mathcal{G}' \rightarrow \mathcal{F}'$$

defined by

$${}_g \langle \widetilde{L}_u f, g \rangle_{\mathcal{G}'} = {}_{\mathcal{F}} \langle f, (\widetilde{L}_u)^\dagger g \rangle_{\mathcal{F}'}, \quad \forall f \in \mathcal{F}, \forall g \in \mathcal{G}'$$

on the other hand, for each $\phi, \psi \in \mathcal{D}$, one has:

$$\begin{aligned} {}_{\mathcal{D}'} \langle L_u \psi, \phi \rangle_{\mathcal{D}} &= {}_{\mathcal{D}'} \langle u \psi, \phi \rangle_{\mathcal{D}} = {}_{\mathcal{D}'} \langle u, \psi^* \phi \rangle_{\mathcal{D}} = {}_{\mathcal{D}'} \langle \overline{u^*, (\psi^* \phi)^*} \rangle_{\mathcal{D}} \\ &= {}_{\mathcal{D}} \langle \phi^* \psi, u^* \rangle_{\mathcal{D}'} = {}_{\mathcal{D}} \langle \psi, \phi u^* \rangle_{\mathcal{D}'} = {}_{\mathcal{D}} \langle \psi, L_{u^*} \phi \rangle_{\mathcal{D}'}, \end{aligned}$$

thus $(L_u)^\dagger = L_{u^*}$. If $g \in \mathcal{G}'$, there exists a net $\omega_\epsilon \in \mathcal{D}$ converging to g in the topology of \mathcal{G}' . By definition of continuous extension one has:

$${}_{\mathcal{D}} \langle \psi, L_{u^*} \omega_\epsilon \rangle_{\mathcal{D}'} \rightarrow {}_{\mathcal{F}} \langle \psi, \widetilde{L}_{u^*} f \rangle_{\mathcal{F}'}$$

and so, by definition: $(\widetilde{L}_u)^\dagger = \widetilde{L}_{u^*}$. The equality $(\widetilde{L}_u)^\dagger = (\widetilde{L}_u^\dagger)$ follows easily. \square

Let now \mathcal{T} be a multiplication framework satisfying the following properties:

- A1) If $\mathcal{F} \in \mathcal{T}$, then $*$: $\mathcal{F} \rightarrow \mathcal{F}$, continuously.
- A2) If $\mathcal{F} \in \mathcal{T}$ and $\phi \in \mathcal{D}$, then $\phi \mathcal{F} \subset \mathcal{F}$, and the map $T_\phi : \mathcal{F} \rightarrow \mathcal{F}$, defined by:

$$u \in \mathcal{F} \mapsto \phi u \in \mathcal{F}$$
 is continuous in \mathcal{F} .
- A3) If $\mathcal{F} \in \mathcal{T}$, $u \in \mathcal{F}$, the map $L_u : \phi \in \mathcal{D} \rightarrow \phi u \in \mathcal{F}$ is continuous from \mathcal{D} into \mathcal{F} .

Remark 3.5. If A1) and A2) are satisfied, then (h1) and (h2) extend to any pair $\mathcal{F}, \mathcal{F}'$ of dual interspaces of \mathcal{T} , due to the density of \mathcal{D} in any interspace. That is

- (h1)' ${}_{\mathcal{F}} \langle u, v \rangle_{\mathcal{F}'} = {}_{\mathcal{F}} \langle \overline{u^*, v^*} \rangle_{\mathcal{F}'}, \quad \forall u \in \mathcal{F}, v \in \mathcal{F}'$
- (h2)' ${}_{\mathcal{F}} \langle u \phi, v \rangle_{\mathcal{F}'} = {}_{\mathcal{F}} \langle u, \phi^* v \rangle_{\mathcal{F}'}, \quad \forall u \in \mathcal{F}, v \in \mathcal{F}', \phi \in \mathcal{D}$.

Making use of these facts, we have

Proposition 3.6. *Let \mathcal{D} possess an approximate identity and let \mathcal{T} be a multiplication framework satisfying A1) and A2). If $u, v \in \mathcal{D}'$ and $L_u \circ L_v$ is well defined with respect to \mathcal{T} , then also $L_v \circ L_u$ is well defined and*

$$L_u \circ L_v = L_v \circ L_u.$$

Proof. $L_u \circ L_v$ is well defined, then there exist $\mathcal{E}, \mathcal{F}, \mathcal{G} \in \mathcal{T}$ such that:

$$\begin{aligned} \tilde{L}_v &: \mathcal{E} \rightarrow \mathcal{F} \\ \tilde{L}_u &: \mathcal{F} \rightarrow \mathcal{G} \end{aligned}$$

continuously. For each $\phi, \psi \in \mathcal{D}$, if (η_ϵ) is the approximate identity of \mathcal{D} :

$$\begin{aligned} {}_{\mathcal{D}'} \langle L_u \circ L_v \phi, \psi \rangle_{\mathcal{D}} &= \lim_{\epsilon} {}_{\mathcal{D}'} \langle L_u \circ L_v \phi, \psi \eta_\epsilon \rangle_{\mathcal{D}} = \lim_{\epsilon} {}_{\mathcal{F}} \langle L_v \phi, L_u^\dagger \psi \eta_\epsilon \rangle_{\mathcal{F}'} \\ &= \lim_{\epsilon} {}_{\mathcal{F}} \langle v \phi, u^* \psi \eta_\epsilon \rangle_{\mathcal{F}'} = \lim_{\epsilon} {}_{\mathcal{F}} \langle \psi^* v \phi, u^* \eta_\epsilon \rangle_{\mathcal{F}'} = \lim_{\epsilon} {}_{\mathcal{F}} \langle \psi^* v, u^* \eta_\epsilon \phi^* \rangle_{\mathcal{F}'} \\ &= {}_{\mathcal{F}} \langle \psi^* v, u^* \phi^* \rangle_{\mathcal{F}'} = {}_{\mathcal{F}} \langle (\psi v^*)^*, (u \phi)^* \rangle_{\mathcal{F}'} = {}_{\mathcal{F}'} \langle u \phi, v^* \psi \rangle_{\mathcal{F}} = {}_{\mathcal{F}'} \langle L_u \phi, L_v^\dagger \psi \rangle_{\mathcal{F}}. \end{aligned}$$

This implies that $L_v \circ L_u$ is well defined. From the previous equalities we get:

$${}_{\mathcal{D}'} \langle (L_u \circ L_v) \phi, \psi \rangle_{\mathcal{D}} = {}_{\mathcal{F}'} \langle L_u \phi, L_v^\dagger \psi \rangle_{\mathcal{F}} = {}_{\mathcal{D}'} \langle (L_v \circ L_u) \phi, \psi \rangle_{\mathcal{D}} \quad \forall \phi, \psi \in \mathcal{D}$$

and therefore $L_u \circ L_v = L_v \circ L_u$. □

The equality $L_u \circ L_v = L_v \circ L_u$ stated in the previous Proposition strictly depends on the fact that L_u and L_v are multiplication operators, but it does not imply that $L_u \circ L_v$ is itself a multiplication operator. Thus, in what follows we will give conditions for the product $L_u \circ L_v$ to be itself a distribution, in the sense that there exist $w \in \mathcal{D}'$ such that $L_u \circ L_v = L_w$.

Lemma 3.7. *Let \mathcal{T} be a multiplication framework satisfying the properties A2), A3) and let $v \in \mathcal{A} \in \mathcal{T}$ and $u \in \mathcal{D}'$. Let L_v, L_u be the corresponding multiplication operators from \mathcal{D} into \mathcal{D}' . If L_u has a continuous extension $\tilde{L}_u : \mathcal{A} \rightarrow \mathcal{C} \in \mathcal{T}$, then there exist $C > 0$ and a seminorm $p_\gamma^{\mathcal{C}'}$ in the directed family defining the topology of \mathcal{C}' such that for each $\phi, \psi \in \mathcal{D}$:*

$$(4) \quad |{}_{\mathcal{D}'} \langle \tilde{L}_u \cdot L_v \phi, \psi \rangle_{\mathcal{D}}| \leq C p_\gamma^{\mathcal{C}'}(\psi \phi^*).$$

Proof. Indeed, there exist $C_1 > 0$ and two seminorms $p_{\delta_1}^{\mathcal{A}}$ and $p_{\delta_2}^{\mathcal{A}'}$ in the directed families generating, respectively, the topologies of \mathcal{A} and \mathcal{A}' such that:

$$\begin{aligned} |{}_{\mathcal{D}'} \langle \tilde{L}_u \cdot L_v \phi, \psi \rangle_{\mathcal{D}}| &= |{}_c \langle \tilde{L}_u \cdot L_v \phi, \psi \rangle_{c'}| = |{}_A \langle L_v \phi, \tilde{L}_u^\dagger \psi \rangle_{A'}| = |{}_A \langle v \phi, u^* \psi \rangle_{A'}| \\ &= |{}_A \langle v, u^* \psi \phi^* \rangle_{A'}| \leq C_1 p_{\delta_1}^{\mathcal{A}}(v) p_{\delta_2}^{\mathcal{A}'}(\tilde{L}_u^\dagger(\psi \phi^*)). \end{aligned}$$

Since the adjoint map $\tilde{L}_u^\dagger : \mathcal{C}' \rightarrow \mathcal{A}'$ is continuous, there exists $C_2 > 0$ and a seminorm $p_\gamma^{\mathcal{C}'}$ on \mathcal{C}' such that:

$$C_1 p_{\delta_1}^{\mathcal{A}}(v) p_{\delta_2}^{\mathcal{A}'}(\tilde{L}_u^\dagger(\psi \phi^*)) \leq C_1 p_{\delta_1}^{\mathcal{A}}(v) C_2 p_\gamma^{\mathcal{C}'}(\psi \phi^*) = C p_\gamma^{\mathcal{C}'}(\psi \phi^*),$$

where C depends, in general, on u, v . □

Proposition 3.8. *Let \mathcal{T} be a multiplication framework satisfying the properties A2), A3). For fixed $v \in \mathcal{A}$ and $u \in \mathcal{D}'$, let L_v, L_u be the respective multiplication maps from \mathcal{D} into \mathcal{D}' . If L_u has a continuous extension, $\tilde{L}_u : \mathcal{A} \rightarrow \mathcal{C} \in \mathcal{T}$, then there exists $w \in \mathcal{C}$ such that $L_w = \tilde{L}_u \cdot L_v : \mathcal{D} \rightarrow \mathcal{C}$.*

Proof. We need to prove that the conditions (i), (ii) of Proposition 3.2 are satisfied.

First we prove (i), that is

$$L_u \cdot L_v(\phi\psi) = \phi L_u \cdot L_v\psi \quad \forall \phi, \psi \in \mathcal{D}.$$

Indeed, for each $\phi, \psi, \xi \in \mathcal{D}$, we have:

$$\begin{aligned} {}_{\mathcal{D}'}\langle L_u \cdot L_v(\phi\psi), \xi \rangle_{\mathcal{D}} &= {}_c\langle \tilde{L}_u \cdot L_v(\phi\psi), \xi \rangle_{\mathcal{C}'} = {}_{\mathcal{A}}\langle L_v(\phi\psi), \tilde{L}_u^\dagger \xi \rangle_{\mathcal{A}'} \\ {}_{\mathcal{A}}\langle v(\phi\psi), u^* \xi \rangle_{\mathcal{A}'} &= {}_{\mathcal{A}}\langle v\psi, u^* \xi \phi^* \rangle_{\mathcal{A}'} = {}_{\mathcal{A}}\langle L_v\psi, \tilde{L}_u^\dagger \xi \phi^* \rangle_{\mathcal{A}'} = {}_c\langle \tilde{L}_u \cdot L_v\psi, \xi \phi^* \rangle_{\mathcal{C}'} \\ &= {}_c\langle \phi \tilde{L}_u \cdot L_v\psi, \xi \rangle_{\mathcal{C}'} = {}_{\mathcal{D}'}\langle \phi L_u \cdot L_v\psi, \xi \rangle_{\mathcal{D}}. \end{aligned}$$

In order to prove (ii) of Proposition 3.2, we make use of the inequality (4) stated in Lemma 3.7. Since \mathcal{D} is continuously embedded in \mathcal{C}' , there exists $C_3 > 0$ and $\delta \in K$ such that $p_\gamma^{\mathcal{C}'}(\psi\phi^*) \leq C_3 p_\delta^{\mathcal{D}}(\psi\phi^*)$. Then, we have:

$$|{}_{\mathcal{D}'}\langle \tilde{L}_u \cdot L_v\phi, \psi \rangle_{\mathcal{D}}| \leq C_4 p_\delta^{\mathcal{D}}(\psi\phi^*)$$

for some $C_4 > 0$. Then, by Proposition 3.2, there exists $w \in \mathcal{D}'$ such that:

$$L_w = \tilde{L}_u \cdot L_v.$$

In order to prove that $w \in \mathcal{C}$, it is enough to show that w is a continuous functional on \mathcal{D} as a dense subspace of \mathcal{C}' . Let (η_ϵ) be an approximate identity of \mathcal{D} : then $\eta_\epsilon\phi \rightarrow \phi$ in the topology of \mathcal{D} , and therefore also in \mathcal{C}' , which is weaker. Using the inequality (4), there exists $C > 0$ and a seminorm $p_\gamma^{\mathcal{C}'}$ on \mathcal{C}' such that:

$$\begin{aligned} |{}_{\mathcal{D}'}\langle w, \phi \rangle_{\mathcal{D}}| &= \lim_\epsilon |{}_{\mathcal{D}'}\langle w, \eta_\epsilon\phi \rangle_{\mathcal{D}}| = \lim_\epsilon |{}_{\mathcal{D}'}\langle \eta_\epsilon w, \phi \rangle_{\mathcal{D}}| = \lim_\epsilon |{}_c\langle L_w \eta_\epsilon, \phi \rangle_{\mathcal{C}'}| \\ &\leq C \lim_\epsilon p_\gamma^{\mathcal{C}'}(\eta_\epsilon\phi) = C p_\gamma^{\mathcal{C}'}(\phi). \end{aligned}$$

This proves that w can be identified with a continuous functional on \mathcal{C}' . □

The next corollary summarizes the results of Propositions 3.6 and 3.8:

Corollary 3.9. *Let \mathcal{T} be a multiplication framework satisfying the properties A1), A2), A3), and let $v \in \mathcal{A}$ and $u \in \mathcal{D}'$, with L_v, L_u the corresponding multiplication maps. If $L_u \in \mathcal{C}(\mathcal{A}, \mathcal{C})$, then it is possible to define a product $w = u \cdot v$ with $w \in \mathcal{C}$. The multiplication defined in this way is commutative.*

A special case of Corollary 3.9 is the following

Corollary 3.10. *Let \mathcal{T} be a multiplication framework satisfying the properties A1), A2), A3) and $u, v \in \mathcal{D}'$. If there exists an interspace $\mathcal{M} \in \mathcal{T}$ such that $u \in \mathcal{M}$ and $v \in \mathcal{M}'$ then $L_u \circ L_v$ is well defined as a multiplication map, i.e, there exists $w \in \mathcal{D}'$ such that $L_w = L_u \circ L_v$. This product is commutative.*

Proof. This is once more an application of Proposition 3.2. The proof is similar to that of Proposition 3.8, so we omit it. □

Let now $u \in \mathcal{D}'$ and \mathcal{T} be a multiplication framework. In general, u need not belong to any proper interspace $\mathcal{F} \in \mathcal{T}$. This is, however, needed to apply the corollaries 3.9 and 3.10. For this reason we put:

$$\mathcal{D}'_{\mathcal{T}} = \bigcup \left\{ \mathcal{F} : \mathcal{F} \in \mathcal{T}, \mathcal{D} \subset \mathcal{F} \subset \mathcal{D}' \right\}.$$

At the light of the previous discussion, we get

Proposition 3.11. *Let \mathcal{T} be a multiplication framework satisfying the properties A1), A2), A3). Then $\mathcal{D}'_{\mathcal{T}}$ can be identified with a commutative partial *-algebra*

$$\mathcal{A} = \{L_u, u \in \mathcal{D}'_{\mathcal{T}} \subset \mathcal{L}(\mathcal{D}, \mathcal{D}')\},$$

where the multiplication is that defined in $\mathcal{L}(\mathcal{D}, \mathcal{D}')$ by the multiplication framework \mathcal{T} .

Remark 3.12. We notice that in the Propositions given above, the assumption that \mathcal{T} is a multiplication framework having the properties A1), A2) and A3), can be replaced with the requirement that \mathcal{T} is a generating family of interspaces enjoying the same properties. This is due to the fact that if \mathcal{T} is a generating family of interspaces with the properties A1), A2) and A3), then the generated multiplication framework $\widehat{\mathcal{T}}$ has again the properties A1), A2) and A3).

Now we turn to the generating family of interspaces constituted by the Banach spaces

$$L_A^{s,\alpha} := \{u \in \mathcal{D}' : J^s u \in E_{\alpha}\}$$

and we ask ourselves under which conditions the multiplication framework \mathcal{T} they generate satisfies the conditions A1), A2) and A3). As we shall see, for this to be true additional assumptions on the operator A and on the family of spaces $\{E_\alpha\}$ should be added.

We then assume that the family of Banach spaces $\{E_\alpha\}$ used in the construction satisfies the following additional requirements:

- (E1) $*$ is a continuous map from E_α onto itself, for each α ;
- (E2) for every $\alpha \in I$, there exists a seminorm $p_\gamma^{\mathcal{D}}$ and $C > 0$ such that

$$\|\phi\psi\|_\alpha \leq Cp_\gamma^{\mathcal{D}}(\phi)\|\psi\|_\alpha, \quad \forall \phi, \psi \in \mathcal{D}.$$

Due to the assumptions made at the beginning of this section, $*$ is a bounded conjugate linear operator in \mathcal{H} . So, if $*$ strongly commutes with A , in the sense that it commutes with the spectral family of A , then it commutes also with $J^s = (1 + A^2)^{s/2}$. From this fact and from (E1), it follows easily that $*$ maps each $L_A^{s,\alpha}$ in itself continuously. Thus \mathcal{T} satisfies the condition A1).

The situation for the other two conditions is more involved and requires even stronger assumptions.

We will only discuss the case where the relationship between A and the family of multiplication operators T_ϕ , $\phi \in \mathcal{D}$ is described by the following commutation relation:

$$(5) \quad [T_\phi, A]f = -T_{A\phi}f, \quad f \in \mathcal{D}$$

that reduces to the (extended) canonical commutation relation:

$$\left[\phi(x), i \frac{d}{dx} \right] f = -i \frac{d\phi}{dx} f$$

when $\mathcal{D} = C_0^\infty(\mathbb{R})$ and $A = i \frac{d}{dx}$.

Lemma 3.13. *If (5) holds, then*

$$(6) \quad [A^n, T_\phi]f = \sum_{k=1}^n \binom{n}{k} T_{A^k\phi} A^{n-k} f, \quad f \in \mathcal{D}.$$

Proof. It is simple induction argument, based on straightforward calculations and well-known properties of the binomial coefficients. □

Lemma 3.14. For every $s \in \mathbb{N}$, the norms

$$\|f\|_{s,\alpha} := \|J^s f\|_\alpha$$

and

$$\|f\|_{s,\alpha}^0 := \sum_{k=0}^s \|A^k f\|_\alpha$$

define equivalent topologies on \mathcal{D} .

Proof. The statement is proved again by induction on s , taking into account the condition d1). □

Proposition 3.15. Assume that, for each $\alpha \in I$, there exists $C_\alpha > 0$ such that

$$\|J^{iy} F\|_\alpha \leq C_\alpha \|F\|_\alpha, \quad \forall y \in \mathbb{R}, F \in E_\alpha.$$

Let $s_0, s_1 \in \mathbb{R}$ and $\alpha \in I$. Put $s_t = ts_1 + (1 - t)s_0$, $0 \leq t \leq 1$. Then the spaces $L_A^{s_t, \alpha}$ are the interpolating spaces between $L_A^{s_0, \alpha}$ and $L_A^{s_1, \alpha}$.

Proof. Instead of proving the statement for the spaces $L_A^{s, \alpha}$, we prove it for the spaces $\mathcal{L}_A^{s, \alpha}$, defined in (3). The final result is obtained by the identification of Lemma 2.7. The proof is made with standard methods of complex interpolation theory. We only give a sketch.

For shortness we put $X = \mathcal{D}^\infty(A)$. Let now $\phi \in X$ and put $s_t = ts_1 - (1 - t)s_0$. We will show that for each $t \in [0, 1]$ the interpolating norm $\|\cdot\|^{(t)}$ is equivalent to $\|\cdot\|_{s_t, \alpha}$ on X . The statement will then follow from the density of X in each $\mathcal{L}_A^{s, \alpha}$.

Let

$$f(z) = J^{s_t - z s_1 - (1-z)s_0} \phi.$$

Then $f(z) \in X$, for each $z \in \mathbb{C}$.

$$\begin{aligned} \|f(iy)\|_{s_0, \alpha} &= \|J^{i(s_0 - s_1)y} J^{s_t} \phi\|_\alpha \leq C_\alpha \|\phi\|_{s_t, \alpha} \\ \|f(1 + iy)\|_{s_1, \alpha} &= \|J^{i(s_0 - s_1)y} J^{s_t} \phi\|_\alpha \leq C_\alpha \|\phi\|_{s_t, \alpha}. \end{aligned}$$

Since $f(t) = \phi$, one obtains for the interpolating norm $\|\cdot\|^{(t)}$ the relation $\|\cdot\|^{(t)} \leq C_\alpha \|\cdot\|_{s_t, \alpha}$. On the other hand, if $\mathcal{F}(X)$ denotes the class of continuous functions on $S = \{z \in \mathbb{C} : 0 \leq \Re z \leq 1\}$ analytic on the interior of S and satisfying the smoothness conditions required in interpolation theory (see, e.g., [14, IX.4, Appendix] or [15]) and $f \in \mathcal{F}(X)$, with the choice

$$g(z) = J^{-s_t + z s_1 + (1-z)s_0} \phi,$$

putting $H(z) = \langle f(z), g(z) \rangle$, by Hadamard’s three line theorem [14, IX.4, Appendix], one finds

$$|H(t)| \leq C_{\bar{\alpha}} \|\phi\|_{-s_t, \bar{\alpha}} \|f\|$$

where $\|f\| := \sup_{t \in \mathbb{R}} \{\|f(it)\|_{s_0}, \|f(1+it)\|_{s_1}\}$.

This in turn implies that $\frac{1}{C_{\bar{\alpha}}} \|\psi\|_{s_t, \alpha} \leq \|\psi\|^{(t)}$, for every $\psi \in X$. So finally,

$$\frac{1}{C_{\bar{\alpha}}} \|\psi\|_{s_t, \alpha} \leq \|\psi\|^{(t)} \leq C_{\alpha} \|\psi\|_{s_t, \alpha}, \quad \forall \psi \in \mathcal{D}^{\infty}(A).$$

□

In the following Proposition, we will suppose, as before, that the topology of \mathcal{D} is defined by a directed family of seminorms $\{p_{\gamma}^{\mathcal{D}}\}$.

Proposition 3.16. *The following statements hold:*

- (i) *For each $s \in \mathbb{Z}$, $\alpha \in I$, there exists a seminorm $p_{\gamma(s, \alpha)}^{\mathcal{D}}$ and $C \equiv C_{s, \alpha} > 0$ such that*

$$(7) \quad \|T_{\phi} F\|_{s, \alpha} \leq C p_{\gamma(s, \alpha)}^{\mathcal{D}}(\phi) \|F\|_{s, \alpha}, \quad \forall F \in L_A^{s, \alpha}.$$

- (ii) *If for each $\alpha \in I$ there exists $C_{\alpha} > 0$ such that*

$$\|J^{iy} F\|_{\alpha} \leq C_{\alpha} \|F\|_{\alpha}, \quad \forall y \in \mathbb{R}, F \in E_{\alpha},$$

then the statement (i) holds also for $s \in \mathbb{R}$.

Proof. For $s \in \mathbb{N}$, making use of the norm $\|\cdot\|_{s, \alpha}^0$, by Lemma 3.13 one can write, for $\phi, \psi \in \mathcal{D}$:

$$(8) \quad \|T_{\phi} \psi\|_{s, \alpha}^0 = \sum_{k=0}^s \|A^k T_{\phi} \psi\|_{\alpha} = \sum_{k=0}^s \|T_{\phi} A^k \psi\|_{\alpha} + \sum_{j=0}^k \binom{k}{j} \|T_{A^j \phi} A^{k-j} \psi\|_{\alpha}.$$

Taking into account (E2) and the continuity in \mathcal{D} of all powers of A , with some straightforward computation, it is easily seen that the right hand side of (8) can be estimated by a term

$$C p_{\gamma}^{\mathcal{D}}(\phi) \sum_{k=0}^s \|A^k \psi\|_{\alpha},$$

and this implies the result in the case $F = \psi \in \mathcal{D}$. The general case is obtained taking into account the density of \mathcal{D} .

If $s \in \mathbb{Z}$, $s < 0$ the statement follows by an easy duality argument.

Finally for $s \in \mathbb{R}$, under the assumptions made, by Proposition 3.15, we can use an interpolation argument. So, if $s_0, s_1 \in \mathbb{Z}$, by the Calderon-Lions theorem [14, Theorem IX.20], the continuity of T_ϕ in $L_A^{s_0, \alpha}$ and in $L_A^{s_1, \alpha}$ implies the continuity in each $L_A^{s_t, \alpha}$ with $s_t = ts_1 + (1-t)s_0$, and since

$$\|T_\phi F\|_{s_0, \alpha} \leq C_{s_0, \alpha} p_{\gamma(s_0, \alpha)}^{\mathcal{D}}(\phi) \|F\|_{s_0, \alpha}, \quad \forall F \in L_A^{s_0, \alpha}$$

and

$$\|T_\phi F\|_{s_1, \alpha} \leq C_{s_1, \alpha} p_{\gamma(s_1, \alpha)}^{\mathcal{D}}(\phi) \|F\|_{s_1, \alpha}, \quad \forall F \in L_A^{s_1, \alpha},$$

the norm $\|T_\phi\|_{(s_t, \alpha), (s_t, \alpha)}$ of T_ϕ , as bounded operator from $L_A^{s_t, \alpha}$ into itself can be estimated as follows

$$\begin{aligned} & \|T_\phi\|_{(s_t, \alpha), (s_t, \alpha)} \\ & \leq \left(C_{s_0, \alpha} p_{\gamma(s_0, \alpha)}^{\mathcal{D}}(\phi) \|F\|_{s_0, \alpha} \right)^{1-t} \left(C_{s_1, \alpha} p_{\gamma(s_1, \alpha)}^{\mathcal{D}}(\phi) \|F\|_{s_1, \alpha} \right)^t \\ & \leq \tilde{C} p_{\tilde{\gamma}}^{\mathcal{D}}(\phi) \end{aligned}$$

taking into account that the set of seminorms $\{p_\gamma^{\mathcal{D}}\}$ is directed. □

In conclusion, under quite reasonable assumptions, the family of spaces $\{L_A^{s, \alpha}\}$ generates a multiplication framework satisfying the conditions A1), A2) and A3) and Proposition 3.11 can be applied.

Remark 3.17. The inequality (7) actually says something more than what we asked with the conditions A2), A3), because it implies joint continuity of the multiplication.

§4. The Case of Tempered Distributions

Let us consider the rigged Hilbert space:

$$\mathcal{S}(\mathbb{R}) \hookrightarrow L^2(\mathbb{R}) \hookrightarrow \mathcal{S}'(\mathbb{R}).$$

As is known, $\mathcal{S}(\mathbb{R})$ coincides with the space of C^∞ -vectors of the operator $B = -\frac{d^2}{dx^2} + x^2$; i.e.

$$\mathcal{D}^\infty(B) = \mathcal{S}(\mathbb{R})$$

and the topology t_B (defined as in Section 3.1) is equivalent to the usual topology of the Schwartz space $\mathcal{S}(\mathbb{R})$.

To begin with, we take as A the operator P defined on the Sobolev space $W^{1,2}(\mathbb{R})$ by

$$(Pf)(x) = -if'(x), \quad f \in W^{1,2}(\mathbb{R}),$$

where f' stands for the weak derivative. As is known, the operator P is self-adjoint on $W^{1,2}(\mathbb{R})$ and

$$\mathcal{D}^\infty(P) = \{f \in C^\infty(\mathbb{R}) : f^{(k)} \in L^2(\mathbb{R}), \forall k \in \mathbb{N}\}.$$

Clearly, $\mathcal{S}(\mathbb{R}) \subset \mathcal{D}^\infty(P)$. It is well known that the usual topology of $\mathcal{S}(\mathbb{R})$ is finer than the one induced on it by t_P . Furthermore, the operator $(1 + P^2)^{\frac{1}{2}}$ leaves $\mathcal{S}(\mathbb{R})$ invariant and it is continuous on it. Moreover, $\mathcal{S}(\mathbb{R})$ is a core for any power of $(1 + P^2)^{\frac{1}{2}}$. Hence the conditions d1)-d3) are all satisfied.

The spaces $L^p(\mathbb{R})$ with $p > 1$ will play the role of the $\{E_\alpha\}$'s in the previous construction.

The family $\{L^p(\mathbb{R})\}_{p>1}$ is compatible with $P = -i\frac{d}{dx}$. Indeed $U(t) = e^{iPt}$ and $e^{iPt}f(s) = f(t + s)$. By Lebesgue theorem (see, e.g. [16, Lemma IV.3]), if $f \in L^p(\mathbb{R})$ $1 < p < \infty$ then:

$$\lim_{t \rightarrow 0} \|f(s + t) - f(s)\|_{L^p} = 0.$$

The spaces $L_P^{s,p}$ are defined by

$$L_P^{s,p} = \{u \in \mathcal{S}' : (1 + P^2)^{\frac{s}{2}}u \in L^p(\mathbb{R})\}.$$

With the help of the Fourier transform, one can prove that, if u is a tempered distribution,

$$u \in L_P^{s,p} \quad \text{if, and only if,} \quad \mathcal{F}^{-1}\left(\left(1 + |\xi|^2\right)^{-\frac{s}{2}}\mathcal{F}u\right) \in L^p(\mathbb{R}).$$

The condition on the right hand side defines the so called Bessel potential spaces $L^{s,p}(\mathbb{R})$, which as is known reduce to the Sobolev space $W^{s,p}$ for integer s .

These spaces generate a multiplication framework \mathcal{T} which has the properties A1), A2) and A3) required in our construction (we refer to [17, 18] for the properties of these spaces; inequalities analogous to (7) have been given in [18] and [19]). Then they can be used to reformulate the abstract results of Section 3 in the concrete case of tempered distributions.

So, for instance, an immediate application of Corollary 3.9 yields:

Proposition 4.1. *Let $v \in L^{t,q}(\mathbb{R})$ and $u \in \mathcal{S}'$. Assume that there exist $s, t \in \mathbb{R}$ and $p, q \in [1, \infty[$ such that $L_u \in \mathcal{C}(L^{s,p}(\mathbb{R}), L^{t,q}(\mathbb{R}))$. Then the product $w = u \cdot v$ exists and $w \in L^{t,q}(\mathbb{R})$.*

As a consequence of Proposition 3.11, the set \mathcal{S}'_T of tempered distributions belonging to some Bessel potential space $L^{s,p}(\mathbb{R})$ is a partial *-algebra with respect to the partial multiplication inherited by $\mathcal{L}(\mathcal{S}, \mathcal{S}')$.

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