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Higher Arithmetic K-Theory

By

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Abstract

A concrete definition of higher K-theory in Arakelov geometry is given. The K-theory defined in this paper is a higher extension of the arithmetic K_0-group of an arithmetic variety defined by Gillet and Soulé. Products and direct images in this K-theory are discussed.

§1. Introduction

The aim of this paper is to provide a new definition of higher K-theory in Arakelov geometry and to show that it enjoys the same formal properties as the higher algebraic K-theory of schemes.

Let X be a proper arithmetic variety, namely, a regular scheme which is flat and proper over $\mathbb{Z}$, the ring of integers. In the research on the arithmetic Chern character of a hermitian vector bundle on X, Gillet and Soulé defined the arithmetic K_0-group $\widehat{K}_0(X)$ of X [9]. It can be viewed as an analogue in Arakelov geometry of the K_0-group of vector bundles on a scheme.

After the advent of $\widehat{K}_0(X)$, its higher extension was discussed in [6, 7, 14]. In these papers one common thing was suggested that higher arithmetic K-theory should be obtained as the homotopy group of the homotopy fiber of the Beilinson's regulator map. To be more precise, there should exist a group $KM_n(X)$ for each $n \geq 0$ fitting into the long exact sequence

$$\cdots \to K_{n+1}(X) \xrightarrow{\rho} \bigoplus_p H_{\mathcal{D}}^{2p-n-1}(X, \mathbb{R}(p)) \to KM_n(X) \to K_n(X) \to \cdots,$$

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where $H^n_{\mathcal{D}}(X, \mathbb{R}(p))$ is the real Deligne cohomology of X and ρ is the Beilinson's regulator map.

To get the homotopy fiber, a simplicial description of the regulator map is necessary. And it has already been given by Burgos and Wang in [6]. For a compact complex manifold M, they introduced an exact cube of hermitian vector bundles on M and associated with it a differential form called a *higher Bott-Chern form*. This gives a homomorphism of complexes

$$\mathrm{ch} : \mathbb{Z}\widehat{S}_*(M) \to \mathcal{D}^*(M, p)[2p+1]$$

from the homology complex $\mathbb{Z}\widehat{S}_*(M)$ of the S-construction of the category of hermitian vector bundles on M to the complex $\mathcal{D}^*(M, p)$ computing the real Deligne cohomology of M defined in [4]. It is the main theorem of [6] that the following map coincides with the Beilinson's regulator map:

$$\rho : K_n(M) \simeq \pi_{n+1}(\widehat{S}(M)) \overset{Hurewicz}{\longrightarrow} H_{n+1}(\mathbb{Z}\widehat{S}_*(M)) \overset{H(\mathrm{ch})}{\longrightarrow} H^{2p-n}_{\mathcal{D}}(M, \mathbb{R}(p)).$$

Applying this to the complex manifold $X(\mathbb{C})$ associated with X, we can obtain a simplicial description of the regulator map for X.

In this paper, we will give another definition of higher arithmetic K-theory for a proper arithmetic variety. One of remarkable features of our arithmetic K-theory is that it is given as an extension of the algebraic K-theory by the cokernel of the regulator map.

Before explaining our method, let us recall the definition of $\widehat{K}_0(X)$ by Gillet and Soulé [9]. For a proper arithmetic variety X, let $\mathcal{A}^{p,p}(X)$ be the space of real (p,p)-forms ω on $X(\mathbb{C})$ such that $\overline{F}^*_\infty \omega = (-1)^p \omega$ for the complex conjugation F_∞ on $X(\mathbb{C})$ and let $\widetilde{\mathcal{A}}(X) = \oplus_p \mathcal{A}^{p,p}(X)/(\mathrm{Im}\,\partial + \mathrm{Im}\,\overline{\partial})$. Then $\widehat{K}_0(X)$ is defined as a factor group of the free abelian group generated by pairs $(\overline{E}, \omega)$ where $\overline{E}$ is a hermitian vector bundle on X and $\omega \in \widetilde{\mathcal{A}}(X)$. Relations on pairs are given by each short exact sequence $\mathcal{E} : 0 \to \overline{E'} \to \overline{E} \to \overline{E''} \to 0$ and $\omega', \omega'' \in \widetilde{\mathcal{A}}(X)$ as follows:

$$(\overline{E'}, \omega') + (\overline{E''}, \omega'') = (\overline{E}, \omega' + \omega'' + \widetilde{\mathrm{ch}}(\mathcal{E})),$$

where $\widetilde{\mathrm{ch}}(\mathcal{E})$ is the Bott-Chern secondary characteristic class of $\mathcal{E}$.

The above definition of $\widehat{K}_0(X)$ can be rephrased in terms of loops and homotopies on $|\widehat{S}(X)|$, the topological realization of the S-construction of the category of hermitian vector bundles on X. Consider a pair (l, ω), where l is a pointed simplicial loop on $|\widehat{S}(X)|$ and $\omega \in \widetilde{\mathcal{A}}(X)$. Two pairs (l, ω) and (l', ω') are said to be homotopy equivalent if there is a cellular homotopy $H : (S^1 \times I)/(\{*\} \times I) \to |\widehat{S}(X)|$ from l to l' such that the Bott-Chern secondary characteristic class $\widetilde{\mathrm{ch}}(H)$ of H, which is defined in a natural way, is equal to $\omega' - \omega$. Let $\widehat{\pi}_1(|\widehat{S}(X)|, \widetilde{\mathrm{ch}})$ denote the set of all equivalence classes of such pairs. Then it carries the structure of an abelian group and the map

$$\widehat{K}_0(X) \to \widehat{\pi}_1(|\widehat{S}(X)|, \widetilde{\mathrm{ch}})$$

given by $(\overline{E}, \omega) \mapsto (l_{\overline{E}}, -\omega)$, where $l_{\overline{E}}$ is the simplicial loop on $|\widehat{S}(X)|$ determined by $\overline{E}$, is proved to be bijective.

Let us generalize this observation to higher homotopy groups in a general setting. Take a pointed CW-complex T and a homomorphism

$$\rho : C_*(T) \to W_*$$

from the reduced homology complex of T to a chain complex of abelian groups W_*. Let S^n denote the n-dimensional sphere and consider a pair (f, ω) of a pointed cellular map $f : S^n \to T$ and $\omega \in \widetilde{W}_n = W_n / \operatorname{Im} \partial$. Two pairs (f, ω) and (f', ω') are said to be homotopy equivalent if there is a pointed cellular homotopy $H : (S^n \times I)/(\{*\} \times I) \to T$ from f to f' such that the image of the fundamental chain of H by ρ is equal to $(-1)^{n+1}(\omega' - \omega)$. This actually gives an equivalence relation on the set of such pairs. The set of homotopy equivalence classes has the structure of a group and it becomes an abelian group when $n \geq 2$. This group is denoted by $\widehat{\pi}_n(T, \rho)$ and called the n-th homotopy group of T modified by ρ. We define the n-th arithmetic K-theory of a proper arithmetic variety X as the $(n+1)$-th homotopy group of $|\widehat{S}(X)|$ modified by the higher Bott-Chern form:

$$\widehat{K}_n(X) = \widehat{\pi}_{n+1}(|\widehat{S}(X)|, \mathrm{ch}).$$

We will show that $\widehat{K}_n(X)$ possesses the same properties as the usual higher K-theory of schemes. More precisely, we will show the following:

(1) *Fundamental exact sequence*:

$$K_{n+1}(X) \to \widetilde{\mathcal{D}}_{n+1}(X) \to \widehat{K}_n(X) \to K_n(X) \to 0,$$

where $\widetilde{\mathcal{D}}_{n+1}(X) = \mathcal{D}_{n+1}(X)/\operatorname{Im} d_{\mathcal{D}}$. In the case of $\widehat{K}_0(X)$, this exact sequence has been obtained in [9].

(2) *Chern class map*:

$$\mathrm{ch}_n : \widehat{K}_n(X) \to \mathcal{D}_n(X).$$

If we denote $KM_n(X) = \operatorname{Ker} \mathrm{ch}_n$, then we can obtain the long exact sequence

$$\cdots \to K_{n+1}(X) \xrightarrow{\rho} \bigoplus_p H_{\mathcal{D}}^{2p-n-1}(X, \mathbb{R}(p)) \to KM_n(X) \to K_n(X) \xrightarrow{\rho} \cdots.$$

We show that $KM_n(X)$ is canonically isomorphic to the homotopy fiber of the Bott-Chern form.

(3) *Arakelov K-theory*: Fix an F_∞-invariant Kähler metric h_X on $X(\mathbb{C})$. The pair $\overline{X} = (X, h_X)$ is called an Arakelov variety. We can define Arakelov K-group of $\overline{X}$ as

$$K_n(\overline{X}) = \left\{ x \in \widehat{K}_n(X); \mathrm{ch}_n(x) \text{ is a harmonic form with respect to } h_X \right\}.$$

We have the exact sequence

$$K_{n+1}(X) \xrightarrow{\rho} \underset{p}{\oplus} H_{\mathcal{D}}^{2p-n-1}(X, \mathbb{R}(p)) \to K_n(\overline{X}) \to K_n(X) \to 0.$$

(4) *Products*: $\widehat{K}_*(X)$ has a product

$$\widehat{K}_n(X) \times \widehat{K}_m(X) \to \widehat{K}_{n+m}(X).$$

It does not admit the associative law. But if we restrict this to $K_*(\overline{X})$, it becomes associative. It is shown that the product is graded commutative up to 2-torsion.

(5) *Functoriality*: For arbitrary morphism $f : X \to Y$, we can define pull back map

$$\widehat{f}^* : \widehat{K}_n(Y) \to \widehat{K}_n(X).$$

It is compatible with product. Suppose that f is smooth and projective, and fix a Kähler metric on the relative tangent bundle of $f(\mathbb{C}) : X(\mathbb{C}) \to Y(\mathbb{C})$. Then we can define direct image homomorphism

$$\widehat{f}_* : \widehat{K}_n(X) \to \widehat{K}_n(Y).$$

The projection formula for $\widehat{f}_*$ and $\widehat{f}^*$ holds.

From the above properties, we can obtain a non-canonical decomposition of $\widehat{K}_n(X)$ into three summands:

$$\widehat{K}_n(X) \simeq K_n(X) \oplus (\mathcal{D}_{n+1}(X)/\operatorname{Ker} d_{\mathcal{D}}) \oplus \left(\underset{p}{\oplus} H_{\mathcal{D}}^{2p-n-1}(X, \mathbb{R}(p))/\operatorname{Im}\rho \right).$$

The Bass' conjecture says that the first summand is a finitely generated abelian group. The second one is an infinite dimensional $\mathbb{R}$-vector space, and the Beilinson's conjectures imply that the third one becomes a real torus.

Let us describe the organization of the paper: In §2 we introduce some materials used in the paper, such as S-construction, exact cubes and higher

Bott-Chern forms. In §3 we propose the notion of modified homotopy groups. In §4 we give the definition of the higher arithmetic K-group $\widehat{K}_*(X)$ and deduce the fundamental exact sequence. We also define the Arakelov K-group. In §5 we prove a product formula for higher Bott-Chern forms. It provides an alternative proof of the fact that the regulator map respects the products. In §6, we discuss product in higher arithmetic K-theory. In §7, we define a direct image homomorphism in higher arithmetic K-theory. To do this we employ the higher analytic torsion form of an exact hermitian cube defined by Roessler [13]. Moreover we establish the projection formula.

§2. Preliminaries

§2.1. Conventions on complexes

Let us first settle some conventions on complexes. By *complex* of an abelian category $\mathfrak{A}$, we mean a family of objects $\{A^k\}_{k\in\mathbb{Z}}$ with differential $d_A : A^k \to A^{k+1}$. For a complex A^* and $n \in \mathbb{Z}$, the *n-th translation* $A[n]^*$ is defined as $A[n]^k = A^{n+k}$ and $d_{A[n]} = (-1)^n d_A$.

By *chain complex* we mean a family of objects $\{A_k\}_{k\geq 0}$ with boundary $\partial_A : A_k \to A_{k-1}$. For a complex $A^* = (A^k, d_A)$ such that $A^k = 0$ for $k > 0$, we can define a chain complex A_* as $A_k = A^{-k}$ and $d_A = \partial_A$. The n-th translation $A[n]_*$ of a chain complex A_* for $n \geq 0$ is defined as $A[n]_k = A_{k-n}$ and $\partial_{A[n]} = (-1)^n \partial_A$.

§2.2. S-construction

In this subsection we recall the S-construction developed by Waldhausen [15]. Throughout this paper, we assume that any small exact category has a distinguished zero object denoted by 0. Let $[n]$ be the finite ordered set $\{0, 1, \ldots, n\}$ and $\mathrm{Ar}[n]$ the category of arrows of $[n]$. For a small exact category $\mathfrak{A}$, let $S_n\mathfrak{A}$ be the set of functors $E : \mathrm{Ar}[n] \to \mathfrak{A}$ satisfying the following conditions for $E_{i,j} = E(i \leq j)$:

(1) $E_{i,i} = 0$ for any $0 \leq i \leq n$.

(2) For any $i \leq j \leq k$, $E_{i,j} \to E_{i,k} \to E_{j,k}$ is a short exact sequence of $\mathfrak{A}$.

For example, $S_0\mathfrak{A} = \{0\}$, $S_1\mathfrak{A}$ is the set of objects of $\mathfrak{A}$ and $S_2\mathfrak{A}$ is the set of short exact sequences of $\mathfrak{A}$. The functor $S\mathfrak{A} : [n] \mapsto S_n\mathfrak{A}$ becomes a simplicial set with the base point given by $0 \in S_0\mathfrak{A}$. It is shown in [15] that $S\mathfrak{A}$ is homotopy equivalent to the Quillen's Q-construction of $\mathfrak{A}$. Therefore the $(n +$

1)-th homotopy group $\pi_{n+1}(S\mathfrak{A},0)$ is isomorphic to $K_i(\mathfrak{A})$, the algebraic K-theory of $\mathfrak{A}$.

§2.3. Exact n-cubes

Let us recall the notion of an exact n-cube. For more details, see [5, 6]. Let $\langle -1,0,1\rangle$ be the ordered set consisting of three elements. An *n-cube* of a small exact category $\mathfrak{A}$ is a covariant functor from the n-th power of $\langle -1,0,1\rangle$ to $\mathfrak{A}$. For an n-cube $\mathcal{F}$, we denote by $\mathcal{F}_{\alpha_1,\dots,\alpha_n}$ the image of an object $(\alpha_1,\dots,\alpha_n)$ of $\langle -1,0,1\rangle^n$. For integers i and j satisfying $1\le i\le n$ and $-1\le j\le 1$, an $(n-1)$-cube $\partial_i^j\mathcal{F}$ is given by $(\partial_i^j\mathcal{F})_{\alpha_1,\dots,\alpha_{n-1}} = \mathcal{F}_{\alpha_1,\dots,\alpha_{i-1},j,\alpha_i,\dots,\alpha_{n-1}}$. It is called a *face* of $\mathcal{F}$. For an object α of $\langle -1,0,1\rangle^{n-1}$ and an integer i satisfying $1\le i\le n$, a 1-cube $\partial_{i^c}^{\alpha}\mathcal{F}$ called an *edge* of $\mathcal{F}$ is

$$\mathcal{F}_{\alpha_1,\dots,\alpha_{i-1},-1,\alpha_i,\dots,\alpha_{n-1}} \to \mathcal{F}_{\alpha_1,\dots,\alpha_{i-1},0,\alpha_i,\dots,\alpha_{n-1}} \to \mathcal{F}_{\alpha_1,\dots,\alpha_{i-1},1,\alpha_i,\dots,\alpha_{n-1}}.$$

An n-cube $\mathcal{F}$ is said to be *exact* if all edges of $\mathcal{F}$ are short exact sequences.

Let $C_n\mathfrak{A}$ denote the set of all exact n-cubes of $\mathfrak{A}$. If $\mathcal{F}$ is an exact n-cube, then any face $\partial_i^j\mathcal{F}$ is also exact. Hence ∂_i^j induces a map

$$\partial_i^j : C_n\mathfrak{A} \to C_{n-1}\mathfrak{A}.$$

Let $\mathcal{F}$ be an exact n-cube of $\mathfrak{A}$. For an integer i satisfying $1\le i\le n+1$, let $s_i^1\mathcal{F}$ be an exact $(n+1)$-cube such that its edge $\partial_{i^c}^{\alpha}(s_i^1\mathcal{F})$ is $\mathcal{F}_\alpha \xrightarrow{\text{id}} \mathcal{F}_\alpha \to 0$. Similarly, let $s_i^{-1}\mathcal{F}$ be an exact $(n+1)$-cube such that $\partial_{i^c}^{\alpha}(s_i^{-1}\mathcal{F})$ is $0\to\mathcal{F}_\alpha \xrightarrow{\text{id}} \mathcal{F}_\alpha$. An exact cube written as $s_i^j\mathcal{F}$ is said to be *degenerate*.

Let $\mathbb{Z}C_n\mathfrak{A}$ be the free abelian group generated by $C_n\mathfrak{A}$ and $D_n\subset\mathbb{Z}C_n\mathfrak{A}$ the subgroup generated by all degenerate exact n-cubes. Let $\widetilde{\mathbb{Z}}C_n\mathfrak{A} = \mathbb{Z}C_n\mathfrak{A}/D_n$ and

$$\partial = \sum_{i=1}^{n}\sum_{j=-1}^{1}(-1)^{i+j+1}\partial_i^j : \widetilde{\mathbb{Z}}C_n\mathfrak{A} \to \widetilde{\mathbb{Z}}C_{n-1}\mathfrak{A}.$$

Then $\widetilde{\mathbb{Z}}C_*\mathfrak{A} = (\widetilde{\mathbb{Z}}C_n\mathfrak{A},\partial)$ becomes a chain complex.

In [6, §4.4], an exact $(n-1)$-cube $\mathrm{Cub}(E)$ for any $E\in S_n\mathfrak{A}$ is constructed and it is shown that $E\mapsto\mathrm{Cub}(E)$ induces a homomorphism of complexes

$$\mathrm{Cub} : \mathbb{Z}S_*\mathfrak{A}[1] \to \widetilde{\mathbb{Z}}C_*\mathfrak{A}.$$

§2.4. Higher Bott-Chern forms

In this subsection we recall higher Bott-Chern forms developed by Burgos and Wang. For more details, see [5, 6]. First we introduce the recipient of higher Bott-Chern forms. Let M be a compact complex algebraic manifold, namely, the analytic space consisting of all $\mathbb{C}$-valued points of a smooth proper algebraic variety over $\mathbb{C}$. Let $\mathcal{E}^p_{\mathbb{R}}(M)$ be the space of real smooth differential forms of degree p on M and $\mathcal{E}^p(M) = \mathcal{E}^p_{\mathbb{R}}(M) \otimes_{\mathbb{R}} \mathbb{C}$. Let $\mathcal{E}^{p,q}(M)$ be the space of complex differential forms of type (p,q) on M. Set

$$\mathcal{D}^n(M,p) = \begin{cases} \mathcal{E}^{n-1}_{\mathbb{R}}(M)(p-1) \cap \bigoplus\limits_{\substack{p'+q'=n-1 \\ p'<p, q'<p}} \mathcal{E}^{p',q'}(M), & n < 2p, \\ \mathcal{E}^{2p}_{\mathbb{R}}(M)(p) \cap \mathcal{E}^{p,p}(M) \cap \operatorname{Ker} d, & n = 2p, \\ 0, & n > 2p \end{cases}$$

and define a differential $d_{\mathcal{D}} : \mathcal{D}^n(M,p) \to \mathcal{D}^{n+1}(M,p)$ by

$$d_{\mathcal{D}}(\omega) = \begin{cases} -\pi(d\omega), & n < 2p-1, \\ -2\partial\bar{\partial}\omega, & n = 2p-1, \\ 0, & n > 2p-1, \end{cases}$$

where $\pi : \mathcal{E}^n(M) \to \mathcal{D}^n(M,p)$ is the canonical projection. Then it is shown in [4, Thm. 2.6] that the pair $(\mathcal{D}^*(M,p), d_{\mathcal{D}})$ is a complex of $\mathbb{R}$-vector spaces with

$$H^n(\mathcal{D}^*(M,p), d_{\mathcal{D}}) \simeq H^n_{\mathcal{D}}(M, \mathbb{R}(p))$$

for $n \le 2p$.

By a *hermitian vector bundle* $\overline{E} = (E,h)$ on M we mean an algebraic vector bundle E on M with a smooth hermitian metric h. Let $K_{\overline{E}}$ denote the curvature form of the unique connection on $\overline{E}$ that is compatible with both the metric and the complex structure. Let us write

$$\mathrm{ch}_0(\overline{E}) = \mathrm{Tr}(\exp(-K_{\overline{E}})) \in \bigoplus_p \mathcal{D}^{2p}(M,p).$$

An *exact hermitian n-cube* on M is an exact n-cube made of hermitian vector bundles on M. Let $\mathcal{F} = \{\overline{E}_\alpha\}$ be an exact hermitian n-cube on M. We call $\mathcal{F}$ an *emi-n-cube* if the metric on any $\overline{E}_\alpha$ with $\alpha_i = 1$ coincides with the metric induced from $\overline{E}_{\alpha_1,\dots,\alpha_{i-1},0,\alpha_{i+1},\dots,\alpha_n}$ for the surjection $\overline{E}_{\alpha_1,\dots,\alpha_{i-1},0,\alpha_{i+1},\dots,\alpha_n} \to \overline{E}_\alpha$.

For an emi-1-cube $\mathcal{E} : \overline{E}_{-1} \to \overline{E}_0 \to \overline{E}_1$, a canonical way of constructing a hermitian vector bundle $\mathrm{tr}_1\,\mathcal{E}$ on $M \times \mathbb{P}^1$ connecting $\overline{E}_0$ with $\overline{E}_{-1} \oplus \overline{E}_1$ is given in [6]. More precisely, if $(x:y)$ denotes the homogeneous coordinate of $\mathbb{P}^1$ and $z = x/y$, then $\mathrm{tr}_1\,\mathcal{E}$ is a hermitian vector bundle on $M \times \mathbb{P}^1$ satisfying the following conditions:

$$\mathrm{tr}_1\,\mathcal{E}|_{z=0} \simeq \overline{E}_0,\ \mathrm{tr}_1\,\mathcal{E}|_{z=\infty} \simeq \overline{E}_{-1} \oplus \overline{E}_1.$$

For an emi-n-cube $\mathcal{F}$, let $\mathrm{tr}_1(\mathcal{F})$ be an emi-$(n-1)$-cube on $M \times \mathbb{P}^1$ given by $\mathrm{tr}_1(\mathcal{F})_\alpha = \mathrm{tr}_1(\partial^\alpha_{n^c}(\mathcal{F}))$ for $\alpha \in \langle -1, 0, 1\rangle^{n-1}$, and $\mathrm{tr}_n(\mathcal{F})$ a hermitian vector bundle on $M \times (\mathbb{P}^1)^n$ given by

$$\mathrm{tr}_n(\mathcal{F}) = \overbrace{\mathrm{tr}_1\,\mathrm{tr}_1 \ldots \mathrm{tr}_1}^{n \text{ times}}(\mathcal{F}).$$

Let $\pi_i : (\mathbb{P}^1)^n \to \mathbb{P}^1$ be the i-th projection and $z_i = \pi_i^* z$. For an integer i satisfying $1 \leq i \leq n$,

$$S_n^i = \sum_{\sigma \in \mathfrak{S}_n} (-1)^\sigma \log |z_{\sigma(1)}|^2 \frac{dz_{\sigma(2)}}{z_{\sigma(2)}} \wedge \cdots \wedge \frac{dz_{\sigma(i)}}{z_{\sigma(i)}} \wedge \frac{d\bar{z}_{\sigma(i+1)}}{\bar{z}_{\sigma(i+1)}} \wedge \cdots \wedge \frac{d\bar{z}_{\sigma(n)}}{\bar{z}_{\sigma(n)}},$$

which is a differential form with logarithmic poles on $(\mathbb{P}^1)^n$. The *Bott-Chern form* of an emi-n-cube $\mathcal{F}$ is

$$\mathrm{ch}_n(\mathcal{F}) = \frac{1}{(2\pi\sqrt{-1})^n} \int_{(\mathbb{P}^1)^n} \mathrm{ch}_0(\mathrm{tr}_n(\mathcal{F})) \wedge T_n \in \bigoplus_p \mathcal{D}^{2p-n}(M, p),$$

where

$$T_n = \frac{(-1)^n}{2n!} \sum_{i=1}^n (-1)^i S_n^i.$$

A process to produce an emi-n-cube $\lambda\mathcal{F}$ from an arbitrary exact hermitian n-cube $\mathcal{F}$ is given in [6]. By virtue of this process, we can extend the definition of the Bott-Chern form to an arbitrary exact hermitian n-cube.

Definition 2.1. The *Bott-Chern form* of an exact hermitian n-cube $\mathcal{F}$ is an element of $\oplus_p \mathcal{D}^{2p-n}(M, p)$ given as follows:

$$\mathrm{ch}_n(\mathcal{F}) = \frac{1}{(2\pi\sqrt{-1})^n} \int_{(\mathbb{P}^1)^n} \mathrm{ch}_0(\mathrm{tr}_n(\lambda\mathcal{F})) \wedge T_n.$$

Theorem 2.2 ([6]). *Let $\widehat{\mathcal{P}}(M)$ denote the category of hermitian vector bundles on M and let $\widetilde{\mathbb{Z}}\widehat{C}_*(M) = \widetilde{\mathbb{Z}}C_*\widehat{\mathcal{P}}(M)$. Then $\mathcal{F} \mapsto \mathrm{ch}_n(\mathcal{F})$ induces a homomorphism*

$$\mathrm{ch} : \widetilde{\mathbb{Z}}\widehat{C}_*(M) \to \bigoplus_p \mathcal{D}^*(M, p)[2p].$$

Moreover, the following map

$$K_n(M) = \pi_{n+1}(\widehat{S}(M)) \overset{Hurewicz}{\longrightarrow} H_{n+1}(\mathbb{Z}\widehat{S}_*(M))$$
$$\overset{\text{Cub}}{\longrightarrow} H_n(\widetilde{\mathbb{Z}}\widehat{C}_*(M)) \overset{\text{ch}}{\to} \bigoplus_p H_{\mathcal{D}}^{2p-n}(M, \mathbb{R}(p))$$

coincides with the Beilinson's regulator map.

§3. Modified Homotopy Groups

§3.1. Definition of modified homotopy groups

In this section we develop a general framework used later in this paper. Let I be the closed interval $[0, 1]$ equipped with the usual CW-complex structure. Throughout this paper we identify the n-dimensional sphere S^n with $I^n/\partial I^n$. Therefore S^n consists of two cells and any point of S^n except the base point is expressed by an n-tuple of real numbers $(t_1, \ldots, t_n)$ with $0 < t_i < 1$.

Let T be a pointed CW-complex and $* \in T$ the base point. Let $\mathrm{sk}_n(T)$ be the n-th skeleton of T when $n \geq 0$ and $\mathrm{sk}_{-1}(T) = \{*\}$. For $n \geq 0$, let us write $C_n(T) = H_n(\mathrm{sk}_n(T), \mathrm{sk}_{n-1}(T); \mathbb{Z})$, the n-th relative homology group of the pair $(\mathrm{sk}_n(T), \mathrm{sk}_{n-1}(T))$. Let $\partial : C_n(T) \to C_{n-1}(T)$ be the connecting homomorphism for the triple $(\mathrm{sk}_n(T), \mathrm{sk}_{n-1}(T), \mathrm{sk}_{n-2}(T))$. Then $(C_*(T), \partial)$ is a chain complex whose homology group is isomorphic to the reduced homology group of T.

Suppose that a chain complex of abelian groups (W_*, ∂) and a homomorphism of chain complexes $\rho : C_*(T) \to W_*$ are given. Let $\widetilde{W}_n = W_n/\operatorname{Im}\partial$. Let us consider a pair (f, ω) of a pointed cellular map $f : S^n \to T$ and $\omega \in \widetilde{W}_{n+1}$. A *cellular homotopy* from one pair (f, ω) to another pair (f', ω') is a pointed cellular map $H : (S^n \times I)/(\{*\} \times I) \to T$ satisfying the following:

(1) $H(x, 0) = f(x)$ and $H(x, 1) = f'(x)$.
(2) Let $[S^n \times I] \in C_{n+1}(S^n \times I)$ denote the fundamental chain of $S^n \times I$, where the orientation on $S^n \times I$ is inherited from the canonical orientation of the interval I. Then

$$\omega' - \omega = (-1)^{n+1}\rho H_*([S^n \times I]).$$

It can be shown that the cellular homotopy gives an equivalence relation on the set of pairs. Two pairs are said to be *homotopy equivalent* if there exists a cellular homotopy between them. We denote by $\widehat{\pi}_n(T, \rho)$ the set of all homotopy equivalence classes of pairs.

Let us define a multiplication on the set $\widehat{\pi}_n(T,\rho)$. Let $T\vee T=\{(x,y)\in T\times T; x=*\text{ or } y=*\}$. Then we can define a natural map $T\vee T\to T$ by $(x,*)\mapsto x$ and $(*,y)\mapsto y$. A comultiplication map $\mu: S^n\to S^n\vee S^n$ is given by

$$\mu(t_1,\dots,t_n)=\begin{cases}((t_1,t_2,\dots,2t_n),*)\,, & 0<t_n\le\frac{1}{2},\\ (*,(t_1,t_2,\dots,2t_n-1))\,, & \frac{1}{2}\le t_n<1,\end{cases}$$

and a homotopy inverse map $\nu: S^n\to S^n$ by $\nu(t_1,\dots,t_{n-1},t_n)=(t_1,\dots,t_{n-1}, 1-t_n)$. For two pointed cellular maps $f,g: S^n\to T$, let us write

$$f\cdot g: S^n\xrightarrow{\mu} S^n\vee S^n\xrightarrow{f\vee g} T\vee T\to T,$$

and

$$f^{-1}: S^n\xrightarrow{\nu} S^n\xrightarrow{f} T.$$

A multiplication of two pairs (f,ω) and (g,τ) is

$$(f,\omega)\cdot(g,\tau)=(f\cdot g,\omega+\tau).$$

It is easy to show that the multiplication $\cdot$ is compatible with the homotopy equivalence relation on pairs. Hence it gives rise to a multiplication on $\widehat{\pi}_n(T,\rho)$.

Let us next verify the associativity of the multiplication. For three pointed cellular maps $f,g,h: S^n\to T$, a cellular homotopy $H_1:(S^n\times I)/(\{*\}\times I)\to T$ from $(f\cdot g)\cdot h$ to $f\cdot(g\cdot h)$ is given as follows:

$$H_1(t_1,\dots,t_{n-1},t_n,u)=\begin{cases}f(t_1,\dots,t_{n-1},\frac{4t_n}{u+1}), & 0<t_n\le\frac{u+1}{4},\\ g(t_1,\dots,t_{n-1},4t_n-u-1), & \frac{u+1}{4}\le t_n\le\frac{u+2}{4},\\ h(t_1,\dots,t_{n-1},\frac{4t_n-2-u}{2-u}), & \frac{u+2}{4}\le t_n<1.\end{cases}$$

Since the image of H_1 is contained in $sk_n(T)$, we have $(H_1)_*([S^n\times I])=0$ in $C_{n+1}(T)$. Hence H_1 becomes a cellular homotopy from $((f,\omega)\cdot(g,\tau))\cdot(h,\eta)$ to $(f,\omega)\cdot((g,\tau)\cdot(h,\eta))$ for any $\omega,\tau,\eta\in\widetilde{W}_{n+1}$.

Finally we show the existence of unit and inverse with respect to the multiplication $\cdot$. Let $0: S^n\to T$ be the map given by $0(S^n)=\{*\}$. For a pointed cellular map $f: S^n\to T$, a homotopy H_2 from $f\cdot 0$ to f is given as follows:

$$H_2(t_1,\dots,t_{n-1},t_n,u)=\begin{cases}f(t_1,\dots,t_{n-1},\frac{2t_n}{u+1}), & 0<t_n\le\frac{u+1}{2},\\ *, & \frac{u+1}{2}\le t_n<1.\end{cases}$$

A homotopy H_3 from $0 \cdot f$ to f can be given in a similar form. Moreover, a homotopy H_4 from $f \cdot f^{-1}$ to 0 is given as follows:

$$H_4(t_1, \ldots, t_{n-1}, t_n, u) = \begin{cases} f(t_1, \ldots, t_{n-1}, \frac{2t_n}{1-u}), & 0 < t_n \leq \frac{1-u}{2}, \\ *, & \frac{1-u}{2} \leq t_n \leq \frac{1+u}{2}, \\ f(t_1, \ldots, t_{n-1}, \frac{-2t_n+2}{1-u}), & \frac{u+1}{2} \leq t_n < 1. \end{cases}$$

A homotopy H_5 from $f^{-1} \cdot f$ to 0 can be given in a similar form. These homotopies are all cellular and their images are contained in $sk_n(T)$. Hence $(f, \omega) \cdot (0,0)$ and $(0,0) \cdot (f, \omega)$ are homotopy equivalent to (f, ω), and $(f, \omega) \cdot (f^{-1}, -\omega)$ and $(f^{-1}, -\omega) \cdot (f, \omega)$ are homotopy equivalent to $(0,0)$.

Theorem 3.1. *For $n \geq 1$, the multiplication $\cdot$ gives the structure of a group on $\widehat{\pi}_n(T, \rho)$ and when $n \geq 2$, it becomes commutative.*

Proof. The former part has already been proved. When $n \geq 2$, for two pointed cellular maps $f, g : S^n \to T$, $f \cdot g$ is homotopy equivalent to $g \cdot f$. A homotopy between them is described in every textbook of homotopy theory, and it is easy to see that the image of this homotopy is also contained in $sk_n(T)$. Hence $(f \cdot g, 0)$ is homotopy equivalent to $(g \cdot f, 0)$. □

Definition 3.2. The group $\widehat{\pi}_n(T, \rho)$ is called the n-th homotopy group of T modified by the homomorphism ρ.

Let $\zeta : \widehat{\pi}_n(T, \rho) \to \pi_n(T)$ denote the surjection obtained by forgetting elements of $\widetilde{W}_{n+1}$. Then we have the following:

Theorem 3.3. *There is an exact sequence*

$$\pi_{n+1}(T) \xrightarrow{\widetilde{\rho}} \widetilde{W}_{n+1} \xrightarrow{a} \widehat{\pi}_n(T, \rho) \xrightarrow{\zeta} \pi_n(T) \to 0,$$

where the map $\widetilde{\rho}$ is given by

$$\widetilde{\rho} : \pi_{n+1}(T) \xrightarrow{Hurewicz} H_{n+1}(T) \xrightarrow{H_{n+1}(\rho)} H_{n+1}(W_*) \subset \widetilde{W}_{n+1}$$

and the map a by $a(\omega) = [(0, \omega)] \in \widehat{\pi}_n(T, \rho)$.

Proof. The cellular approximation theorem implies that $\operatorname{Im} a = \operatorname{Ker} \zeta$. Hence we have only to show that $\operatorname{Ker} a = \operatorname{Im} \widetilde{\rho}$. For $\omega \in \widetilde{W}_{n+1}$, the pair $(0, \omega)$ is homotopy equivalent to $(0,0)$ if and only if there is a cellular homotopy $H : (S^n \times I)/(\{*\} \times I) \to T$ from 0 to 0 such that $(-1)^{n+1} \rho H_*([S^n \times I]) = \omega$.

Since $H(S^n \times \partial I) = \{*\}$, H gives a pointed cellular map $H' : S^{n+1} \to T$. Then ω is equal to the image of $(-1)^{n+1}[H'] \in \pi_{n+1}(T)$ by $\widetilde{\rho}$, therefore $\operatorname{Ker} a \subset \operatorname{Im} \widetilde{\rho}$. The opposite inclusion $\operatorname{Im} \widetilde{\rho} \subset \operatorname{Ker} a$ can be verified by regarding a pointed cellular map $S^{n+1} \to T$ as a cellular homotopy from 0 to 0. □

§3.2. A homomorphism from a modified homotopy group

For a pair (f, ω) as in the previous subsection, let $\rho(f, \omega) = \rho f_*([S^n]) + \partial\omega \in W_n$.

Proposition 3.4. *The above $\rho(f, \omega)$ gives rise to a homomorphism*

$$\rho : \widehat{\pi}_n(T, \rho) \to W_n$$

and $\operatorname{Im} \rho$ *is contained in* $\operatorname{Ker}(\partial : W_n \to W_{n-1})$.

Proof. If $H : (S^n \times I)/(\{*\} \times I) \to T$ is a cellular homotopy from (f, ω) to (f', ω'), then

$$\partial H_*([S^n \times I]) = (-1)^n (f'_*([S^n]) - f_*([S^n]))$$

in $C_n(T)$ and $\rho H_*([S^n \times I]) = (-1)^{n+1}(\omega' - \omega)$. Hence we have

$$\begin{aligned} \rho(f, \omega) &= \rho f_*([S^n]) + \partial\omega \\ &= \rho f'_*([S^n]) + (-1)^{n+1} \partial \rho H_*([S^n \times I])) + \partial\omega \\ &= \rho f'_*([S^n]) + \partial(\omega' - \omega) + \partial\omega \\ &= \rho(f', \omega'), \end{aligned}$$

therefore $\rho(f, \omega)$ gives rise to a homomorphism from $\widehat{\pi}_n(T, \rho)$. The inclusion $\operatorname{Im} \rho \subset \operatorname{Ker}(\partial : W_n \to W_{n-1})$ is obvious. □

The exact sequence in Theorem 3.3 implies the following corollaries:

Corollary 3.5. *There is an exact sequence*

$$\pi_{n+1}(T) \xrightarrow{\widetilde{\rho}} H_{n+1}(W_*) \xrightarrow{a} \widehat{\pi}_n(T, \rho) \xrightarrow{\zeta \oplus \rho} \pi_n(T) \oplus \operatorname{Ker} \partial \xrightarrow{cl} H_n(W_*) \to 0,$$

where $\operatorname{Ker} \partial = \operatorname{Ker}(\partial : W_n \to W_{n-1})$ *and* $cl(x, \omega) = \widetilde{\rho}(x) - [\omega]$.

Corollary 3.6. *For $n \geq 1$, let*

$$\widehat{\pi}_n(T, \rho)_0 = \operatorname{Ker}(\rho : \widehat{\pi}_n(T, \rho) \to W_n).$$

Then there is a long exact sequence

$$\cdots \xrightarrow{\zeta} \pi_{n+1}(T) \xrightarrow{\widetilde{\rho}} H_{n+1}(W_*) \xrightarrow{a} \widehat{\pi}_n(T, \rho)_0 \xrightarrow{\zeta} \pi_n(T) \xrightarrow{\widetilde{\rho}} \cdots .$$

§3.3. Comparison to the homotopy group of the homotopy fiber of ρ

In this subsection we show that $\widehat{\pi}_n(T,\rho)_0$ is canonically isomorphic to the n-th homotopy group of the homotopy fiber of the map ρ. Here we work with the category of simplicial sets, not with the category of CW-complexes. Let us first recall Dold-Kan correspondence. See [11] for a concrete account.

Let A be a simplicial abelian group. Then we obtain the chain complex associated to A by $A_* = (A_n, \partial = \sum_i (-1)^i \partial_i)$. We can define another chain complex NA_* called the normalized chain complex of A. It is a subcomplex of A_* such that the inclusion is a quasi-isomorphism.

For a chain complex W_* of abelian groups, we can construct a simplicial abelian group $\Gamma(W_*)$. The group of n-th simplexes of $\Gamma(W_*)$ is the direct sum of W_n with the subgroup generated by degenerate simplexes, and the canonical projection

$$\varphi : \Gamma(W_*)_* \to W_*$$

is a homomorphism of chain complexes. The homotopy group of $\Gamma(W_*)$ is canonically isomorphic to the homology group of W_*:

$$\pi_n(\Gamma(W_*)) \simeq H_n(W_*).$$

Dold-Kan correspondence [11, Cor. III. 2.3] says that the functors N and Γ are mutually inverse.

Suppose $T = |K|$, the topological realization of a pointed simplicial set K. Let $\mathbb{Z}K$ denote the simplicial abelian group spanned by K and $\mathbb{Z}K_*$ denote the chain complex associated to $\mathbb{Z}K$. We can regard ρ as the homomorphism from $\mathbb{Z}K_*$:

$$\rho : \mathbb{Z}K_* \twoheadrightarrow C_*(|K|) \to W_*.$$

Then we have the map of simplicial sets

$$\rho^\sharp : K \to \mathbb{Z}K = \Gamma(N\mathbb{Z}K_*) \hookrightarrow \Gamma(\mathbb{Z}K_*) \xrightarrow{\Gamma(\rho)} \Gamma(W_*).$$

Lemma 3.7. *The map*

$$\mathbb{Z}K_* \xrightarrow{\rho_*^\sharp} \Gamma(W_*)_* \xrightarrow{\varphi} W_*$$

coincides with ρ.

Proof. We have $\mathbb{Z}K_* = N\mathbb{Z}K_* \oplus D_*$ and $\Gamma(W_*)_* = W_* \oplus D'_*$ where D_* and D'_* are subcomplexes generated by degenerate elements. Since $\rho_*^\sharp$ comes from the map of simplicial sets, it is described as

$$\mathbb{Z}K_* = N\mathbb{Z}K_* \oplus D_* \xrightarrow{\rho\oplus\psi} W_* \oplus D'_* = \Gamma(W_*)_*,$$

where $\psi = \rho^\sharp_*|_{D_*}$.

For $x \in \mathbb{Z}K_*$, take the decomposition $x = x_N + x_D$, where $x_N \in N\mathbb{Z}K_*$ and $x_D \in D_*$. Then $\varphi\rho^\sharp_*(x) = \rho(x_N) = \rho(x) - \rho(x_D)$. Since ρ factors through $C_*(|K|)$, we have $\rho(x_D) = 0$, hence $\varphi\rho^\sharp_*(x) = \rho(x)$. □

Theorem 3.8. *Let* $\mathrm{Fib}_{\rho^\sharp}$ *denote the homotopy fiber of* $\rho^\sharp$. *Then for* $n \geq 1$ *there exists a canonical isomorphism*

$$\pi_n(\mathrm{Fib}_{\rho^\sharp}) \simeq \widehat{\pi}_n(|K|, \rho)_0$$

making the following diagram commutative up to multiplication by ± 1:

$$\begin{array}{ccccccccc}
\pi_{n+1}(K) & \longrightarrow & \pi_{n+1}(\Gamma(W_*)) & \longrightarrow & \pi_n(\mathrm{Fib}_{\rho^\sharp}) & \longrightarrow & \pi_n(K) & \longrightarrow & H_n(W_*) \\
\downarrow{\scriptstyle \mathrm{id}} & & \downarrow & & \downarrow & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow \\
\pi_{n+1}(|K|) & \xrightarrow{\widetilde{\rho}} & H_{n+1}(W_*) & \xrightarrow{a} & \widehat{\pi}_n(|K|, \rho)_0 & \xrightarrow{\zeta} & \pi_n(|K|) & \xrightarrow{\widetilde{\rho}} & H_n(W_*).
\end{array}$$

Proof. Take a Kan complex K' with an anodyne extension $K \to K'$. Since $\Gamma(W_*)$ is also a Kan complex by [11, Lem. I. 3.4], there is an extension ${\rho'}^\sharp : K' \to \Gamma(W_*)$ of $\rho^\sharp$. Let ρ' be the homomorphism of chain complexes given as follows:

$$\mathbb{Z}K'_* \xrightarrow{{\rho'}^\sharp_*} \Gamma(W_*)_* \xrightarrow{\varphi} W_*.$$

It is easily shown that the image of any degenerate simplex of K' by ρ' is zero. Hence ρ' gives the homomorphism $C_*(|K'|) \to W_*$. By Lemma 3.7 we have the commutative diagram

$$\begin{array}{ccc}
\mathbb{Z}K_* & \overset{\rho}{\searrow} & \\
\downarrow & & W_*. \\
\mathbb{Z}K'_* & \underset{\rho'}{\nearrow} &
\end{array}$$

Therefore we may assume that K itself is a Kan complex.

Let $\Delta[1]$ be the simplicial set represented by $[1]$ and fix $\{0\} \in \Delta[1]_0 = \mathrm{Hom}([0],[1])$ as the base point. Let $\mathcal{H}om_\bullet(\Delta[1], \Gamma(W_*))$ denote the function complex from $\Delta[1]$ to $\Gamma(W_*)$ preserving the base point. Define F to be the cartesian product of the following diagram:

$$\begin{array}{ccc} & & \mathcal{H}om_{\bullet}(\Delta[1], \Gamma(W_*)) \\ & & \big\downarrow {\scriptstyle i_1^{\sharp}} \\ K & \xrightarrow{\rho^{\sharp}} & \Gamma(W_*), \end{array}$$

where $i_1^{\sharp}$ is the map taking composite with the injection $i_1 : \{1\} \to \Delta[1]$. Then the topological realization of F is homotopy equivalent to $\mathrm{Fib}_{\rho^{\sharp}}$.

Let $\Delta[n]$ be the simplicial set represented by $[n]$ and $\partial\Delta[n]$ its boundary. Note that the topological realization of $\Delta[n]/\partial\Delta[n]$ is the n-dimensional sphere S^n with the usual cellular decomposition into two cells. Since F is a Kan complex, the homotopy group of F is given by the set of all maps from $\Delta[n]/\partial\Delta[n]$ to F modulo simplicial homotopy.

Take a map of pointed simplicial sets $f : \Delta[n]/\partial\Delta[n] \to F$. Then we have two maps:

$$\begin{aligned} f_1 &: \Delta[n]/\partial\Delta[n] \xrightarrow{f} F \xrightarrow{\pi_1} K, \\ f_2 &: \Delta[n]/\partial\Delta[n] \xrightarrow{f} F \xrightarrow{\pi_2} \mathcal{H}om_{\bullet}(\Delta[1], \Gamma(W_*)), \end{aligned}$$

where π_1 and π_2 are the projections. Let

$$|f_1| : S^n = |\Delta[n]/\partial\Delta[n]| \to |K|$$

be the topological realization of f_1 and

$$f_2^{\sharp} : (\Delta[n] \times \Delta[1])/(\partial\Delta[n] \times \Delta[1]) \cup (\Delta[n] \times \{0\}) \to \Gamma(W_*)$$

be the map corresponding to f_2. Let $[\Delta[n] \times \Delta[1]]$ be the fundamental chain and $\omega_{f_2} = \varphi f_{2*}^{\sharp}([\Delta[n] \times \Delta[1]]) \in W_{n+1}$. Then

$$\begin{aligned} \partial\omega_{f_2} &= \varphi f_{2*}^{\sharp}(\partial[\Delta[n] \times \Delta[1]]) \\ &= (-1)^n \varphi f_{2*}^{\sharp}([\Delta[n] \times \{1\}]) \\ &= (-1)^n \varphi \rho^{\sharp} f_{1*}([\Delta[n]]) \\ &= (-1)^n \rho f_{1*}([\Delta[n]]). \end{aligned}$$

Hence the pair $(|f_1|, (-1)^{n+1}\omega_{f_2})$ gives an element of $\widehat{\pi}_n(|K|, \rho)_0$.

Let $f, g : \Delta[n]/\partial\Delta[n] \to F$ be maps of pointed simplicial sets and

$$H : (\Delta[n] \times \Delta[1])/(\partial\Delta[n] \times \Delta[1]) \to F$$

a homotopy from f to g. Let

$$H_1 : (\Delta[n] \times \Delta[1])/(\partial\Delta[n] \times \Delta[1]) \xrightarrow{H} F \xrightarrow{\pi_1} K,$$
$$H_2 : (\Delta[n] \times \Delta[1])/(\partial\Delta[n] \times \Delta[1]) \xrightarrow{H} F \xrightarrow{\pi_2} \mathcal{H}om_\bullet(\Delta[1], \Gamma(W_*)).$$

The map H_1 is a homotopy from f_1 to g_1. Let

$H_2^\sharp : (\Delta[n] \times \Delta[1] \times \Delta[1])/(\partial\Delta[n] \times \Delta[1] \times \Delta[1]) \cup (\Delta[n] \times \Delta[1] \times \{0\}) \to \Gamma(W_*)$

the map corresponding to H_2. If we denote $\omega_{H_2} = \varphi H^\sharp_{2*}([\Delta[n] \times \Delta[1] \times \Delta[1]]) \in W_{n+2}$, then

$$\begin{aligned}\partial\omega_{H_2} &= (-1)^n \varphi H^\sharp_{2*}([\Delta[n] \times \{1\} \times \Delta[1]] - [\Delta[n] \times \{0\} \times \Delta[1]] \\ &\qquad -[\Delta[n] \times \Delta[1] \times \{1\}]) \\ &= (-1)^n(\omega_{g_2} - \omega_{f_2}) + (-1)^{n+1}\rho H_{1*}([\Delta[n] \times \Delta[1]]).\end{aligned}$$

Hence

$$[(|f_1|, (-1)^{n+1}\omega_{f_2})] = [(|g_1|, (-1)^{n+1}\omega_{g_2})],$$

which tells that $f \mapsto [(|f_1|, (-1)^{n+1}\omega_{f_2})]$ gives rise to a map

$$\pi_n(F) \to \widehat{\pi}_n(|K|, \rho)_0.$$

Next we show that the above map is a homomorphism of groups. Let $f, g, h : \Delta[n]/\partial\Delta[n] \to F$ be maps of pointed simplicial sets and $\sigma : \Delta[n+1] \to F$ a map such that $\partial_{n-1}\sigma = f, \partial_{n+1}\sigma = g, \partial_n\sigma = h$ and $\partial_j\sigma$ is the map collapsing to the base point for $j \le n-2$. Then $[f] + [g] = [h]$ in $\pi_n(F)$ and any sum in $\pi_n(F)$ is described in this way. Let

$$\sigma_1 : \Delta[n+1] \xrightarrow{\sigma} F \xrightarrow{\pi_1} K,$$
$$\sigma_2 : \Delta[n+1] \xrightarrow{\sigma} F \xrightarrow{\pi_2} \mathcal{H}om_\bullet(\Delta[1], \Gamma(W_*))$$

and

$$\sigma_2^\sharp : (\Delta[n+1] \times \Delta[1])/(\Delta[n+1] \times \{0\}) \to \Gamma(W_*)$$

the map corresponding to σ_2. If we denote $\omega_{\sigma_2} = \varphi\sigma^\sharp_{2*}([\Delta[n+1] \times \Delta[1]]) \in W_{n+2}$, then

$$\begin{aligned}\partial\omega_{\sigma_2} &= \varphi\sigma^\sharp_{2*}([\partial\Delta[n+1] \times \Delta[1]] + (-1)^{n+1}[\Delta[n+1] \times \{1\}]) \\ &= (-1)^{n-1}(\omega_{f_2} + \omega_{g_2} - \omega_{h_2}) + (-1)^{n+1}\rho\sigma_{1*}([\Delta[n+1]]).\end{aligned}$$

Hence σ_1 is a homotopy from $f_1 \cdot g_1$ to h_1 such that

$$\rho\sigma_{1*}([\Delta[n+1]]) = \omega_{h_2} - \omega_{f_2} - \omega_{g_2} + (-1)^{n+1}\partial\omega_{\sigma_2}.$$

Hence

$$[(|h_1|, (-1)^{n+1}\omega_{h_2})] = [(|f_1|, (-1)^{n+1}\omega_{f_2})] + [(|g_1|, (-1)^{n+1}\omega_{g_2})],$$

which tells that the map $\pi_n(F) \to \widehat{\pi}_n(|K|, \rho)_0$ is a homomorphism of groups.

It is obvious that the diagram

$$\begin{array}{ccc} \pi_n(F) & \longrightarrow & \pi_n(K) \\ \downarrow & & \downarrow \\ \widehat{\pi}_n(|K|, \rho)_0 & \xrightarrow{\zeta} & \pi_n(|K|) \end{array}$$

is commutative. Consider the following diagram:

$$\begin{array}{ccc} \pi_{n+1}(\Gamma(W_*)) & \longrightarrow & \pi_n(F) \\ \downarrow & & \downarrow \\ H_{n+1}(W_*) & \xrightarrow{a} & \widehat{\pi}_n(|K|, \rho)_0. \end{array}$$

In the above, the upper horizontal arrow is obtained from the map of simplicial sets

$$\mathcal{H}om_\bullet(\Delta[1]/\partial\Delta[1], \Gamma(W_*)) \longrightarrow F = K \underset{\Gamma(W_*)}{\times} \mathcal{H}om_\bullet(\Delta[1], \Gamma(W_*))$$

given by $\gamma \mapsto (*, \widetilde{\gamma})$, where $*$ is the base point of K and $\widetilde{\gamma}$ is the element of $\mathcal{H}om_\bullet(\Delta[1], \Gamma(W_*))$ given by γ. Take a map of pointed simplicial sets

$$\omega : \Delta[n]/\partial\Delta[n] \to \mathcal{H}om_\bullet(\Delta[1]/\partial\Delta[1], \Gamma(W_*))$$

and let

$$\omega^\sharp : \Delta[n] \times \Delta[1]/\partial(\Delta[n] \times \Delta[1]) \to \Gamma(W_*)$$

be the map corresponding to ω. Then the image of

$$[\omega^\sharp] \in \pi_{n+1}(\Gamma(W_*)) \simeq \pi_n(\mathcal{H}om_\bullet(\Delta[1]/\partial\Delta[1], \Gamma(W_*)))$$

by the map

$$\pi_{n+1}(\Gamma(W_*)) \to \pi_n(F) \to \widehat{\pi}_n(|K|, \rho)_0$$

is $[(0, (-1)^{n+1}\varphi\omega_*^\sharp([\Delta[n] \times \Delta[1]]))]$. This shows that the above diagram is commutative up to multiplication by $(-1)^{n+1}$.

It is obvious from the five lemma that $\pi_n(F) \to \widehat{\pi}_n(|K|, \rho)_0$ is an isomorphism. $\square$

§3.4. A functoriality of modified homotopy groups

Let T and T' be pointed CW-complexes and let W_* and W'_* be chain complexes. Let $\alpha : T \to T'$ be a pointed cellular map and let $\rho : C_*(T) \to W_*$, $\rho' : C_*(T') \to W'_*$ and $\beta : W_* \to W'_*$ be homomorphisms of chain complexes that make the diagram

$$\begin{array}{ccc} C_*(T) & \xrightarrow{\alpha_*} & C_*(T') \\ \downarrow{\scriptstyle\rho} & & \downarrow{\scriptstyle\rho'} \\ W_* & \xrightarrow{\beta} & W'_* \end{array}$$

commutative up to a homotopy Φ. In other words, there is a homomorphism $\Phi : C_*(T) \to W'_{*+1}$ satisfying $\rho'\alpha_* - \beta\rho = \partial\Phi + \Phi\partial$.

Proposition 3.9. *Under the above notations, we can define a homomorphism*

$$(\alpha, \beta, \Phi)_* : \widehat{\pi}_n(T, \rho) \to \widehat{\pi}_n(T', \rho')$$

by $[(f, \omega)] \mapsto [(\alpha f, \beta(\omega) - \Phi f_*([S^n]))]$. *This homomorphism enjoys the following functorial property: Let* $\alpha : T \to T'$ *and* $\alpha' : T' \to T''$ *be pointed cellular maps and let* $\beta : W_* \to W'_*$ *and* $\beta' : W'_* \to W''_*$ *be homomorphisms of chain complexes. We assume that the squares*

$$\begin{array}{ccccc} C_*(T) & \xrightarrow{\alpha_*} & C_*(T') & \xrightarrow{\alpha'_*} & C_*(T'') \\ \downarrow{\scriptstyle\rho} & & \downarrow{\scriptstyle\rho'} & & \downarrow{\scriptstyle\rho''} \\ W_* & \xrightarrow{\beta} & W'_* & \xrightarrow{\beta'} & W''_* \end{array}$$

are commutative up to homotopies Φ *and* Φ' *respectively. Then*

$$(\alpha', \beta', \Phi')_*(\alpha, \beta, \Phi)_* = (\alpha'\alpha, \beta'\beta, \beta'\Phi + \Phi'\alpha_*)_* : \widehat{\pi}_n(T, \rho) \to \widehat{\pi}_n(T'', \rho'').$$

Proof. Let $f, f' : S^n \to T$ be pointed cellular maps and $\omega, \omega' \in \widetilde{W}_{n+1}$. If $H : (S^n \times I)/(\{*\} \times I) \to T$ is a cellular homotopy from (f, ω) to (f', ω'), then

$$\begin{aligned} &(-1)^{n+1}\rho'\alpha_* H_*([S^n \times I]) \\ &\qquad = (-1)^{n+1}\beta\rho H_*([S^n \times I]) + (-1)^{n+1}\partial\Phi H_*([S^n \times I]) \\ &\qquad\quad +(-1)^{n+1}\Phi\partial H_*([S^n \times I]) \\ &\qquad \equiv (\beta(\omega') - \Phi f'_*([S^n])) - (\beta(\omega) - \Phi f_*([S^n])) \end{aligned}$$

modulo $\operatorname{Im}\partial$. This tells that the map $\alpha H : (S^n \times I)/(\{*\} \times I) \to T'$ is a cellular homotopy from $(\alpha f, \beta(\omega) - \Phi f_*([S^n]))$ to $(\alpha f', \beta(\omega') - \Phi f'_*([S^n]))$.

Hence $(\alpha, \beta, \Phi)_*$ is well-defined. The functorial property can be shown by an easy calculation. □

Proposition 3.10. *Under the above notations, we have a commutative diagram*

$$\begin{array}{ccc} \widehat{\pi}_n(T,\rho) & \xrightarrow{\ \rho\ } & W_n \\ \Big\downarrow{\scriptstyle (\alpha,\beta,\Phi)_*} & & \Big\downarrow{\scriptstyle \beta} \\ \widehat{\pi}_n(T',\rho') & \xrightarrow{\ \rho'\ } & W'_n. \end{array}$$

Proof. For a pointed cellular map $f : S^n \to T$,

$$\rho'\alpha_* f_*([S^n]) - \beta\rho f_*([S^n]) = \partial\Phi f_*([S^n]).$$

Hence

$$\begin{aligned} \rho'(\alpha,\beta,\Phi)_*([(f,\omega)]) &= \rho'([(\alpha f, \beta(\omega) - \Phi f_*([S^n]))]) \\ &= \rho'\alpha_* f_*([S^n]) + \partial(\beta(\omega) - \Phi f_*([S^n])) \\ &= \beta(\rho f_*([S^n]) + \partial\omega) \\ &= \beta\rho([(f,\omega)]). \end{aligned}$$

□

§4. Definition of Arithmetic K-groups

§4.1. Triviality of the Bott-Chern form of a degenerate element

In this subsection we prove that the Bott-Chern form of a degenerate element of $S_*\mathfrak{A}$ vanishes. We begin with the following lemma:

Lemma 4.1. *For any $E \in S_n\mathfrak{A}$, we have*

$$\begin{aligned} \mathrm{Cub}(s_0 E) &= s_1^{-1}\,\mathrm{Cub}(E), \\ \mathrm{Cub}(s_n E) &= s_n^{1}\,\mathrm{Cub}(E) \end{aligned}$$

and if $1 \leq i \leq n-1$, then we have

$$\mathrm{Cub}(s_i E) = \tau_i\,\mathrm{Cub}(s_i E),$$

where $\tau_i \in \mathfrak{S}_n$ is the transposition of i and $i+1$.

Proof. In order to prove the lemma, we use [6, Prop. 4.5], in which all faces of $\mathrm{Cub}(E)$ for $E \in S_n\mathfrak{A}$ are described. Using this proposition, we can show that

$$\partial_1^{-1}\,\mathrm{Cub}(s_0E) = 0,$$
$$\partial_1^{0}\,\mathrm{Cub}(s_0E) = \partial_1^{1}\,\mathrm{Cub}(s_0E) = \mathrm{Cub}(E).$$

Hence $\mathrm{Cub}(s_0E) = s_1^{-1}\,\mathrm{Cub}(E)$. The second identity can be shown in a similar way. The last identity follows from

$$\partial_i^{j}\,\mathrm{Cub}(s_iE) = \partial_{i+1}^{j}\,\mathrm{Cub}(s_iE)$$

for $1 \le i \le n-1$ and $-1 \le j \le 1$, which can be shown also by using [6, Prop. 4.5]. □

Let $\mathfrak{S}_n$ denote the n-th symmetric group. For $\sigma \in \mathfrak{S}_n$ and an exact n-cube $\mathcal{F}$ of a small exact category $\mathfrak{A}$, let $\sigma\mathcal{F}$ be an exact n-cube defined by $(\sigma\mathcal{F})_{\alpha_1,\dots,\alpha_n} = \mathcal{F}_{\alpha_{\sigma(1)},\alpha_{\sigma(2)},\dots,\alpha_{\sigma(n)}}$. Let $S_n \subset \mathbb{Z}C_n\mathfrak{A}$ be the subgroup generated by exact n-cubes $\mathcal{F}$ such that $\tau_i\mathcal{F} = \mathcal{F}$ for some integer i with $1 \le i \le n-1$. Set

$$\mathrm{Cub}_n(\mathfrak{A}) = \mathbb{Z}C_n\mathfrak{A}/(D_n + S_n).$$

Lemma 4.2. *We have $\partial S_n \subset S_{n-1}$. Hence $\mathrm{Cub}_*(\mathfrak{A}) = (\mathrm{Cub}_n(\mathfrak{A}), \partial)$ becomes a chain complex.*

Proof. Let $\mathcal{F}$ be an exact n-cube satisfying $\tau_i\mathcal{F} = \mathcal{F}$. If $k < i$, then $\partial_k^j\mathcal{F} = \partial_k^j\tau_i\mathcal{F} = \tau_{i-1}\partial_k^j\mathcal{F}$ and if $k > i+1$, then $\partial_k^j\mathcal{F} = \partial_k^j\tau_i\mathcal{F} = \tau_i\partial_k^j\mathcal{F}$. Furthermore, $\tau_i\mathcal{F} = \mathcal{F}$ implies that $\partial_i^j\mathcal{F} = \partial_{i+1}^j\mathcal{F}$. Hence

$$\partial\mathcal{F} = \sum_{k\ne i,i+1}\sum_{j=-1}^{1}(-1)^{k+j+1}\partial_k^j\mathcal{F} \in S_{n-1}.$$

□

Lemma 4.3. *Let $\mathcal{F}$ be an exact hermitian n-cube on a complex algebraic manifold M. For any $\sigma \in \mathfrak{S}_n$, there is a canonical isometry $\sigma(\lambda\mathcal{F}) \simeq \lambda(\sigma\mathcal{F})$.*

Proof. As seen in [6, §3], the emi-n-cube $\lambda\mathcal{F}$ is written as $\lambda_n\cdots\lambda_2\lambda_1\mathcal{F}$, where each λ_i is an endomorphism of the chain complex $\widetilde{\mathbb{Z}}\widehat{C}_*(M)$, and it is easy to see that $\sigma(\lambda_i\mathcal{F}) = \lambda_{\sigma(i)}(\sigma\mathcal{F})$. Hence it is sufficient to show the existence of a canonical isometry $\lambda_i\lambda_j \simeq \lambda_j\lambda_i$. For simplicity, we prove it only in the case of $n = 2$.

For an exact hermitian 2-cube $\mathcal{F} = \{\overline{E_{i,j}}\}$, $\lambda_2\lambda_1\mathcal{F}$ is given as follows:

$$\begin{array}{ccccc}
\overline{E_{-1,-1}} \oplus \overline{E_{1,-1}} \oplus \overline{E_{-1,1}} \oplus \overline{E_{1,1}} & \longrightarrow & \overline{E_{-1,0}} \oplus \overline{E_{1,0}} \oplus \overline{E'_{-1,1}} \oplus \overline{E'_{1,1}} & \longrightarrow & \overline{E'_{-1,1}} \oplus \overline{E'_{1,1}} \\
\downarrow & & \downarrow & & \downarrow \\
\overline{E_{0,-1}} \oplus \overline{E'_{1,-1}} \oplus \overline{E_{0,1}} \oplus \overline{E'_{1,1}} & \longrightarrow & \overline{E_{0,0}} \oplus \overline{E'_{1,0}} \oplus \overline{E'_{0,1}} \oplus \overline{E'_{1,1}} & \longrightarrow & \overline{E'_{0,1}} \oplus \overline{E'_{1,1}} \\
\downarrow & & \downarrow & & \downarrow \\
\overline{E'_{1,-1}} \oplus \overline{E'_{1,1}} & \longrightarrow & \overline{E'_{1,0}} \oplus \overline{E'_{1,1}} & \longrightarrow & \overline{E'_{1,1}},
\end{array}$$

where $\overline{E'_{i,j}}$ is the same vector bundle as $\overline{E_{i,j}}$ equipped with the metric induced from $\overline{E_{0,0}}$. On the other hand, $\lambda_1\lambda_2\mathcal{F}$ is given as follows:

$$\begin{array}{ccccc}
\overline{E_{-1,-1}} \oplus \overline{E_{-1,1}} \oplus \overline{E_{1,-1}} \oplus \overline{E_{1,1}} & \longrightarrow & \overline{E_{-1,0}} \oplus \overline{E'_{-1,1}} \oplus \overline{E_{1,0}} \oplus \overline{E'_{1,1}} & \longrightarrow & \overline{E'_{-1,1}} \oplus \overline{E'_{1,1}} \\
\downarrow & & \downarrow & & \downarrow \\
\overline{E_{0,-1}} \oplus \overline{E_{0,1}} \oplus \overline{E'_{1,-1}} \oplus \overline{E'_{1,1}} & \longrightarrow & \overline{E_{0,0}} \oplus \overline{E'_{0,1}} \oplus \overline{E'_{1,0}} \oplus \overline{E'_{1,1}} & \longrightarrow & \overline{E'_{0,1}} \oplus \overline{E'_{1,1}} \\
\downarrow & & \downarrow & & \downarrow \\
\overline{E'_{1,-1}} \oplus \overline{E'_{1,1}} & \longrightarrow & \overline{E'_{1,0}} \oplus \overline{E'_{1,1}} & \longrightarrow & \overline{E'_{1,1}}.
\end{array}$$

Hence an isometry $\lambda_2\lambda_1\mathcal{F} \simeq \lambda_1\lambda_2\mathcal{F}$ is given by appropriate permutations of direct summands. □

Theorem 4.4. *The Bott-Chern form of a degenerate element of $\widehat{S}_n(M)$ is zero.*

Proof. For an integer i with $1 \leq i \leq n-1$, let $t_i : (\mathbb{P}^1)^n \to (\mathbb{P}^1)^n$ denote the involution interchanging the i-th and the $(i+1)$-th components. Then by [13, Prop. 2.1] and Lemma 4.3, there is an isometry $t_i^* \operatorname{tr}_n(\lambda\mathcal{F}) \simeq \operatorname{tr}_n(\lambda\tau_i\mathcal{F})$. Furthermore, it follows from the definition of T_n that $t_i^* T_n = -T_n$. Hence if $\tau_i\mathcal{F} = \mathcal{F}$, then

$$\begin{aligned}
\operatorname{ch}_n(\mathcal{F}) &= \int_{(\mathbb{P}^1)^n} \operatorname{ch}_0(\operatorname{tr}_n(\lambda\mathcal{F})) \wedge T_n \\
&= \int_{(\mathbb{P}^1)^n} t_i^*(\operatorname{ch}_0(\operatorname{tr}_n(\lambda\mathcal{F})) \wedge T_n) = -\operatorname{ch}_n(\mathcal{F}),
\end{aligned}$$

therefore $\operatorname{ch}_n(\mathcal{F}) = 0$.

By Lemma 4.1, the cube $\operatorname{Cub}(E)$ associated with a degenerate element $E \in \widehat{S}_n(X)$ is either a degenerate cube or a cube satisfying $\tau_i\mathcal{F} = \mathcal{F}$ for some $1 \leq i \leq n-2$. Hence we can say that $\operatorname{ch}_{n-1}(E) = 0$. □

§4.2. Definition of higher arithmetic K-theory

Let X be a proper arithmetic variety. Let $X(\mathbb{C})$ denote the compact complex manifold consisting of $\mathbb{C}$-valued points on X and $F_\infty : X(\mathbb{C}) \to X(\mathbb{C})$ the complex conjugation. The real Deligne cohomology of X is the $\overline{F}_\infty^*$-invariant part of that of $X(\mathbb{C})$:

$$H^n_{\mathcal{D}}(X, \mathbb{R}(p)) = H^n_{\mathcal{D}}(X(\mathbb{C}), \mathbb{R}(p))^{\overline{F}_\infty^* = \mathrm{id}}.$$

Hence if we set

$$\mathcal{D}^n(X, p) = \mathcal{D}^n(X(\mathbb{C}), p)^{\overline{F}_\infty^* = \mathrm{id}},$$

then we have an isomorphism

$$H^n(\mathcal{D}^*(X, p), d_{\mathcal{D}}) \simeq H^n_{\mathcal{D}}(X, \mathbb{R}(p)).$$

By a *hermitian vector bundle* on X we mean a pair $\overline{E} = (E, h)$ of a vector bundle E on X and an F_∞-invariant smooth hermitian metric h on $E(\mathbb{C})$. An *exact hermitian n-cube* on X is an exact n-cube made of hermitian vector bundles on X. Since the Chern form $\mathrm{ch}_0(\overline{E})$ is contained in $\oplus_p \mathcal{D}^{2p}(X, p)$, the Bott-Chern form of an exact hermitian n-cube on X is contained in $\oplus_p \mathcal{D}^{2p-n}(X, p)$.

Let $\widehat{\mathcal{P}}(X)$ be the category of hermitian vector bundles on X, $\widehat{S}(X)$ the S-construction of $\widehat{\mathcal{P}}(X)$, and $\widehat{\mathrm{Cub}}_*(X) = \mathrm{Cub}_*(\widehat{\mathcal{P}}(X))$. If we set $\mathcal{D}_n(X) = \oplus_p \mathcal{D}^{2p-n}(X, p)$, then by Theorem 4.4 we can obtain a homomorphism of chain complexes

$$\mathrm{ch} : C_*(|\widehat{S}(X)|) \xrightarrow{\mathrm{Cub}} \widehat{\mathrm{Cub}}_*(X)[1] \xrightarrow{\mathrm{ch}} \mathcal{D}_*(X)[1].$$

Definition 4.5. The *n-th arithmetic K-group* $\widehat{K}_n(X)$ of X is the $(n+1)$-th homotopy group of $|\widehat{S}(X)|$ modified by ch:

$$\widehat{K}_n(X) = \widehat{\pi}_{n+1}(|\widehat{S}(X)|, \mathrm{ch}).$$

Applying Theorem 3.3 to the present context, we can obtain the following:

Theorem 4.6. *There is an exact sequence*

$$K_{n+1}(X) \to \widetilde{\mathcal{D}}_{n+1}(X) \to \widehat{K}_n(X) \to K_n(X) \to 0,$$

where $\widetilde{\mathcal{D}}_{n+1}(X) = \mathcal{D}_{n+1}(X)/\operatorname{Im} d_{\mathcal{D}}$.

As we mentioned in §1, the 0-th arithmetic K-group has already been defined by Gillet and Soulé in [9]. Let us recall their definition again.

Consider a pair $(\overline{E}, \omega)$ of a hermitian vector bundle $\overline{E}$ on X and $\omega \in \widetilde{\mathcal{D}}_1(X)$. Let $\widehat{\mathcal{K}}_0(X)$ be the abelian group generated by all such pairs modulo the subgroup generated by

$$(\overline{E'}, \omega') + (\overline{E''}, \omega'') - (\overline{E}, \omega' + \omega'' - \mathrm{ch}_1(\mathcal{E}))$$

for all short exact sequences $\mathcal{E} : 0 \to \overline{E'} \to \overline{E} \to \overline{E''} \to 0$ and $\omega', \omega'' \in \widetilde{\mathcal{D}}_1(X)$. We denote by $[(\overline{E}, \omega)]$ the element of $\widehat{\mathcal{K}}_0(X)$ represented by a pair $(\overline{E}, \omega)$.

Strictly speaking, the group $\widehat{\mathcal{K}}_0(X)$ is different from the one defined by Gillet and Soulé up to a constant factor. This results from a difference of Chern-Weil forms. In fact, they think of the Chern-Weil form of $\overline{E}$ as the following real form:

$$\mathrm{Tr}\left(\exp\left(\frac{-K_{\overline{E}}}{2\pi\sqrt{-1}}\right)\right).$$

Hence the (p,p)-part of the above form is equal to $\frac{1}{(2\pi\sqrt{-1})^p}\,\mathrm{ch}_0(\overline{E})^{(p,p)}$. For $\omega = \sum_p \omega_p \in \oplus_p \mathcal{D}^{2p-1}(X,p) = \mathcal{D}_1(X)$, let

$$\Theta(\omega) = \sum_p \frac{2}{(2\pi\sqrt{-1})^{p-1}}\omega_p,$$

then $\Theta(\omega)$ is a real form such that $-\Theta(\mathrm{ch}_1(\mathcal{E}))$ modulo $\mathrm{Im}\,\partial + \mathrm{Im}\,\overline{\partial}$ is the Bott-Chern secondary characteristic class of $\mathcal{E}$. Hence $(\overline{E}, \omega) \mapsto (\overline{E}, \Theta(\omega))$ gives an isomorphism from $\widehat{\mathcal{K}}_0(X)$ to Gillet and Soulé's arithmetic K_0-group of X.

Theorem 4.7. *There is a canonical isomorphism*

$$\widehat{\alpha} : \widehat{\mathcal{K}}_0(X) \simeq \widehat{\pi}_1(|\widehat{S}(X)|, \mathrm{ch}) = \widehat{K}_0(X).$$

Proof. Since $\widehat{S}_1(X)$ is the set of all hermitian vector bundles on X and $\widehat{S}_0(X) = \{*\}$, any hermitian vector bundle $\overline{E}$ on X gives a pointed simplicial loop $l_{\overline{E}} : S^1 \to |\widehat{S}(X)|$. Moreover, any short exact sequence $\mathcal{E} : 0 \to \overline{E'} \to \overline{E} \to \overline{E''} \to 0$ gives a 2-simplex $\Delta_{\mathcal{E}}$ in $\widehat{S}(X)$ whose faces are $\partial_0 \Delta_{\mathcal{E}} = \overline{E''}$, $\partial_1 \Delta_{\mathcal{E}} = \overline{E}$ and $\partial_2 \Delta_{\mathcal{E}} = \overline{E'}$. If we regard $\Delta_{\mathcal{E}}$ as a cellular homotopy from $l_{\overline{E'}} \cdot l_{\overline{E''}}$ to $l_{\overline{E}}$, then $\mathrm{ch}_1((\Delta_{\mathcal{E}})_*[S^1 \times I]) = \mathrm{ch}_1(\mathcal{E})$. Hence we have

$$[(l_{\overline{E'}} \cdot l_{\overline{E''}}, 0)] = [(l_{\overline{E}}, \mathrm{ch}_1(\mathcal{E}))]$$

in $\widehat{\pi}_1(|\widehat{S}(X)|, \mathrm{ch})$. This tells that $(\overline{E}, \omega) \mapsto (l_{\overline{E}}, -\omega)$ gives rise to a homomorphism of groups

$$\widehat{\alpha} : \widehat{\mathcal{K}}_0(X) \to \widehat{\pi}_1(|\widehat{S}(X)|, \mathrm{ch}) = \widehat{K}_0(X).$$

Consider the following commutative diagram:

$$\begin{array}{ccccccccc} K_1(X) & \longrightarrow & \widetilde{\mathcal{D}}_1(X) & \xrightarrow{a} & \widehat{\mathcal{K}}_0(X) & \longrightarrow & K_0(X) & \longrightarrow & 0 \\ \downarrow{\scriptstyle -\mathrm{id}} & & \downarrow{\scriptstyle -\mathrm{id}} & & \downarrow{\scriptstyle \widehat{\alpha}} & & \downarrow{\scriptstyle \mathrm{id}} & & \\ K_1(X) & \longrightarrow & \widetilde{\mathcal{D}}_1(X) & \xrightarrow{a} & \widehat{K}_0(X) & \longrightarrow & K_0(X) & \longrightarrow & 0. \end{array}$$

The upper sequence is exact by [9, Theorem 6.2] and the lower one is exact by Theorem 4.6. Hence $\widehat{\alpha}$ is bijective by the five lemma. □

Let $\varphi : X \to Y$ be a morphism of proper arithmetic varieties. Then we have a commutative diagram

$$\begin{array}{ccc} C_*(|\widehat{S}(Y)|) & \xrightarrow{\mathrm{ch}} & \mathcal{D}_*(Y)[1] \\ \downarrow{\scriptstyle \varphi^*} & & \downarrow{\scriptstyle \varphi^*} \\ C_*(|\widehat{S}(X)|) & \xrightarrow{\mathrm{ch}} & \mathcal{D}_*(X)[1]. \end{array}$$

Hence we obtain a pull back homomorphism

$$\widehat{\varphi}^* : \widehat{K}_n(Y) \to \widehat{K}_n(X)$$

by $\widehat{\varphi}^*([(f,\omega)]) = [(\varphi^* f, \varphi^* \omega)]$.

Let

$$\mathrm{ch}_n : \widehat{K}_n(X) \to \mathcal{D}_n(X)$$

be the map introduced in §3.2, that is,

$$\mathrm{ch}_n([(f,\omega)]) = \mathrm{ch}_n(f) - d_{\mathcal{D}}\omega,$$

where $\mathrm{ch}_n(f) = \mathrm{ch}(f_*([S^{n+1}])) \in \mathcal{D}_n(X)$. We call it the *Chern form map*. Applying Corollary 3.5, Corollary 3.6 and Theorem 3.8 to the present situation, we can obtain the following corollaries:

Corollary 4.8. *There is an exact sequence*

$$K_{n+1}(X) \xrightarrow{\rho} \bigoplus_p H_{\mathcal{D}}^{2p-n-1}(X, \mathbb{R}(p)) \to \widehat{K}_n(X) \xrightarrow{(\zeta, \mathrm{ch}_n)} K_n(X) \oplus \mathrm{Ker}\, d_{\mathcal{D}} \xrightarrow{cl} \bigoplus_p H_{\mathcal{D}}^{2p-n}(X, \mathbb{R}(p)) \to 0,$$

where $cl(x,\omega) = \rho(x) - [\omega]$ and ρ is the Beilinson's regulator map.

Corollary 4.9. *Let $KM_n(X)$ denote the kernel of the Chern form map. Then there is a long exact sequence*

$$\cdots \to K_{n+1}(X) \xrightarrow{\rho} \underset{p}{\oplus} H_{\mathcal{D}}^{2p-n-1}(X, \mathbb{R}(p)) \to KM_n(X) \to K_n(X) \xrightarrow{\rho} \cdots .$$

Moreover, $KM_n(X)$ is canonically isomorphic to the $(n+1)$-th homotopy group of the homotopy fiber of

$$\mathrm{ch}^\sharp : \widehat{S}(X) \to \Gamma(\mathcal{D}_*(X)[1]),$$

where $\mathrm{ch}^\sharp$ *is the map of pointed simplicial sets constructed from* ch *in the way as shown in* §3.3.

We conclude this subsection by calculating the higher arithmetic K-theory of the ring of integers. Let K be an algebraic number field and $\mathcal{O}_K$ its ring of integers. Let $X = \operatorname{Spec} \mathcal{O}_K$. Since $X(\mathbb{C})$ is zero-dimensional, $\mathcal{D}_{2n}(X) = 0$ if $n > 0$ and $\mathcal{D}_{2n+1}(X)$ is the recipient of the regulator map for $K_{2n+1}(\mathcal{O}_K)$. Hence the exact sequence of Theorem 4.6 implies

$$\widehat{K}_{2n+1}(\mathcal{O}_K) \simeq K_{2n+1}(\mathcal{O}_K)$$

and

$$0 \to \operatorname{Coker}\left(\rho : K_{2n+1}(\mathcal{O}_K) \to \left(\underset{\sigma : K \hookrightarrow \mathbb{C}}{\oplus} \mathbb{R}(n)\right)^{\overline{F}_\infty = \mathrm{id}}\right)$$
$$\to \widehat{K}_{2n}(\mathcal{O}_K) \to K_{2n}(\mathcal{O}_K) \to 0,$$

where $K_{2n}(\mathcal{O}_K)$ is a finite abelian group and the Borel's theorem [3] says that $\operatorname{Coker} \rho$ is a quotient of a finite dimensional $\mathbb{R}$-vector space by a lattice.

§4.3. Arakelov K-theory

Let M be a compact algebraic Kähler manifold with a Kähler metric h_M. Let $\mathcal{H}_{\mathbb{R}}^n(M)$ be the space of real harmonic forms on M with respect to h_M and $\mathcal{H}^{p,q}(M)$ the space of harmonic forms of type (p, q). Set

$$\mathcal{H}_{\mathcal{D}}^n(M, p) = \begin{cases} \mathcal{H}_{\mathbb{R}}^{n-1}(M)(p-1) \cap \underset{\substack{p'+q'=n-1 \\ p'<p, q'<p}}{\oplus} \mathcal{H}^{p',q'}(M), & n < 2p, \\ \mathcal{H}_{\mathbb{R}}^{2p}(M)(p) \cap \mathcal{H}^{p,p}(M), & n = 2p. \end{cases}$$

Then the short exact sequence

$$0 \to F^p H^{n-1}(M, \mathbb{C}) \to H^{n-1}(M, \mathbb{R}(p-1)) \to H_{\mathcal{D}}^n(M, \mathbb{R}(p)) \to 0$$

for $n < 2p$ or the short exact sequence

$$0 \to H^{2p}_{\mathcal{D}}(M, \mathbb{R}(p)) \to F^p H^{2p}(M, \mathbb{C}) \to H^{2p}(M, \mathbb{R}(p-1)) \to 0$$

yields an isomorphism

$$H^n_{\mathcal{D}}(M, \mathbb{R}(p)) \simeq \mathcal{H}^n_{\mathcal{D}}(M, p)$$

for $n \leq 2p$.

Let us return to the arithmetic situation. An *Arakelov variety* is a pair $\overline{X} = (X, h_X)$ of an arithmetic variety X and an F_∞-invariant Kähler metric h_X on $X(\mathbb{C})$. We now assume that X is proper over $\mathbb{Z}$. Let $\mathcal{H}_n(X)$ denote the space of harmonic forms with respect to h_X in $\mathcal{D}_n(X)$, that is,

$$\mathcal{H}_n(X) = \bigoplus_p \mathcal{H}^{2p-n}_{\mathcal{D}}(X(\mathbb{C}), p)^{\overline{F}^*_\infty = \mathrm{id}}.$$

Then there is an isomorphism $H_n(\mathcal{D}_*(X), d_{\mathcal{D}}) \simeq \mathcal{H}_n(X)$, which implies the following:

Proposition 4.10. *There is an orthogonal decomposition*

$$\mathrm{Ker}\, d_{\mathcal{D}} = \mathrm{Im}\, d_{\mathcal{D}} \oplus \mathcal{H}_n(X)$$

in $\mathcal{D}_n(X)$.

Definition 4.11. The subgroup $K_n(\overline{X}) = (\mathrm{ch}_n)^{-1}(\mathcal{H}_n(X))$ of $\widehat{K}_n(X)$ is called the *n-th Arakelov K-group* of $\overline{X} = (X, h_X)$.

Theorem 4.12. *There is an exact sequence*

$$K_{n+1}(X) \xrightarrow{\rho} \oplus_p H^{2p-n-1}_{\mathcal{D}}(X, \mathbb{R}(p)) \to K_n(\overline{X}) \to K_n(X) \to 0.$$

Proof. This is derived from the fact that $[(0, \omega)] \in K_n(\overline{X})$ if and only if $d_{\mathcal{D}}\omega = 0$, which follows from Proposition 4.10. □

Künnemann has constructed a section of the inclusion from the Arakelov Chow group to the arithmetic Chow group in [12]. We adapt his method to the inclusion $K_n(\overline{X}) \hookrightarrow \widehat{K}_n(X)$. Let $\mathcal{H} : \mathcal{D}_n(X) \to \mathcal{H}_n(X)$ be the orthogonal projection with respect to the L_2-inner product. Let (f, ω) be a pair of a pointed cellular map $f : S^{n+1} \to |\widehat{S}(X)|$ and $\omega \in \widetilde{\mathcal{D}}_{n+1}(X)$. Then we can take $\omega_\sharp \in \widetilde{\mathcal{D}}_{n+1}(X)$ such that $\mathrm{ch}_n(f) - d_{\mathcal{D}}\omega_\sharp$ is harmonic and $\mathcal{H}(\omega_\sharp) = \mathcal{H}(\omega)$. Existence and uniqueness of $\omega_\sharp$ follow from Proposition 4.10.

If (f, ω) is homotopy equivalent to (f', ω'), then

$$\mathrm{ch}_n(f') - d_{\mathcal{D}}(\omega' - \omega_\sharp - \omega) = \mathrm{ch}_n(f) - d_{\mathcal{D}}\omega_\sharp$$

and $\mathcal{H}(\omega' - \omega_\sharp - \omega) = \mathcal{H}(\omega')$. Hence $\omega'_\sharp = \omega' - \omega_\sharp - \omega$, therefore $(f, \omega_\sharp)$ is homotopy equivalent to $(f', \omega'_\sharp)$. Hence we can obtain a section

$$\sigma : \widehat{K}_n(X) \to K_n(\overline{X})$$

by $\sigma([(f, \omega)]) = [(f, \omega_\sharp)]$. The map σ is called the *harmonic projection* of $\widehat{K}_n(X)$.

§5. A Product Formula for Higher Bott-Chern Forms

§5.1. A product formula

We begin this section by recalling the multiplicative structure on $\mathcal{D}^n(M, p)$ for a compact complex algebraic manifold M introduced in [4]. Let

$$\bullet : \mathcal{D}^n(M, p) \otimes \mathcal{D}^m(M, q) \to \mathcal{D}^{m+n}(M, p+q)$$

be a homomorphism given by

$$x \bullet y = (-1)^n(\partial x^{(p-1,n-p)} - \overline{\partial} x^{(n-p,p-1)}) \wedge y + x \wedge (\partial y^{(q-1,m-q)} - \overline{\partial} y^{(m-q,q-1)})$$

if $n < 2p$ and $m < 2q$ and $x \bullet y = x \wedge y$ if $n = 2p$ or $m = 2q$. Here $x^{(\alpha,\beta)}$ is the (α, β)-part of the differential form x. Then it satisfies $d_{\mathcal{D}}(x \bullet y) = d_{\mathcal{D}}x \bullet y + (-1)^n x \bullet d_{\mathcal{D}}y$ and $x \bullet y = (-1)^{nm} y \bullet x$. Moreover, it induces the product in the real Deligne cohomology defined in [1].

The higher Bott-Chern forms are not compatible with products, that is, $\mathrm{ch}_{n+m}(\mathcal{F} \otimes \mathcal{G})$ is not equal to $\mathrm{ch}_n(\mathcal{F}) \bullet \mathrm{ch}_m(\mathcal{G})$ in general. But since the Beilinson's regulator $K_n(M) \to H^{2p-n}_{\mathcal{D}}(M, \mathbb{R}(p))$ respects the products, it is quite natural to expect that the difference $\mathrm{ch}_{n+m}(\mathcal{F} \otimes \mathcal{G}) - \mathrm{ch}_n(\mathcal{F}) \bullet \mathrm{ch}_m(\mathcal{G})$ is written in terms of exact forms.

Let us introduce another operation on $\mathcal{D}^n(M, p)$. For integers i and j satisfying $1 \le i \le n$ and $1 \le j \le m$, let

$$a^{n,m}_{i,j} = 1 - 2\binom{n+m}{n}^{-1} \sum_{\alpha=0}^{i-1} \binom{n+m-i-j+1}{n-\alpha}\binom{i+j-1}{\alpha},$$

where $\binom{a}{b} = \frac{(a+b)!}{a!b!}$. When $b < 0$ or $a < b$, $\binom{a}{b}$ is assumed to be zero.

Lemma 5.1. *We have* $a_{i,j}^{n,m} = -a_{n-i+1,m-j+1}^{n,m}$ *and* $a_{i,j}^{n,m} = -a_{j,i}^{m,n}$.

Proof. Let us recall the following formula on binomial coefficients:

$$\sum_{\alpha=0}^{a} \binom{b}{a-\alpha}\binom{c}{\alpha} = \binom{b+c}{a}.$$

Using this identity, we have

$$\begin{aligned} a_{i,j}^{n,m} + a_{n-i+1,m-j+1}^{n,m} &= 2 - 2\binom{n+m}{n}^{-1} \sum_{\alpha=0}^{i-1} \binom{n+m-i-j+1}{n-\alpha}\binom{i+j-1}{\alpha} \\ &\quad - 2\binom{n+m}{n}^{-1} \sum_{\alpha=0}^{n-i} \binom{i+j-1}{n-\alpha}\binom{n+m-i-j+1}{\alpha} \\ &= 2 - 2\binom{n+m}{n}^{-1} \sum_{\alpha=0}^{n} \binom{n+m-i-j+1}{n-\alpha}\binom{i+j-1}{\alpha} \\ &= 0. \end{aligned}$$

Furthermore,

$$\begin{aligned} a_{j,i}^{m,n} &= 1 - 2\binom{n+m}{m}^{-1} \sum_{\alpha=0}^{j-1} \binom{m+n-j-i+1}{m-\alpha}\binom{j+i-1}{\alpha} \\ &= -1 + 2\binom{n+m}{m}^{-1} \sum_{\alpha=j}^{m} \binom{m+n-j-i+1}{m-\alpha}\binom{j+i-1}{\alpha} \\ &= -1 + 2\binom{n+m}{m}^{-1} \sum_{\alpha=j}^{m} \binom{m+n-j-i+1}{n-j-i+1+\alpha}\binom{j+i-1}{j+i-1-\alpha}. \end{aligned}$$

If we put $\beta = i + j - 1 - \alpha$, then

$$\begin{aligned} a_{j,i}^{m,n} &= -1 + 2\binom{n+m}{n}^{-1} \sum_{\beta=0}^{i-1} \binom{m+n-i-j+1}{n-\beta}\binom{i+j-1}{\beta} \\ &= -a_{i,j}^{n,m}. \end{aligned}$$

□

For $x \in \mathcal{D}^{2p-n}(M,p)$ and $y \in \mathcal{D}^{2q-m}(M,q)$ with $n, m \geq 1$, we define another operation $x \triangle y$ as follows:

$$x \triangle y = \sum_{\substack{1 \leq i \leq n \\ 1 \leq j \leq m}} a_{i,j}^{n,m} x^{(p-n+i-1,p-i)} \wedge y^{(q-m+j-1,q-j)}.$$

If $n = 0$ or $m = 0$, then $x \vartriangle y$ is defined to be zero. The first claim of Lemma 5.1 implies that $x \vartriangle y \in \mathcal{D}^{2(p+q)-n-m-1}(M, p+q)$ and the second claim implies that $x \vartriangle y = (-1)^{nm+n+m} y \vartriangle x$.

Theorem 5.2. *Let $\mathcal{F}$ (resp. $\mathcal{G}$) be an exact hermitian n-cube (resp. m-cube) on M, then*

$$\begin{aligned}\mathrm{ch}_{n+m}(\mathcal{F} \otimes \mathcal{G}) - \mathrm{ch}_n(\mathcal{F}) \bullet \mathrm{ch}_m(\mathcal{G}) &= (-1)^{n+1} d_{\mathcal{D}}(\mathrm{ch}_n(\mathcal{F}) \vartriangle \mathrm{ch}_m(\mathcal{G})) \\ &+ (-1)^n \mathrm{ch}_{n-1}(\partial\mathcal{F}) \vartriangle \mathrm{ch}_m(\mathcal{G}) - \mathrm{ch}_n(\mathcal{F}) \vartriangle \mathrm{ch}_{m-1}(\partial\mathcal{G}).\end{aligned}$$

§5.2. Proof of Theorem 5.2

Let us first prepare some notations. For differential forms $u_1, \ldots, u_n$ on M, let $(u_1, \ldots, u_n)^{(\alpha,\beta)}$ be the (α, β)-part of $du_1 \wedge \cdots \wedge du_n$. When u_i is a (p_i, p_i)-form, let

$$(u_1, \ldots, u_n)^{(i)} = \sum_p (u_1, \ldots, u_n)^{(p+i, p+n-i)}$$

and

$$S_n^i(u_1, \ldots, u_n) = (i-1)!(n-i)! \sum_{\alpha=1}^{n} (-1)^{\alpha+1} u_\alpha (u_1, \ldots, \widehat{u_\alpha}, \ldots, u_n)^{(i-1)}.$$

Then $S_n^i(u_1, \ldots, u_n) \in \mathcal{D}_n(M)$ if $u_i \in \mathcal{D}_1(M)$. If we take u_i as $\log |z_i|^2$, $S_n^i(\log |z_1|^2, \ldots, \log |z_n|^2)$ is nothing but S_n^i introduced in §2.4.

Lemma 5.3. *If u_i is a (p_i, p_i)-form on M, then*

$$\begin{aligned}\partial S_n^i(u_1, \ldots, u_n) &= i!(n-i)!(u_1, \ldots, u_n)^{(i)} \\ &+ (n-i) \sum_{\alpha=1}^{n} (-1)^\alpha \partial\overline{\partial} u_\alpha S_{n-1}^i(u_1, \ldots, \widehat{u_\alpha}, \ldots, u_n)\end{aligned}$$

and

$$\begin{aligned}\overline{\partial} S_n^i(u_1, \ldots, u_n) &= (i-1)!(n-i+1)!(u_1, \ldots, u_n)^{(i-1)} \\ &- (i-1) \sum_{\alpha=1}^{n} (-1)^\alpha \partial\overline{\partial} u_\alpha S_{n-1}^{i-1}(u_1, \ldots, \widehat{u_\alpha}, \ldots, u_n).\end{aligned}$$

Proof. We have

$$
\begin{aligned}
\partial S_n^i(u_1,\ldots,u_n) &= (i-1)!(n-i)!\sum_{\alpha=1}^{n}(-1)^{\alpha+1}\partial\left(u_\alpha(u_1,\ldots,\widehat{u_\alpha},\ldots,u_n)^{(i-1)}\right)\\
&= (i-1)!(n-i)!\sum_{\alpha=1}^{n}(-1)^{\alpha+1}\partial u_\alpha(u_1,\ldots,\widehat{u_\alpha},\ldots,u_n)^{(i-1)}\\
&\quad + (i-1)!(n-i)!\sum_{\alpha=1}^{n}(-1)^{\alpha+1}u_\alpha\\
&\quad \times\left(\sum_{\beta<\alpha}(-1)^{\beta-1}\partial\overline{\partial}u_\beta(u_1,\ldots,\widehat{u_\beta},\ldots,\widehat{u_\alpha},\cdots,u_n)^{(i-1)}\right.\\
&\qquad\qquad \left.+\sum_{\alpha<\beta}(-1)^{\beta}\partial\overline{\partial}u_\beta(u_1,\ldots,\widehat{u_\alpha},\ldots,\widehat{u_\beta},\ldots,u_n)^{(i-1)}\right)\\
&= i!(n-i)!(u_1,\ldots,u_n)^{(i)}\\
&\quad + (i-1)!(n-i)!\sum_{\beta=1}^{n}(-1)^{\beta}\partial\overline{\partial}u_\beta\\
&\quad \times\left(\sum_{\alpha<\beta}(-1)^{\alpha+1}u_\alpha(u_1,\ldots,\widehat{u_\alpha},\ldots,\widehat{u_\beta},\ldots,u_n)^{(i-1)}\right.\\
&\qquad\qquad \left.+\sum_{\beta<\alpha}(-1)^{\alpha}u_\alpha(u_1,\ldots,\widehat{u_\beta},\ldots,\widehat{u_\alpha},\ldots,u_n)^{(i-1)}\right)\\
&= i!(n-i)!(u_1,\ldots,u_n)^{(i)} + (n-i)\sum_{\beta=1}^{n}(-1)^{\beta}\partial\overline{\partial}u_\beta S_{n-1}^i(u_1,\ldots,\widehat{u_\beta},\ldots,u_n).
\end{aligned}
$$

The second identity can be proved in a similar way. □

Lemma 5.4. *For (p_i,p_i)-forms u_i and (q_j,q_j)-forms v_j,*

$$
\begin{aligned}
&S_{n+m}^k(u_1,\ldots,u_n,v_1,\ldots,v_m)\\
&\quad=\sum_{i=1}^{k}\frac{(k-1)!(n+m-k)!}{(n-i)!(i-1)!}S_n^i(u_1,\ldots,u_n)\wedge(v_1,\ldots,v_m)^{(k-i)}\\
&\qquad+(-1)^n\sum_{j=1}^{k}\frac{(k-1)!(n+m-k)!}{(m-j)!(j-1)!}(u_1,\ldots,u_n)^{(k-j)}\wedge S_m^j(v_1,\ldots,v_m).
\end{aligned}
$$

Hence

$$\sum_{k=1}^{n+m}(-1)^k S^k_{n+m}(u_1,\dots,u_n,v_1,\dots,v_m)$$
$$=\sum_{\substack{1\le i\le n\\0\le j\le m}}(-1)^{i+j}\frac{(n+m-i-j)!(i+j-1)!}{(n-i)!(i-1)!}S^i_n(u_1,\dots,u_n)\wedge(v_1,\dots,v_m)^{(j)}$$
$$+\sum_{\substack{0\le i\le n\\1\le j\le m}}(-1)^{n+i+j}\frac{(n+m-i-j)!(i+j-1)!}{(m-j)!(j-1)!}(v_1,\dots,v_n)^{(i)}\wedge S^j_m(v_1,\dots,v_m).$$

Proof. We have

$$S^k_{n+m}(u_1,\dots,u_n,v_1,\dots,v_m)$$
$$=(k-1)!(n+m-k)!\sum_{\alpha=1}^{n}(-1)^{\alpha+1}u_\alpha(u_1,\dots,\widehat{u_\alpha},\dots,u_n,v_1,\dots,v_m)^{(k-1)}$$
$$+(-1)^n(k-1)!(n+m-k)!\sum_{\beta=1}^{n}(-1)^{\beta+1}v_\beta$$
$$\times(u_1,\dots,u_n,v_1,\dots,\widehat{v_\beta},\dots,v_m)^{(k-1)}$$
$$=(k-1)!(n+m-k)!\sum_{\alpha=1}^{n}(-1)^{\alpha+1}u_\alpha$$
$$\times\left(\sum_{i=1}^{k}(u_1,\dots,\widehat{u_\alpha},\dots,u_n)^{(i-1)}\wedge(v_1,\dots,v_m)^{(k-i)}\right)$$
$$+(-1)^n(k-1)!(n+m-k)!\sum_{\beta=1}^{n}(-1)^{\beta+1}v_\beta$$
$$\times\left(\sum_{j=1}^{k}(u_1,\dots,u_n)^{(k-j)}\wedge(v_1,\dots,\widehat{v_\beta},\dots,v_m)^{(j-1)}\right)$$
$$=\sum_{i=1}^{k}\tfrac{(k-1)!(n+m-k)!}{(i-1)!(n-i)!}S^i_n(u_1,\dots,u_n)\wedge(v_1,\dots,v_m)^{(k-i)}$$
$$+(-1)^n\sum_{j=1}^{k}\tfrac{(k-1)!(n+m-k)!}{(j-1)!(m-j)!}(u_1,\dots,u_n)^{(k-j)}\wedge S^j_m(v_1,\dots,v_m).$$

□

If we assume that u_i and v_j are in $\mathcal{D}_1(M)$, then by Lemma 5.3 we have

$$
\begin{aligned}
&d\left(\sum_{i=1}^{n}(-1)^i S_n^i(u_1,\dots,u_n)\vartriangle\sum_{j=1}^{m}(-1)^j S_m^j(v_1,\dots,v_m)\right)\\
&=\sum_{\substack{1\le i\le n\\1\le j\le m}}(-1)^{i+j}a_{i,j}^{n,m}dS_n^i(u_1,\dots,u_n)\wedge S_m^j(v_1,\dots,v_m)\\
&\quad+(-1)^{n+1}\sum_{\substack{1\le i\le n\\1\le j\le m}}(-1)^{i+j}a_{i,j}^{n,m}S_n^i(u_1,\dots,u_n)\wedge dS_m^j(v_1,\dots,v_m)\\
&=\sum_{\substack{0\le i\le n\\1\le j\le m}}(-1)^{i+j}i!(n-i)!(a_{i,j}^{n,m}-a_{i+1,j}^{n,m})(u_1,\dots,u_n)^{(i)}\wedge S_m^j(v_1,\dots,v_m)\\
&\quad+\sum_{\substack{1\le i\le n-1\\1\le j\le m}}(-1)^{i+j}((n-i)a_{i,j}^{n,m}+ia_{i+1,j}^{n,m})\\
&\qquad\times\left(\sum_{\alpha=1}^{n}(-1)^\alpha\partial\overline{\partial}u_\alpha S_{n-1}^i(u_1,\dots,\widehat{u_\alpha},\dots,u_n)\wedge S_m^j(v_1,\dots,v_m)\right)\\
&\quad+(-1)^{n+1}\sum_{\substack{1\le i\le n\\0\le j\le m}}(-1)^{i+j}j!(m-j)!(a_{i,j}^{n,m}-a_{i,j+1}^{n,m})\\
&\qquad\times S_n^i(u_1,\dots,u_n)\wedge(v_1,\dots,v_m)^{(j)}\\
&\quad+(-1)^{n+1}\sum_{\substack{1\le i\le n\\1\le j\le m-1}}(-1)^{i+j}((m-j)a_{i,j}^{n,m}+ja_{i,j+1}^{n,m})\\
&\qquad\times\left(S_n^i(u_1,\dots,u_n)\wedge\sum_{\beta=1}^{m}(-1)^\beta\partial\overline{\partial}v_\beta S_{m-1}^j(v_1,\dots,\widehat{v_\beta},\dots,v_m)\right).
\end{aligned}
$$

Let us compute the coefficients of the above expression. Since $\binom{n+m-i-j+1}{n-\alpha}=\binom{n+m-i-j}{n-\alpha}+\binom{n+m-i-j}{n-1-\alpha}$ and $\binom{i+j}{\alpha}=\binom{i+j-1}{\alpha}+\binom{i+j-1}{\alpha-1}$,

$$
\begin{aligned}
a_{i,j}^{n,m}-a_{i+1,j}^{n,m}&=1-2\binom{n+m}{n}^{-1}\sum_{\alpha=0}^{i-1}\binom{n+m-i-j+1}{n-\alpha}\binom{i+j-1}{\alpha}\\
&\quad-1+2\binom{n+m}{n}^{-1}\sum_{\alpha=0}^{i}\binom{n+m-i-j}{n-\alpha}\binom{i+j}{\alpha}\\
&=2\binom{n+m}{n}^{-1}\binom{n+m-i-j}{n-i}\binom{i+j-1}{i}
\end{aligned}
$$

for $1 \leq i \leq n-1$ and

$$\begin{aligned}
a_{i,j}^{n,m} - a_{i,j+1}^{n,m} &= 1 - 2\binom{n+m}{n}^{-1} \sum_{\alpha=0}^{i-1} \binom{n+m-i-j+1}{n-\alpha}\binom{i+j-1}{\alpha} \\
&\quad - 1 + 2\binom{n+m}{n}^{-1} \sum_{\alpha=0}^{i-1} \binom{n+m-i-j}{n-\alpha}\binom{i+j}{\alpha} \\
&= -2\binom{n+m}{n}^{-1}\binom{n+m-i-j}{n-i}\binom{i+j-1}{i-1} \\
&= -2\binom{n+m}{n}^{-1}\binom{n+m-i-j}{m-j}\binom{i+j-1}{j}
\end{aligned}$$

for $1 \leq j \leq m-1$. It follows from the definition of $a_{i,j}^{n,m}$ and Lemma 5.1 that

$$\begin{aligned}
a_{1,j}^{n,m} &= 1 - 2\binom{n+m}{n}^{-1}\binom{n+m-j}{n}, \\
a_{n,j}^{n,m} &= -1 + 2\binom{n+m}{n}^{-1}\binom{n+j-1}{n}, \\
a_{i,1}^{n,m} &= -1 + 2\binom{n+m}{m}^{-1}\binom{n+m-i}{m}, \\
a_{i,m}^{n,m} &= 1 - 2\binom{n+m}{m}^{-1}\binom{m+i-1}{m}.
\end{aligned}$$

Moreover, by Lemma A.1 we have

$$\begin{aligned}
&(n-i)a_{i,j}^{n,m} + i a_{i+1,j}^{n,m} \\
&\quad = n - 2\binom{n+m}{n}^{-1}\left((n-i)\sum_{\alpha=0}^{i-1}\binom{n+m-i-j+1}{n-\alpha}\binom{i+j-1}{\alpha} + i\sum_{\alpha=0}^{i}\binom{n+m-i-j}{n-\alpha}\binom{i+j}{\alpha}\right) \\
&\quad = n - 2n\binom{n+m-1}{n-1}^{-1}\sum_{\alpha=0}^{i-1}\binom{n+m-i-j}{n-1-\alpha}\binom{i+j-1}{\alpha} \\
&\quad = n a_{i,j}^{n-1,m}
\end{aligned}$$

and

$$\begin{aligned}
&(m-j)a_{i,j}^{n,m} + j a_{i,j+1}^{n,m} \\
&\quad = m - 2\binom{n+m}{n}^{-1}\left((m-j)\sum_{\alpha=0}^{i-1}\binom{n+m-i-j+1}{n-\alpha}\binom{i+j-1}{\alpha} + j\sum_{\alpha=0}^{i-1}\binom{n+m-i-j}{n-\alpha}\binom{i+j}{\alpha}\right) \\
&\quad = m - 2m\binom{n+m-1}{n}^{-1}\sum_{\alpha=0}^{i-1}\binom{n+m-i-j}{n-\alpha}\binom{i+j-1}{\alpha} \\
&\quad = m a_{i,j}^{n,m-1}.
\end{aligned}$$

These computations imply that

$$
\begin{aligned}
& d\left(\sum_{i=1}^{n}(-1)^i S_n^i(u_1,\dots,u_n) \vartriangle \sum_{j=1}^{n}(-1)^j S_m^j(v_1,\dots,v_m)\right) \\
&\quad = 2\binom{n+m}{n}^{-1} \sum_{\substack{0\le i\le n\\ 1\le j\le m}} (-1)^{i+j} i!(n-i)!\binom{n+m-i-j}{n-i}\binom{i+j-1}{i} \\
&\qquad\qquad \times (u_1,\dots,u_n)^{(i)} \wedge S_m^j(v_1,\dots,v_m) \\
&\qquad + n \sum_{\substack{1\le i\le n-1\\ 1\le j\le m}} (-1)^{i+j} a_{i,j}^{n-1,m} \sum_{\alpha=1}^{n}(-1)^\alpha \partial\overline{\partial} u_\alpha \\
&\qquad\qquad \times S_{n-1}^i(u_1,\dots,\widehat{u_\alpha},\dots,u_n) \wedge S_m^j(v_1,\dots,v_m) \\
&\qquad + 2(-1)^n\binom{n+m}{n}^{-1} \sum_{\substack{1\le i\le n\\ 0\le j\le m}} (-1)^{i+j} j!(m-j)!\binom{n+m-i-j}{m-j}\binom{i+j-1}{j} \\
&\qquad\qquad \times S_n^i(u_1,\dots,u_n) \wedge (v_1,\dots,v_m)^{(j)} \\
&\qquad + (-1)^{n+1} m \sum_{\substack{1\le i\le n\\ 1\le j\le m-1}} (-1)^{i+j} a_{i,j}^{n,m-1} S_n^i(u_1,\dots,u_n) \\
&\qquad\qquad \wedge \sum_{\beta=1}^{m}(-1)^\beta \partial\overline{\partial} v_\beta S_{m-1}^j(v_1,\dots,\widehat{v_\beta},\dots,v_m) \\
&\qquad - \sum_{1\le j\le m} (-1)^j n!(u_1,\dots,u_n)^{(0)} \wedge S_m^j(v_1,\dots,v_m) \\
&\qquad - \sum_{1\le j\le m} (-1)^{n+j} n!(u_1,\dots,u_n)^{(n)} \wedge S_m^j(v_1,\dots,v_m) \\
&\qquad - \sum_{1\le i\le n} (-1)^{n+i} m! S_n^i(u_1,\dots,u_n) \wedge (v_1,\dots,v_m)^{(0)} \\
&\qquad - \sum_{1\le i\le n} (-1)^{n+m+i} m! S_n^i(u_1,\dots,u_n) \wedge (v_1,\dots,v_m)^{(m)} \\
&\quad = 2\binom{n+m}{n}^{-1} \sum_{\substack{0\le i\le n\\ 1\le j\le m}} (-1)^{i+j} \tfrac{(n+m-i-j)!(i+j-1)!}{(m-j)!(j-1)!} \\
&\qquad\qquad \times (u_1,\dots,u_n)^{(i)} \wedge S_m^j(v_1,\dots,v_m) \\
&\qquad + n \sum_{\substack{1\le i\le n-1\\ 1\le j\le m}} (-1)^{i+j} a_{i,j}^{n-1,m} \sum_{\alpha=1}^{n}(-1)^\alpha \partial\overline{\partial} u_\alpha
\end{aligned}
$$

$$\times S^i_{n-1}(u_1,\dots,\widehat{u_\alpha},\dots,u_n)\wedge S^j_m(v_1,\dots,v_m)$$
$$+2(-1)^n\binom{n+m}{n}^{-1}\sum_{\substack{1\le i\le n\\0\le j\le m}}(-1)^{i+j}\tfrac{(n+m-i-j)!(i+j-1)!}{(n-i)!(i-1)!}$$
$$\times S^i_n(u_1,\dots,u_n)\wedge(v_1,\dots,v_m)^{(j)}$$
$$+(-1)^{n+1}m\sum_{\substack{1\le i\le n\\1\le j\le m-1}}(-1)^{i+j}a^{n,m-1}_{i,j}S^i_n(u_1,\dots,u_n)$$
$$\wedge\sum_{\beta=1}^m(-1)^\beta\partial\overline{\partial}v_\beta S^j_{m-1}(v_1,\dots,\widehat{v_\beta},\dots,v_m)$$
$$-\sum_{1\le j\le m}(-1)^j\left(\overline{\partial}S^1_n(u_1,\dots,u_n)+(-1)^n\partial S^n_n(u_1,\dots,u_n)\right)\wedge S^j_m(v_1,\dots,v_m)$$
$$-\sum_{1\le i\le n}(-1)^{n+i}S^i_n(u_1,\dots,u_n)\wedge\left(\overline{\partial}S^1_m(v_1,\dots,v_m)+(-1)^m\partial S^m_m(v_1,\dots,v_m)\right).$$

Applying Lemma 5.4 to the above, we can obtain the following:

Proposition 5.5. *For $u_i\in\mathcal{D}_1(M)$ and $v_j\in\mathcal{D}_1(M)$, we have*

$$d\left(\sum_{i=1}^n(-1)^iS^i_n(u_1,\dots,u_n)\vartriangle\sum_{j=1}^m(-1)^jS^j_m(v_1,\dots,v_m)\right)$$
$$=2(-1)^n\binom{n+m}{n}^{-1}\sum_{k=1}^{n+m}(-1)^kS^k_{n+m}(u_1,\dots,u_n,v_1,\dots,v_m)$$
$$+n\sum_{\substack{1\le i\le n-1\\1\le j\le m}}(-1)^{i+j}a^{n-1,m}_{i,j}\sum_{\alpha=1}^n(-1)^\alpha\partial\overline{\partial}u_\alpha$$
$$\times S^i_{n-1}(u_1,\dots,\widehat{u_\alpha},\dots,u_n)\wedge S^j_m(v_1,\dots,v_m)$$
$$+(-1)^{n+1}m\sum_{\substack{1\le i\le n\\1\le j\le m-1}}(-1)^{i+j}a^{n,m-1}_{i,j}S^i_n(u_1,\dots,u_n)$$
$$\wedge\sum_{\beta=1}^m(-1)^\beta\partial\overline{\partial}v_\beta S^j_{m-1}(v_1,\dots,\widehat{v_\beta},\dots,v_m)$$
$$+(-1)^{n+1}\left(\sum_{i=1}^n(-1)^iS^i_n(u_1,\dots,u_n)\right)\bullet\left(\sum_{j=1}^m(-1)^jS^j_m(v_1,\dots,v_m)\right).$$

Let us return to the proof of Theorem 5.2. We may assume that $\mathcal{F}$ and $\mathcal{G}$ are emi-cubes. For $s<t$, let $\pi_1:(\mathbb{P}^1)^t\to(\mathbb{P}^1)^s$ denote the projection

given by $(x_1,\dots,x_t)\mapsto(x_1,\dots,x_s)$ and let $\pi_2:(\mathbb{P}^1)^t\to(\mathbb{P}^1)^s$ denote the projection given by $(x_1,\dots,x_t)\mapsto(x_{t-s+1},\dots,x_t)$. Let $u_i=\log|z_i|^2$ for $i=1,\dots,n$ and $v_j=\log|z_{n+j}|^2$ for $j=1,\dots,m$. If we regard $\partial\overline{\partial}\log|z_i|^2$ as $-2\pi\sqrt{-1}(\delta_{\{z_i=0\}}-\delta_{\{z_i=\infty\}})$, then the above identity is still valid as currents on $(\mathbb{P}^1)^n$. Hence we have

$$
\begin{aligned}
&d_{\mathcal{D}}(\mathrm{ch}_n(\mathcal{F})\vartriangle\mathrm{ch}_m(\mathcal{G}))=-d(\mathrm{ch}_n(\mathcal{F})\vartriangle\mathrm{ch}_m(\mathcal{G}))\\
&=\tfrac{(-1)^{n+m+1}}{4n!m!(2\pi\sqrt{-1})^{n+m}}\int_{(\mathbb{P}^1)^{n+m}}\pi_1^*\mathrm{ch}_0(\mathrm{tr}_n\mathcal{F})\wedge\pi_2^*\mathrm{ch}_0(\mathrm{tr}_m\mathcal{G})\\
&\qquad\wedge d\left(\left(\sum_{i=1}^n(-1)^i\pi_1^*S_n^i\right)\vartriangle\left(\sum_{j=1}^m(-1)^j\pi_2^*S_m^j\right)\right)\\
&=\tfrac{(-1)^{m+1}}{2(n+m)!(2\pi\sqrt{-1})^{n+m}}\int_{(\mathbb{P}^1)^{n+m}}\mathrm{ch}_0(\mathrm{tr}_{n+m}(\mathcal{F}\otimes\mathcal{G}))\wedge\sum_{k=1}^{n+m}(-1)^kS_{n+m}^k\\
&\quad+\tfrac{(-1)^{n+m+1}}{4(n-1)!m!(2\pi\sqrt{-1})^{n+m-1}}\int_{(\mathbb{P}^1)^{n+m-1}}\mathrm{ch}_0(\mathrm{tr}_{n+m-1}(\partial\mathcal{F}\otimes\mathcal{G}))\\
&\qquad\wedge\sum_{\substack{1\le i\le n-1\\1\le j\le m}}(-1)^{i+j}a_{i,j}^{n-1,m}\pi_1^*S_{n-1}^i\wedge\pi_2^*S_m^j\\
&\quad+\tfrac{(-1)^m}{4n!(m-1)!(2\pi\sqrt{-1})^{n+m-1}}\int_{(\mathbb{P}^1)^{n+m-1}}\mathrm{ch}_0(\mathrm{tr}_{n+m-1}(\mathcal{F}\otimes\partial\mathcal{G}))\\
&\qquad\wedge\sum_{\substack{1\le i\le n\\1\le j\le m-1}}(-1)^{i+j}a_{i,j}^{n,m-1}\pi_1^*S_n^i\wedge\pi_2^*S_{m-1}^j\\
&\quad+\tfrac{(-1)^m}{4n!m!(2\pi\sqrt{-1})^{n+m}}\int_{(\mathbb{P}^1)^{n+m}}\mathrm{ch}_0(\mathrm{tr}_{n+m}(\mathcal{F}\otimes\mathcal{G}))\\
&\qquad\wedge\left(\left(\sum_{i=1}^n(-1)^i\pi_1^*S_n^i\right)\bullet\left(\sum_{j=1}^m(-1)^j\pi_2^*S_m^j\right)\right)\\
&=(-1)^{n+1}\mathrm{ch}_{n+m}(\mathcal{F}\otimes\mathcal{G})+\mathrm{ch}_{n-1}(\partial\mathcal{F})\vartriangle\mathrm{ch}_m(\mathcal{G})\\
&\quad+(-1)^{n+1}\mathrm{ch}_n(\mathcal{F})\vartriangle\mathrm{ch}_m(\partial\mathcal{G})+(-1)^n\mathrm{ch}_n(\mathcal{F})\bullet\mathrm{ch}_m(\mathcal{G}).
\end{aligned}
$$

□

§6. Product

§6.1. Notations on bisimplicial sets

A *bisimplicial set* is a contravariant functor from the category of pairs of finite ordered sets to the category of sets. The product $S\times T$ and the reduced product $S \wedge T$ of two simplicial sets S, T are examples of bisimplicial sets. The topological realization $|S|$ of a bisimplicial set S is defined in a similar way to that of a simplicial set.

For a bisimplicial set S, let $\Delta(S)$ denote the simplicial set given by $[n] \mapsto S([n],[n])$. Then its topological realization $|\Delta(S)|$ is a subdivision of $|S|$. Hence the identity map $|S| \to |\Delta(S)|$ is cellular, although the inverse is not.

§6.2. Product in higher K-theory

In this subsection we review the product in higher algebraic K-theory by means of the S-construction [15]. For a small exact category $\mathfrak{A}$, let $S_n S_m \mathfrak{A}$ be the set of functors

$$E : \mathrm{Ar}[n] \times \mathrm{Ar}[m] \to \mathfrak{A},\ (i \le j, \alpha \le \beta) \mapsto E_{(i,j)\times(\alpha,\beta)}$$

satisfying the following conditions:

(1) $E_{(i,i)\times(\alpha,\beta)} = 0$ and $E_{(i,j)\times(\alpha,\alpha)} = 0$.
(2) For any $i \le j \le k$ and $\alpha \le \beta$, $E_{(i,j)\times(\alpha,\beta)} \to E_{(i,k)\times(\alpha,\beta)} \to E_{(j,k)\times(\alpha,\beta)}$ is a short exact sequence of $\mathfrak{A}$.
(3) For any $i \le j$ and $\alpha \le \beta \le \gamma$, $E_{(i,j)\times(\alpha,\beta)} \to E_{(i,j)\times(\alpha,\gamma)} \to E_{(i,j)\times(\beta,\gamma)}$ is a short exact sequence of $\mathfrak{A}$.

Then $([n],[m]) \mapsto S_n S_m \mathfrak{A}$ is a bisimplicial set. Let us denote it by $S^{(2)}\mathfrak{A}$. The natural identification $S_1 S_m \mathfrak{A} = S_m \mathfrak{A}$ yields a map of bisimplicial sets

$$S^1 \wedge S\mathfrak{A} \to S^{(2)}\mathfrak{A},$$

and its adjoint map $|S\mathfrak{A}| \to \Omega|S^{(2)}\mathfrak{A}|$ is proved to be a homotopy equivalence.

When $\mathfrak{A}$ is equipped with tensor product, we can define a map of bisimplicial sets

$$m : S\mathfrak{A} \wedge S\mathfrak{A} \to S^{(2)}\mathfrak{A}$$

by $m(E,F)_{(i,j)\times(\alpha,\beta)} = E_{i,j} \otimes F_{\alpha,\beta}$. This induces a pairing

$$m_* : \pi_{n+1}(|S\mathfrak{A}|) \times \pi_{m+1}(|S\mathfrak{A}|) \to \pi_{n+m+2}(|S^{(2)}\mathfrak{A}|).$$

Combining this with the isomorphisms $K_n(\mathfrak{A}) \simeq \pi_{n+1}(|S\mathfrak{A}|) \simeq \pi_{n+2}(|S^{(2)}\mathfrak{A}|)$ yields the product in higher algebraic K-theory $K_*(\mathfrak{A})$.

§6.3. G-construction

In [8], Gillet and Grayson have constructed a simplicial set $G\mathfrak{A}$ associated with a small exact category $\mathfrak{A}$ that is homotopy equivalent to the loop space of the S-construction $S\mathfrak{A}$. In this subsection we recall their construction.

Let $G_n\mathfrak{A}$ be the set of pairs (E^+, E^-) of $E^+, E^- \in S_{n+1}\mathfrak{A}$ with $\partial_0 E^+ = \partial_0 E^-$. Then $[n] \mapsto G_n\mathfrak{A}$ becomes a simplicial set by $\partial_k(E^+, E^-) = (\partial_{k+1}E^+, \partial_{k+1}E^-)$ and $s_k(E^+, E^-) = (s_{k+1}E^+, s_{k+1}E^-)$. We fix $0 = (0, 0) \in G_0\mathfrak{A}$ as the base point of $G\mathfrak{A}$.

Let $\Delta[1]$ be the simplicial set represented by $[1]$. Let ι_k denote the element of $\Delta[1]_n$ given by

$$\iota_k(i) = \begin{cases} 0, & i < k, \\ 1, & i \geq k. \end{cases}$$

Then $\Delta[1]_n = \{\iota_0, \iota_1, \ldots, \iota_{n+1}\}$. Let

$$\chi_n^{\pm} : \Delta[1]_n \times G_n\mathfrak{A} \to S_n\mathfrak{A}$$

be the maps given by

$$\chi_n^{\pm}(\iota_k, (E^+, E^-)) = \begin{cases} \partial_0 E^{\pm}, & k = 0, \\ (s_0)^{k-1}(\partial_1)^k E^{\pm}, & k \geq 1. \end{cases}$$

Then $\chi^{\pm} = \{\chi_n^{\pm}\} : \Delta(\Delta[1] \times G\mathfrak{A}) \to S\mathfrak{A}$ are maps of simplicial sets such that $\chi^{\pm}(\{0\} \times G\mathfrak{A}) = *$ and $\chi^+|_{\{1\}\times G\mathfrak{A}} = \chi^-|_{\{1\}\times G\mathfrak{A}}$.

Let T^1 be the simplicial set given by the following cocartesian square:

$$\begin{array}{ccccc} & & \Delta[1] & & \\ & \nearrow & & \searrow & \\ \{0\} \cup \{1\} & & & & T^1. \\ & \searrow & & \nearrow & \\ & & \Delta[1] & & \end{array}$$

We fix 0 as the base point of T^1. The topological realization of T^1 is the barycentric subdivision of the circle $S^1 = I/\partial I$. Gluing the maps $\chi^{\pm}$, we obtain a map of simplicial sets

$$\chi : \Delta(T^1 \wedge G\mathfrak{A}) \to S\mathfrak{A}.$$

It is the main theorem of [8] that the adjoint map $|G\mathfrak{A}| \to \Omega|S\mathfrak{A}|$ to $|\chi|$ is a homotopy equivalence . Therefore we have an isomorphism $\pi_i(|G\mathfrak{A}|, 0) \simeq K_i(\mathfrak{A})$.

We next introduce a description of the product in K-theory by means of G-construction. Let

$$G_n G_m \mathfrak{A} = \{(E^{++}, E^{+-}, E^{-+}, E^{--}); E^{\pm\pm} \in S_{n+1} S_{m+1} \mathfrak{A},\\ \partial_0 E^{+\pm} = \partial_0 E^{-\pm}, \partial_0' E^{\pm+} = \partial_0' E^{\pm-}\},$$

where ∂_0 is the boundary map on the first factor of the bisimplicial set $S^{(2)}\mathfrak{A}$ and ∂_0' is the boundary map on the second factor. Then $([n],[m]) \mapsto G_n G_m \mathfrak{A}$ becomes a bisimplicial set. Let us denote it by $G^{(2)}\mathfrak{A}$. Let $R : G_n \mathfrak{A} \to G_0 G_n \mathfrak{A}$ be the map given by $R(E^+, E^-) = (E^+, E^-, 0, 0)$. Then it is shown in [8] that R induces a homotopy equivalence $R : G\mathfrak{A} \to G^{(2)}\mathfrak{A}$.

Let us define a map of bisimplicial sets

$$m^G : G\mathfrak{A} \wedge G\mathfrak{A} \to G^{(2)}\mathfrak{A}$$

by $m^G(E,F)^{\pm\pm} = E^\pm \otimes F^\pm$ for $E = (E^+, E^-) \in G_n\mathfrak{A}$ and $F = (F^+, F^-) \in G_m\mathfrak{A}$. Then the pairing

$$m_*^G : \pi_n(|G\mathfrak{A}|, 0) \times \pi_m(|G\mathfrak{A}|, 0) \to \pi_{n+m}(|G^{(2)}\mathfrak{A}|, 0)$$

induces the product in $K_*(\mathfrak{A})$.

Finally, let us define an exact cube associated with an element of $G\mathfrak{A}$ or $G^{(2)}\mathfrak{A}$. The map χ yields a homomorphism of chain complexes

$$\mathrm{Cub} : C_*(|G\mathfrak{A}|) \xrightarrow{\chi_*} C_*(|S\mathfrak{A}|)[-1] \xrightarrow{\mathrm{Cub}} \mathrm{Cub}_*(\mathfrak{A}).$$

Let us define $\mathrm{Cub}(E) \in \mathrm{Cub}_n(\mathfrak{A})$ associated with $E = (E^+, E^-) \in G_n\mathfrak{A}$ as the image of $[E] \in C_*(|G\mathfrak{A}|)$ by the above map. In other words, $\mathrm{Cub}(E) = \mathrm{Cub}(E^+) - \mathrm{Cub}(E^-)$. Similarly, we define an exact $(n+m)$-cube associated with $E = (E^{\pm\pm}) \in G_n G_m \mathfrak{A}$ by

$$\mathrm{Cub}(E) = \mathrm{Cub}(E^{++}) - \mathrm{Cub}(E^{+-}) - \mathrm{Cub}(E^{-+}) + \mathrm{Cub}(E^{--}),$$

where $\mathrm{Cub}(E^{\pm\pm})$ is the image of the element of $E^{\pm\pm} \in S_{n+1}S_{m+1}\mathfrak{A}$ by the homomorphism

$$S_{n+1}S_{m+1}\mathfrak{A} \to \mathrm{Cub}_n(S_{m+1}\mathfrak{A}) \to \mathrm{Cub}_n(\mathrm{Cub}_m(\mathfrak{A})) = \mathrm{Cub}_{n+m}(\mathfrak{A}).$$

When $E = (E^{\pm\pm})$ is degenerate, the associated cube $\mathrm{Cub}(E)$ is zero in $\mathrm{Cub}_*(\mathfrak{A})$ by Lemma 4.1. Hence $E = (E^{\pm\pm}) \mapsto \mathrm{Cub}(E)$ induces a homomorphism

$$\mathrm{Cub} : C_*(|G^{(2)}\mathfrak{A}|) \to \mathrm{Cub}_*(\mathfrak{A}).$$

Proposition 6.1. *The following diagram is commutative*:

$$\begin{array}{ccc} C_*(|G\mathfrak{A}|)\otimes C_*(|G\mathfrak{A}|) & \xrightarrow{\mathrm{Cub}\otimes\mathrm{Cub}} & \mathrm{Cub}_*(\mathfrak{A})\otimes \mathrm{Cub}_*(\mathfrak{A}) \\ \downarrow{\scriptstyle m_*^G} & & \downarrow{\scriptstyle\otimes} \\ C_*(|G^{(2)}\mathfrak{A}|) & \xrightarrow{\mathrm{Cub}} & \mathrm{Cub}_*(\mathfrak{A}) \\ \uparrow{\scriptstyle R_*} & & \uparrow{\scriptstyle\mathrm{id}} \\ C_*(|G\mathfrak{A}|) & \xrightarrow{\mathrm{Cub}} & \mathrm{Cub}_*(\mathfrak{A}). \end{array}$$

§6.4. Pairing $\widehat{\mathcal{K}}_0 \times \widehat{K}_n \to \widehat{K}_n$

For a proper arithmetic variety X, let $\widehat{G}(X) = G(\widehat{\mathcal{P}}(X))$, the G-construction of the category of hermitian vector bundles on X. Then there is a homomorphism of chain complexes

$$\mathrm{ch} : C_*(|\widehat{G}(X)|) \xrightarrow{\mathrm{Cub}} \widehat{\mathrm{Cub}}_*(X) \xrightarrow{\mathrm{ch}} \mathcal{D}_*(X).$$

Proposition 6.2. *The map $\chi : \Delta(T^1 \wedge \widehat{G}(X)) \to \widehat{S}(X)$ yields an isomorphism*

$$\widehat{\chi}_* : \widehat{\pi}_n(|\widehat{G}(X)|, \mathrm{ch}) \simeq \widehat{\pi}_{n+1}(|\widehat{S}(X)|, \mathrm{ch})$$

by $[(f, \omega)] \mapsto [(\chi(1\wedge f), -\omega)]$. *Hence for* $n \geq 1$ *there is a canonical isomorphism*

$$\widehat{K}_n(X) \simeq \widehat{\pi}_n(|\widehat{G}(X)|, \mathrm{ch}).$$

Proof. It is obvious that the map $(f, \omega) \mapsto (\chi(1 \wedge f), -\omega)$ gives rise to a homomorphism of the modified homotopy groups. Consider the following commutative diagram:

$$\begin{array}{ccccccccc} \pi_{n+1}(|\widehat{G}(X)|) & \xrightarrow{\rho} & \widetilde{\mathcal{D}}_{n+1}(X) & \longrightarrow & \widehat{\pi}_n(|\widehat{G}(X)|, \mathrm{ch}) & \longrightarrow & \pi_n(|\widehat{G}(X)|) & \longrightarrow & 0 \\ \downarrow{\scriptstyle -\chi_*} & & \downarrow{\scriptstyle -\mathrm{id}} & & \downarrow{\scriptstyle \widehat{\chi}_*} & & \downarrow{\scriptstyle \chi_*} & & \\ \pi_{n+2}(|\widehat{S}(X)|) & \xrightarrow{\rho} & \widetilde{\mathcal{D}}_{n+1}(X) & \longrightarrow & \widehat{\pi}_{n+1}(|\widehat{S}(X)|, \mathrm{ch}) & \longrightarrow & \pi_{n+1}(|\widehat{S}(X)|) & \longrightarrow & 0, \end{array}$$

where the upper and lower sequences are exact by Theorem 3.3. Hence the proposition follows from the five lemma. □

If we set $\widehat{G}^{(2)}(X) = G^{(2)}(\widehat{\mathcal{P}}(X))$, then we have

$$\mathrm{ch} : C_*(|\widehat{G}^{(2)}(X)|) \xrightarrow{\mathrm{Cub}} \widehat{\mathrm{Cub}}_*(X) \xrightarrow{\mathrm{ch}} \mathcal{D}_*(X)$$

and the following square is commutative by Proposition 6.1:

$$\begin{array}{ccc} C_*(|\widehat{G}(X)|) & \xrightarrow{\text{ch}} & \mathcal{D}_*(X) \\ \Big\downarrow R_* & & \Big\downarrow \text{id} \\ C_*(|\widehat{G}^{(2)}(X)|) & \xrightarrow{\text{ch}} & \mathcal{D}_*(X). \end{array}$$

Hence R induces an isomorphism

$$\widehat{R}_* : \widehat{\pi}_n(|\widehat{G}(X)|, \text{ch}) \simeq \widehat{\pi}_n(|\widehat{G}^{(2)}(X)|, \text{ch}).$$

The product $\widehat{\mathcal{K}}_0(X) \times \widehat{\mathcal{K}}_0(X) \to \widehat{\mathcal{K}}_0(X)$ given in [9] is written as follows:

$$[(\overline{E}, \omega)] \times [(\overline{F}, \tau)] = [(\overline{E} \otimes \overline{F}, \text{ch}_0(\overline{E}) \bullet \tau + \omega \bullet \text{ch}_0(\overline{F}) + \omega \bullet d_{\mathcal{D}}\tau)],$$

and it makes $\widehat{\mathcal{K}}_0(X)$ a commutative associative algebra. To construct product in higher arithmetic K-theory, we will use the G-construction. However, since we have not had any expression of $\widehat{K}_0(X)$ by means of the G-construction, we have to distinguish the cases including $\widehat{K}_0(X)$ from the general case.

Let $(\overline{E}, \eta)$ be a pair of a hermitian vector bundle $\overline{E}$ on X and $\eta \in \widetilde{\mathcal{D}}_1(X)$ and let (f, ω) be a pair of a pointed cellular map $f : S^n \to |\widehat{G}(X)|$ and $\omega \in \widetilde{\mathcal{D}}_{n+1}(X)$. Let us define a product of these pairs by

$$(\overline{E}, \eta) \times (f, \omega) = (\overline{E} \otimes f, \text{ch}_0(\overline{E}) \bullet \omega + \eta \bullet \text{ch}_n(f) + \eta \bullet d_{\mathcal{D}}\omega),$$

where $\overline{E} \otimes f : S^n \xrightarrow{f} |\widehat{G}(X)| \xrightarrow{\overline{E}\otimes} |\widehat{G}(X)|$.

Theorem 6.3. *The above product gives rise to a pairing*

$$\times : \widehat{\mathcal{K}}_0(X) \times \widehat{K}_n(X) \to \widehat{K}_n(X).$$

Proof. To prove the theorem, we have to show that $(\overline{E}, \eta) \times (f, \omega)$ is compatible with the equivalence relations for $\widehat{\mathcal{K}}_0(X)$ and $\widehat{K}_n(X)$. Let us first show the compatibility with the relation for $\widehat{K}_n(X)$.

Let $H : (S^n \times I)/(\{*\} \times I) \to |\widehat{G}(X)|$ be a cellular homotopy from (f, ω) to (f', ω'). We write $\text{ch}_{n+1}(H)$ for $\text{ch} \circ H_*([S^n \times I]) \in \mathcal{D}_{n+1}(X)$. Then $\omega' - \omega = (-1)^{n+1} \text{ch}_{n+1}(H)$ and the map

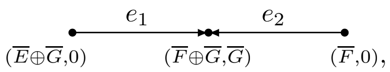
$$\overline{E} \otimes H : (S^n \times I)/(\{*\} \times I) \xrightarrow{H} |\widehat{G}(X)| \xrightarrow{\overline{E}\otimes} |\widehat{G}(X)|$$

is a cellular homotopy from $\overline{E} \otimes f$ to $\overline{E} \otimes f'$. Furthermore, by Proposition 5.2 we have

$$\text{ch}_{n+1}(\overline{E} \otimes H) = \text{ch}_0(\overline{E}) \bullet \text{ch}_{n+1}(H) = (-1)^{n+1} \text{ch}_0(\overline{E}) \bullet (\omega' - \omega).$$

This tells that $\overline{E} \otimes H$ is a cellular homotopy from $(\overline{E}, \eta) \times (f, \omega)$ to $(\overline{E}, \eta) \times (f', \omega')$.

Next we show the compatibility with the relation for $\widehat{\mathcal{K}}_0(X)$. Let $\mathcal{E} : 0 \to \overline{E} \to \overline{F} \to \overline{G} \to 0$ be a short exact sequence of hermitian vector bundles on X. Consider the following 1-dimensional subcomplex of $|\widehat{G}(X)|$:



where

$$e_1 = \left(\begin{array}{ccc} \overline{E} \oplus \overline{G} & \longrightarrow & \overline{F} \oplus \overline{G} \\ & & \downarrow \\ & & \overline{G} \end{array} \quad \begin{array}{ccc} 0 & \longrightarrow & \overline{G} \\ & & \downarrow \\ & & \overline{G} \end{array} \right),$$

$$e_2 = \left(\begin{array}{ccc} \overline{F} & \longrightarrow & \overline{F} \oplus \overline{G} \\ & & \downarrow \\ & & \overline{G} \end{array} \quad \begin{array}{ccc} 0 & \longrightarrow & \overline{G} \\ & & \downarrow \\ & & \overline{G} \end{array} \right).$$

We denote by $\iota_{\mathcal{E}} : I \to |\widehat{G}(X)|$ a cellular map such that $\iota_{\mathcal{E}}(I) = e_1 e_2^{-1}$.

For a pointed cellular map $f : S^n \to |\widehat{G}(X)|$, let

$$H : (S^n \times I)/(\{*\} \times I) \xrightarrow{T} (I \times S^n)/(I \times \{*\}) \xrightarrow{\iota_{\mathcal{E}} \wedge f} |\widehat{G}(X)| \wedge |\widehat{G}(X)| \xrightarrow{m^G} |\widehat{G}^{(2)}(X)|,$$

where $T(s,t) = (t,s)$ for $t \in S^n$ and $s \in I$. If $H_0(s) = H(s,0)$, then H_0 is written as

$$S^n \xrightarrow{f} |\widehat{G}(X)| \xrightarrow{\iota_{\overline{E}\oplus\overline{G}} \wedge \mathrm{id}} |\widehat{G}(X)| \wedge |\widehat{G}(X)| \xrightarrow{m^G} |\widehat{G}^{(2)}(X)|,$$

where $\iota_{\overline{E}\oplus\overline{G}} : S^0 \to |\widehat{G}(X)|$ is the pointed map determined by $(\overline{E} \oplus \overline{G}, 0) \in \widehat{G}_0(X)$. Since the diagram

$$\begin{array}{ccc} |\widehat{G}(X)| & \xrightarrow{\iota_{\overline{E}\oplus\overline{G}} \wedge \mathrm{id}} & |\widehat{G}(X)| \wedge |\widehat{G}(X)| \\ \downarrow{\scriptstyle (\overline{E}\oplus\overline{G})\otimes} & & \downarrow{\scriptstyle m^G} \\ |\widehat{G}(X)| & \xrightarrow{R} & |\widehat{G}^{(2)}(X)| \end{array}$$

is commutative, we have $H_0 = R((\overline{E} \oplus \overline{G}) \otimes f)$. If $H_1(s) = H(s,1)$, then we can show that $H_1 = R(\overline{F} \otimes f)$ in the same way. Moreover, Proposition 5.2 implies that

$$\begin{aligned}\mathrm{ch}_{n+1}(H) &= (-1)^n \mathrm{ch}_{n+1}(m_*^G(\iota_{\mathcal{E}} \wedge f)_*([I \times S^n])) \\ &\equiv (-1)^n \mathrm{ch}_1(\iota_{\mathcal{E}}) \bullet \mathrm{ch}_n(f) \\ &= (-1)^n \mathrm{ch}_1(\mathcal{E}) \bullet \mathrm{ch}_n(f)\end{aligned}$$

modulo $\operatorname{Im} d_{\mathcal{D}}$. Hence H is a cellular homotopy from $(R((\overline{E} \oplus \overline{G}) \otimes f), \mathrm{ch}_1(\mathcal{E}) \bullet \mathrm{ch}_n(f))$ to $(R(\overline{F} \otimes f), 0)$. Since $\widehat{R}_* : \widehat{\pi}_n(|\widehat{G}(X)|, \mathrm{ch}) \to \widehat{\pi}_n(|\widehat{G}^{(2)}(X)|, \mathrm{ch})$ is bijective,

$$[((\overline{E} \oplus \overline{G}) \otimes f, \mathrm{ch}_1(\mathcal{E}) \bullet \mathrm{ch}_n(f))] = [(\overline{F} \otimes f, 0)]$$

in $\widehat{\pi}_n(|\widehat{G}(X)|, \mathrm{ch})$.

The short exact sequence $\mathcal{E}$ gives the relation

$$[(\overline{E}, 0)] + [(\overline{G}, 0)] = [(\overline{F}, -\mathrm{ch}_1(\mathcal{E}))]$$

in $\widehat{\mathcal{K}}_0(X)$. We have

$$[(\overline{E}, 0) \times (f, \omega)] \cdot [(\overline{G}, 0) \times (f, \omega)] = [((\overline{E} \otimes f) \cdot (\overline{G} \otimes f), (\mathrm{ch}_0(\overline{E}) + \mathrm{ch}_0(\overline{G})) \bullet \omega)]$$

and

$$\begin{aligned}&[(\overline{F}, -\mathrm{ch}_1(\mathcal{E})) \times (f, \omega)] \\ &\quad = [(\overline{F} \otimes f, \mathrm{ch}_0(\overline{F}) \bullet \omega - \mathrm{ch}_1(\mathcal{E}) \bullet \mathrm{ch}_n(f) - d_{\mathcal{D}} \mathrm{ch}_1(\mathcal{E}) \bullet \omega)] \\ &\quad = [((\overline{E} \oplus \overline{G}) \otimes f, (\mathrm{ch}_0(\overline{E}) + \mathrm{ch}_0(\overline{G})) \bullet \omega)]\end{aligned}$$

in $\widehat{\pi}_n(|\widehat{G}(X)|, \mathrm{ch})$. Hence Theorem 6.3 follows from Lemma 6.4 and Lemma 6.5 below. □

Lemma 6.4. *For a pointed cellular map $f : S^n \to |\widehat{G}(X)|$ and two hermitian vector bundles $\overline{E}, \overline{G}$ on X,*

$$[((\overline{E} \otimes f) \oplus (\overline{G} \otimes f), 0)] = [((\overline{E} \otimes f) \cdot (\overline{G} \otimes f), 0)]$$

in $\widehat{\pi}_n(|\widehat{G}(X)|, \mathrm{ch})$.

Proof. Let us first describe the map $(\overline{E} \otimes f) \oplus (\overline{G} \otimes f)$ explicitly. Since f is a pointed cellular map, the map

$$S^n \overset{\Delta}{\hookrightarrow} S^n \times S^n \overset{(\overline{E} \otimes f) \times (\overline{G} \otimes f)}{\longrightarrow} |\Delta(\widehat{G}(X) \times \widehat{G}(X))|$$

is also a pointed cellular map. Moreover, the direct sum of hermitian vector bundles induces a map of simplicial sets $\oplus : \Delta(\widehat{G}(X) \times \widehat{G}(X)) \to \widehat{G}(X)$. Then the map $(\overline{E}\otimes f)\oplus(\overline{G}\otimes f)$ is expressed as the composition of these two cellular maps, that is,

$$(\overline{E}\otimes f)\oplus(\overline{G}\otimes f): S^n \overset{\Delta}{\hookrightarrow} S^n\times S^n \xrightarrow{(\overline{E}\otimes f)\times(\overline{G}\otimes f)} |\Delta(\widehat{G}(X)\times\widehat{G}(X))| \xrightarrow{\oplus} |\widehat{G}(X)|.$$

Consider the homomorphism of chain complexes

$$\mathrm{ch}\oplus\mathrm{ch}: C_*(|\Delta(\widehat{G}(X)\times\widehat{G}(X))|)\to\mathcal{D}_*(X)\oplus\mathcal{D}_*(X)$$

given by $(E,F)\mapsto(\mathrm{ch}_n(E),\mathrm{ch}_n(F))$ for $E,F\in\widehat{G}_n(X)$ and the inclusion

$$in_1 \text{ (resp. } in_2\text{)}: \widehat{G}(X)\to\Delta(\widehat{G}(X)\times\widehat{G}(X))$$

given by $in_1(t)=(t,*)$ (resp. $in_2(t)=(*,t)$). Then we have the following commutative diagram:

$$\begin{array}{ccc} C_*(|\widehat{G}(X)|) & \xrightarrow{\mathrm{ch}} & \mathcal{D}_*(X) \\ \Big\downarrow {\scriptstyle in_{1*} \text{ (resp. } in_{2*})} & & \Big\downarrow {\scriptstyle in_1 \text{ (resp. } in_2)} \\ C_*(|\Delta(\widehat{G}(X)\times\widehat{G}(X))|) & \xrightarrow{\mathrm{ch}\oplus\mathrm{ch}} & \mathcal{D}_*(X)\oplus\mathcal{D}_*(X), \end{array}$$

where the right vertical arrow is $in_1(\omega)=(\omega,0)$ (resp. $in_2(\omega)=(0,\omega)$). On the other hand, the projection

$$pr_1 \text{ (resp. } pr_2\text{)}: \Delta(\widehat{G}(X)\times\widehat{G}(X))\to\widehat{G}(X)$$

given by $pr_1(x,y)=x$ (resp. $pr_2(x,y)=y$) is also a map of simplicial sets and we have the following commutative diagram:

$$\begin{array}{ccc} C_*(|\Delta(\widehat{G}(X)\times\widehat{G}(X))|) & \xrightarrow{\mathrm{ch}\oplus\mathrm{ch}} & \mathcal{D}_*(X)\oplus\mathcal{D}_*(X) \\ \Big\downarrow {\scriptstyle pr_{1*} \text{ (resp. } pr_{2*})} & & \Big\downarrow {\scriptstyle pr_1 \text{ (resp. } pr_2)} \\ C_*(|\widehat{G}(X)|) & \xrightarrow{\mathrm{ch}} & \mathcal{D}_*(X), \end{array}$$

where the right vertical arrow is $pr_1(\omega,\tau)=\omega$ (resp. $pr_2(\omega,\tau)=\tau$). Hence we have four homomorphisms between the modified homotopy groups

$$\widehat{\pi}_n(|\Delta(\widehat{G}(X)\times\widehat{G}(X))|,\mathrm{ch}\oplus\mathrm{ch}) \underset{\widehat{in_j}_*}{\overset{\widehat{pr_j}_*}{\rightleftarrows}} \widehat{\pi}_n(|\widehat{G}(X)|,\mathrm{ch})$$

that induce an isomorphism

$$\widehat{\pi}_n(|\widehat{G}(X)|,\mathrm{ch})\oplus\widehat{\pi}_n(|\widehat{G}(X)|,\mathrm{ch})\simeq\widehat{\pi}_n(|\Delta(\widehat{G}(X)\times\widehat{G}(X))|,\mathrm{ch}\oplus\mathrm{ch})$$

by $(x,y) \mapsto \widehat{in}_{1*}(x) \cdot \widehat{in}_{2*}(y)$. The inverse of it is $\widehat{pr}_{1*} \oplus \widehat{pr}_{2*}$. Since

$$\widehat{pr}_{1*}([(((\overline{E} \otimes f) \times (\overline{G} \otimes f))\Delta, 0)]) = [(\overline{E} \otimes f, 0)],$$
$$\widehat{pr}_{2*}([(((\overline{E} \otimes f) \times (\overline{G} \otimes f))\Delta, 0)]) = [(\overline{G} \otimes f, 0)],$$

we have

$$[(((\overline{E} \otimes f) \times (\overline{G} \otimes f))\Delta, 0)] = \widehat{in}_{1*}([(\overline{E} \otimes f, 0)]) \cdot \widehat{in}_{2*}([(\overline{G} \otimes f, 0)])$$

in $\widehat{\pi}_*(|\Delta(\widehat{G}(X) \times \widehat{G}(X))|, \mathrm{ch} \oplus \mathrm{ch})$.

The commutative diagram

$$\begin{array}{ccc} C_*(|\Delta(\widehat{G}(X) \times \widehat{G}(X))|) & \xrightarrow{\mathrm{ch}\oplus\mathrm{ch}} & \mathcal{D}_*(X) \oplus \mathcal{D}_*(X) \\ \downarrow \oplus_* & & \downarrow + \\ C_*(|\widehat{G}(X)|) & \xrightarrow{\mathrm{ch}} & \mathcal{D}_*(X) \end{array}$$

implies a homomorphism

$$\widehat{\oplus}_* : \widehat{\pi}_n(|\Delta(\widehat{G}(X) \times \widehat{G}(X))|, \mathrm{ch} \oplus \mathrm{ch}) \to \widehat{\pi}_n(|\widehat{G}(X)|, \mathrm{ch}).$$

Since $\widehat{\oplus}_* \widehat{in}_{j*}$ is the identity homomorphism, we have

$$\begin{aligned} [((\overline{E} \otimes f) \oplus (\overline{G} \otimes f), 0)] &= \widehat{\oplus}_* \big([(((\overline{E} \otimes f) \times (\overline{G} \otimes f))\Delta, 0)]\big) \\ &= [(\overline{E} \otimes f, 0)] \cdot [(\overline{G} \otimes f, 0)] \\ &= [((\overline{E} \otimes f) \cdot (\overline{G} \otimes f), 0)]. \end{aligned}$$

□

Lemma 6.5. *In the same notations as in Lemma* 6.4, *we have*

$$[((\overline{E} \oplus \overline{G}) \otimes f, 0)] = [((\overline{E} \otimes f) \oplus (\overline{G} \otimes f), 0)]$$

in $\widehat{\pi}_n(|\widehat{G}(X)|, \mathrm{ch})$.

Proof. Consider the following diagram:

$$\begin{array}{ccccc} \Delta(\widehat{G}(X) \times \widehat{G}(X)) & \xrightarrow{(\overline{E}\otimes)\times(\overline{G}\otimes)} & \Delta(\widehat{G}(X) \times \widehat{G}(X)) & & \\ \uparrow \Delta & & \downarrow \oplus & & \\ \widehat{G}(X) & \xrightarrow{(\overline{E}\oplus\overline{G})\otimes} & \widehat{G}(X) & \xrightarrow{R} & \widehat{G}^{(2)}(X). \end{array}$$

Let $\alpha_0 : \widehat{G}(X) \to \widehat{G}^{(2)}(X)$ be the upper map of the diagram and α_1 the lower map. Then for $P = (P^{\pm}) \in \widehat{G}_n(X)$, $\alpha_0(P)$ and $\alpha_1(P)$ are elements of $\widehat{G}_0\widehat{G}_n(X)$ written as follows:

$$\alpha_0(P) = ((\overline{E} \otimes P^+) \oplus (\overline{G} \otimes P^+), (\overline{E} \otimes P^-) \oplus (\overline{G} \otimes P^-), 0, 0),$$
$$\alpha_1(P) = ((\overline{E} \oplus \overline{G}) \otimes P^+, (\overline{E} \oplus \overline{G}) \otimes P^-, 0, 0).$$

The canonical isometries $(\overline{E} \otimes P^{\pm}) \oplus (\overline{G} \otimes P^{\pm}) \simeq (\overline{E} \oplus \overline{G}) \otimes P^{\pm}$ give an element of $\widehat{G}_1\widehat{G}_n(X)$ whose Bott-Chern form is zero. Collecting these elements for all $P = (P^{\pm})$ provides a map of bisimplicial sets $\Psi : \Delta[1] \times \widehat{G}(X) \to \widehat{G}^{(2)}(X)$ such that $\Psi(0, s) = \alpha_0(s)$ and $\Psi(1, s) = \alpha_1(s)$. Therefore for any pointed cellular map $f : S^n \to |\widehat{G}(X)|$,

$$H : (S^n \times I)/(\{*\} \times I) \xrightarrow{T} (I \times S^n)/(I \times \{*\}) \xrightarrow{\mathrm{id} \times f} (I \times |\widehat{G}(X)|)/(I \times \{*\})$$
$$\xrightarrow{|\Psi|} |\widehat{G}^{(2)}(X)|$$

is a cellular homotopy from $R((\overline{E} \otimes f) \oplus (\overline{G} \otimes f))$ to $R((\overline{E} \oplus \overline{G}) \otimes f)$ such that $\mathrm{ch}_{n+1}(H) = 0$. Since $\widehat{R}_* : \widehat{\pi}_n(|\widehat{G}(X)|, \mathrm{ch}) \to \widehat{\pi}_n(|\widehat{G}^{(2)}(X)|, \mathrm{ch})$ is bijective, we have

$$[((\overline{E} \otimes f) \oplus (\overline{G} \otimes f), 0)] = [((\overline{E} \oplus \overline{G}) \otimes f, 0)].$$

□

We can define a pairing $\widehat{K}_n(X) \times \widehat{\mathcal{K}}_0(X) \to \widehat{K}_n(X)$ by

$$[(f, \omega)] \times [(\overline{E}, \eta)] = [(f \otimes \overline{E}, (-1)^n \mathrm{ch}_n(f) \bullet \eta + \omega \bullet \mathrm{ch}_0(\overline{E}) + \omega \bullet d_{\mathcal{D}}\eta)],$$

where $f \otimes \overline{E} : S^n \xrightarrow{f} |\widehat{G}(X)| \xrightarrow{\otimes \overline{E}} |\widehat{G}(X)|$. Combining these pairings with the isomorphism $\widehat{\alpha} : \widehat{\mathcal{K}}_0(X) \simeq \widehat{K}_0(X)$, we can obtain a pairing

$$\times : \widehat{K}_n(X) \times \widehat{K}_m(X) \to \widehat{K}_{n+m}(X)$$

when $n = 0$ or $m = 0$.

§6.5. Pairing of higher arithmetic K-theory

In this subsection we define a pairing $\widehat{K}_n(X) \times \widehat{K}_m(X) \to \widehat{K}_{n+m}(X)$ in the case of $n, m \geq 1$. For two pointed cellular maps $f : S^n \to |\widehat{G}(X)|$ and $g : S^m \to |\widehat{G}(X)|$, let

$$f \times g : S^{n+m} = S^n \wedge S^m \xrightarrow{f \wedge g} |\widehat{G}(X)| \wedge |\widehat{G}(X)| \xrightarrow{m^G} |\widehat{G}^{(2)}(X)|.$$

For $\omega \in \widetilde{\mathcal{D}}_{n+1}(X)$ and $\tau \in \widetilde{\mathcal{D}}_{m+1}(X)$, a product of pairs (f, ω) and (g, τ) is defined by

$$\begin{aligned}&(f,\omega)\times(g,\tau)\\&=(f\times g,(-1)^n \operatorname{ch}_n(f)\bullet\tau+\omega\bullet\operatorname{ch}_m(g)+\omega\bullet d_{\mathcal{D}}\tau+(-1)^n\operatorname{ch}_n(f)\vartriangle\operatorname{ch}_m(g)).\end{aligned}$$

Proposition 6.6. *The above product gives rise to a pairing*

$$\widehat{m}^G_* : \widehat{\pi}_n(|\widehat{G}(X)|,\mathrm{ch})\times\widehat{\pi}_m(|\widehat{G}(X)|,\mathrm{ch})\to\widehat{\pi}_{n+m}(|\widehat{G}^{(2)}(X)|,\mathrm{ch}).$$

Proof. For a cellular homotopy H from (f, ω) to (f', ω'), let $\widetilde{H}$ be a cellular map given by

$$\widetilde{H} : (S^{n+m}\times I)/(\{*\}\times I)\longrightarrow |\widehat{G}(X)|\wedge|\widehat{G}(X)|\xrightarrow{m^G}|\widehat{G}^{(2)}(X)|,$$

where the first map is $(s_1, s_2, t) \mapsto (H(s_1, t), g(s_2))$ for $s_1 \in S^n, s_2 \in S^m$ and $t \in I$. Then $\widetilde{H}$ is a homotopy from $f \times g$ to $f' \times g$. Theorem 5.2 and Proposition 6.1 imply that

$$\begin{aligned}&\operatorname{ch}_{n+m+1}(\widetilde{H})\\&\quad=(-1)^m\operatorname{ch}_{n+m+1}(m^G_*(H\times g)_*([S^n\times I\times S^m]))\\&\quad\equiv(-1)^m\operatorname{ch}_{n+1}(H)\bullet\operatorname{ch}_m(g)+(-1)^{n+m+1}\operatorname{ch}_n(\partial H_*([S^n\times I]))\vartriangle\operatorname{ch}_m(g)\\&\quad=(-1)^{n+m+1}(\omega'-\omega)\bullet\operatorname{ch}_m(g)+(-1)^{m+1}(\operatorname{ch}_n(f')-\operatorname{ch}_n(f))\vartriangle\operatorname{ch}_m(g)\\&\quad=(-1)^{n+m+1}(\omega'\bullet\operatorname{ch}_m(g)+(-1)^n\operatorname{ch}_n(f')\vartriangle\operatorname{ch}_m(g))\\&\qquad-(-1)^{n+m+1}(\omega\bullet\operatorname{ch}_m(g)+(-1)^n\operatorname{ch}_n(f)\vartriangle\operatorname{ch}_m(g))\end{aligned}$$

modulo $\operatorname{Im} d_{\mathcal{D}}$. This tells that the map $\widetilde{H}$ is a cellular homotopy from $(f, \omega)\times(g, \tau)$ to $(f', \omega') \times (g, \tau)$.

If H' is a cellular homotopy from (g, τ) to (g', τ'), we can show in the same way that the map

$$(S^{n+m}\times I)/(\{*\}\times I)\xrightarrow{f\wedge H'}|\widehat{G}(X)|\wedge|\widehat{G}(X)|\xrightarrow{m^G}|\widehat{G}^{(2)}(X)|$$

is a cellular homotopy from $(f, \omega) \times (g, \tau)$ to $(f, \omega) \times (g', \tau')$. □

Definition 6.7. For $n, m \geq 1$, we define a product in higher arithmetic K-theory

$$\times : \widehat{K}_n(X)\times\widehat{K}_m(X)\to\widehat{K}_{n+m}(X)$$

by the following homomorphism:

$$\begin{aligned}\widehat{\pi}_n(|\widehat{G}(X)|,\mathrm{ch})\times\widehat{\pi}_m(|\widehat{G}(X)|,\mathrm{ch})&\xrightarrow{\widehat{m}^G_*}\widehat{\pi}_{n+m}(|\widehat{G}^{(2)}(X)|,\mathrm{ch})\\&\xrightarrow{\widehat{R}^{-1}_*}\widehat{\pi}_{n+m}(|\widehat{G}(X)|,\mathrm{ch}).\end{aligned}$$

Proposition 6.8. *The Chern form map respects the products, that is, we have*

$$\mathrm{ch}_{n+m}(x \times y) = \mathrm{ch}_n(x) \bullet \mathrm{ch}_m(y)$$

for $x \in \widehat{K}_n(X)$ *and* $y \in \widehat{K}_m(X)$.

Proof. Assume $n, m \geq 1$. Define the Chern form map on $\widehat{\pi}_{n+m}(|\widehat{G}^{(2)}(X)|, \mathrm{ch})$ by

$$\mathrm{ch}_{n+m}([(f, \omega)]) = \mathrm{ch}_{n+m}(f_*([S^{n+m}])) + d_{\mathcal{D}}\omega \in \mathcal{D}_{n+m}(X).$$

Then $\mathrm{ch}_{n+m}(\widehat{R}_*(x)) = \mathrm{ch}_{n+m}(x)$ for any $x \in \widehat{\pi}_{n+m}(|\widehat{G}(X)|, \mathrm{ch})$. Hence it is sufficient to show that $\mathrm{ch}_{n+m}(\widehat{m}_*^G(x, y)) = \mathrm{ch}_n(x) \bullet \mathrm{ch}_m(y)$.

For $x = [(f, \omega)]$ and $y = [(g, \tau)]$, Theorem 5.2 implies that

$$\begin{aligned}\mathrm{ch}_{n+m}(\widehat{m}_*^G(x, y)) &= \mathrm{ch}_{n+m}(f \times g) + d_{\mathcal{D}}\left((-1)^n \mathrm{ch}_n(f) \bullet \tau + \omega \bullet \mathrm{ch}_m(g)\right.\\ &\qquad \left.+\omega \bullet d_{\mathcal{D}}\tau + (-1)^n \mathrm{ch}_n(f) \vartriangle \mathrm{ch}_m(g)\right)\\ &= (\mathrm{ch}_n(f) + d_{\mathcal{D}}\omega) \bullet (\mathrm{ch}_m(g) + d_{\mathcal{D}}\tau)\\ &= \mathrm{ch}_n(x) \bullet \mathrm{ch}_m(y).\end{aligned}$$

The case where $n = 0$ or $m = 0$ is trivial. □

Remark 1. The map Cub : $\widehat{S}_{n+1}\widehat{S}_{m+1}(X) \to \widehat{\mathrm{Cub}}_{n+m}(X)$ gives rise to a map

$$\mathrm{Cub} : C_*(|\widehat{S}^{(2)}(X)|)[-2] \to \widehat{\mathrm{Cub}}_*(X)$$

and the tensor product of hermitian vector bundles induces a map

$$C_*(|\widehat{S}(X)|)[-1] \otimes C_*(|\widehat{S}(X)|)[-1] \to C_*(|\widehat{S}^{(2)}(X)|)[-2].$$

But both of them are not compatible with the differentials. So it seems impossible to the author to define a product in $\widehat{K}_*(X)$ by using the S-construction.

Remark 2. In [6], another complex $\mathcal{H}^*_{TW}(X, p)$ computing real Deligne cohomology and higher Bott-Chern form with values in this complex are introduced. In the same argument as in §4.1, we can prove that this Bott-Chern form of any degenerate element of $\widehat{S}(X)$ is zero. Hence we have

$$\mathrm{ch}_{TW} : C_*(|\widehat{S}(X)|) \to \bigoplus_p \mathcal{H}^{2p-*}_{TW}(X, p),$$

and we can define a new version of higher arithmetic K-theory:

$$\widehat{K}^{TW}_n(X) = \widehat{\pi}_{n+1}(|\widehat{S}(X)|, \mathrm{ch}_{TW}).$$

The complex $\mathcal{H}^*_{TW}(X,p)$ is much bigger than $\mathcal{D}^*(X,p)$, therefore $\widehat{K}^{TW}_n(X)$ is not isomorphic to $\widehat{K}_n(X)$ even in the case of $n=0$.

The advantage of working with $\mathcal{H}^*_{TW}(X,p)$ rather than $\mathcal{D}^*(X,p)$ is the multiplicative property of ch_{TW}. In fact, it is proved in [6, §6] that $\mathcal{H}^*_{TW}(X,p)$ is equipped with graded commutative and associative product and ch_{TW} respects the product structures on the both sides. Hence in this case we do not need to deal with the operation $\triangle$ which compensates for the lack of compatibility of Bott-Chern forms with products, and we can define product in $\widehat{K}^{TW}_*(X)$ in a simpler form. Moreover, it can be proved that the product in $\widehat{K}^{TW}_*(X)$ satisfies the associative law.

§6.6. The commutativity of the product

In this subsection we discuss the commutativity of the product in $\widehat{K}_*(X)$. When $n=0$ or $m=0$, it is easy to prove that the product $\widehat{K}_n(X)\times\widehat{K}_m(X)\to\widehat{K}_{n+m}(X)$ is commutative. So we concentrate to the case of $n,m\geq 1$.

For a small exact category $\mathfrak{A}$, let $L: G_n\mathfrak{A}\to G_nG_0\mathfrak{A}$ be the map given by $L(E^+,E^-)=(E^+,0,E^-,0)$. Then it induces a homotopy equivalence $L: G\mathfrak{A}\to G^{(2)}\mathfrak{A}$ and it is homotopy equivalent to the map R. Similarly to the case of the map R, we can obtain an isomorphism

$$\widehat{L}_*:\widehat{\pi}_n(|\widehat{G}(X)|,\mathrm{ch})\simeq\widehat{\pi}_n(|\widehat{G}^{(2)}(X)|,\mathrm{ch}).$$

Definition 6.9. For $n,m\geq 1$, we define a new product

$$\underset{L}{\times}:\widehat{K}_n(X)\times\widehat{K}_m(X)\to\widehat{K}_{n+m}(X)$$

by

$$\widehat{\pi}_n(|\widehat{G}(X)|,\mathrm{ch})\times\widehat{\pi}_m(|\widehat{G}(X)|,\mathrm{ch})\overset{\widehat{m}^G_*}{\to}\widehat{\pi}_m(|\widehat{G}^{(2)}(X)|,\mathrm{ch})\overset{\widehat{L}^{-1}_*}{\to}\widehat{\pi}_m(|\widehat{G}(X)|,\mathrm{ch}).$$

Let us compare this new product with the one given in the previous section. Let $T:\widehat{S}_n\widehat{S}_m(X)\to\widehat{S}_m\widehat{S}_n(X)$ be the switching map $T(E)_{(i,j)\times(\alpha,\beta)}=E_{(\alpha,\beta)\times(i,j)}$. Then the map

$$\coprod_{n,m}\widehat{G}_n\widehat{G}_m(X)\times\Delta^n\times\Delta^m\to\coprod_{n,m}\widehat{G}_m\widehat{G}_n(X)\times\Delta^m\times\Delta^n$$

given by $(E^{\pm\pm},t_1,t_2)\mapsto(T(E^{\pm\pm}),t_2,t_1)$ induces an involution $\mathcal{T}$ on $|\widehat{G}^{(2)}(X)|$.

Lemma 6.10. *The diagram*

$$\begin{array}{ccc} C_*(|\widehat{G}^{(2)}(X)|) & \xrightarrow{\mathrm{ch}} & \mathcal{D}_*(X) \\ \downarrow{\scriptstyle \mathcal{T}_*} & & \downarrow{\scriptstyle \mathrm{id}} \\ C_*(|\widehat{G}^{(2)}(X)|) & \xrightarrow{\mathrm{ch}} & \mathcal{D}_*(X) \end{array}$$

is commutative. Hence we can obtain an isomorphism

$$\widehat{\mathcal{T}}_* : \widehat{\pi}_n(|\widehat{G}^{(2)}(X)|, \mathrm{ch}) \simeq \widehat{\pi}_n(|\widehat{G}^{(2)}(X)|, \mathrm{ch})$$

by $[(f,\omega)] \mapsto [(\mathcal{T}f,\omega)]$.

Proof. If we denote by $[E]$ the element of $C_*(|\widehat{G}^{(2)}(X)|)$ determined by $E \in \widehat{G}_n\widehat{G}_m(X)$, then $\mathcal{T}_*([E]) = (-1)^{nm}[T(E)]$. Hence we have

$$\begin{aligned} \mathrm{ch}_{n+m}(\mathcal{T}_*([E])) &= (-1)^{nm}\,\mathrm{ch}_{n+m}(\mathrm{Cub}(T(E))) \\ &= (-1)^{nm}\,\mathrm{ch}_{n+m}(T_{n,m}(\mathrm{Cub}(E))), \end{aligned}$$

where $T_{n,m}(\mathcal{F})$ for an exact hermitian $(n+m)$-cube $\mathcal{F}$ is given by $T_{n,m}(\mathcal{F})_{\alpha_1,\dots,\alpha_{n+m}} = \mathcal{F}_{\alpha_{n+1},\dots,\alpha_{n+m},\alpha_1,\dots,\alpha_n}$. Then it is easy to see that $\mathrm{ch}_{n+m}(T_{n,m}(\mathcal{F})) = (-1)^{nm}\,\mathrm{ch}_{n+m}(\mathcal{F})$. Hence we can say that $\mathrm{ch}_{n+m}(\mathcal{T}_*([E])) = \mathrm{ch}_{n+m}(E)$. □

Proposition 6.11. *Let $x \in \widehat{K}_n(X)$ and $y \in \widehat{K}_m(X)$ with $n,m \geq 1$. Then we have*

$$x \times y = (-1)^{nm} y \underset{L}{\times} x.$$

Proof. For two pointed CW-complexes S_1 and S_2, let $T : S_1\wedge S_2 \to S_2\wedge S_1$ denote the map given by $T(s_1,s_2) = (s_2,s_1)$. For two pointed cellular maps $f : S^n \to |\widehat{G}(X)|$ and $g : S^m \to |\widehat{G}(X)|$, we consider the following diagram:

$$\begin{array}{ccccc} S^n\wedge S^m & \xrightarrow{f\wedge g} & |\widehat{G}(X)|\wedge|\widehat{G}(X)| & \xrightarrow{m^G} & |\widehat{G}^{(2)}(X)| \\ \downarrow{\scriptstyle T} & & \downarrow{\scriptstyle T} & & \downarrow{\scriptstyle \mathcal{T}} \\ S^m\wedge S^n & \xrightarrow{g\wedge f} & |\widehat{G}(X)|\wedge|\widehat{G}(X)| & \xrightarrow{m^G} & |\widehat{G}^{(2)}(X)|. \end{array}$$

The left square is obviously commutative, but the right one is not. In fact, for $E \in \widehat{G}_n(X)$ and $F \in \widehat{G}_m(X)$ we have $\mathcal{T}m^G(E,F)_{(i,j)\times(\alpha,\beta)} = E_{\alpha,\beta}\otimes F_{i,j}$ and $m^GT(E,F)_{(i,j)\times(\alpha,\beta)} = F_{i,j}\otimes E_{\alpha,\beta}$. Hence a homotopy from $\mathcal{T}m^G$ to m^GT is given by means of the canonical isometry $\overline{P}\otimes\overline{Q} \simeq \overline{Q}\otimes\overline{P}$. Hence we can show that $[(\mathcal{T}(f\times g),0)] = [((g\times f)T,0)]$ in the same way as the proof of Lemma 6.5.

If $x = [(f,\omega)]$ and $y = [(g,\tau)]$, then we have

$$\begin{aligned}
&\widehat{\mathcal{T}}_* \widehat{m}^G_*([(f,\omega)],[(g,\tau)]) \\
&\quad = [(\mathcal{T}(f\times g), (-1)^n \operatorname{ch}_n(f)\bullet\tau + \omega\bullet\operatorname{ch}_m(g) + \omega\bullet d_{\mathcal{D}}\tau \\
&\qquad + (-1)^n \operatorname{ch}_n(f)\vartriangle\operatorname{ch}_m(g))] \\
&\quad = [((g\times f)T, (-1)^{nm}\tau\bullet\operatorname{ch}_n(f) + (-1)^{(n+1)m}\operatorname{ch}_m(g)\bullet\omega \\
&\qquad + (-1)^{nm}\tau\bullet d_{\mathcal{D}}\omega + (-1)^{(n+1)m}\operatorname{ch}_m(g)\vartriangle\operatorname{ch}_n(f))].
\end{aligned}$$

Since $T : S^{n+m}\to S^{n+m}$ is homotopic to $(-1)^{nm}\operatorname{id}_{S^{n+m}}$, we have

$$\widehat{\mathcal{T}}_* \widehat{m}^G_*([(f,\omega)],[(g,\tau)]) = (-1)^{nm}\widehat{m}^G_*([(g,\tau)],[(f,\omega)])$$

in $\widehat{\pi}_{n+m}(|\widehat{G}^{(2)}(X)|,\mathrm{ch})$. Hence

$$\begin{aligned}
(-1)^{nm}\widehat{L}_*([(g,\tau)]\underset{L}{\times}[(f,\omega)]) &= (-1)^{nm}\widehat{m}^G_*([(g,\tau)],[(f,\omega)]) \\
&= \widehat{\mathcal{T}}_*\widehat{m}^G_*([(f,\omega)],[(g,\tau)]) \\
&= \widehat{\mathcal{T}}_*\widehat{R}_*([(f,\omega)]\times[(g,\tau)]) \\
&= \widehat{L}_*([(f,\omega)]\times[(g,\tau)]).
\end{aligned}$$

Since $\widehat{L}_*$ is bijective, we have completed the proof. □

Proposition 6.12. *For $x\in\widehat{\pi}_n(|\widehat{G}(X)|,\mathrm{ch})$, $\widehat{R}_*(x)-\widehat{L}_*(x)$ is contained in $\operatorname{Im}(\widetilde{\mathcal{D}}_{n+1}(X)\to\widehat{\pi}_n(|\widehat{G}^{(2)}(X)|,\mathrm{ch}))$ and $2(\widehat{R}_*(x)-\widehat{L}_*(x))=0$. In particular, for $x\in\widehat{K}_n(X)$ and $y\in\widehat{K}_m(X)$ with $n,m\geq 1$, $x\times y - x\underset{L}{\times} y$ is contained in $\operatorname{Im}(\widetilde{\mathcal{D}}_{n+m+1}(X)\to\widehat{K}_{n+m}(X))$ and $2(x\times y - x\underset{L}{\times} y)=0$.*

Proof. Since R is homotopy equivalent to L, $\widehat{R}_*(x)-\widehat{L}_*(x)$ is contained in $\operatorname{Im}(\widetilde{\mathcal{D}}_{n+1}(X)\to\widehat{\pi}_n(|\widehat{G}^{(2)}(X)|,\mathrm{ch}))$. If $f : S^n\to|\widehat{G}(X)|$ is a pointed cellular map, then there is a pointed cellular map

$$H : (S^n\times I)/(\{*\}\times I)\to|\widehat{G}^{(2)}(X)|$$

such that $H(s,0)=Rf(s)$ and $H(s,1)=Lf(s)$. Let

$$H' = \mathcal{T}H : (S^n\times I)/(\{*\}\times I)\to|\widehat{G}^{(2)}(X)|,$$

then we have $H'(s,0)=Lf(s)$ and $H'(s,1)=Rf(s)$. The commutative square in Lemma 6.10 implies that $\operatorname{ch}_{n+1}(H')=\operatorname{ch}_{n+1}(H)$. Gluing the maps H and H' on the boundaries, we obtain a cellular map

$$H\cup H' : (S^n\times T^1)/(\{*\}\times T^1)\to|\widehat{G}^{(2)}(X)|,$$

where T^1 is the barycentric subdivision of S^1.

Lemma 6.13. *If $n \geq 1$, there is a surjection*

$$p : S^{n+1} \to (S^n \times S^1)/(\{*\} \times S^1)$$

such that $p^{-1}((S^n - \{*\}) \times S^1) \to (S^n - \{*\}) \times S^1$ is a homeomorphism.

Proof. We describe the space S^{n+1} as follows:

$$S^{n+1} = \left\{(z, t_1, \ldots, t_n) \in \mathbb{C} \times \mathbb{R}^n; |z|^2 + t_1^2 + \cdots + t_n^2 = 1\right\}.$$

Let $S^{n-1} = \{(0, t_1, \ldots, t_n) \in S^{n+1}\}$. Then the map $S^{n+1} \setminus S^{n-1} \to B^n \times S^1$ given by

$$(z, t_1, \ldots, t_n) \mapsto \left((t_1, \ldots, t_n), \frac{z}{\sqrt{1 - t_1^2 - \cdots - t_n^2}}\right)$$

is a homeomorphism, where $B^n = \{(t_1, \ldots, t_n) \in \mathbb{R}^n; t_1^2 + \cdots + t_n^2 < 1\}$. Since $(S^n \times S^1)/(\{*\} \times S^1)$ is the one-point compactification of $B^n \times S^1$, this homeomorphism can be extended to the map $p : S^{n+1} \to (S^n \times S^1)/(\{*\} \times S^1)$ which satisfies the above condition. □

Let us return to the proof of Proposition 6.12. Since T^1 is the barycentric subdivision of S^1, the Bott-Chern form of the map

$$F : S^{n+1} \xrightarrow{p} (S^n \times T^1)/(\{*\} \times T^1) \xrightarrow{H \cup H'} |\widehat{G}^{(2)}(X)|$$

is $2\,\mathrm{ch}_{n+1}(H)$ up to sign. Therefore $2\,\mathrm{ch}_{n+1}(H)$ is contained in the image of $\pi_{n+1}(|\widehat{G}^{(2)}(X)|) \to \widetilde{\mathfrak{D}}_{n+1}(X)$. Hence $2[(0, \mathrm{ch}_{n+1}(H))] = 0$ in $\widehat{\pi}_n(|\widehat{G}^{(2)}(X)|, \mathrm{ch})$ by Theorem 3.3. For $x = [(f, \omega)] \in \widehat{\pi}_n(|\widehat{G}(X)|, \mathrm{ch})$,

$$\begin{aligned} \widehat{R}_*(x) - \widehat{L}_*(x) &= [(Rf, 0)] - [(Lf, 0)] \\ &= (-1)^{n+1}[(0, \mathrm{ch}_{n+1}(H))], \end{aligned}$$

therefore $2(\widehat{R}_*(x) - \widehat{L}_*(x)) = 0$. □

Combining Proposition 6.11 with Proposition 6.12 yields the following:

Theorem 6.14. *Let $x \in \widehat{K}_n(X)$ and $y \in \widehat{K}_m(X)$. Then $x \times y - (-1)^{nm} y \times x$ is a 2-torsion element contained in $\mathrm{Im}(\widetilde{\mathfrak{D}}_{n+m+1}(X) \to \widehat{K}_{n+m}(X))$. Hence the product in $\widehat{K}_*(X)$ is graded commutative up to 2-torsion.*

§6.7. The lack of the associativity

In this subsection we discuss the associativity of the product in $\widehat{K}_*(X)$. Let $\widehat{G}^{(3)}(X)$ be the trisimplicial set given by taking $\widehat{G}$ three times. Then the tensor product of hermitian vector bundles gives the following maps:

$$m^G : \widehat{G}^{(2)}(X) \wedge \widehat{G}(X) \to \widehat{G}^{(3)}(X),$$
$$m^G : \widehat{G}(X) \wedge \widehat{G}^{(2)}(X) \to \widehat{G}^{(3)}(X).$$

Let $R : \widehat{G}(X) \to \widehat{G}^{(3)}(X)$ be a homotopy equivalent map given by $R(E)^{++\pm} = E^{\pm}$ and $R(E)^{+-\pm} = R(E)^{-+\pm} = R(E)^{--\pm} = 0$ for $E = (E^+, E^-) \in \widehat{G}_n(X)$. Under the above notations, the following diagram

$$\begin{array}{ccc} \widehat{G}(X) \wedge \widehat{G}(X) \wedge \widehat{G}(X) & \xrightarrow{m^G \wedge 1} & \widehat{G}^{(2)}(X) \wedge \widehat{G}(X) \\ \downarrow{\scriptstyle 1\wedge m^G} & & \downarrow{\scriptstyle m^G} \\ \widehat{G}(X) \wedge \widehat{G}^{(2)}(X) & \xrightarrow{m^G} & \widehat{G}^{(3)}(X) \end{array}$$

is commutative up to a homotopy arising from the natural isometry $(\overline{E} \otimes \overline{F}) \otimes \overline{G} \simeq \overline{E} \otimes (\overline{F} \otimes \overline{G})$. This commutative diagram implies the associativity of the product in usual algebraic K-theory $K_*(X)$.

For two pointed cellular maps $f : S^n \to |\widehat{G}^{(2)}(X)|$ and $g : S^m \to |\widehat{G}(X)|$, let

$$f \times g : S^{n+m} \xrightarrow{f \wedge g} |\widehat{G}^{(2)}(X)| \wedge |\widehat{G}(X)| \xrightarrow{\widehat{m}^G} |\widehat{G}^{(3)}(X)|.$$

We define a pairing

$$\widehat{m}^G_* : \widehat{\pi}_n(|\widehat{G}^{(2)}(X)|, \mathrm{ch}) \times \widehat{\pi}_m(|\widehat{G}(X)|, \mathrm{ch}) \to \widehat{\pi}_{n+m}(|\widehat{G}^{(3)}(X)|, \mathrm{ch})$$

by

$$([(f,\omega)], [(g,\tau)]) \mapsto$$
$$[(f \times g, (-1)^n \operatorname{ch}_n(f) \bullet \tau + \omega \bullet \operatorname{ch}_m(g) + \omega \bullet d_{\mathcal{D}}\tau + (-1)^n \operatorname{ch}_n(f) \vartriangle \operatorname{ch}_m(g))].$$

The well-definedness of the pairing can be verified in the same way as the proof of Proposition 6.6. We can also define a pairing

$$\widehat{m}^G_* : \widehat{\pi}_n(|\widehat{G}(X)|, \mathrm{ch}) \times \widehat{\pi}_m(|\widehat{G}^{(2)}(X)|, \mathrm{ch}) \to \widehat{\pi}_{n+m}(|\widehat{G}^{(3)}(X)|, \mathrm{ch})$$

by the same expression as above. Then the associativity of the product in

$\widehat{K}_*(X)$ is equivalent to the commutativity of the following diagram:

$$
\begin{array}{ccc}
 & \widehat{\pi}_n(|\widehat{G}(X)|,\mathrm{ch})\times\widehat{\pi}_m(|\widehat{G}(X)|,\mathrm{ch})\times\widehat{\pi}_l(|\widehat{G}(X)|,\mathrm{ch}) & \\
\swarrow {\scriptstyle \widehat{m}_*^G\times\mathrm{id}} & & {\scriptstyle \mathrm{id}\times\widehat{m}_*^G} \searrow \\
\widehat{\pi}_{n+m}(|\widehat{G}^{(2)}(X)|,\mathrm{ch})\times\widehat{\pi}_l(|\widehat{G}(X)|,\mathrm{ch}) & & \widehat{\pi}_n(|\widehat{G}(X)|,\mathrm{ch})\times\widehat{\pi}_{m+l}(|\widehat{G}^{(2)}(X)|,\mathrm{ch}) \\
{\scriptstyle \widehat{m}_*^G} \searrow & & \swarrow {\scriptstyle \widehat{m}_*^G} \\
 & \widehat{\pi}_{n+m+l}(|\widehat{G}^{(3)}(X)|,\mathrm{ch}). &
\end{array}
$$

However, the diagram is not commutative. Take $[(f,\omega)] \in \widehat{\pi}_n(|\widehat{G}(X)|,\mathrm{ch})$, $[(g,\tau)] \in \widehat{\pi}_m(|\widehat{G}(X)|,\mathrm{ch})$ and $[(h,\eta)] \in \widehat{\pi}_l(|\widehat{G}(X)|,\mathrm{ch})$. Then in the same way as the proof of Lemma 6.5, we can prove the identity

$$[((f\times g)\times h,0)] = [(f\times(g\times h),0)]$$

in $\widehat{\pi}_{n+m+l}(|\widehat{G}^{(3)}(X)|,\mathrm{ch})$. Hence an easy calculation implies the following:

Proposition 6.15. *We have*

$$
\begin{aligned}
&\widehat{m}_*^G\left(\widehat{m}_*^G([(f,\omega)],[(g,\tau)]),[(h,\eta)]\right) - \widehat{m}_*^G\left([(f,\omega)],\widehat{m}_*^G([(g,\tau)],[(h,\eta)])\right)\\
&\qquad = [(0,r(f,g,h,\omega,\tau,\eta))]
\end{aligned}
$$

in $\widehat{\pi}_{n+m+l}(|\widehat{G}^{(3)}(X)|,\mathrm{ch})$*, where*

$$
\begin{aligned}
&r(f,g,h,\omega,\tau,\eta)\\
&\quad = (-1)^n((\mathrm{ch}_n(f)+d_{\mathcal{D}}\omega)\bullet\tau)\bullet(\mathrm{ch}_l(h)+d_{\mathcal{D}}\eta)\\
&\qquad\qquad - (-1)^n(\mathrm{ch}_n(f)+d_{\mathcal{D}}\omega)\bullet(\tau\bullet(\mathrm{ch}_l(h)+d_{\mathcal{D}}\eta))\\
&\qquad + (-1)^{n+m}\,\mathrm{ch}_{n+m}(f\times g)\bullet\eta + (-1)^n(\mathrm{ch}_n(f)\vartriangle\mathrm{ch}_m(g))\bullet d_{\mathcal{D}}\eta\\
&\qquad\qquad - (-1)^{n+m}\,\mathrm{ch}_n(f)\bullet(\mathrm{ch}_m(g)\bullet\eta)\\
&\qquad + (\omega\bullet\mathrm{ch}_m(g))\bullet\mathrm{ch}_l(h) - (-1)^{n+m}d_{\mathcal{D}}\omega\bullet(\mathrm{ch}_m(g)\vartriangle\mathrm{ch}_l(h))\\
&\qquad - \omega\bullet\mathrm{ch}_{m+l}(g\times h) + (\omega\bullet\mathrm{ch}_m(g))\bullet d_{\mathcal{D}}\eta - (-1)^{n+m}d_{\mathcal{D}}\omega\bullet(\mathrm{ch}_m(g)\bullet\eta)\\
&\qquad + (-1)^n(\mathrm{ch}_n(f)\vartriangle\mathrm{ch}_m(g))\bullet\mathrm{ch}_l(h) + (-1)^{n+m}\,\mathrm{ch}_{n+m}(f\times g)\vartriangle\mathrm{ch}_l(h)\\
&\qquad - (-1)^{n+m}\,\mathrm{ch}_n(f)\bullet(\mathrm{ch}_m(g)\vartriangle\mathrm{ch}_l(h)) - (-1)^n\,\mathrm{ch}_n(f)\vartriangle\mathrm{ch}_{m+l}(g\times h).
\end{aligned}
$$

Lemma 6.16. *Assume* $nml \geq 1$ *and let* $\alpha \in \mathcal{D}_n(X)$, $\beta \in \mathcal{D}_m(X)$ *and* $\gamma \in \mathcal{D}_l(X)$.
1) *We have*

$$\begin{aligned}(\alpha \bullet \beta) \bullet \gamma - \alpha \bullet (\beta \bullet \gamma) \\ &= (-1)^{n+m}(\partial\alpha^{(-1,-n)} + \overline{\partial}\alpha^{(-n,-1)}) \wedge (\partial\beta^{(-1,-m)} + \overline{\partial}\beta^{(-m,-1)}) \wedge \gamma \\ &\quad - \alpha \wedge (\partial\beta^{(-1,-m)} + \overline{\partial}\beta^{(-m,-1)}) \wedge (\partial\gamma^{(-1,-l)} + \overline{\partial}\gamma^{(-l,-1)}).\end{aligned}$$

2) *If* $d_{\mathcal{D}}\alpha = d_{\mathcal{D}}\beta = 0$, *then*

$$(\alpha \bullet \beta) \bullet \gamma - \alpha \bullet (\beta \bullet \gamma) \equiv \begin{cases} -\alpha \wedge d\beta \wedge d_{\mathcal{D}}\gamma, & l \geq 2, \\ 0, & l = 1 \end{cases}$$

modulo $\operatorname{Im} d_{\mathcal{D}}$.
3) *If* $d_{\mathcal{D}}\beta = d_{\mathcal{D}}\gamma = 0$, *then*

$$(\alpha \bullet \beta) \bullet \gamma - \alpha \bullet (\beta \bullet \gamma) \equiv \begin{cases} (-1)^{n+m} d_{\mathcal{D}}\alpha \wedge d\beta \wedge \gamma, & n \geq 2, \\ 0, & n = 1 \end{cases}$$

modulo $\operatorname{Im} d_{\mathcal{D}}$.
4) *If* $d_{\mathcal{D}}\alpha = d_{\mathcal{D}}\gamma = 0$, *then*

$$(\alpha \bullet \beta) \bullet \gamma - \alpha \bullet (\beta \bullet \gamma) \equiv \begin{cases} (-1)^{m} \alpha \wedge d d_{\mathcal{D}}\beta \wedge \gamma, & m \geq 2, \\ 0, & m = 1 \end{cases}$$

modulo $\operatorname{Im} d_{\mathcal{D}}$.

Proof. The identity in 1) follows from an easy calculation. If $l \geq 2$ and $d_{\mathcal{D}}\alpha = d_{\mathcal{D}}\beta = 0$, then

$$\begin{aligned}(\alpha \bullet \beta) \bullet \gamma - \alpha \bullet (\beta \bullet \gamma) \\ &= (-1)^{n+m} d\alpha \wedge d\beta \wedge \gamma - \alpha \wedge d\beta \wedge (d\gamma + d_{\mathcal{D}}\gamma) \\ &= (-1)^{n+m} d(\alpha \wedge d\beta \wedge \gamma) - \alpha \wedge d\beta \wedge d_{\mathcal{D}}\gamma.\end{aligned}$$

The form $\alpha \wedge d\beta \wedge \gamma$ is contained in $\mathcal{D}^{2(p+q+r)-n-m-l-1}(X, p+q+r)$ and $d_{\mathcal{D}}(\alpha \wedge d\beta \wedge \gamma) = -d(\alpha \wedge d\beta \wedge \gamma)$. Hence we have

$$(\alpha \bullet \beta) \bullet \gamma - \alpha \bullet (\beta \bullet \gamma) \equiv -\alpha \wedge d\beta \wedge d_{\mathcal{D}}\gamma$$

modulo $\operatorname{Im} d_{\mathcal{D}}$. When $l = 1$, we have

$$(\alpha \bullet \beta) \bullet \gamma - \alpha \bullet (\beta \bullet \gamma) = (-1)^{n+m+1} d_{\mathcal{D}}(\alpha \wedge d\beta \wedge \gamma).$$

Hence 2) holds. The identities in 3) and 4) can be proved in the same way. □

Let us calculate $r(f, g, h, \omega, \tau, \eta)$ by using Lemma 6.16. If $nml \geq 1$, then

$$\begin{aligned}&(-1)^n((\mathrm{ch}_n(f) + d_{\mathfrak{D}}\omega) \bullet \tau) \bullet (\mathrm{ch}_l(h) + d_{\mathfrak{D}}\eta)\\&\quad - (-1)^n(\mathrm{ch}_n(f) + d_{\mathfrak{D}}\omega) \bullet (\tau \bullet (\mathrm{ch}_l(h) + d_{\mathfrak{D}}\eta))\\&\qquad \equiv (-1)^{n+m+1}(\mathrm{ch}_n(f) + d_{\mathfrak{D}}\omega) \wedge dd_{\mathfrak{D}}\tau \wedge (\mathrm{ch}_l(h) + d_{\mathfrak{D}}\eta)\end{aligned}$$

and

$$\begin{aligned}&(\omega \bullet \mathrm{ch}_m(g)) \bullet d_{\mathfrak{D}}\eta - (-1)^{n+m} d_{\mathfrak{D}}\omega \bullet (\mathrm{ch}_m(g) \bullet \eta)\\&\quad \equiv (-1)^{n+m+1} d_{\mathfrak{D}}\omega \wedge d\,\mathrm{ch}_m(g) \wedge d_{\mathfrak{D}}\eta\end{aligned}$$

modulo $\mathrm{Im}\, d_{\mathfrak{D}}$. Since

$$\mathrm{ch}_{n+m}(f \times g) = \mathrm{ch}_n(f) \bullet \mathrm{ch}_m(g) + (-1)^{n+1} d_{\mathfrak{D}}(\mathrm{ch}_n(f) \vartriangle \mathrm{ch}_m(g))$$

by Theorem 5.2, we have

$$\begin{aligned}&(-1)^{n+m}\,\mathrm{ch}_{n+m}(f \times g) \bullet \eta + (-1)^n(\mathrm{ch}_n(f) \vartriangle \mathrm{ch}_m(g)) \bullet d_{\mathfrak{D}}\eta\\&\quad - (-1)^{n+m}\,\mathrm{ch}_n(f) \bullet (\mathrm{ch}_m(g) \bullet \eta)\\&\quad \equiv (-1)^{n+m}(\mathrm{ch}_n(f) \bullet \mathrm{ch}_m(g)) \bullet \eta - (-1)^{n+m}\,\mathrm{ch}_n(f) \bullet (\mathrm{ch}_m(g) \bullet \eta)\\&\quad \equiv (-1)^{n+m+1}\,\mathrm{ch}_n(f) \wedge d\,\mathrm{ch}_m(g) \wedge d_{\mathfrak{D}}\eta\end{aligned}$$

modulo $\mathrm{Im}\, d_{\mathfrak{D}}$. In the same way we have

$$\begin{aligned}&(\omega \bullet \mathrm{ch}_m(g)) \bullet \mathrm{ch}_l(h) - \omega \bullet \mathrm{ch}_{m+l}(g \times h) - (-1)^{n+m} d_{\mathfrak{D}}\omega \bullet (\mathrm{ch}_m(g) \vartriangle \mathrm{ch}_l(h))\\&\quad \equiv (-1)^{n+m+1} d_{\mathfrak{D}}\omega \wedge d\,\mathrm{ch}_m(g) \wedge \mathrm{ch}_l(h)\end{aligned}$$

modulo $\mathrm{Im}\, d_{\mathfrak{D}}$. As for the last four terms, we have the following:

Proposition 6.17. *If $nml \geq 1$, then we have*

$$\begin{aligned}&(-1)^n(\mathrm{ch}_n(f) \vartriangle \mathrm{ch}_m(g)) \bullet \mathrm{ch}_l(h) - (-1)^{n+m}\,\mathrm{ch}_n(f) \bullet (\mathrm{ch}_m(g) \vartriangle \mathrm{ch}_l(h))\\&\quad + (-1)^{n+m}\,\mathrm{ch}_{n+m}(f \times g) \vartriangle \mathrm{ch}_l(h) - (-1)^n\,\mathrm{ch}_n(f) \vartriangle \mathrm{ch}_{m+l}(g \times h)\\&\qquad \equiv (-1)^{n+m+1}\,\mathrm{ch}_n(f) \wedge d\,\mathrm{ch}_m(g) \wedge \mathrm{ch}_l(h)\end{aligned}$$

modulo $\mathrm{Im}\, d_{\mathfrak{D}}$.

We will prove this proposition in §6.9. Substituting these identities into that in Proposition 6.15 yields that

$$r(f, g, h, \omega, \tau, \eta) \equiv (-1)^{n+m+1}(\mathrm{ch}_n(f) + d_{\mathfrak{D}}\omega) \wedge d(\mathrm{ch}_m(g) + d_{\mathfrak{D}}\tau) \wedge (\mathrm{ch}_l(h) + d_{\mathfrak{D}}\eta)$$

modulo $\mathrm{Im}\, d_{\mathfrak{D}}$.

Theorem 6.18. *The product in higher arithmetic K-theory does not satisfy the associative law. In fact, if* $x \in \widehat{K}_n(X), y \in \widehat{K}_m(X)$ *and* $z \in \widehat{K}_l(X)$ *for* $nml \geq 1$, *we have*

$$(x \times y) \times z - x \times (y \times z) = [(0, (-1)^{n+m+1} \operatorname{ch}_n(x) \wedge d \operatorname{ch}_m(y) \wedge \operatorname{ch}_l(z))]$$

in $\widehat{K}_{n+m+l}(X)$. *Hence* $(x \times y) \times z = x \times (y \times z)$ *holds when* $nml = 0$ *or* $y \in K_m(\overline{X})$ *or* $x = y = z$.

Proof. When $nml \geq 1$, we have already proved this identity. The identity $(x \times y) \times z = x \times (y \times z)$ in the case of $nml = 0$ follows from the definition of the product and Lemma 6.16. □

§6.8. Product in Arakelov K-theory

For a proper Arakelov variety $\overline{X} = (X, h_X)$, let us define a pairing

$$K_n(\overline{X}) \times K_m(\overline{X}) \to K_{n+m}(\overline{X})$$

by $(x, y) \mapsto \sigma(x \times y)$, where σ is the harmonic projection defined in §4.3.

Theorem 6.19. *The above pairing makes* $K_*(\overline{X})$ *a graded associative algebra. That is to say, it follows that*

$$\sigma(\sigma(x \times y) \times z) = \sigma(x \times \sigma(y \times z))$$

for $x, y, z \in K_*(\overline{X})$.

Proof. This identity is obvious when $nml = 0$, so we may assume that $nml \geq 1$. We first prove the identity

$$\sigma(\sigma(x \times y) \times z) = \sigma((x \times y) \times z)$$

for $x \in K_n(\overline{X}), y \in K_m(\overline{X})$ and $z \in K_l(\overline{X})$. It follows from the definition of σ that $\sigma(x \times y) = x \times y + [(0, \alpha)]$ where $\alpha \in \mathcal{D}_{n+m+1}(X)$ with $\mathcal{H}(\alpha) = 0$. Then we have

$$\sigma(x \times y) \times z = (x \times y) \times z + [(0, \alpha \bullet \operatorname{ch}_l(z))],$$

therefore

$$\sigma(\sigma(x \times y) \times z) = (x \times y) \times z + [(0, \alpha \bullet \operatorname{ch}_l(z) + \beta)]$$

where $\beta \in \mathcal{D}_{n+m+l+1}(X)$ with $\mathcal{H}(\beta) = 0$. Let α' be the sum of $(p-1, p-n-m)$-part of α and α'' the sum of $(p-n-m, p-1)$-part of α. Since $\mathrm{ch}_l(z)$ is harmonic,

$$\alpha \bullet \mathrm{ch}_l(z) = (-1)^{n+m+1}(\partial\alpha' - \overline{\partial}\alpha'') \wedge \mathrm{ch}_l(z).$$

Since $\partial\alpha' \wedge \mathrm{ch}_l(z)$ is ∂-exact and $\overline{\partial}\alpha'' \wedge \mathrm{ch}_l(z)$ is $\overline{\partial}$-exact, we have $\mathcal{H}(\alpha \bullet \mathrm{ch}_l(z)) = 0$, so $\mathcal{H}(\alpha \bullet \mathrm{ch}_l(z) + \beta) = 0$. Therefore $\sigma(\sigma(x \times y) \times z) = \sigma((x \times y) \times z)$. In the same way, we can show that $\sigma(x \times \sigma(y \times z)) = \sigma(x \times (y \times z))$. Hence by Thm. 6.18 we can obtain the desired identity. □

§6.9. Proof of Proposition 6.17

For $\omega \in \mathcal{D}_n(X)$ and for an integer i with $1 \leq i \leq n$, set

$$\omega^{(-i,-n+i-1)} = \sum_p \omega^{(p-i,p-n+i-1)},$$

where $\omega^{(p-i,p-n+i-1)}$ is the $(p-i, p-n+i-1)$-part of ω. Then for $\omega \in \mathcal{D}_n(X)$ and $\tau \in \mathcal{D}_m(X)$, we can write $\omega \vartriangle \tau$ as follows:

$$\omega \vartriangle \tau = \sum a_{i,j}^{n,m} \omega^{(-i,-n+i-1)} \wedge \tau^{(-j,-m+j-1)}.$$

Set

$$\begin{aligned}\Phi =& (-1)^n(\mathrm{ch}_n(f) \vartriangle \mathrm{ch}_m(g)) \bullet \mathrm{ch}_l(h) - (-1)^{n+m} \mathrm{ch}_n(f) \bullet (\mathrm{ch}_m(g) \vartriangle \mathrm{ch}_l(h)) \\ &+ (-1)^{n+m} \mathrm{ch}_{n+m}(f \times g) \vartriangle \mathrm{ch}_l(h) - (-1)^n \mathrm{ch}_n(f) \vartriangle \mathrm{ch}_{m+l}(g \times h),\end{aligned}$$

and let $\Phi(f)$ (resp. $\Phi(g)$ and $\Phi(h)$) be the part of Φ including the derivatives of $\mathrm{ch}_n(f)$ (resp. $\mathrm{ch}_m(g)$ and $\mathrm{ch}_l(h)$). In other words, $\Phi = \Phi(f) + \Phi(g) + \Phi(h)$ such that

$$\begin{aligned}\Phi(f) =& (-1)^{m+1}(\partial \mathrm{ch}_n(f)^{(-1,-n)} - \overline{\partial} \mathrm{ch}_n(f)^{(-n,-1)}) \wedge (\mathrm{ch}_m(g) \vartriangle \mathrm{ch}_l(h)) \\ &+ (-1)^m \left((\partial \mathrm{ch}_n(f)^{(-1,-n)} - \overline{\partial} \mathrm{ch}_n(f)^{(-n,-1)}) \wedge \mathrm{ch}_m(g)\right) \vartriangle \mathrm{ch}_l(h) \\ &+ (-1)^m \left(\sum_{\substack{1\leq i\leq n \\ 1\leq j\leq m}} a_{i,j}^{n,m} d \,\mathrm{ch}_n(f)^{(-n+i-1,-i)} \wedge \mathrm{ch}_m(g)^{(-m+j-1,-j)} \right) \vartriangle \mathrm{ch}_l(h),\end{aligned}$$

$$\begin{aligned}\Phi(g) =& (-1)^{n+m}(\mathrm{ch}_n(f) \wedge (\partial \mathrm{ch}_m(g)^{(-1,-m)} - \overline{\partial} \mathrm{ch}_m(g)^{(-m,-1)})) \vartriangle \mathrm{ch}_l(h) \\ &+ (-1)^{n+m+1} \left(\sum_{\substack{1\leq i\leq n \\ 1\leq j\leq m}} a_{i,j}^{n,m} \mathrm{ch}_n(f)^{(-n+i-1,-i)} \wedge d \,\mathrm{ch}_m(g)^{(-m+j-1,-j)} \right)\end{aligned}$$

$$\begin{aligned}
&\triangle \operatorname{ch}_l(h)\\
&+(-1)^{n+m+1}\operatorname{ch}_n(f)\triangle((\partial\operatorname{ch}_m(g)^{(-1,-m)}-\overline{\partial}\operatorname{ch}_m(g)^{(-m,-1)})\wedge\operatorname{ch}_l(h))\\
&+(-1)^{n+m+1}\operatorname{ch}_n(f)\triangle\left(\sum_{\substack{1\le j\le m\\1\le k\le l}}a^{m,l}_{j,k}d\operatorname{ch}_m(g)^{(-m+j-1,-j)}\wedge\operatorname{ch}_l(h)^{(-l+k-1,-k)}\right)
\end{aligned}$$

and

$$\begin{aligned}
\Phi(h)&=(-1)^n(\operatorname{ch}_n(f)\triangle\operatorname{ch}_m(g))\wedge(\partial\operatorname{ch}_l(h)^{(-1,-l)}-\overline{\partial}\operatorname{ch}_l(h)^{(-l,-1)})\\
&+(-1)^{n+1}\operatorname{ch}_n(f)\triangle\left(\operatorname{ch}_m(g)\wedge(\partial\operatorname{ch}_l(h)^{(-1,-l)}-\overline{\partial}\operatorname{ch}_l(h)^{(-l,-1)})\right)\\
&+(-1)^{n}\operatorname{ch}_n(f)\triangle\left(\sum_{\substack{1\le j\le m\\1\le k\le l}}a^{m,l}_{j,k}\operatorname{ch}_m(g)^{(-m+j-1,-j)}\wedge d\operatorname{ch}_l(h)^{(-l+k-1,-k)}\right).
\end{aligned}$$

Let us first calculate $\Phi(f)$. It follows from $d_{\mathcal{D}}(\operatorname{ch}_n(f))=0$ that $\partial\operatorname{ch}_n(f)^{(-n+i-1,-i)}=-\overline{\partial}\operatorname{ch}_n(f)^{(-n+i,-i-1)}$ for $1\le i\le n-1$. Then $\Phi(f)$ is expressed as follows:

$$\begin{aligned}
\Phi(f)&=b^{n,m,l}_{0,j,k}\sum_{\substack{1\le j\le m\\1\le k\le l}}\overline{\partial}\operatorname{ch}_n(f)^{(-n,-1)}\wedge\operatorname{ch}_m(g)^{(-m+j-1,-j)}\wedge\operatorname{ch}_l(h)^{(-l+k-1,-k)}\\
&+\sum_{\substack{1\le i\le n\\1\le j\le m\\1\le k\le l}}b^{n,m,l}_{i,j,k}\partial\operatorname{ch}_n(f)^{(-n+i-1,-i)}\wedge\operatorname{ch}_m(g)^{(-m+j-1,-j)}\wedge\operatorname{ch}_l(h)^{(-l+k-1,-k)}
\end{aligned}$$

where

$$\begin{aligned}
b^{n,m,l}_{0,j,k}&=(-1)^m a^{m,l}_{j,k}+(-1)^{m+1}a^{n+m,l}_{j,k}+(-1)^m a^{n+m,l}_{j,k}\times a^{n,m}_{1,j}\\
&=(-1)^m a^{m,l}_{j,k}+2(-1)^{m+1}\binom{n+m}{n}^{-1}\binom{n+m-j}{n}a^{n+m,l}_{j,k},\\
b^{n,m,l}_{n,m,k}&=(-1)^{m+1}a^{m,l}_{j,k}+(-1)^m a^{n+m,l}_{n+j,k}+(-1)^m a^{n+m,l}_{n+j,k}\times a^{n,m}_{n,j}\\
&=(-1)^{m+1}a^{m,l}_{j,k}+2(-1)^m\binom{n+m}{n}^{-1}\binom{n-j+1}{n}a^{n+m,l}_{n+j,k},
\end{aligned}$$

and

$$\begin{aligned}
b^{n,m,l}_{i,j,k}&=(-1)^{m+1}a^{n+m,l}_{i+j,k}\times a^{n,m}_{i+1,j}+(-1)^m a^{n+m,l}_{i+j,k}\times a^{n,m}_{i,j}\\
&=2(-1)^m\binom{n+m}{n}^{-1}\binom{n+m-i-j}{n-i}\binom{i+j-1}{i}a^{n+m,l}_{i+j,k}
\end{aligned}$$

for $1\le i\le n-1$.

Lemma 6.20. *If $1 \leq j \leq m$ and $1 \leq k \leq l$, then*

$$\binom{n+m}{n}^{-1} \sum_{i=0}^{n} \binom{n+m-i-j}{n-i} \binom{i+j-1}{i} a_{i+j,k}^{n+m,l} = a_{j,k}^{m,l}.$$

Proof. By Lemma A.2 and Lemma A.3, we have

$$\begin{aligned}
&\binom{n+m}{n}^{-1} \sum_{i=0}^{n} \binom{n+m-i-j}{n-i} \binom{i+j-1}{i} a_{i+j,k}^{n+m,l} \\
&\quad = \binom{n+m}{n}^{-1} \sum_{i=0}^{n} \binom{n+m-i-j}{n-i} \binom{i+j-1}{i} \\
&\qquad - 2\binom{n+m}{n}^{-1} \binom{n+m+l}{n+m}^{-1} \sum_{i=0}^{n} \binom{n+m-i-j}{n-i} \binom{i+j-1}{i} \\
&\qquad\qquad \times \sum_{\alpha=0}^{i+j-1} \binom{n+m+l-i-j-k+1}{n+m-\alpha} \binom{i+j+k-1}{\alpha} \\
&\quad = 1 - 2\binom{m+l}{m}^{-1} \sum_{\alpha=0}^{j-1} \binom{m+l-j-k+1}{m-\alpha} \binom{j+k-1}{\alpha} \\
&\quad = a_{j,k}^{m,l}.
\end{aligned}$$

□

Let

$$c_{i,j,k}^{n,m,l} = (-1)^m a_{j,k}^{m,l} - 2(-1)^m \binom{n+m}{n}^{-1} \sum_{\alpha=0}^{i-1} \binom{n+m-\alpha-j}{n-\alpha} \binom{\alpha+j-1}{\alpha} a_{\alpha+j,k}^{n+m,l}.$$

Then we have

$$\begin{aligned}
c_{1,j,k}^{n,m,l} &= b_{0,j,k}^{n,m,l}, \\
c_{i,j,k}^{n,m,l} - c_{i+1,j,k}^{n,m,l} &= b_{i,j,k}^{n,m,l}
\end{aligned}$$

for $1 \leq i \leq n-1$ and by Lemma 6.20,

$$c_{n,j,k}^{n,m,l} = b_{n,j,k}^{n,m,l}.$$

Let Ψ be a differential form given by

$$\Psi = \sum_{\substack{1 \leq i \leq n \\ 1 \leq j \leq m \\ 1 \leq k \leq l}} c_{i,j,k}^{n,m,l} \operatorname{ch}_n(f)^{(-n+i-1,-i)} \wedge \operatorname{ch}_m(g)^{(-m+j-1,-j)} \wedge \operatorname{ch}_l(h)^{(-l+k-1,-k)}.$$

Lemma 6.21. *It follows that* $c^{n,m,l}_{n-i+1,m-j+1,l-k+1} = c^{n,m,l}_{i,j,k}$. *Hence* Ψ *is contained in* $\mathcal{D}_{n+m+l+2}(X)$.

Proof. We have

$$\begin{aligned}
&c^{n,m,l}_{n-i+1,m-j+1,l-k+1}\\
&\quad = (-1)^m a^{m,l}_{m-j+1,l-k+1}\\
&\qquad - 2(-1)^m \binom{n+m}{n}^{-1} \sum_{\alpha=0}^{n-i} \binom{n-\alpha+j-1}{n-\alpha}\binom{\alpha+m-j}{\alpha} a^{n+m,l}_{\alpha+m-j+1,l-k+1}\\
&\quad = (-1)^{m+1} a^{m,l}_{j,k} + 2(-1)^m \binom{n+m}{n}^{-1} \sum_{\alpha=0}^{n-i} \binom{n-\alpha+j-1}{n-\alpha}\binom{\alpha+m-j}{\alpha} a^{n+m,l}_{n-\alpha+j,k}\\
&\quad = (-1)^{m+1} a^{m,l}_{j,k} + 2(-1)^m \binom{n+m}{n}^{-1} \sum_{\beta=i}^{n} \binom{\beta+j-1}{\beta}\binom{n-\beta+m-j}{n-\beta} a^{n+m,l}_{\beta+j,k}.
\end{aligned}$$

Hence Lemma 6.20 implies that

$$\begin{aligned}
&c^{n,m,l}_{n-i+1,m-j+1,l-k+1}\\
&\quad = (-1)^{m+1} a^{m,l}_{j,k} + 2(-1)^m\\
&\qquad \times \left(a^{m,l}_{j,k} - \binom{n+m}{n}^{-1} \sum_{\beta=0}^{i-1} \binom{\beta+j-1}{\beta}\binom{n-\beta+m-j}{n-\beta} a^{n+m,l}_{\beta+j,k} \right)\\
&\quad = (-1)^m a^{m,l}_{j,k} - 2(-1)^m \binom{n+m}{n}^{-1} \sum_{\beta=0}^{i-1} \binom{n+m-\beta-j}{n-\beta}\binom{\beta+j-1}{\beta} a^{n+m,l}_{\beta+j,k}\\
&\quad = c^{n,m,l}_{i,j,k}.
\end{aligned}$$

□

Let us denote the parts of $d\Psi$ including the derivatives of $\mathrm{ch}_n(f), \mathrm{ch}_m(g)$ and $\mathrm{ch}_l(h)$ by $\Psi(f)$, $\Psi(g)$ and $\Psi(h)$ respectively. Then $d\Psi = \Psi(f)+\Psi(g)+\Psi(h)$ and

$$\begin{aligned}
\Psi(f) = \sum_{\substack{1\le i\le n\\ 1\le j\le m\\ 1\le k\le l}} & c^{n,m,l}_{i,j,k}\, d\,\mathrm{ch}_n(f)^{(-n+i-1,-i)}\\
&\wedge \mathrm{ch}_m(g)^{(-m+j-1,-j)} \wedge \mathrm{ch}_l(h)^{(-l+k-1,-k)}
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{\substack{1\le j\le m\\1\le k\le l}} c^{n,m,l}_{1,j,k}\overline{\partial}\operatorname{ch}_n(f)^{(-n,-1)}\wedge\operatorname{ch}_m(g)^{(-m+j-1,-j)}\wedge\operatorname{ch}_l(h)^{(-l+k-1,-k)}\\
&\quad+\sum_{\substack{1\le i\le n-1\\1\le j\le m\\1\le k\le l}}(c^{n,m,l}_{i,j,k}-c^{n,m,l}_{i+1,j,k})\partial\operatorname{ch}_n(f)^{(-n+i-1,-i)}\\
&\qquad\qquad\wedge\operatorname{ch}_m(g)^{(-m+j-1,-j)}\wedge\operatorname{ch}_l(h)^{(-l+k-1,-k)}\\
&\quad+\sum_{\substack{1\le j\le m\\1\le k\le l}} c^{n,m,l}_{n,j,k}\partial\operatorname{ch}_n(f)^{(-1,-n)}\wedge\operatorname{ch}_m(g)^{(-m+j-1,-j)}\wedge\operatorname{ch}_l(h)^{(-l+k-1,-k)}\\
&=\Phi(f).
\end{aligned}
$$

Let us express $\Phi(h)-\Psi(h)$ as follows:

$$
\begin{aligned}
&\Phi(h)-\Psi(h)\\
&\quad= d^{n,m,l}_{i,j,0}\sum_{\substack{1\le i\le n\\1\le j\le m}}\operatorname{ch}_n(f)^{(-n+i-1,-i)}\wedge\operatorname{ch}_m(g)^{(-m+j-1,-j)}\wedge\overline{\partial}\operatorname{ch}_l(h)^{(-l,-1)}\\
&\quad+\sum_{\substack{1\le i\le n\\1\le j\le m\\1\le k\le l}} d^{n,m,l}_{i,j,k}\operatorname{ch}_n(f)^{(-n+i-1,-i)}\wedge\operatorname{ch}_m(g)^{(-m+j-1,-j)}\wedge\partial\operatorname{ch}_l(h)^{(-l+k-1,-k)}.
\end{aligned}
$$

Lemma 6.22. *It follows that* $d^{n,m,l}_{i,j,k}=0$, *therefore* $\Phi(h)-\Psi(h)=0$.

Proof. When $1\le k\le l-1$,

$$
\begin{aligned}
d^{n,m,l}_{i,j,k} &= (-1)^n a^{n,m+l}_{i,j+k}\times(a^{m,l}_{j,k}-a^{m,l}_{j,k+1})-(-1)^{n+m}(c^{n,m,l}_{i,j,k}-c^{n,m,l}_{i,j,k+1})\\
&=(-1)^{n+1}(1-a^{n,m+l}_{i,j+k})(a^{m,l}_{j,k}-a^{m,l}_{j,k+1})\\
&\quad+2(-1)^n\binom{n+m}{n}^{-1}\sum_{\alpha=0}^{i-1}\binom{n+m-\alpha-j}{n-\alpha}\binom{\alpha+j-1}{\alpha}(a^{n+m,l}_{\alpha+j,k}-a^{n+m,l}_{\alpha+j,k+1})\\
&=4(-1)^n\binom{n+m+l}{n}^{-1}\binom{m+l}{l}^{-1}\binom{m+l-j-k}{m-j}\binom{j+k-1}{j-1}\\
&\qquad\times\sum_{\alpha=0}^{i-1}\binom{n+m+l-i-j-k+1}{n-\alpha}\binom{i+j+k-1}{\alpha}\\
&\quad-4(-1)^n\binom{n+m+l}{l}^{-1}\binom{n+m}{n}^{-1}\\
&\qquad\times\sum_{\alpha=0}^{i-1}\binom{n+m+l-\alpha-j-k}{n+m-\alpha-j}\binom{\alpha+j+k-1}{\alpha+j-1}\binom{n+m-\alpha-j}{n-\alpha}\binom{\alpha+j-1}{\alpha}.
\end{aligned}
$$

Since

$$\binom{n+m+l}{l}\binom{n+m}{n} = \binom{n+m+l}{n}\binom{m+l}{l},$$
$$\binom{n+m+l-\alpha-j-k}{n+m-\alpha-j}\binom{n+m-\alpha-j}{n-\alpha} = \binom{n+m+l-\alpha-j-k}{m+l-j-k}\binom{m+l-j-k}{m-j},$$
$$\binom{\alpha+j+k-1}{\alpha+j-1}\binom{\alpha+j-1}{\alpha} = \binom{\alpha+j+k-1}{\alpha}\binom{j+k-1}{j-1},$$

we have

$$\begin{aligned} d^{n,m,l}_{i,j,k} &= 4(-1)^n\binom{n+m+l}{n}^{-1}\binom{m+l}{l}^{-1}\binom{m+l-j-k}{m-j}\binom{j+k-1}{j-1} \\ &\quad\times\left(\sum_{\alpha=0}^{i-1}\binom{n+m+l-i-j-k+1}{n-\alpha}\binom{i+j+k-1}{\alpha} - \sum_{\alpha=0}^{i-1}\binom{n+m+l-\alpha-j-k}{n-\alpha}\binom{\alpha+j+k-1}{\alpha}\right) \\ &= 0 \end{aligned}$$

by Lemma A.3. When $k = 0$,

$$\begin{aligned} d^{n,m,l}_{i,j,0} &= (-1)^{n+1}a^{n,m}_{i,j} + (-1)^n a^{n,m+l}_{i,j}\times(1+a^{m,l}_{j,1}) - (-1)^{n+m}c^{n,m,l}_{i,j,1} \\ &= (-1)^{n+1}\left((1-2\binom{n+m}{n}^{-1}\sum_{\alpha=0}^{i-1}\binom{n+m-i-j+1}{n-\alpha}\binom{i+j-1}{\alpha}\right) \\ &\quad + 2(-1)^n\binom{m+l}{m}^{-1}\binom{m+l-j}{l} \\ &\quad\quad\times\left(1-2\binom{n+m+l}{n}^{-1}\sum_{\alpha=0}^{i-1}\binom{n+m+l-i-j+1}{n-\alpha}\binom{i+j-1}{\alpha}\right) \\ &\quad - (-1)^n\left(-1+2\binom{m+l}{l}^{-1}\binom{m+l-j}{l}\right) \\ &\quad + 2(-1)^n\binom{n+m}{n}^{-1}\sum_{\alpha=0}^{i-1}\binom{n+m-\alpha-j}{n-\alpha}\binom{\alpha+j-1}{\alpha} \\ &\quad\quad\times\left(-1+2\binom{n+m+l}{l}^{-1}\binom{n+m+l-\alpha-j}{l}\right). \end{aligned}$$

Since

$$\binom{n+m+l-\alpha-j}{l}\binom{n+m-\alpha-j}{n-\alpha} = \binom{n+m+l-\alpha-j}{n-\alpha}\binom{m+l-j}{l},$$

we have

$$\begin{aligned} d^{n,m,l}_{i,j,0} &= 2(-1)^n\binom{n+m}{n}^{-1} \\ &\quad\quad\times\left(\sum_{\alpha=0}^{i-1}\binom{n+m-i-j+1}{n-\alpha}\binom{i+j-1}{\alpha} - \sum_{\alpha=0}^{i-1}\binom{n+m-\alpha-j}{n-\alpha}\binom{\alpha+j-1}{\alpha}\right) \\ &\quad + 4(-1)^{n+1}\binom{n+m+l}{n}^{-1}\binom{m+l}{l}^{-1}\binom{m+l-j}{l} \\ &\quad\quad\times\left(\sum_{\alpha=0}^{i-1}\binom{n+m+l-i-j+1}{n-\alpha}\binom{i+j-1}{\alpha} - \sum_{\alpha=0}^{i-1}\binom{n+m+l-\alpha-j}{n-\alpha}\binom{\alpha+j-1}{\alpha}\right) \end{aligned}$$

$$= 0$$

by Lemma A.3. We can prove that $d^{n,m,l}_{i,j,l} = 0$ in the same way. □

We finally calculate $\Phi(g) - \Psi(g)$. Let us express it as follows:

$$\begin{aligned}
&\Phi(g) - \Psi(g) \\
&\quad = e^{n,m,l}_{i,0,k} \sum_{\substack{1\le i\le n\\1\le k\le l}} \mathrm{ch}_n(f)^{(-n+i-1,-i)} \wedge \overline{\partial}\,\mathrm{ch}_m(g)^{(-m,-1)} \wedge \mathrm{ch}_l(h)^{(-l+k-1,-k)} \\
&\qquad + \sum_{\substack{1\le i\le n\\1\le j\le m\\1\le k\le l}} e^{n,m,l}_{i,j,k}\,\mathrm{ch}_n(f)^{(-n+i-1,-i)} \wedge \partial\,\mathrm{ch}_m(g)^{(-m+j-1,-j)} \\
&\qquad\qquad \wedge \mathrm{ch}_l(h)^{(-l+k-1,-k)}.
\end{aligned}$$

Lemma 6.23. *When $1 \le j \le m-1$, $e^{n,m,l}_{i,j,k} = 0$ and $e^{n,m,l}_{i,0,k} = e^{n,m,l}_{i,m,k} = (-1)^{n+m+1}$.*

Proof. When $1 \le j \le m-1$,

$$\begin{aligned}
e^{n,m,l}_{i,j,k} &= (-1)^{n+m+1} a^{n+m,l}_{i+j,k} \times (a^{n,m}_{i,j} - a^{n,m}_{i,j+1}) \\
&\quad + (-1)^{n+m+1} a^{n,m+l}_{i,j+k} \times (a^{m,l}_{j,k} - a^{m,l}_{j+1,k}) \\
&\quad + (-1)^n (c^{n,m,l}_{i,j,k} - c^{n,m,l}_{i,j+1,k}) \\
&= (-1)^{n+m+1} a^{n+m,l}_{i+j,k} \times (a^{n,m}_{i,j} - a^{n,m}_{i,j+1}) \\
&\quad + (-1)^{n+m+1} (a^{n,m+l}_{i,j+k} - 1) \times (a^{m,l}_{j,k} - a^{m,l}_{j+1,k}) \\
&\quad + 2(-1)^{n+m+1} \binom{n+m}{n}^{-1} \sum_{\alpha=0}^{i-1} \binom{n+m-\alpha-j}{n-\alpha} \binom{\alpha+j-1}{\alpha} a^{n+m,l}_{\alpha+j,k} \\
&\quad - 2(-1)^{n+m+1} \binom{n+m}{n}^{-1} \sum_{\alpha=0}^{i-1} \binom{n+m-\alpha-j-1}{n-\alpha} \binom{\alpha+j}{\alpha} a^{n+m,l}_{\alpha+j+1,k}.
\end{aligned}$$

Hence we have

$$\begin{aligned}
&e^{n,m,l}_{i+1,j,k} - e^{n,m,l}_{i,j,k} \\
&\quad = (-1)^{n+m+1} a^{n+m,l}_{i+j+1,k} \times (a^{n,m}_{i+1,j} - a^{n,m}_{i+1,j+1}) \\
&\qquad - (-1)^{n+m+1} a^{n+m,l}_{i+j,k} \times (a^{n,m}_{i,j} - a^{n,m}_{i,j+1}) \\
&\qquad + (-1)^{n+m+1} (a^{n,m+l}_{i+1,j+k} - a^{n,m+l}_{i,j+k})(a^{m,l}_{j,k} - a^{m,l}_{j+1,k}) \\
&\qquad + 2(-1)^{n+m+1} \binom{n+m}{n}^{-1} \Bigg(\binom{n+m-i-j}{n-i} \binom{i+j-1}{i} a^{n+m,l}_{i+j,k}
\end{aligned}$$

$$
\begin{aligned}
&-\tbinom{n+m-i-j-1}{n-i}\tbinom{i+j}{i}a^{n+m,l}_{i+j+1,k}\Big)\\
&=2(-1)^{n+m}\tbinom{n+m}{n}^{-1}\tbinom{n+m-i-j-1}{n-i-1}\tbinom{i+j}{i}a^{n+m,l}_{i+j+1,k}\\
&\quad-2(-1)^{n+m}\tbinom{n+m}{n}^{-1}\tbinom{n+m-i-j}{n-i}\tbinom{i+j-1}{i-1}a^{n+m,l}_{i+j,k}\\
&\quad+(-1)^{n+m+1}(a^{n,m+l}_{i+1,j+k}-a^{n,m+l}_{i,j+k})(a^{m,l}_{j,k}-a^{m,l}_{j+1,k})\\
&\quad+2(-1)^{n+m+1}\tbinom{n+m}{n}^{-1}\Big(\tbinom{n+m-i-j}{n-i}\tbinom{i+j-1}{i}a^{n+m,l}_{i+j,k}\\
&\quad-\tbinom{n+m-i-j-1}{n-i}\tbinom{i+j}{i}a^{n+m,l}_{i+j+1,k}\Big)\\
&=2(-1)^{n+m}\tbinom{n+m}{n}^{-1}\tbinom{n+m-i-j}{n-i}\tbinom{i+j}{i}(a^{n+m,l}_{i+j+1,k}-a^{n+m,l}_{i+j,k})\\
&\quad+(-1)^{n+m+1}(a^{n,m+l}_{i+1,j+k}-a^{n,m+l}_{i,j+k})(a^{m,l}_{j,k}-a^{m,l}_{j+1,k})\\
&=-4(-1)^{n+m}\tbinom{n+m}{n}^{-1}\tbinom{n+m-i-j}{n-i}\tbinom{i+j}{i}\tbinom{n+m+l}{n+m}^{-1}\tbinom{n+m+l-i-j-k}{n+m-i-j}\tbinom{i+j+k-1}{i+j}\\
&\quad+4(-1)^{n+m}\tbinom{n+m+l}{n}^{-1}\tbinom{n+m+l-i-j-k}{n-i}\tbinom{i+j+k-1}{i}\tbinom{m+l}{m}^{-1}\tbinom{m+l-j-k}{m-j}\tbinom{j+k-1}{j}\\
&=0
\end{aligned}
$$

and

$$
\begin{aligned}
e^{n,m,l}_{1,j,k}&=(-1)^{n+m+1}a^{n+m,l}_{j+1,k}\times(a^{n,m}_{1,j}-a^{n,m}_{1,j+1})\\
&\quad+(-1)^{n+m+1}(a^{n,m+l}_{1,j+k}-1)(a^{m,l}_{j,k}-a^{m,l}_{j+1,k})\\
&\quad+2(-1)^{n+m+1}\tbinom{n+m}{n}^{-1}\Big(\tbinom{n+m-j}{n}a^{n+m,l}_{j,k}-\tbinom{n+m-j-1}{n}a^{n+m,l}_{j+1,k}\Big)\\
&=(-1)^{n+m+1}(a^{n,m+l}_{1,j+k}-1)(a^{m,l}_{j,k}-a^{m,l}_{j+1,k})\\
&\quad-2(-1)^{n+m}\tbinom{n+m}{n}^{-1}\tbinom{n+m-j}{n}(a^{n+m,l}_{j,k}-a^{n+m,l}_{j+1,k})\\
&=4(-1)^{n+m}\tbinom{n+m+l}{n}^{-1}\tbinom{n+m+l-j-k}{n}\tbinom{m+l}{m}^{-1}\tbinom{m+l-j-k}{m-j}\tbinom{j+k-1}{j}\\
&\quad-4(-1)^{n+m}\tbinom{n+m}{n}^{-1}\tbinom{n+m-j}{n}\tbinom{n+m+l}{n+m}^{-1}\tbinom{n+m+l-j-k}{n+m-j}\tbinom{j+k-1}{j}\\
&=0.
\end{aligned}
$$

Hence $e^{n,m,l}_{i,j,k}=0$ if $1\le j\le m-1$. On the other hand, when $j=0$,

$$
\begin{aligned}
e^{n,m,l}_{i,0,k}&=(-1)^{n+m+1}a^{n+m,l}_{i,k}\times(1+a^{n,m}_{i,1})+(-1)^{n+m}a^{n,m+l}_{i,k}\\
&\quad\times(1-a^{m,l}_{1,k})-(-1)^{n+1}c^{n,m,l}_{i,1,k}\\
&=2(-1)^{n+m+1}\tbinom{n+m}{n}^{-1}\tbinom{n+m-i}{m}a^{n+m,l}_{i,k}\\
&\quad+2(-1)^{n+m}\tbinom{m+l}{m}^{-1}\tbinom{m+l-k}{m}a^{n,m+l}_{i,k}\\
&\quad+(-1)^{n+m}a^{m,l}_{1,k}-2(-1)^{n+m}\tbinom{n+m}{n}^{-1}\sum_{\alpha=0}^{i-1}\tbinom{n+m-\alpha-1}{n-\alpha}a^{n+m,l}_{\alpha+1,k}.
\end{aligned}
$$

Hence we have

$$
\begin{aligned}
&e^{n,m,l}_{i+1,0,k} - e^{n,m,l}_{i,0,k} \\
&\quad = -2(-1)^{n+m}\binom{n+m}{n}^{-1}\binom{n+m-i-1}{m}a^{n+m,l}_{i+1,k} \\
&\qquad + 2(-1)^{n+m}\binom{n+m}{n}^{-1}\binom{n+m-i}{m}a^{n+m,l}_{i,k} \\
&\qquad + 2(-1)^{n+m}\binom{m+l}{m}^{-1}\binom{m+l-k}{m}(a^{n,m+l}_{i+1,k} - a^{n,m+l}_{i,k}) \\
&\qquad - 2(-1)^{n+m}\binom{n+m}{n}^{-1}\binom{n+m-i-1}{n-i}a^{n+m,l}_{i+1,k} \\
&\quad = 2(-1)^{n+m+1}\binom{n+m}{n}^{-1}\binom{n+m-i}{m}(a^{n+m,l}_{i+1,k} - a^{n+m,l}_{i,k}) \\
&\qquad + 2(-1)^{n+m}\binom{m+l}{m}^{-1}\binom{m+l-k}{m}(a^{n,m+l}_{i+1,k} - a^{n,m+l}_{i,k}) \\
&\quad = 4(-1)^{n+m}\binom{n+m}{n}^{-1}\binom{n+m-i}{m}\binom{n+m+l}{n+m}^{-1}\binom{n+m+l-i-k}{n+m-i}\binom{i+k-1}{i} \\
&\qquad - 4(-1)^{n+m}\binom{m+l}{m}^{-1}\binom{m+l-k}{m}\binom{n+m+l}{n}^{-1}\binom{n+m+l-i-k}{n-i}\binom{i+k-1}{i} \\
&\quad = 0
\end{aligned}
$$

and

$$
\begin{aligned}
e^{n,m,l}_{1,0,k} &= 2(-1)^{n+m+1}\binom{n+m}{n}^{-1}\binom{n+m-1}{m}a^{n+m,l}_{1,k} \\
&\quad + 2(-1)^{n+m}\binom{m+l}{m}^{-1}\binom{m+l-k}{m}a^{n,m+l}_{1,k} \\
&\quad + (-1)^{n+m}a^{m,l}_{1,k} - 2(-1)^{n+m}\binom{n+m}{n}^{-1}\binom{n+m-1}{n}a^{n+m,l}_{1,k} \\
&= 2(-1)^{n+m+1}\left(1 - 2\binom{n+m+l}{n+m}^{-1}\binom{n+m+l-k}{n+m}\right) \\
&\quad + (-1)^{n+m}\left(1 - 2\binom{m+l}{m}^{-1}\binom{m+l-k}{m}\right) \\
&\quad + 2(-1)^{n+m}\binom{m+l}{m}^{-1}\binom{m+l-k}{m}\left(1 - 2\binom{n+m+l}{n}^{-1}\binom{n+m+l-k}{n}\right) \\
&= (-1)^{n+m+1} + 4(-1)^{n+m}\binom{n+m+l}{n+m}^{-1}\binom{n+m+l-k}{n+m} \\
&\quad - 4(-1)^{n+m}\binom{m+l}{m}^{-1}\binom{m+l-k}{m}\binom{n+m+l}{n}^{-1}\binom{n+m+l-k}{n} \\
&= (-1)^{n+m+1}.
\end{aligned}
$$

Hence $e^{n,m,l}_{i,0,k} = (-1)^{n+m+1}$. We can prove that $e^{n,m,l}_{i,m,k} = (-1)^{n+m+1}$ in the same way. □

Let us return to the proof of Proposition 6.17. By the above calculations, we have

$$\begin{aligned}
&\Phi - d\Psi \\
&= (-1)^{n+m+1} \sum_{\substack{1\le i\le n \\ 1\le k\le l}} \mathrm{ch}_n(f)^{(-n+i-1,-i)} \\
&\quad \wedge \left(\overline{\partial}\,\mathrm{ch}_m(g)^{(-m,-1)} + \partial\,\mathrm{ch}_m(g)^{(-1,-m)}\right) \wedge \mathrm{ch}_l(h)^{(-l+k-1,-k)} \\
&= (-1)^{n+m+1}\,\mathrm{ch}_n(f) \wedge d\,\mathrm{ch}_m(g) \wedge \mathrm{ch}_l(h).
\end{aligned}$$

Since $\Psi \in \mathcal{D}_{n+m+l+2}(X)$ and $d_{\mathcal{D}}\Psi = -d\Psi$, we have completed the proof. □

§7. Direct Images

§7.1. Higher analytic torsion forms

We start this section by recalling the higher analytic torsion forms defined by Bismut and Köhler [2]. We fix some notations.

Let $\varphi : M \to N$ be a smooth projective morphism of compact complex algebraic manifolds. Let $T\varphi$ be the relative tangent bundle of φ and we fix a smooth hermitian metric h_φ on $T\varphi$ that induces a Kähler metric on each fiber $\varphi^{-1}(y)$ for $y \in N$. The pair (φ, h_φ) is called a *Kähler fibration.* A real closed $(1,1)$-form Ω on M is called a *Kähler form* with respect to h_φ if the restriction of Ω to $\varphi^{-1}(y)$ is an associated Kähler form. Let us write $\overline{T\varphi} = (T\varphi, h_\varphi)$ and denote by $\mathrm{Td}(\overline{T\varphi})$ the Todd polynomial for $\overline{T\varphi}$.

Let $\overline{E}$ be a φ-acyclic hermitian vector bundle on M, that is, $\overline{E}$ is a hermitian vector bundle on M such that the higher direct image $R^i\varphi_*\overline{E}$ is trivial for $i > 0$. Then the direct image $\varphi_*\overline{E}$ becomes a vector bundle and is equipped with the L_2-hermitian metric. The Grothendieck-Riemann-Roch theorem says that the two closed forms

$$\frac{1}{(2\pi\sqrt{-1})^{\dim(M/N)}} \int_{M/N} \mathrm{Td}(\overline{T\varphi})\,\mathrm{ch}_0(\overline{E}) \text{ and } \mathrm{ch}_0(\varphi_*\overline{E})$$

give the same cohomology class. The *higher analytic torsion form* $T(\overline{E}, \varphi, \Omega) \in \mathcal{D}_1(N)$ is a homotopy between these forms, namely,

$$d_{\mathcal{D}}T(\overline{E}, \varphi, \Omega) = \mathrm{ch}_0(\varphi_*\overline{E}) - \frac{1}{(2\pi\sqrt{-1})^{\dim(M/N)}} \int_{M/N} \mathrm{Td}(\overline{T\varphi})\,\mathrm{ch}_0(\overline{E}).$$

Dependence of $T(\overline{E}, \varphi, \Omega)$ on a Kähler form has been discussed in [2]. Following their argument, for two Kähler forms Ω and Ω' giving the same hermitian metric on $T\varphi$, we can obtain $\mu(\overline{E}, \Omega, \Omega') \in \mathcal{D}_2(N)$ such that

$$d_{\mathcal{D}}\mu(\overline{E}, \Omega, \Omega') = T(\overline{E}, \varphi, \Omega) - T(\overline{E}, \varphi, \Omega').$$

Finally let us discuss the compatibility of $T(\overline{E}, \varphi, \Omega)$ and $\mu(\overline{E}, \Omega, \Omega')$ with the pull back for a closed immersion. Consider the following cartesian square:

$$\begin{array}{ccc} M' & \xrightarrow{j} & M \\ \big\downarrow{\scriptstyle \varphi'} & & \big\downarrow{\scriptstyle \varphi} \\ N' & \xrightarrow{i} & N, \end{array}$$

where i and j are closed immersions and φ is a Kähler fibration with respect to a smooth hermitian metric h_φ on $T\varphi$. Then it follows that $T\varphi' \simeq j^*T\varphi$, therefore a hermitian metric $h_{\varphi'}$ on $T\varphi'$ with which φ' becomes a Kähler fibration is induced from h_φ. If Ω is a Kähler form with respect to h_φ, then $j^*\Omega$ is a Kähler form with respect to $h_{\varphi'}$.

Take a φ-acyclic hermitian vector bundle $\overline{E}$ on M. Then it is obvious that the ingredients of the definitions of $T(\overline{E}, \varphi, \Omega)$ and $\mu(\overline{E}, \Omega, \Omega')$ such as the Bismut superconnection and the number operator are compatible with the pull back for the immersions i and j. Hence we have

$$i^*T(\overline{E}, \varphi, \Omega) = T(j^*\overline{E}, \varphi', j^*\Omega),$$
$$i^*\mu(\overline{E}, \Omega, \Omega') = \mu(j^*\overline{E}, j^*\Omega, j^*\Omega').$$

§7.2. Higher analytic torsion forms for cubes

In this subsection we introduce the higher analytic torsion form of an exact hermitian n-cube defined by Roessler [13].

Let $\varphi : M \to N, T\varphi$ and h_φ be as in the previous subsection. Let $\mathcal{F}$ be an exact hermitian n-cube made of φ-acyclic vector bundles on M. Then $\lambda\mathcal{F}$ is also made of φ-acyclic vector bundles and there is a canonical isomorphism $\varphi_*(\mathrm{tr}_n\,\lambda\mathcal{F}) \simeq \lambda\,\mathrm{tr}_n\,\varphi_*\mathcal{F}$. When we put the L_2-metrics on the both sides, however, this isomorphism does not preserve the metrics. In [13, §3.1], Roessler has constructed a hermitian vector bundle connecting these metrics. Namely, he has defined a hermitian vector bundle $\overline{h(\mathcal{F})}$ on $N \times (\mathbb{P}^1)^{n+1}$ satisfying the following conditions:

$$\overline{h(\mathcal{F})}|_{X\times\{0\}\times(\mathbb{P}^1)^n} = \varphi_*(\mathrm{tr}_n\,\lambda\mathcal{F}),$$
$$\overline{h(\mathcal{F})}|_{X\times\{\infty\}\times(\mathbb{P}^1)^n} = \lambda\,\mathrm{tr}_n\,\varphi_*\mathcal{F}$$

and

$$\overline{h(\mathcal{F})}|_{X\times(\mathbb{P}^1)^i\times\{0\}\times(\mathbb{P}^1)^{n-i}} = \overline{h(\partial_i^0\mathcal{F})},$$
$$\overline{h(\mathcal{F})}|_{X\times(\mathbb{P}^1)^i\times\{\infty\}\times(\mathbb{P}^1)^{n-i}} = \overline{h(\partial_i^{-1}\mathcal{F})}\oplus\overline{h(\partial_i^1\mathcal{F})}$$

for $1 \le i \le n$. Let us write

$$T_1(\mathcal{F},\varphi) = \frac{(-1)^n}{2(2\pi\sqrt{-1})^{n+1}(n+1)!}\int_{(\mathbb{P}^1)^{n+1}} \mathrm{ch}_0(\overline{h(\mathcal{F})})\sum_{i=1}^{n+1}(-1)^i S^i_{n+1} \in \mathcal{D}_{n+1}(N).$$

Take a Kähler form Ω with respect to h_φ and

$$T_2(\mathcal{F},\varphi,\Omega) = \frac{(-1)^{n+1}}{(2\pi\sqrt{-1})^n(n+1)!}\int_{(\mathbb{P}^1)^n}\sum_{i=1}^{n+1}(-1)^i S^i_{n+1}(\mathcal{F}) \in \mathcal{D}_{n+1}(N)$$

where

$$S^i_{n+1}(\mathcal{F}) = S^i_{n+1}(T(\mathrm{tr}_n\,\lambda\mathcal{F},\varphi,\Omega),\log|z_1|^2,\ldots,\log|z_n|^2).$$

Theorem 7.1 ([13, Thm. 3.6])**.** *We have*

$$d_{\mathcal{D}}T_1(\mathcal{F},\varphi) + T_1(\partial\mathcal{F},\varphi) = \mathrm{ch}_n(\varphi_*\mathcal{F}) - \frac{(-1)^n}{2(2\pi\sqrt{-1})^n n!}\int_{(\mathbb{P}^1)^n}\mathrm{ch}_0(\varphi_*\,\mathrm{tr}_n\,\lambda\mathcal{F})\sum_{i=1}^{n}(-1)^i S^i_n$$

and

$$d_{\mathcal{D}}T_2(\mathcal{F},\varphi,\Omega) + T_2(\partial\mathcal{F},\varphi,\Omega) = \frac{(-1)^n}{2(2\pi\sqrt{-1})^n n!}\int_{(\mathbb{P}^1)^n}\mathrm{ch}_0(\varphi_*\,\mathrm{tr}_n\,\lambda\mathcal{F})\sum_{i=1}^{n}(-1)^i S^i_n - \frac{1}{(2\pi\sqrt{-1})^{\dim(M/N)}}\int_{M/N}\mathrm{Td}(\overline{T\varphi})\,\mathrm{ch}_n(\mathcal{F}).$$

Hence if we write $T(\mathcal{F},\varphi,\Omega) = T_1(\mathcal{F},\varphi) + T_2(\mathcal{F},\varphi,\Omega)$, then

$$d_{\mathcal{D}}T(\mathcal{F},\varphi,\Omega) + T(\partial\mathcal{F},\varphi,\Omega) = \mathrm{ch}_n(\varphi_*\mathcal{F}) - \frac{1}{(2\pi\sqrt{-1})^{\dim(M/N)}}\int_{M/N}\mathrm{Td}(\overline{T\varphi})\,\mathrm{ch}_n(\mathcal{F}).$$

Let us discuss dependence of $T(\mathcal{F},\varphi,\Omega)$ on a Kähler form Ω. For $u_i \in \mathcal{D}_1(M)$, let

$$C_n(u_1,\ldots,u_n) = \frac{1}{2^n}\sum_{\sigma\in\mathfrak{S}_n}(-1)^{\mathrm{sgn}\,\sigma}u_{\sigma(1)}\bullet\left(u_{\sigma(2)}\bullet\left(\cdots u_{\sigma(k)}\cdots\right)\right).$$

Then it is easy to show that

$$C_n(u_1,\dots,u_n)=\frac{(-1)^n}{2}\sum_{i=1}^{n}(-1)^i S_n^i(u_1,\dots,u_n).$$

Lemma 7.2. *For $u_1\in\mathcal{D}_2(M)$ and $u_i\in\mathcal{D}_1(M)$ with $2\le i\le n$, let*

$$C_n(u_1,\dots,u_n)=\frac{1}{2^n}\sum_{j=1}^{n}(-1)^{j+1}\sum_{\substack{\sigma\in\mathfrak{S}_n\\ \sigma(j)=1}}(-1)^{\operatorname{sgn}\sigma}u_{\sigma(1)}\bullet(u_{\sigma(2)}\bullet(\cdots u_{\sigma(j)}\cdots)).$$

Then we have

$$\begin{aligned}&d_{\mathcal{D}}C_n(u_1,u_2,\dots,u_n)\\ &\quad=C_n(d_{\mathcal{D}}u_1,u_2,\dots,u_n)+\frac{n}{2}\sum_{k=2}^{n}(-1)^k(d_{\mathcal{D}}u_k)\bullet C_{n-1}(u_1,u_2,\dots,\widehat{u_k},\cdots,u_n).\end{aligned}$$

Proof. Since $d_{\mathcal{D}}(u\bullet v)=d_{\mathcal{D}}u\bullet v+(-1)^{\deg u}u\bullet d_{\mathcal{D}}v$, we have

$$\begin{aligned}&d_{\mathcal{D}}C_n(u_1,u_2,\dots,u_n)\\ &=\frac{1}{2^n}\sum_{j=1}^{n}\sum_{\substack{\sigma\in\mathfrak{S}_n\\ \sigma(j)=1}}(-1)^{\operatorname{sgn}\sigma}\sum_{i<j}(-1)^{i+j}d_{\mathcal{D}}u_{\sigma(i)}(u_{\sigma(1)}\bullet(\cdots\widehat{u_{\sigma(i)}}\cdots u_{\sigma(j)}\cdots))\\ &\quad+\frac{1}{2^n}\sum_{j=1}^{n}\sum_{\substack{\sigma\in\mathfrak{S}_n\\ \sigma(j)=1}}(-1)^{\operatorname{sgn}\sigma}u_{\sigma(1)}\bullet(\cdots d_{\mathcal{D}}u_{\sigma(j)}\cdots)\\ &\quad+\frac{1}{2^n}\sum_{j=1}^{n}\sum_{\substack{\sigma\in\mathfrak{S}_n\\ \sigma(j)=1}}(-1)^{\operatorname{sgn}\sigma}\sum_{j<i}(-1)^{i+j+1}d_{\mathcal{D}}u_{\sigma(i)}(u_{\sigma(1)}\bullet(\cdots u_{\sigma(j)}\cdots\widehat{u_{\sigma(i)}}\cdots))\\ &=\frac{1}{2^n}\sum_{j=1}^{n}\sum_{\substack{\sigma\in\mathfrak{S}_n\\ \sigma(j)=1}}(-1)^{\operatorname{sgn}\sigma}u_{\sigma(1)}\bullet(\cdots d_{\mathcal{D}}u_{\sigma(j)}\cdots)\\ &\quad+\frac{1}{2^n}\sum_{k=2}^{n}\sum_{i<j}\sum_{\substack{\sigma\in\mathfrak{S}_n\\ \sigma(j)=1\\ \sigma(i)=k}}(-1)^{\operatorname{sgn}\sigma}(-1)^{i+j}d_{\mathcal{D}}u_k(u_{\sigma(1)}\bullet(\cdots\widehat{u_{\sigma(i)}}\cdots u_{\sigma(j)}\cdots))\\ &\quad+\frac{1}{2^n}\sum_{k=2}^{n}\sum_{j<i}\sum_{\substack{\sigma\in\mathfrak{S}_n\\ \sigma(j)=1\\ \sigma(i)=k}}(-1)^{\operatorname{sgn}\sigma}(-1)^{i+j+1}d_{\mathcal{D}}u_k(u_{\sigma(1)}\bullet(\cdots u_{\sigma(j)}\cdots\widehat{u_{\sigma(i)}}\cdots))\end{aligned}$$

$$= C_n(d_{\mathcal{D}}u_1, u_2, \ldots, u_n) + \frac{n}{2}\sum_{k=2}^{n}(-1)^k (d_{\mathcal{D}}u_k) C_{n-1}(u_1, \ldots, \widehat{u_k}, \ldots, u_n).$$

□

Proposition 7.3. *Let Ω and Ω' be Kähler forms with respect to a smooth hermitian metric h_φ on $T\varphi$. For an exact hermitian n-cube $\mathcal{F}$ made of φ-acyclic vector bundles on M, let us write $\mu(\mathcal{F})$ for $\mu(\mathrm{tr}_n \lambda\mathcal{F}, \Omega, \Omega') \in \mathcal{D}_2(N \times (\mathbb{P}^1)^n)$. Then we have*

$$T(\mathcal{F}, \varphi, \Omega) - T(\mathcal{F}, \varphi, \Omega') \equiv \frac{-2}{(2\pi\sqrt{-1})^{n-1} n!}\int_{(\mathbb{P}^1)^{n-1}} C_n(\mu(\partial\mathcal{F}), \log|z_1|^2, \ldots, \log|z_{n-1}|^2)$$

modulo $\mathrm{Im}\, d_{\mathcal{D}}$.

Proof. It follows from the definition that

$$T_2(\mathcal{F}, \varphi, \Omega) = \frac{2}{(2\pi\sqrt{-1})^n (n+1)!}\int_{(\mathbb{P}^1)^n} C_{n+1}(T(\mathrm{tr}_n \lambda\mathcal{F}, \varphi, \Omega), \log|z_1|^2, \ldots, \log|z_n|^2).$$

Then by Lemma 7.2 we have

$$\begin{aligned}
&T(\mathcal{F}, \varphi, \Omega) - T(\mathcal{F}, \varphi, \Omega') = T_2(\mathcal{F}, \varphi, \Omega) - T_2(\mathcal{F}, \varphi, \Omega') \\
&= \frac{2}{(2\pi\sqrt{-1})^n (n+1)!} \\
&\quad \int_{(\mathbb{P}^1)^n} C_{n+1}(T(\mathrm{tr}_n \lambda\mathcal{F}, \varphi, \Omega) - T(\mathrm{tr}_n \lambda\mathcal{F}, \varphi, \Omega'), \log|z_1|^2, \ldots, \log|z_n|^2) \\
&= \frac{2}{(2\pi\sqrt{-1})^n (n+1)!}\int_{(\mathbb{P}^1)^n} C_{n+1}(d_{\mathcal{D}}\mu(\mathcal{F}), \log|z_1|^2, \cdots, \log|z_n|^2) \\
&= \frac{2}{(2\pi\sqrt{-1})^n (n+1)!}\int_{(\mathbb{P}^1)^n} d_{\mathcal{D}} C_{n+1}(\mu(\mathcal{F}), \log|z_1|^2, \ldots, \log|z_n|^2) \\
&\quad - \frac{1}{(2\pi\sqrt{-1})^n n!}\sum_{k=1}^{n}(-1)^{k-1} \\
&\quad \int_{(\mathbb{P}^1)^n} d_{\mathcal{D}} \log|z_k|^2 C_n(\mu(\mathcal{F}), \log|z_1|^2, \ldots, \widehat{\log|z_k|}^2, \ldots, \log|z_n|^2) \\
&= \frac{2}{(2\pi\sqrt{-1})^n (n+1)!} d_{\mathcal{D}}\left(\int_{(\mathbb{P}^1)^n} C_{n+1}(\mu(\mathcal{F}), \log|z_1|^2, \ldots, \log|z_n|^2)\right) \\
&\quad - \frac{2}{(2\pi\sqrt{-1})^{n-1} n!}\int_{(\mathbb{P}^1)^{n-1}} C_n(\mu(\partial\mathcal{F}), \log|z_1|^2, \ldots, \log|z_{n-1}|^2).
\end{aligned}$$

□

§7.3. Definition of direct image homomorphism

In this subsection, we apply the results obtained so far to an arithmetic situation and define a direct image homomorphism in higher arithmetic K-theory. Let $\varphi : X \to Y$ be a smooth projective morphism of proper arithmetic varieties. We fix an F_∞-invariant smooth hermitian metric h_φ on $T\varphi(\mathbb{C})$ such that $(\varphi(\mathbb{C}), h_\varphi)$ is a Kähler fibration, and take an anti-F_∞-invariant Kähler form Ω on $X(\mathbb{C})$ with respect to h_φ. Let $\widehat{S}(\varphi\text{-ac})$ denote the S-construction of the category of φ-acyclic hermitian vector bundles on X. Then the natural inclusion $\widehat{S}(\varphi\text{-ac}) \to \widehat{S}(X)$ is a homotopy equivalence, and the direct image of a φ-acyclic hermitian vector bundle with the L_2-metric gives a map of simplicial sets $\varphi_* : \widehat{S}(\varphi\text{-ac}) \to \widehat{S}(Y)$.

Proposition 7.4. *If E is a degenerate element of $\widehat{S}_{n+1}(\varphi\text{-}ac)$, then $T(\mathrm{Cub}(E), \varphi, \Omega) = 0$.*

The proof is similar to that of Theorem 4.4, so we omit it. By virtue of this proposition, taking higher analytic torsion forms yields a homomorphism

$$T(\ \varphi, \Omega) : C_*(|\widehat{S}(\varphi\text{-ac})|) \to \mathcal{D}_*(Y).$$

In particular, the higher analytic torsion form of a pointed cellular map $f : S^{n+1} \to |\widehat{S}(\varphi\text{-ac})|$ is defined by $T(f, \varphi, \Omega) = T(f_*([S^{n+1}]), \varphi, \Omega)$. We abbreviate $T(f, \varphi, \Omega)$ to $T(f)$ if the morphism φ and the Kähler form Ω are fixed.

Let $\varphi_! : \mathcal{D}_*(X) \to \mathcal{D}_*(Y)$ be the map given by

$$\varphi_! \omega = \frac{1}{(2\pi\sqrt{-1})^{\dim(X/Y)}} \int_{X(\mathbb{C})/Y(\mathbb{C})} \mathrm{Td}(\overline{T\varphi})\omega.$$

Then by Theorem 7.1 the diagram

$$\begin{array}{ccc} C_*(|\widehat{S}(\varphi\text{-ac})|) & \xrightarrow{\varphi_*} & C_*(|\widehat{S}(Y)|) \\ \big\downarrow \mathrm{ch} & & \big\downarrow \mathrm{ch} \\ \mathcal{D}_*(X)[1] & \xrightarrow{\varphi_!} & \mathcal{D}_*(Y)[1] \end{array}$$

is commutative up to the homotopy $-T(\ , \varphi, \Omega)$. Hence by Proposition 3.9 we can obtain a homomorphism

$$\widehat{\varphi}(\Omega)_* : \widehat{\pi}_{n+1}(|\widehat{S}(\varphi\text{-ac})|, \mathrm{ch}) \to \widehat{\pi}_{n+1}(|\widehat{S}(Y)|, \mathrm{ch})$$

by $[(f, \omega)] \mapsto [(\varphi_* f, \varphi_! \omega + T(f, \varphi, \Omega))]$.

If Ω' is another anti-F_∞-invariant Kähler form with respect to h_φ, then it follows from Proposition 7.3 that $T(f,\varphi,\Omega) \equiv T(f,\varphi,\Omega')$ modulo $\operatorname{Im} d_{\mathcal{D}}$ for any pointed cellular map $f : S^{n+1} \to |\widehat{S}(\varphi\text{-ac})|$. Hence the homomorphism $\widehat{\varphi}(\Omega)_*$ depends only on the hermitian metric h_φ and does not concern the Kähler form Ω.

Summing up the arguments in this subsection leads to the following:

Theorem 7.5. *Let $\varphi : X \to Y$ be a smooth projective morphism of proper arithmetic varieties. We fix an F_∞-invariant metric h_φ on $T\varphi$ such that $(\varphi(\mathbb{C}), h_\varphi)$ is a Kähler fibration. Then we can define a direct image homomorphism $\widehat{\varphi}(h_\varphi)_* : \widehat{K}_n(X) \to \widehat{K}_n(Y)$ by*

$$\widehat{\pi}_{n+1}(|\widehat{S}(X)|, \mathrm{ch}) \simeq \widehat{\pi}_{n+1}(|\widehat{S}(\varphi\text{-}ac)|, \mathrm{ch}) \xrightarrow{\widehat{\varphi}(\Omega)_*} \widehat{\pi}_{n+1}(|\widehat{S}(Y)|, \mathrm{ch}),$$

where Ω is an anti-F_∞-invariant Kähler from on $X(\mathbb{C})$ with respect to h_φ.

When $n = 0$, the isomorphism $\widehat{\alpha} : \widehat{\mathcal{K}}_0(X) \to \widehat{K}_0(X)$ gives an identification between the direct image homomorphism defined above and $\varphi_! : \widehat{\mathcal{K}}_0(X) \to \widehat{\mathcal{K}}_0(Y)$ in [10].

Proposition 3.10 implies that the diagram

$$\begin{array}{ccc} \widehat{K}_n(X) & \xrightarrow{\mathrm{ch}_n} & \mathcal{D}_*(X) \\ \Big\downarrow{\scriptstyle \widehat{\varphi}(h_\varphi)_*} & & \Big\downarrow{\scriptstyle \varphi_!} \\ \widehat{K}_n(Y) & \xrightarrow{\mathrm{ch}_n} & \mathcal{D}_*(Y) \end{array}$$

is commutative. In particular, we can obtain a direct image homomorphism in KM-groups

$$\widehat{\varphi}(h_\varphi)_* : KM_n(X) \to KM_n(Y).$$

Finally, we give a description of the direct image homomorphism by means of the G-construction. Given a pointed cellular map $f : S^n \to |\widehat{G}(X)|$ for $n \geq 1$, let $T(f,\varphi,\Omega) = T(\chi_* f_*([S^n]), \varphi, \Omega)$. Then we can obtain a homomorphism $\widehat{\varphi}(\Omega)_* : \widehat{\pi}_n(|\widehat{G}(\varphi\text{-ac})|, \mathrm{ch}) \to \widehat{\pi}_n(|\widehat{G}(Y)|, \mathrm{ch})$ by

$$\widehat{\varphi}(\Omega)_*([(f,\omega)]) = [(\varphi_* f, \varphi_! \omega - T(f,\varphi,\Omega))],$$

and it satisfies the following commutative diagram:

$$\begin{array}{ccc} \widehat{\pi}_n(|\widehat{G}(\varphi\text{-ac})|, \mathrm{ch}) & \xrightarrow{\widehat{\varphi}(\Omega)_*} & \widehat{\pi}_n(|\widehat{G}(Y)|, \mathrm{ch}) \\ \Big\downarrow{\scriptstyle \widehat{\chi}_*} & & \Big\downarrow{\scriptstyle \widehat{\chi}_*} \\ \widehat{\pi}_{n+1}(|\widehat{S}(\varphi\text{-ac})|, \mathrm{ch}) & \xrightarrow{\widehat{\varphi}(\Omega)_*} & \widehat{\pi}_{n+1}(|\widehat{S}(Y)|, \mathrm{ch}). \end{array}$$

Hence the direct image homomorphism in $\widehat{K}_*(X)$ can also be given as follows:

$$\widehat{\pi}_n(|\widehat{G}(X)|,\mathrm{ch}) \simeq \widehat{\pi}_n(|\widehat{G}(\varphi\text{-ac})|,\mathrm{ch}) \stackrel{\widehat{\varphi}(\Omega)_*}{\longrightarrow} \widehat{\pi}_n(|\widehat{G}(Y)|,\mathrm{ch}).$$

§7.4. The projection formula

In this subsection we prove the projection formula in higher arithmetic K-theory. We first consider the case of $\widehat{\mathcal{K}}_0$-groups. Let $\varphi : X \to Y$, h_φ and Ω be as in the last subsection. Let $\overline{E}$ be a hermitian vector bundle on Y and $\overline{F}$ a φ-acyclic hermitian vector bundle on X. Then the canonical isomorphism $\varphi_*(\varphi^*\overline{E}\otimes\overline{F}) \simeq \overline{E}\otimes\varphi_*\overline{F}$ preserves the metrics.

Let $\varphi_!$ denote the direct image homomorphism in $\widehat{\mathcal{K}}_0$ given in [10]. For $\omega \in \widetilde{\mathcal{D}}_1(Y)$ and $\tau \in \widetilde{\mathcal{D}}_1(X)$, we have

$$\begin{aligned}
&\varphi_!(\widehat{\varphi}^*(\overline{E},\omega)\times(\overline{F},\tau)) \\
&\qquad = \varphi_!((\varphi^*\overline{E})\otimes\overline{F}, \varphi^*\omega\wedge\mathrm{ch}_0(\overline{F}) + \varphi^*\mathrm{ch}_0(\overline{E})\wedge\tau + \varphi^* d_{\mathcal{D}}\omega\wedge\tau) \\
&\qquad = (\varphi_*(\varphi^*\overline{E}\otimes\overline{F}),\eta),
\end{aligned}$$

where

$$\begin{aligned}
\eta = {} & \frac{1}{(2\pi\sqrt{-1})^{\dim(X/Y)}}\int_{X(\mathbb{C})/Y(\mathbb{C})} \mathrm{Td}(\overline{T\varphi})\wedge\big(\varphi^*\omega\wedge\mathrm{ch}_0(\overline{F}) \\
& +\varphi^*\mathrm{ch}_0(\overline{E})\wedge\tau + \varphi^* d_{\mathcal{D}}\omega\wedge\tau\big) \\
& - T(\varphi^*\overline{E}\otimes\overline{F}) \\
= {} & \frac{1}{(2\pi\sqrt{-1})^{\dim(X/Y)}}(\mathrm{ch}_0(\overline{E}) + d_{\mathcal{D}}\omega)\wedge\int_{X(\mathbb{C})/Y(\mathbb{C})}\mathrm{Td}(\overline{T\varphi})\wedge\tau \\
& + \omega\wedge(\mathrm{ch}_0(\varphi_*\overline{F}) - d_{\mathcal{D}}T(\overline{F})) - T(\varphi^*\overline{E}\otimes\overline{F}).
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
(\overline{E},\omega)\times\varphi_!(\overline{F},\tau) &= (\overline{E},\omega)\otimes(\varphi_*\overline{F},\varphi_!\tau - T(\overline{F})) \\
&= (\overline{E}\otimes\varphi_*\overline{F},\eta'),
\end{aligned}$$

where

$$\begin{aligned}
\eta' = {} & \frac{1}{(2\pi\sqrt{-1})^{\dim(X/Y)}}(\mathrm{ch}_0(\overline{E}) + d_{\mathcal{D}}\omega)\wedge\int_{X(\mathbb{C})/Y(\mathbb{C})}\mathrm{Td}(\overline{T\varphi})\wedge\tau \\
& + \omega\wedge\mathrm{ch}_0(\varphi_*\overline{F}) - (\mathrm{ch}_0(\overline{E}) + d_{\mathcal{D}}\omega)\wedge T(\overline{F}).
\end{aligned}$$

Comparing these identities, we have

$$\eta - \eta' = -T(\varphi^*\overline{E}\otimes\overline{F}) + \mathrm{ch}_0(\overline{E})\wedge T(\overline{F}) + d_{\mathcal{D}}(\omega\wedge T(\overline{F})).$$

Hence the projection formula in $\widehat{\mathcal{K}}_0$-groups is reduced to the following proposition:

Proposition 7.6. *Under the above notations, we have*

$$T(\varphi^*\overline{E}\otimes\overline{F}) = \mathrm{ch}_0(\overline{E})\wedge T(\overline{F}).$$

Proof. Let $\mathcal{E}$ be the infinite dimensional vector bundle on N consisting of smooth sections of $\Lambda^* T^{*(1,0)}\varphi\otimes\overline{F}$. Let B_u and N_u denote the Bismut superconnection and the number operator on $\mathcal{E}$ respectively. Let $\mathcal{E}'$ be the infinite dimensional vector bundle consisting of smooth sections of $\Lambda^* T^{*(1,0)}\varphi\otimes(\varphi^*\overline{E}\otimes\overline{F})$. Let B'_u and N'_u denote the Bismut superconnection and the number operator on $\mathcal{E}'$ respectively. Then we have a canonical isometry $\mathcal{E}'\simeq\overline{E}\otimes\mathcal{E}$ and under this identification, we have $B'_u = 1\otimes B_u + \nabla_{\overline{E}}\otimes 1$ and $N'_u = 1\otimes N_u$. Substituting these into the definition of $T(\varphi^*\overline{E}\otimes\overline{F})$ in [2] yields the desired identity. $\square$

Let us move on to the higher case. We assume that $n, m\geq 1$. Consider the following diagram:

$$\begin{array}{ccc}
\widehat{G}(Y)\wedge\widehat{G}(\varphi\text{-ac}) & \xrightarrow{m^G(\varphi^*\wedge 1)} & \widehat{G}^{(2)}(\varphi\text{-ac}) \\
\Big\downarrow{\scriptstyle 1\wedge\varphi_*} & & \Big\downarrow{\scriptstyle\varphi_*} \\
\widehat{G}(Y)\wedge\widehat{G}(Y) & \xrightarrow{m^G} & \widehat{G}^{(2)}(Y).
\end{array}$$

This diagram is commutative up to a homotopy arising from the isometry $\varphi_*(\varphi^*\overline{E}\otimes\overline{F})\simeq\overline{E}\otimes\varphi_*\overline{F}$. Hence for two pointed cellular maps $f: S^n\to|\widehat{G}(Y)|$ and $g: S^m\to|\widehat{G}(\varphi\text{-ac})|$, $[(\varphi_*(\varphi^* f\times g), 0)] = [(f\times\varphi_* g, 0)]$. For $\omega\in\widetilde{\mathcal{D}}_{n+1}(Y)$ and $\tau\in\widetilde{\mathcal{D}}_{m+1}(X)$,

$$\begin{aligned}
\varphi(\Omega)_*(\varphi^*(f,\omega)\times(g,\tau)) &= \varphi(\Omega)_*(\varphi^* f\times g, (-1)^n\varphi^*\mathrm{ch}_n(f)\bullet\tau + \varphi^*\omega\bullet\mathrm{ch}_m(g) \\
&\quad + (-1)^n d_{\mathcal{D}}\varphi^*\omega\bullet\tau + (-1)^n\varphi^*\mathrm{ch}_n(f)\vartriangle\mathrm{ch}_m(g)) \\
&= (\varphi_*(\varphi^* f\times g), \eta),
\end{aligned}$$

where

$$\begin{aligned}
\eta = {} & \frac{(-1)^n}{(2\pi\sqrt{-1})^{\dim(X/Y)}}(\mathrm{ch}_n(f) + d_{\mathcal{D}}\omega)\bullet\int_{X(\mathbb{C})/Y(\mathbb{C})}\mathrm{Td}(\overline{T\varphi})\wedge\tau \\
& + \omega\bullet(\mathrm{ch}_m(\varphi_* g) - d_{\mathcal{D}} T(g)) \\
& + (-1)^n\mathrm{ch}_n(f)\vartriangle(\mathrm{ch}_m(\varphi_* g) - d_{\mathcal{D}} T(g)) - T(\varphi^* f\times g).
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}(f,\omega)\times\varphi(h_\varphi)_*(g,\tau) &= (f,\omega)\times(\varphi_* g,\varphi_!\tau - T(g))\\ &= (f\times\varphi_* g,\eta'),\end{aligned}$$

where

$$\begin{aligned}\eta' = {} & \frac{(-1)^n}{(2\pi\sqrt{-1})^{\dim(X/Y)}}(\mathrm{ch}_n(f)+d_{\mathcal{D}}\omega)\bullet\int_{X(\mathbb{C})/Y(\mathbb{C})}\mathrm{Td}(\overline{T\varphi})\wedge\tau\\ & -(-1)^n(\mathrm{ch}_n(f)+d_{\mathcal{D}}\omega)\bullet T(g)\\ & +\omega\bullet\mathrm{ch}_n(\varphi_* g)+(-1)^n\,\mathrm{ch}_n(f)\bigtriangleup\mathrm{ch}_m(\varphi_* g).\end{aligned}$$

Hence we have

$$\eta-\eta'\equiv(-1)^{n+1}\,\mathrm{ch}_n(f)\bigtriangleup d_{\mathcal{D}}T(g)-T(\varphi^* f\times g)+(-1)^n\,\mathrm{ch}_n(f)\bullet T(g)$$

modulo $\mathrm{Im}\, d_{\mathcal{D}}$. Thus the projection formula in higher arithmetic K-theory is reduced to the following proposition:

Proposition 7.7. *For an exact hermitian n-cube $\mathcal{F}$ on Y and an exact hermitian m-cube $\mathcal{G}$ made of φ-acyclic vector bundles on X, we have*

$$\begin{aligned}d_{\mathcal{D}}(\mathrm{ch}_n(\mathcal{F})\bigtriangleup T(\mathcal{G})) = {} & -T(\varphi^*\mathcal{F}\otimes\mathcal{G})+(-1)^n\,\mathrm{ch}_n(\mathcal{F})\bullet T(\mathcal{G})\\ & +\mathrm{ch}_{n+1}(\partial\mathcal{F})\bigtriangleup T(\mathcal{G})+(-1)^{n-1}\,\mathrm{ch}_n(\mathcal{F})\bigtriangleup d_{\mathcal{D}}T(\mathcal{G}).\end{aligned}$$

Proof. We will prove the following identities:

$$\begin{aligned}d_{\mathcal{D}}(\mathrm{ch}_n(\mathcal{F})\bigtriangleup T_1(\mathcal{G})) = {} & -T_1(\varphi^*\mathcal{F}\times\mathcal{G})+(-1)^n\,\mathrm{ch}_n(\mathcal{F})\bullet T_1(\mathcal{G})\\ & +\mathrm{ch}_{n-1}(\partial\mathcal{F})\bigtriangleup T_1(\mathcal{G})+(-1)^{n+1}\,\mathrm{ch}_n(\mathcal{F})\bigtriangleup d_{\mathcal{D}}T_1(\mathcal{G}),\\ d_{\mathcal{D}}(\mathrm{ch}_n(\mathcal{F})\bigtriangleup T_2(\mathcal{G})) = {} & -T_2(\varphi^*\mathcal{F}\times\mathcal{G})+(-1)^n\,\mathrm{ch}_n(\mathcal{F})\bullet T_2(\mathcal{G})\\ & +\mathrm{ch}_{n-1}(\partial\mathcal{F})\bigtriangleup T_2(\mathcal{G})+(-1)^{n+1}\,\mathrm{ch}_n(\mathcal{F})\bigtriangleup d_{\mathcal{D}}T_2(\mathcal{G}).\end{aligned}$$

These identities can be proved in the same way, so we will prove only the latter one.

For $t<s$, let $\pi_1:(\mathbb{P}^1)^s\to(\mathbb{P}^1)^t$ denote the projection $\pi_1(x_1,\dots,x_s)=(x_1,\dots,x_t)$ and $\pi_2:(\mathbb{P}^1)^s\to(\mathbb{P}^1)^t$ denote the projection $\pi_2(x_1,\dots,x_s)=(x_{s-t+1},\dots,x_s)$. Then Proposition 5.5 implies that

$$\begin{aligned}& d(\mathrm{ch}_n(\mathcal{F})\bigtriangleup T_2(\mathcal{G}))\\ & =\tfrac{(-1)^{n+m+1}}{(2\pi\sqrt{-1})^{n+m-1}n!(m+1)!}\int_{(\mathbb{P}^1)^{n+m}}\pi_1^*\,\mathrm{ch}_0(\mathrm{tr}_n\,\lambda\mathcal{F})\end{aligned}$$

$$\wedge\, d\left(\sum_{\substack{1\le i\le n\\ 1\le j\le m+1}} (-1)^{i+j}\pi_1^* S_n^i \vartriangle \pi_2^* S_{m+1}^j(\mathcal{G})\right)$$

$$= \tfrac{(-1)^{n+m+1}}{(2\pi\sqrt{-1})^{n+m}(n+m+1)!}\int_{(\mathbb{P}^1)^{n+m}} \pi_1^*\operatorname{ch}_0(\operatorname{tr}_n \lambda\mathcal{F}) \sum_{k=1}^{n+m+1}(-1)^k$$

$$S_{n+m+1}^k(\pi_2^* T(\operatorname{tr}_m \lambda\mathcal{G}), \log|t_1|^2,\dots,\log|t_{n+m}|^2)$$

$$+ \tfrac{(-1)^{n+m+1}}{2(2\pi\sqrt{-1})^{n+m-1}(n-1)!(m+1)!}\int_{(\mathbb{P}^1)^{n+m-1}} \pi_1^*\operatorname{ch}_0(\operatorname{tr}_{n-1}\lambda\partial\mathcal{F})$$

$$\wedge \sum_{\substack{1\le i\le n-1\\ 1\le j\le m+1}} (-1)^{i+j} a_{i,j}^{n-1,m+1}\pi_1^* S_{n-1}^i \wedge \pi_2^* S_{m+1}^j(\mathcal{G})$$

$$+ \tfrac{(-1)^{m+1}}{2(2\pi\sqrt{-1})^{n+m} n!m!}\int_{(\mathbb{P}^1)^{n+m}} \pi_1^*\operatorname{ch}_0(\operatorname{tr}_n\lambda\mathcal{F})$$

$$\wedge \sum_{\substack{1\le i\le n\\ 1\le j\le m}} (-1)^{i+j} a_{i,j}^{n,m}\pi_1^* S_n^i \wedge \pi_2^*\left(\partial\overline{\partial} T(\operatorname{tr}_m\lambda\mathcal{G})\wedge S_m^j\right)$$

$$+ \tfrac{(-1)^{m+1}}{2(2\pi\sqrt{-1})^{n+m-1} n!m!}\int_{(\mathbb{P}^1)^{n+m-1}} \pi_1^*\operatorname{ch}_0(\operatorname{tr}_n\lambda\mathcal{F})$$

$$\wedge \sum_{\substack{1\le i\le n\\ 1\le j\le m}} (-1)^{i+j} a_{i,j}^{n,m}\pi_1^* S_n^i \wedge \pi_2^* S_m^j(\partial\mathcal{G})$$

$$+ \tfrac{(-1)^{m}}{2(2\pi\sqrt{-1})^{n+m} n!(m+1)!}\int_{(\mathbb{P}^1)^{n+m}} \pi_1^*\operatorname{ch}_0(\operatorname{tr}_n\lambda\mathcal{F})$$

$$\wedge \sum_{\substack{1\le i\le n\\ 1\le j\le m+1}} (-1)^{i+j}\pi_1^* S_n^i \bullet \pi_2^* S_{m+1}^j(\mathcal{G}).$$

By Proposition 7.6, we have

$$\begin{aligned}
&\pi_1^*\operatorname{ch}_0(\operatorname{tr}_n\lambda\mathcal{F}) S_{n+m+1}^k(\pi_2^* T(\operatorname{tr}_m\lambda\mathcal{G}),\log|t_1|^2,\dots,\log|t_{n+m}|^2)\\
&\quad= S_{n+m+1}^k(\pi_1^*\operatorname{ch}_0(\operatorname{tr}_n\lambda\mathcal{F})\wedge\pi_2^* T(\operatorname{tr}_m\lambda\mathcal{G}),\log|t_1|^2,\dots,\log|t_{n+m}|^2)\\
&\quad= S_{n+m+1}^k(T(\operatorname{tr}_{n+m}\lambda(\varphi^*\mathcal{F}\otimes\mathcal{G}),\log|t_1|^2,\dots,\log|t_{n+m}|^2)\\
&\quad= S_{n+m+1}^k(\varphi^*\mathcal{F}\otimes\mathcal{G}).
\end{aligned}$$

Moreover,

$$d_{\mathcal{D}} T_2(\mathcal{G}) = \tfrac{(-1)^{m+1}}{(2\pi\sqrt{-1})^m(m+1)!}\int_{(\mathbb{P}^1)^m}\sum_{j=1}^{m}(-1)^{j+1}(\partial S_{m+1}^j(\mathcal{G}) - \overline{\partial} S_{m+1}^{j+1}(\mathcal{G}))$$

$$= \frac{(-1)^{m+1}}{(2\pi\sqrt{-1})^m m!} \int_{(\mathbb{P}^1)^m} \partial\overline{\partial} T(\mathrm{tr}_m \lambda\mathcal{G}) \wedge \left(\sum_{j=1}^m (-1)^j S_m^j \right)$$
$$+ \frac{(-1)^{m+1}}{(2\pi\sqrt{-1})^{m-1} m!} \int_{(\mathbb{P}^1)^{m-1}} \sum_{j=1}^m (-1)^j S_m^j(\partial\mathcal{G}).$$

Hence

$$d_{\mathfrak{D}}(\mathrm{ch}_n(\mathcal{F}) \vartriangle T_2(\mathcal{G})) = -d(\mathrm{ch}_n(\mathcal{F}) \vartriangle T_2(\mathcal{G}))$$
$$= \frac{(-1)^{n+m}}{(2\pi\sqrt{-1})^{n+m}(n+m+1)!} \int_{(\mathbb{P}^1)^{n+m}} \sum_{k=1}^{n+m+1} (-1)^k S_{n+m+1}^k(\varphi^*\mathcal{F} \otimes \mathcal{G})$$
$$+ \frac{(-1)^{n+m}}{2(2\pi\sqrt{-1})^{n+m-1}(n-1)!(m+1)!}$$
$$\int_{(\mathbb{P}^1)^{n+m-1}} \sum_{\substack{1 \le i \le n-1 \\ 1 \le j \le m+1}} (-1)^{i+j} a_{i,j}^{n-1,m+1} \pi_1^*(\mathrm{ch}_0(\mathrm{tr}_{n-1} \lambda\partial\mathcal{F}) S_{n-1}^i) \wedge \pi_2^* S_{m+1}^j(\mathcal{G})$$
$$- \frac{(-1)^m}{2(2\pi\sqrt{-1})^n n!} \sum_{\substack{1 \le i \le n \\ 1 \le j \le m}} (-1)^{i+j} a_{i,j}^{n,m} \left(\int_{(\mathbb{P}^1)^n} \mathrm{ch}_0(\mathrm{tr}_n \lambda\mathcal{F}) S_n^i \right)$$
$$\wedge \left(-\frac{1}{(2\pi\sqrt{-1})^m m!} \int_{(\mathbb{P}^1)^m} \partial\overline{\partial} T(\mathrm{tr}_m \lambda\mathcal{G}) S_m^j - \frac{1}{(2\pi\sqrt{-1})^{m-1} m!} \int_{(\mathbb{P}^1)^{m-1}} S_m^j(\partial\mathcal{G}) \right)$$
$$+ \frac{(-1)^{m+1}}{2(2\pi\sqrt{-1})^{n+m} n!(m+1)!}$$
$$\int_{(\mathbb{P}^1)^{n+m}} \sum_{\substack{1 \le i \le n \\ 1 \le j \le m+1}} (-1)^{i+j} \pi_1^*(\mathrm{ch}_0(\mathrm{tr}_n \lambda\mathcal{F}) S_n^i) \bullet \pi_2^* S_{m+1}^j(\mathcal{G})$$
$$= -T_2(\varphi^*\mathcal{F} \otimes \mathcal{G}) + \mathrm{ch}_{n-1}(\partial\mathcal{F}) \vartriangle T_2(\mathcal{G})$$
$$+ (-1)^{n+1} \mathrm{ch}_n(\mathcal{F}) \vartriangle d_{\mathfrak{D}} T_2(\mathcal{G}) + (-1)^n \mathrm{ch}_n(\mathcal{F}) \bullet T_2(\mathcal{G}).$$

□

Let us consider the case of $n = 0$ and $m > 0$. Let $(\overline{E}, \omega)$ be a pair of a hermitian vector bundle $\overline{E}$ on Y and $\omega \in \widetilde{\mathfrak{D}}_1(Y)$ and let (g, τ) be a pair of a pointed cellular map $g : S^m \to |\widehat{G}(\varphi\text{-ac})|$ and $\tau \in \widetilde{\mathfrak{D}}_{m+1}(X)$. Then we have

$$\widehat{\varphi}(\Omega)_*(\widehat{\varphi}^*(\overline{E}, \omega) \times (g, \tau))$$
$$= (\varphi_*(\varphi^*\overline{E} \otimes g), \varphi_!(\varphi^*\omega \bullet (\mathrm{ch}_m(g) + d_{\mathfrak{D}}\tau)) + \varphi_!(\varphi^* \mathrm{ch}_0(\overline{E}) \bullet \tau)$$
$$- T(\varphi^*\overline{E} \otimes g)$$
$$= (\varphi_*(\varphi^*\overline{E} \otimes g), \omega \bullet \varphi_!(\mathrm{ch}_m(g) + d_{\mathfrak{D}}\tau)) + \mathrm{ch}_0(\overline{E}) \bullet (\varphi_!\tau - T(g)).$$

On the other hand, we have

$$\begin{aligned}(\overline{E},\omega)\times\widehat{\varphi}(\Omega)_*(g,\tau)&\\ &=(\overline{E}\otimes\varphi_* g,\mathrm{ch}_0(\overline{E})\bullet(\varphi_!\tau-T(g))+\omega\bullet\mathrm{ch}_m(\varphi_* g)+\omega\bullet(\varphi_!\tau-T(g))\\ &=(\overline{E}\otimes\varphi_* g,\mathrm{ch}_0(\overline{E})\bullet(\varphi_!\tau-T(g))+\omega\bullet d_{\mathcal{D}}\varphi_!\tau+\omega\bullet\varphi_!\,\mathrm{ch}_m(g)).\end{aligned}$$

Hence we have

$$\widehat{\varphi}(\Omega)_*(\widehat{\varphi}^*([(\overline{E},\omega)])\times[(g,\tau)])=[(\overline{E},\omega)]\times\widehat{\varphi}(\Omega)_*([(g,\tau)]).$$

In the case of $n>0$ and $m=0$, we can prove the projection formula for the pairing $\widehat{K}_n\times\widehat{\mathcal{K}}_0\to\widehat{K}_n$ in the same way. Hence we have the following theorem:

Theorem 7.8. *Let $\varphi : X\to Y$ be a projective smooth morphism of proper arithmetic varieties. Let h_φ be an F_∞-invariant smooth hermitian metric on $T\varphi(\mathbb{C})$ such that $(\varphi(\mathbb{C}),h_\varphi)$ is a Kähler fibration. Then for $y\in\widehat{K}_n(Y)$ and $x\in\widehat{K}_m(X)$,*

$$\widehat{\varphi}(h_\varphi)_*(\widehat{\varphi}^*y\times x)=y\times\widehat{\varphi}(h_\varphi)_*(x).$$

Appendix A. Some Identities Satisfied by Binomial Coefficients

Lemma A.1. (1) *For $0\le k\le i$, we have*

$$\begin{aligned}(n-i)\sum_{\alpha=0}^{k-1}\binom{n+m-i-j+1}{n-\alpha}\binom{i+j-1}{\alpha}+i\sum_{\alpha=0}^{k}\binom{n+m-i-j}{n-\alpha}\binom{i+j}{\alpha}&\\ =(n+m)\sum_{\alpha=0}^{k-1}\binom{n+m-i-j}{n-1-\alpha}\binom{i+j-1}{\alpha}+(i-k)\binom{n+m-i-j}{n-k}\binom{i+j-1}{k}.&\end{aligned}$$

In particular, we have

$$\begin{aligned}(n-i)\sum_{\alpha=0}^{i-1}\binom{n+m-i-j+1}{n-\alpha}\binom{i+j-1}{\alpha}+i\sum_{\alpha=0}^{i}\binom{n+m-i-j}{n-\alpha}\binom{i+j}{\alpha}&\\ =(n+m)\sum_{\alpha=0}^{i-1}\binom{n+m-i-j}{n-1-\alpha}\binom{i+j-1}{\alpha}.&\end{aligned}$$

(2) *For $0\le k\le i$, we have*

$$\begin{aligned}(m-j)\sum_{\alpha=k}^{i-1}\binom{n+m-i-j+1}{n-\alpha}\binom{i+j-1}{\alpha}+j\sum_{\alpha=k}^{i-1}\binom{n+m-i-j}{n-\alpha}\binom{i+j}{\alpha}&\\ =(n+m)\sum_{\alpha=k}^{i-1}\binom{n+m-i-j}{n-\alpha}\binom{i+j-1}{\alpha}-(i-k)\binom{n+m-i-j}{n-k}\binom{i+j-1}{k-1}.&\end{aligned}$$

In particular, we have

$$(m-j)\sum_{\alpha=0}^{i-1}\binom{n+m-i-j+1}{n-\alpha}\binom{i+j-1}{\alpha}+j\sum_{\alpha=0}^{i-1}\binom{n+m-i-j}{n-\alpha}\binom{i+j}{\alpha}$$
$$=(n+m)\sum_{\alpha=0}^{i-1}\binom{n+m-i-j}{n-\alpha}\binom{i+j-1}{\alpha}.$$

Proof. We will prove them by induction on k. When $k=0$, the claim (1) is trivial. If the claim (1) holds for $k-1$, then

$$(n-i)\sum_{\alpha=0}^{k-1}\binom{n+m-i-j+1}{n-\alpha}\binom{i+j-1}{\alpha}+i\sum_{\alpha=0}^{k}\binom{n+m-i-j}{n-\alpha}\binom{i+j}{\alpha}$$
$$=(n+m)\sum_{\alpha=0}^{k-2}\binom{n+m-i-j}{n-1-\alpha}\binom{i+j-1}{\alpha}+(i-k+1)\binom{n+m-i-j}{n-k+1}\binom{i+j-1}{k-1}$$
$$+(n-i)\binom{n+m-i-j+1}{n-k+1}\binom{i+j-1}{k-1}+i\binom{n+m-i-j}{n-k}\binom{i+j}{k}$$
$$=(n+m)\sum_{\alpha=0}^{k-2}\binom{n+m-i-j}{n-1-\alpha}\binom{i+j-1}{\alpha}+(n-k+1)\binom{n+m-i-j}{n-k+1}\binom{i+j-1}{k-1}$$
$$+n\binom{n+m-i-j}{n-k}\binom{i+j-1}{k-1}+i\binom{n+m-i-j}{n-k}\binom{i+j-1}{k}$$
$$=(n+m)\sum_{\alpha=0}^{k-1}\binom{n+m-i-j}{n-1-\alpha}\binom{i+j-1}{\alpha}-(i+j-k)\binom{n+m-i-j}{n-k}\binom{i+j-1}{k-1}$$
$$+i\binom{n+m-i-j}{n-k}\binom{i+j-1}{k}$$
$$=(n+m)\sum_{\alpha=0}^{k-1}\binom{n+m-i-j}{n-1-\alpha}\binom{i+j-1}{\alpha}+(i-k)\binom{n+m-i-j}{n-k}\binom{i+j-1}{k}.$$

Hence the claim (1) holds for k.

The claim (2) for $k=i$ is trivial. If (2) holds for $k+1$, then

$$(m-j)\sum_{\alpha=k}^{i-1}\binom{n+m-i-j+1}{n-\alpha}\binom{i+j-1}{\alpha}+j\sum_{\alpha=k}^{i-1}\binom{n+m-i-j}{n-\alpha}\binom{i+j}{\alpha}$$
$$=(n+m)\sum_{\alpha=k+1}^{i-1}\binom{n+m-i-j}{n-\alpha}\binom{i+j-1}{\alpha}-(i-k-1)\binom{n+m-i-j}{n-k-1}\binom{i+j-1}{k}$$
$$+(m-j)\binom{n+m-i-j+1}{n-k}\binom{i+j-1}{k}+j\binom{n+m-i-j}{n-k}\binom{i+j}{k}$$
$$=(n+m)\sum_{\alpha=k+1}^{i-1}\binom{n+m-i-j}{n-\alpha}\binom{i+j-1}{\alpha}$$

$$
\begin{aligned}
&+ (m-j-i+k+1)\tbinom{n+m-i-j}{n-k-1}\tbinom{i+j-1}{k} + m\tbinom{n+m-i-j}{n-k}\tbinom{i+j-1}{k} \\
&+ j\tbinom{n+m-i-j}{n-k}\tbinom{i+j-1}{k-1} \\
&= (n+m)\sum_{\alpha=k+1}^{i-1}\tbinom{n+m-i-j}{n-\alpha}\tbinom{i+j-1}{\alpha} + (n+m-k)\tbinom{n+m-i-j}{n-k}\tbinom{i+j-1}{k} \\
&+ j\tbinom{n+m-i-j}{n-k}\tbinom{i+j-1}{k-1} \\
&= (n+m)\sum_{\alpha=k}^{i-1}\tbinom{n+m-i-j}{n-\alpha}\tbinom{i+j-1}{\alpha} - (i-k)\tbinom{n+m-i-j}{n-k}\tbinom{i+j-1}{k-1},
\end{aligned}
$$

hence the claim (2) holds for k. □

Lemma A.2. *We have*

$$
\begin{aligned}
&\tbinom{n+m+l}{n}^{-1}\sum_{i=0}^{n}\tbinom{n+m-i-j}{n-i}\tbinom{i+j-1}{i}\sum_{\alpha=0}^{i+j-1}\tbinom{n+m+l-i-j-k+1}{n+m-\alpha}\tbinom{i+j+k-1}{\alpha} \\
&\qquad = \sum_{\alpha=0}^{j-1}\tbinom{m+l-j-k+1}{m-\alpha}\tbinom{j+k-1}{\alpha}.
\end{aligned}
$$

Proof. Let F_n denote the left hand side of the above. Then we have

$$
\begin{aligned}
F_n &= \tbinom{n+m+l}{n}^{-1} \\
&\times\sum_{i=1}^{n}\Bigg(\left(1-\frac{i-1}{n}\right)\tbinom{n+m-i-j+1}{n-i+1}\tbinom{i+j-2}{i-1}\sum_{\alpha=0}^{i+j-2}\tbinom{n+m+l-i-j-k+2}{n+m-\alpha}\tbinom{i+j+k-2}{\alpha} \\
&+ \frac{i}{n}\tbinom{n+m-i-j}{n-i}\tbinom{i+j-1}{i}\sum_{\alpha=0}^{i+j-1}\tbinom{n+m+l-i-j-k+1}{n+m-\alpha}\tbinom{i+j+k-1}{\alpha}\Bigg) \\
&= \frac{1}{n}\tbinom{n+m+l}{n}^{-1}\sum_{i=1}^{n}\tbinom{n+m-i-j}{n-i}\tbinom{i+j-2}{i-1} \\
&\times\Bigg((n+m-i-j+1)\sum_{\alpha=0}^{i+j-2}\tbinom{n+m+l-i-j-k+2}{n+m-\alpha}\tbinom{i+j+k-2}{\alpha} \\
&+(i+j-1)\sum_{\alpha=0}^{i+j-1}\tbinom{n+m+l-i-j-k+1}{n+m-\alpha}\tbinom{i+j+k-1}{\alpha}\Bigg).
\end{aligned}
$$

By Lemma A.1, we have

$$
F_n = \frac{n+m+l}{n}\tbinom{n+m+l}{n}^{-1}\sum_{i=1}^{n}\tbinom{n+m-i-j}{n-i}\tbinom{i+j-2}{i-1}
$$

$$\times \sum_{\alpha=0}^{i+j-2} \binom{n+m+l-i-j-k+1}{n+m-1-\alpha}\binom{i+j+k-1}{\alpha}$$

$$= \binom{n+m+l-1}{n-1}^{-1} \sum_{i=0}^{n-1} \binom{n+m-i-j-1}{n-i-1}\binom{i+j-1}{i} \sum_{\alpha=0}^{i+j-1} \binom{n+m+l-i-j-k}{n+m-1-\alpha}\binom{i+j+k}{\alpha}$$

$$= F_{n-1}.$$

Hence $F_n = F_0$, that is,

$$F_n = \sum_{\alpha=0}^{j-1} \binom{m+l-j-k+1}{m-\alpha}\binom{j+k-1}{\alpha}.$$

□

Lemma A.3. *If $0 \le i \le n$ and $1 \le j \le m$, we have*

$$\sum_{\alpha=0}^{i} \binom{n+m-\alpha-j}{n-\alpha}\binom{\alpha+j-1}{\alpha} = \sum_{\alpha=0}^{i} \binom{n+m-i-j}{n-\alpha}\binom{i+j}{\alpha}.$$

In particular, we have

$$\sum_{\alpha=0}^{n} \binom{n+m-\alpha-j}{n-\alpha}\binom{\alpha+j-1}{\alpha} = \sum_{\alpha=0}^{n} \binom{m-j}{n-\alpha}\binom{n+j}{\alpha} = \binom{n+m}{n}.$$

Proof. We prove the lemma by induction on i. When $i = 0$, the statement of the lemma is clear. If the identity holds for $i - 1$, then

$$\begin{aligned}\sum_{\alpha=0}^{i} \binom{n+m-\alpha-j}{n-\alpha}\binom{\alpha+j-1}{\alpha} &= \sum_{\alpha=0}^{i-1} \binom{n+m-i-j+1}{n-\alpha}\binom{i+j-1}{\alpha} + \binom{n+m-i-j}{n-i}\binom{i+j-1}{i} \\ &= \sum_{\alpha=0}^{i-1} \binom{n+m-i-j}{n-\alpha}\binom{i+j-1}{\alpha} \\ &\quad + \sum_{\alpha=0}^{i-1} \binom{n+m-i-j}{n-1-\alpha}\binom{i+j-1}{\alpha} + \binom{n+m-i-j}{n-i}\binom{i+j-1}{i} \\ &= \sum_{\alpha=0}^{i} \binom{n+m-i-j}{n-\alpha}\binom{i+j-1}{\alpha} + \sum_{\alpha=1}^{i} \binom{n+m-i-j}{n-\alpha}\binom{i+j-1}{\alpha-1} \\ &= \sum_{\alpha=0}^{i} \binom{n+m-i-j}{n-\alpha}\binom{i+j}{\alpha}.\end{aligned}$$

□

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