

# Decomposition Problem on Endomorphisms of Projective Varieties

By

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## Abstract

Let  $Z := X \times Y$  be a product variety of nonsingular projective varieties  $X$  and  $Y$ . Suppose that  $K_Y$  is not nef but  $K_X$  is nef. The aim of this note is to study decomposition problems on an endomorphism  $f: Z \rightarrow Z$  of  $Z$ .

## §1. Introduction

In this paper, we study some kind of decomposition problems concerning endomorphisms of nonsingular projective varieties. By an endomorphism, we mean a surjective morphism  $f: X \rightarrow X$  from a complex variety  $X$  to itself. We begin with a brief background. We first quote an example from our previous paper [2]. Let  $X$  be a nonminimal smooth projective 3-fold with  $\kappa(X) = 0$  which has a nonisomorphic endomorphism  $f: X \rightarrow X$ . Then a suitable finite étale covering  $u: Z \rightarrow X$  of  $X$  is isomorphic to the direct product  $E \times S$  of an elliptic curve  $E$  and a smooth nonminimal algebraic surface  $S$  in which  $S$  is birationally equivalent to an abelian surface or a K3 surface. Furthermore, there exists an endomorphism  $f': Z \rightarrow Z$  with  $u \circ f' = f \circ u$  such that  $f'$  can be decomposed as  $f' = g \times h$  for an endomorphism  $g: E \rightarrow E$  and an isomorphism  $h: S \cong S$  (cf. [2, MAIN THEOREM]). In view of these results, we are naturally led to the following questions.

**Question.** *Let  $X$  be a nonsingular projective variety with nonnegative Kodaira dimension. Suppose that there exists an endomorphism  $f: X \rightarrow X$  which is not an isomorphism.*

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(1) For a generic point  $p \in X$ , let  $S(p)$  be the Zariski closure of the set  $\{f^n(p) \mid n = 1, 2, \dots\}$ . If  $S(p) \neq X$ , then is a suitable finite étale covering  $S'(p)$  of  $S(p)$  an abelian variety?

(2) Furthermore assume that the canonical bundle  $K_X$  of  $X$  is not nef. Then does each extremal rational curve on  $X$  intersect transversally with  $S(p)$ ?

This is a natural question yet to be investigated. As a first step, we shall focus our attention to the following question.

**Question ( $\mathbf{D}_n$ ).** Let  $X$  and  $Y$  be positive-dimensional projective manifolds with nonnegative Kodaira dimension such that the canonical bundle  $K_Y$  of  $Y$  is not nef but  $K_X$  is nef. Here we put  $n := \dim(Y)$ . For each extremal ray  $R$  of  $Y$ , let  $\varphi := \text{Cont}_R: Y \rightarrow Y'$  be the contraction morphism of  $R$  and  $M := \text{Exc}(\varphi)$  the exceptional set of  $\varphi$ . Assume that

(\*) no irreducible component of  $\varphi(M)$  is covered by a family of para-abelian varieties. (Here a nonsingular projective variety  $V$  is called a para-abelian variety if it admits a finite étale covering  $A \rightarrow V$  from an abelian variety  $A$ .)

Suppose that there exists an endomorphism  $f: Z \rightarrow Z$  of the direct product  $Z := X \times Y$ . Then is it true that a suitable power  $f^k$  ( $k > 0$ ) of  $f$  induces an automorphism  $g: Y \rightarrow Y$  of  $Y$  such that  $q \circ f^k = g \circ q$  for the second projection  $q: Z \rightarrow Y$ ?

If the Question ( $\mathbf{D}_n$ ) has an affirmative answer, there exists a relative automorphism  $u$  of  $Z$  over  $Y$  such that  $u \circ f^k = h \times g$  for some endomorphism  $h: X \rightarrow X$  of  $X$  and an isomorphism  $g: Y \cong Y$  of  $Y$ . Note that if the condition (\*) is not satisfied, Question ( $\mathbf{D}_n$ ) does not necessarily have an affirmative answer (cf. Remark 2). The main purpose of this note is to give a partial answer to this question.

### MAIN THEOREM.

Question ( $\mathbf{D}_n$ ) has an affirmative answer for  $n = 2$  and  $3$ .

**Notation.** In this paper, by a smooth projective  $n$ -fold  $X$ , we mean a nonsingular projective manifold of dimension  $n$  defined over the complex number field  $\mathbf{C}$ .

$b_i(X)$ : the  $i$ -th Betti number of  $X$ .

$K_X$ : the canonical bundle of  $X$ .

$\kappa(X)$ : the Kodaira dimension of  $X$ .

$N_1(X) := (\{1\text{-cycles on } X\} / \equiv) \otimes_{\mathbf{Z}} \mathbf{R}$ , where  $\equiv$  denotes the numerical equivalence.

$NE(X) :=$  the smallest convex cone in  $N_1(X)$  containing all effective 1-cycles.

$\overline{NE}(X) :=$  Kleiman-Mori cone of  $X$ , i.e. the closure of  $NE(X)$  in  $N_1(X)$  for the metric topology.

$\rho(X) := \dim_{\mathbb{R}} N_1(X)$ , the Picard number of  $X$ .

$[C]$ : the numerical equivalence class of a 1-cycle  $C$ .

By an extremal ray  $R$  of  $X$ , we mean a  $K_X$ -negative extremal ray of  $\overline{NE}(X)$ . An irreducible curve  $C$  on  $X$  is called an extremal curve if  $[C]$  spans some extremal ray  $R$  of  $\overline{NE}(X)$ .

$g(C)$ : the genus of a smooth curve  $C$ .

Let  $Y$  be a compact complex variety (i.e. a reduced and irreducible complex space). Then:

$\text{Aut}(Y)$  : the complex Lie group of biholomorphic automorphisms of  $Y$ .

$\text{Aut}^0(Y)$  : the identity component of  $\text{Aut}(Y)$ .

$\text{Sur}(Y)$  : the set of surjective holomorphic maps from  $Y$  to itself, which carries a complex space structure (cf. [3].)

For  $f \in \text{Sur}(Y)$ ,  $f^k := f \circ \dots \circ f$  stands for the  $k$ -times composite of  $f$ .

For compact complex spaces  $M$  and  $W$ ,

$\text{Mer}_{\text{dom}}(M, W)$ : the set of dominant meromorphic maps from  $M$  to  $W$ .

### §2. Preliminaries

We begin with an easy lemma.

**Lemma 1.** *Let  $f: V \rightarrow W$  be a surjective morphism between normal projective varieties. Then  $f_*\overline{NE}(V) = \overline{NE}(W)$ .*

*Proof.* Since  $f_*NE(V) \subset NE(W)$ , we have  $f_*\overline{NE}(V) \subset \overline{NE}(W)$  by the continuity of  $f_*$ . If a line bundle  $L$  on  $W$  is positive on  $f_*\overline{NE}(V) \setminus \{0\}$ ,  $f^*L$  is semipositive on  $\overline{NE}(V)$ , that is,  $f^*L$  is nef. Since  $f$  is surjective,  $L$  is also nef. Hence  $f_*\overline{NE}(V) = \overline{NE}(W)$ . □

We prove some facts that provide the key step toward the proof of MAIN THEOREM.

**Theorem 2.** *Let  $Z := X \times Y$  be the direct product of positive-dimensional projective manifolds  $X$  and  $Y$ . Suppose that  $K_Y$  is not nef but  $K_X$  is nef. Let  $q_*: \overline{NE}(Z) \rightarrow \overline{NE}(Y)$  be the surjective map induced by the second projection  $q: Z \rightarrow Y$ . Then we have the following:*

(1) *The push-forward mapping  $q_*$  gives a one-to-one correspondence between the set of extremal rays of  $Z$  and the set of extremal rays of  $Y$ .*

(2) *For each extremal ray  $R$  of  $Z$ , the contraction morphism  $\varphi := \text{Cont}_R: Z \rightarrow Z'$  associated to  $R$  can be decomposed as  $\varphi = \text{id}_X \times \Psi: Z := X \times Y \rightarrow Z' \cong$*

$X \times Y'$ , where  $\Psi := \text{Cont}_{R'}: Y \rightarrow Y'$  is the contraction morphism associated to the extremal ray  $R' := q_*R$  of  $Y$ .

**Lemma 3.** *No extremal curve  $e$  on  $Z$  is contained in a fiber of  $q: Z \rightarrow Y$ .*

*Proof.* Assume the contrary. Let  $p: Z \rightarrow X$  be the first projection. Since  $K_Z \sim p^*K_X + q^*K_Y$ , we have  $(K_Z, e) = (p^*K_X, e) = (K_X, p_*e)$ .

By hypothesis, we have  $(K_Z, e) < 0$  and  $(K_X, p_*e) \geq 0$ . This is a contradiction. □

**Lemma 4.** *Under the same assumption as in Theorem 2, let  $\varphi: Z \rightarrow Z'$  be the contraction morphism associated to the extremal ray  $R$  of  $Z$ . Then there exists a surjective morphism  $g: Z' \rightarrow X$  such that  $p = g \circ \varphi$  for the first projection  $p: Z \rightarrow X$ .*

*Proof.* Since  $\kappa(Z) \geq 0$ ,  $\varphi: Z \rightarrow Z'$  is a birational morphism. Since  $Z'$  is normal, it is sufficient to show that  $p \circ \varphi^{-1}(z')$  is a point for all  $z' \in Z'$ . For  $z' \in Z' \setminus \varphi(\text{Exc}(\varphi))$ , this is clear. For  $z' \in \varphi(\text{Exc}(\varphi))$ , take an extremal rational curve  $e$  on  $\varphi^{-1}(z')$  which spans  $R$  (cf. [9]). By Lemma 3,  $C := q(e)$  is a rational curve on  $Y$ .

*Claim.*  $p(e)$  is a point on  $X$ .

Assume the contrary. Then  $D := p(e)$  is a rational curve on  $X$  and  $e$  is contained in the surface  $S := D \times C \subset X \times Y =: Z$ . Let  $C'$  (resp.  $D'$ ) be the normalization of  $C$  (resp.  $D$ ) and  $e'$  the strict transform of  $e$  by the birational morphism  $\pi: S' := D' \times C' \rightarrow S := D \times C$ . Since  $e'$  moves and sweeps out  $S'$  ( $\cong \mathbf{P}^1 \times \mathbf{P}^1$ ) and  $\varphi \circ \pi(e')$  is a point on  $Z'$ , we have  $\dim \varphi \circ \pi(S') \leq 1$ . If  $\dim \varphi(S) = 1$ , then  $S'$  has three different fiber space structures. This is a contradiction, since  $\rho(S') = 2$ . Hence  $\varphi(S)$  is a point and each fiber  $F$  of  $q|_S: S \rightarrow C$  spans the same extremal ray  $R$ . However, by Lemma 3,  $F$  is not contained in a fiber of  $q: Z \rightarrow Y$ . This is again a contradiction.

Hence  $p(C)$  is a point for any extremal curve  $C$  which spans  $R$ . For arbitrary 2 points  $x, y \in \varphi^{-1}(z')$ , take a chain of irreducible curves  $C_i$ 's on  $\varphi^{-1}(z')$  which connect  $x$  and  $y$ . By the same argument,  $p(C_i)$  is a point. Hence  $p(x) = p(y)$  and  $p \circ \varphi^{-1}(z')$  is a point on  $X$ . □

Using Lemma 4, we now prove Theorem 2.

*Proof of Theorem 2.*

*Step 1.* First we show that for each extremal ray  $R$  of  $Z$ ,  $R' := q_*R$  is also an extremal ray of  $Y$ . Let  $C$  be an extremal curve on  $Z$  which spans  $R$ .

Then, by Lemma 4,  $p(C) = \{o\}$  is a point on  $X$ . Let  $s: Y \rightarrow Z$  be a constant section of  $q: Z \rightarrow Y$  defined by  $s(y) = (o, y)$ ,  $y \in Y$ . Then the ray  $R' := q_*R$  is spanned by an irreducible curve  $C' := q(C)$  and there exists an isomorphism  $q|_C: C \cong C'$  with  $(q|_C)^{-1} = s|_{C'}$ .

We show that  $R'$  is extremal. Assume that  $[C'] = u + v$  for some  $u, v \in \overline{NE}(Y)$ . Since  $[C] = s_*[C'] = s_*u + s_*v$  spans the extremal ray  $R$  of  $Z$ , we have  $s_*u, s_*v \in R$ . Thus we get  $u, v \in q_*R =: R'$ , since  $q_* \circ s_* = \text{id}$ . Moreover,  $(K_Y, C') = (q^*K_Y, C) = (K_Z, C) < 0$ , whence  $R'$  is an extremal ray of  $Y$ .

*Step 2.* Let  $t: Y \rightarrow Z$  be a constant section of  $q: Z \rightarrow Y$  defined by  $t(y) = (o, y)$  for some fixed point  $o \in X$ . Then, for each extremal ray  $R'$  of  $Y$ ,  $t_*R'$  is also an extremal ray of  $Z$ .

*Proof.* Let  $C'$  be an extremal curve on  $Y$  which spans  $R'$  and put  $C := t(C')$ . Then  $(K_Z, C) = (q^*K_Y, C) = (K_Y, C') < 0$ . Hence, by the cone theorem (cf. [9]), we have  $[C] \equiv a[e] + D$  in  $\overline{NE}(Z)$ , where  $a > 0, D \in \overline{NE}(Z)$  and  $e$  is an extremal curve which spans the extremal ray  $R$  of  $Z$ . By step 1,  $e$  is contained in a fiber of  $p: Z \rightarrow X$ . Clearly,  $e$  is numerically equivalent to  $t \circ q(e)$  on  $Z$ . Hence we may assume that  $e \subset p^{-1}(o)$ . Then  $[C'] = [q_*C] \equiv a[e'] + q_*D$ , where  $e' := q(e)$ . Since  $[C']$  spans the extremal ray  $R'$ ,  $e'$  also spans  $R'$ . Since  $[e] = [t_*e']$  spans the ray  $t_*R'$ , we have  $t_*R' = R$ , which is an extremal ray of  $Z$ .

(1) is derived from Step 1 and Step 2.

*Step 3.* Let  $\varphi: Z \rightarrow Z'$  be the contraction morphism associated to an extremal ray  $R$  of  $Z$ . By Step 1,  $R' := q_*R$  is an extremal ray of  $Y$ . Let  $\Psi: Y \rightarrow Y'$  be the contraction morphism associated to  $R'$ . Since  $Z'$  is normal and  $\Psi \circ q \circ \varphi^{-1}(z)$  is a point for all  $z \in Z'$ , there exists a surjective morphism  $h: Z' \rightarrow Y'$  such that  $\Psi \circ q = h \circ \varphi$ . By Lemma 4, there exists a surjective morphism  $g: Z' \rightarrow X$  such that  $p = g \circ \varphi$ . If we put  $u := g \times h: Z' \rightarrow X \times Y'$  and  $v := \text{id}_X \times \Psi: Z = X \times Y \rightarrow X \times Y'$ , then we have  $u \circ \varphi = v$ . Since the relative Picard number  $\rho(Z/X \times Y') = 1$ ,  $u$  is a finite morphism. Note that  $u$  is also a birational morphism and  $X \times Y'$  is normal. Hence  $u$  is an isomorphism by Zariski's main theorem.

**Proposition 5.** *Let  $g: X \cdots \rightarrow W$  be a dominant meromorphic map between compact complex spaces  $X$  and  $W$ . Suppose that  $\text{Mer}_{\text{dom}}(M, W)$  is a finite set for every compact complex space  $M$ . Then, for every dominant meromorphic map  $f: X \cdots \rightarrow X$  from  $X$  onto itself, there exists a suitable power  $f^m$  ( $m > 0$ ) of  $f$  such that  $g \circ f^m = g$ .*

*Proof.* The following simple proof is due to N. Nakayama.

The set  $\{g \circ f^m \mid m = 1, 2, \dots\}$  is contained in a finite set  $\text{Mer}_{\text{dom}}(X, W)$ . Hence there exist  $m, r \in \mathbf{N}$  such that  $(g \circ f^m) \circ f^r = g \circ f^{m+r} = g \circ f^r$ . Since  $f^r : X \cdots \rightarrow X$  is dominant, we have thus shown  $g \circ f^m = g$  as claimed.  $\square$

**Corollary 5.1.** *Let  $W$  be a compact complex variety of general type or Kobayashi hyperbolic type (cf. [4]). Let  $Y$  be a compact complex variety and  $f : Z := Y \times W \cdots \rightarrow Z := Y \times W$  a dominant meromorphic map from  $Z$  onto itself. Then there exists a suitable power  $f^m$  ( $m > 0$ ) of  $f$  such that  $q \circ f^m = q$  for the second projection  $q : Z \rightarrow W$ .*

*Proof.* By [5] and [10],  $\text{Mer}_{\text{dom}}(M, W)$  is a finite set for every compact complex space  $M$ .  $\square$

*Remark 1.*

(1) If we drop the assumption that  $W$  is of general type or Kobayashi hyperbolic, Corollary 5.1 does not necessarily hold. We shall give such an example. Let  $E$  be an elliptic curve and put  $Y = W = E$ ,  $Z := Y \times W$ . Let  $f : Z \rightarrow Z$  be a surjective endomorphism of  $Z$  defined by  $f(y, w) = (y, y + w)$ , for  $y \in Y$ ,  $w \in W$ . Then it is easy to see that there exists no automorphism  $g_n : W \rightarrow W$  such that  $g_n \circ q = q \circ f^n$  for the second projection  $q : Z \rightarrow W$ .

(2) Corollary 5.1 does not necessarily hold unless we take a suitable power  $f^m$  ( $m > 0$ ) of  $f : Z \cdots \rightarrow Z$ . We shall give such an example.

Let  $C$  be a smooth curve of genus  $g(C) \geq 2$  and put  $Y = W = C$ ,  $Z := Y \times W$ . Let  $f : Z \rightarrow Z$  be an automorphism of  $Z$  defined by  $f(y, w) = (w, y)$  for  $y \in Y$ ,  $w \in W$ . Then it is easy to see that there exists no automorphism  $g : W \rightarrow W$  such that  $g \circ q = q \circ f$  for the second projection  $q : Z \rightarrow W$ .

(3) In Proposition 5, any dominant meromorphic (resp. holomorphic) map  $\alpha : W \cdots \rightarrow W$  (resp.  $\beta : W \rightarrow W$ ) from  $W$  onto itself is a bimeromorphic map (resp. an isomorphism).

*Proof.* Both  $\{\alpha^m \mid m = 1, 2, \dots\}$  and  $\{\beta^m \mid m = 1, 2, \dots\}$  are contained in a finite set  $\text{Mer}_{\text{dom}}(W, W)$ . Therefore, there exist positive integers  $m, r, n, t \in \mathbf{N}$  such that  $(\alpha^m) \circ \alpha^r = \alpha^{m+r} = \alpha^r$  and  $(\beta^n) \circ \beta^t = \beta^{n+t} = \beta^t$ . Since the mappings  $\alpha^r : W \cdots \rightarrow W$  and  $\beta^t : W \rightarrow W$  are dominant, we have  $\alpha^m = \text{id}_W$  and  $\beta^n = \text{id}_W$ . Hence  $\deg(\alpha) = \deg(\beta) = 1$  and the claim is derived.  $\square$

**Proposition 6.** *Let  $X$  be a nonuniruled smooth projective variety and  $Y$  a rationally connected projective variety (cf. [6]). Suppose that there exists*

an endomorphism  $f: Z \rightarrow Z$  of the product variety  $Z := X \times Y$ . Then there exist a finite étale covering  $g: X \rightarrow X$  and an endomorphism  $h: Y \rightarrow Y$  such that  $f = g \times h$ . Furthermore, if  $Y$  is nonsingular and  $f: Z \rightarrow Z$  is a finite étale covering, then  $h$  is an isomorphism.

*Proof.* Let  $p: Z \rightarrow X$  be the first projection. Assume that, for some point  $x_0 \in X$ ,  $p \circ f \circ p^{-1}(x_0)$  is not a point on  $X$ . Then, by the rigidity lemma (cf. [7]),  $p \circ f \circ p^{-1}(x)$  is a positive-dimensional variety for all  $x \in X$ . Since  $f: Z \rightarrow Z$  is surjective,  $X$  is covered by a family of rationally connected varieties  $\{p \circ f \circ p^{-1}(x)\}_{x \in X}$ , hence is covered by a family of rational curves. This contradicts the assumption that  $X$  is nonuniruled. Hence  $p \circ f \circ p^{-1}(x)$  is a point for all  $x \in X$  by the rigidity lemma and there exists an endomorphism  $g: X \rightarrow X$  such that  $p \circ f = g \circ p$ . For each  $x \in X$ , if we denote by  $f_x$  the restriction of  $f$  to  $p^{-1}(x) \cong Y$ , we obtain a morphism  $u: X \rightarrow \text{Sur}(Y)$  by  $u(x) := f_x$ . Since  $u(X)$  is a compact subvariety of  $\text{Sur}(Y)$ , it follows from [3, Theorem 3.1] that  $u(X)$  is contained in a left  $\text{Aut}^0(Y)$ -orbit of  $g := f_o$  for a fixed point  $o \in Y$ . Since  $X$  is compact and  $\text{Aut}^0(Y)$  is affine by [1, Lemma 2.5 and Corollary 5.8],  $u$  is a constant map and  $f_x$  is independent of  $x \in X$ . If we put  $f_x =: h \in \text{Sur}(Y)$ , then  $f = g \times h$ . By the next Lemma 7,  $g: X \rightarrow X$  is a finite étale covering. Moreover, if  $Y$  is nonsingular,  $Y$  is simply connected by [6]. Thus the last claim is derived. □

**Lemma 7.** *Let  $f: V \rightarrow V$  be an endomorphism of a smooth nonuniruled projective  $n$ -fold  $V$ . Then  $f$  is a finite étale covering.*

*Proof.* By [2, Lemma 2.3, (1)],  $f: V \rightarrow V$  is a finite morphism. Then  $K_V \sim f^*K_V + R$ , where  $R$  is the ramification divisor of  $f$ . Hence  $K_V \sim (f^k)^*K_V + (f^{k-1})^*R + \dots + f^*R + R$  for all  $k > 0$ . By [8], we have  $(K_V, H^{n-1}) \geq 0$  and  $((f^k)^*K_V, H^{n-1}) \geq 0$  for a sufficiently ample divisor  $H$ , since  $V$  is not uniruled. Assume that  $R \neq 0$ . If we let  $k \rightarrow \infty$ , then  $(K_V, H^{n-1}) = \infty$ , which is a contradiction. Hence  $R = 0$  and the claim is derived. □

### §3. Proof of MAIN THEOREM

We recall from [2] some basic facts; extremal rays of nonsingular projective varieties are preserved by an étale endomorphism.

**Proposition 8** (cf. [2, Proposition 4.2]). *Let  $f: Y \rightarrow X$  be a finite surjective morphism between smooth projective  $n$ -folds with  $\rho(X) = \rho(Y)$ . Then we have the following:*

(1) The pull-back mapping  $f^*: N_1(X) \rightarrow N_1(Y)$  (resp. the push-forward mapping  $f_*: N_1(Y) \rightarrow N_1(X)$ ) is an isomorphism and  $f^*\overline{NE}(X) = \overline{NE}(Y)$  (resp.  $f_*\overline{NE}(Y) = \overline{NE}(X)$ ).

(2) Moreover, if  $f$  is a finite étale covering and the canonical bundle  $K_X$  of  $X$  is not nef, there is a one-to-one correspondence between the set of extremal rays of  $X$  and the set of extremal rays of  $Y$ .

**Proposition 9** (cf. [2, Proposition 4.12]). *Under the same assumption as in Proposition 8, (2), for each extremal ray  $R$  of  $X$ , let  $\varphi := \text{Cont}_R: X \rightarrow X'$  (resp.  $\psi := \text{Cont}_{R'}: Y \rightarrow Y'$ ) be the contraction morphism associated to  $R$  (resp.  $R' := f^*R$ ). Then there exists a unique finite surjective morphism  $f': Y' \rightarrow X'$  such that*

$$(1) \varphi \circ f = f' \circ \psi,$$

(2)  $f'^{-1}(\text{Exc}(\varphi)) = \text{Exc}(\psi)$  and  $f'^{-1}(\varphi(\text{Exc}(\varphi))) = \psi(\text{Exc}(\psi))$  set-theoretically.

Moreover,  $\varphi$  is a birational contraction if and only if  $\psi$  is a birational contraction and  $\varphi$  is a divisorial contraction if and only if  $\psi$  is a divisorial contraction.

*Proof.* We first show that  $f: Y \rightarrow X$  induces a surjective morphism  $f': Y' \rightarrow X'$ . Since  $X'$  is normal, it suffices to show that  $\varphi \circ f \circ \psi^{-1}(y')$  is a point on  $X'$  for all  $y' \in Y'$ . For  $y' \in Y' \setminus \psi(\text{Exc}(\psi))$ , this is clear. For  $y' \in \psi(\text{Exc}(\psi))$ , we have  $\dim \psi^{-1}(y') \geq 1$ . Fix a point  $p \in \psi^{-1}(y')$ . For each  $x \in \psi^{-1}(y')$ , take a chain of irreducible curves  $C_i$ 's on  $\psi^{-1}(y')$  which connect  $x$  and  $p$ . Since  $[C_i]$  spans the extremal ray  $R' := f^*R$  of  $Y$ ,  $[f(C_i)]$  spans the extremal ray  $R = f_*R'$  of  $X$  by Proposition 8. Hence  $\varphi \circ f(C_i)$  is a point and  $\varphi \circ f(x) = \varphi \circ f(p)$  is a fixed point on  $X'$  for all  $x \in \psi^{-1}(y')$ . Thus there exists a surjective morphism  $f': Y' \rightarrow X'$  such that  $\varphi \circ f = f' \circ \psi$ .

Next we show that  $f'$  is a finite morphism. Assume the contrary. Then  $\dim f'^{-1}(x') > 0$  for some point  $x' \in X'$ . For an arbitrary irreducible curve  $\Delta$  in  $f'^{-1}(x')$ , take an irreducible curve  $C'$  on  $Y$  such that  $\psi(C') = \Delta$ . The irreducible curve  $C := f(C')$  on  $X$  is contracted to a point  $x'$  by  $\varphi$  and  $[C]$  spans the extremal ray  $R$  of  $X$ . Hence by Proposition 8,  $[C']$  spans the extremal ray  $R' := f^*R$  of  $Y$  and  $C'$  is contracted to a point by  $\psi$ . This is a contradiction. All the other assertions are clear from the construction.  $\square$

Now we are ready to prove MAIN THEOREM. We shall derive some

sufficient conditions for the *Question* ( $\mathbf{D}_n$ ) to have an affirmative answer.

**Theorem 10.** *Let  $X$  and  $Y$  be positive-dimensional projective manifolds with nonnegative Kodaira dimension. Assume that*

- (1) *the canonical bundle  $K_Y$  of  $Y$  is not nef but  $K_X$  is nef,*
- (2) *there exists some extremal ray  $R'$  of  $Y$  such that the contraction morphism  $\psi = \text{cont}_{R'} : Y \rightarrow Y'$  associated to  $R'$  contracts an irreducible divisor  $E$  on  $Y$  to a point.*

*Let  $f : Z \rightarrow Z$  be an endomorphism of the direct product  $Z := X \times Y$ . Then there exists an automorphism  $g$  of  $Y$  such that  $q \circ f = g \circ q$  for the second projection  $q : Z \rightarrow Y$ .*

*Proof.* Since  $\kappa(Z) \geq 0$ ,  $f : Z \rightarrow Z$  is a finite étale covering by Lemma 7. By Theorem 2, there exists a unique extremal ray  $R$  of  $Z$  such that  $q_*R = R'$ . For each  $n$ ,  $R_n := (f^n)_*R$  (here  $R_0 := R$ ) is an extremal ray of  $Z$  by Proposition 8. Hence, by Proposition 9, there exist a contraction morphism  $\varphi_n := \text{Cont}_{R_n} : Z \rightarrow Z_n$  and a unique finite morphism  $g_n : Z_n \rightarrow Z_{n+1}$  such that  $\varphi_{n+1} \circ f = g_n \circ \varphi_n$ . Then, by Theorem 2, there exists a decomposition  $\varphi_n = \text{id}_X \times \psi_n : Z := X \times Y \rightarrow Z_n \cong X \times Y_n$ , where  $\psi_n : Y \rightarrow Y_n$  is the extremal contraction associated to the extremal ray  $R'_n := q_*R_n$  of  $Y$ . Furthermore,  $\psi_n : Y \rightarrow Y_n$  contracts an irreducible divisor  $E_n$  on  $Y$  to a point  $p_n \in Y_n$  and  $(g_n)^{-1}(X \times \{p_{n+1}\}) = X \times \{p_n\}$  by virtue of Theorem 2 combined with the use of Proposition 9. Hence, by the rigidity lemma, there exists a unique morphism  $h_n : Y_n \rightarrow Y_{n+1}$  such that  $q_{n+1} \circ g_n = h_n \circ q_n$  and  $h_n^{-1}(p_{n+1}) = \{p_n\}$  for the second projection  $q_n : Z_n = X_n \times Y_n \rightarrow Y_n$ . Note that  $\psi_n : Y \rightarrow Y_n$  is a birational morphism and  $\psi_n \circ q = q_n \circ \varphi_n : X \times Y \rightarrow Y_n$ . Thus  $q \circ f \circ q^{-1}(p)$  is a point for general  $p \in Y$ . Again, by the rigidity lemma, there exists a unique endomorphism  $\alpha_n : Y \rightarrow Y$  such that  $q \circ f = \alpha_n \circ q$  and  $\psi_{n+1} \circ \alpha_n = h_n \circ \psi_n$ . Then, combining Theorem 2 and Proposition 9, we see that  $\text{Exc}(\varphi_n) \cong X \times E_n$ ,  $f^{-1}(X \times E_{n+1}) = X \times E_n$ , and thus  $\alpha_n^{-1}(E_{n+1}) = E_n$  set-theoretically. Since  $-E_n$  is  $\psi_n$ -ample,  $(-E_n)^p = -(-E_n|_{E_n})^{p-1} \neq 0$  for all  $n$ , where we put  $p := \dim(Y)$  and  $E_0 := E$ . Moreover, since  $\alpha_n : Y \rightarrow Y$  is a finite étale covering, we have  $\alpha_n^*E_{n+1} \sim E_n$  and

$$(-E)^p = (-E_{n+1})^p \times \prod_{i=0}^n \text{deg}(\alpha_i)$$

for all  $n$ . Therefore,  $\alpha_n : Y \rightarrow Y$  is an isomorphism for a sufficiently large positive integer  $n$ . If we put  $g := \alpha_n : Y \cong Y$ , we have thus shown  $q \circ f = g \circ q$  as claimed.  $\square$

**Theorem 11.** *Let  $X$  and  $Y$  be positive-dimensional projective manifolds with nonnegative Kodaira dimension. Assume that*

- (1) *the canonical bundle  $K_Y$  of  $Y$  is not nef but  $K_X$  is nef,*
- (2) *there exist at most finitely many extremal rays of  $Y$ ,*
- (3) *there exists a contraction morphism  $\psi := \text{Cont}_{R'}: Y \rightarrow Y'$  associated to some extremal ray  $R'$  of  $Y$  such that some irreducible component  $M$  of  $\psi(\text{Exc}(\psi))$  satisfies either of the following conditions :*
  - (a)  *$M$  is a rationally connected variety (cf. [6]), or*
  - (b)  *$\text{Mer}_{\text{dom}}(V, M)$  is a finite set for all compact complex variety  $V$ .*

*Suppose that there exists an endomorphism  $f: Z \rightarrow Z$  of the direct product  $Z := X \times Y$ . Then a suitable power  $f^k$  ( $k > 0$ ) of  $f$  induces an endomorphism  $g: Y \rightarrow Y$  of  $Y$  such that  $q \circ f^k = g \circ q$  for the second projection  $q: Z \rightarrow Y$ . Furthermore, if  $Y'$  is nonsingular in both cases and  $M$  is also nonsingular in the case (3a), then  $g: Y \rightarrow Y$  can be taken to be an isomorphism.*

*Proof.* Since  $\kappa(Z) \geq 0$ ,  $f: Z \rightarrow Z$  is a finite étale covering by Lemma 7. Then, by virtue of Proposition 8 combined with the use of Theorem 2, the push-forward map  $f_*: \overline{NE}(Z) \rightarrow \overline{NE}(Z)$  induces a permutation of the finite set which consists of all the extremal rays of  $Z$ . Hence, for a suitable power  $F := f^k$  ( $k > 0$ ) of  $f$ , we have  $F_*R = R$  for each extremal ray  $R$  of  $Z$ . Hence, by Proposition 9, there exists a unique finite surjective morphism  $F': Z' \rightarrow Z'$  such that  $F' \circ \varphi = \varphi \circ F$  for the contraction morphism  $\varphi := \text{Cont}_R: Z \rightarrow Z'$  associated to the extremal ray  $R$  of  $Z$ . By Theorem 2,  $R' := q_*R$  is also an extremal ray of  $Y$  such that

- (i)  $\varphi = \text{id}_X \times \psi: Z := X \times Y \rightarrow Z' \cong X \times Y'$  for the contraction morphism  $\psi: Y \rightarrow Y'$  associated to  $R'$ , and
- (ii)  $E := \varphi(\text{Exc}(\varphi)) = X \times \psi(\text{Exc}(\psi))$  and  $F'^{-1}(E) = E$ .

Therefore, the finite morphism  $F': Z' \rightarrow Z'$  induces a permutation of the finite set consisting of all the irreducible components of  $E$ .

Suppose that  $R'$  satisfies the assumptions of the theorem. By replacing  $F': Z' \rightarrow Z'$  (hence  $F: Z \rightarrow Z$ ) with a suitable power of  $F'$  (resp.  $F$ ), we may assume that  $F'^{-1}(X \times M) = X \times M$ .

Then by Propositions 5 and 6, if we replace  $F: Z \rightarrow Z$  with a suitable power  $F^n$  ( $n > 0$ ) of  $F$ ,  $F': Z' \rightarrow Z'$  induces an endomorphism  $w: M \rightarrow M$  such that  $(q'|_{X \times M}) \circ (F'|_{X \times M}) = w \circ (q'|_{X \times M})$  for the second projection

$q': X \times Y' \rightarrow Y'$ . By the rigidity lemma, there exists an endomorphism  $h: Y' \rightarrow Y'$  such that  $h \circ q' = q' \circ F'$  and  $h|_M = w$ . Since  $F'^{-1}(X \times M) = X \times M$ ,

we infer that  $h^{-1}(M) = M$ . Since  $\psi: Y \rightarrow Y'$  is a birational morphism,  $q \circ F \circ q^{-1}(y)$  is a point for all  $y \in Y$  by the rigidity lemma. Thus there exists an étale endomorphism  $g: Y \rightarrow Y$  of  $Y$  such that  $g \circ q = q \circ F$  and  $\psi \circ g = h \circ \psi$ .

Now we prove the last claim. Note that:

- (i) in the case (3.b),  $w: M \rightarrow M$  is an isomorphism by Remark 1, (3), and
- (ii) in the case (3.a), if  $M$  is nonsingular and  $w: M \rightarrow M$  is a finite étale covering,  $w: M \rightarrow M$  is an isomorphism by Proposition 6.

If  $Y'$  is nonsingular,  $h: Y' \rightarrow Y'$  is a finite étale covering and hence  $w: M \rightarrow M$  is an isomorphism in both cases. Hence  $\deg(h) = \deg(w) = 1$  and  $h: Y' \rightarrow Y'$  is an isomorphism. Since  $\psi: Y \rightarrow Y'$  is a birational morphism, the étale covering  $g: Y \rightarrow Y$  is an isomorphism. □

As a corollary of these results, we shall prove MAIN THEOREM.

*Proof of MAIN THEOREM.*

- (1) In the case where  $n = 2$ , each extremal ray of  $Y$  is spanned by a unique  $(-1)$ -curve. Hence the claim immediately follows from Theorem 10.
- (2) Next we treat the case where  $n = 3$ .

Since  $Y$  is a nonsingular projective 3-fold with  $\kappa(Y) \geq 0$ , all the extremal contractions are divisorial contractions and there exist only finitely many extremal rays of  $Y$  (cf. [2, Proposition 4.6]).

In [9], extremal divisorial contractions of nonsingular projective 3-folds are classified into 5 types. In 4 cases (called type (E2)~(E5) in [7]) where a prime divisor is contracted to a point, Theorem 10 yields the assertion. Hence we may assume that each extremal ray  $R$  of  $Y$  is of type (E1) (cf. [7]), that is, the contraction morphism  $\text{Cont}_R: Y \rightarrow Y'$  associated to  $R$ , is a birational contraction, which is (the inverse of ) the blow-up along a smooth curve  $C$  on  $Y'$ . By assumption,  $C$  is not an elliptic curve. Since  $Y'$  and  $C$  are nonsingular, the claim immediately follows from Corollary 5.1 and Theorem 11. □

We conclude with a remark concerning the last theorem.

*Remark 2.*

- (1) If we drop the assumption (2), *Question (D<sub>3</sub>)* does not necessarily have an affirmative answer. We shall give such an example:

Let  $E$  be an elliptic curve and  $S$  a nonminimal algebraic surface with  $\kappa(S) \geq 0$ . Put  $X := E, Y := E \times S$  and  $Z := X \times Y (\cong E \times E \times S)$ . Let  $f: Z \rightarrow Z$  be a nonisomorphic endomorphism of  $Z$  defined by  $f(x, y, s) := (2x, x + y, s)$

for  $x \in X$ ,  $y \in Y$  and  $s \in S$ . It is easy to see that for all positive integer  $n$ , there exists no endomorphism  $g_n: Y \rightarrow Y$  such that  $q \circ f^n = g_n \circ q$  for the second projection  $q: Z \rightarrow Y$ .

(2) If the *Question* ( $\mathbf{D}_n$ ) has an affirmative answer, there exists a relative automorphism  $u$  of  $Z$  over  $Y$  such that  $u \circ f^k = h \times g$  for some endomorphism  $h: X \rightarrow X$  of  $X$ . Furthermore, if  $b_1(X) = 0$ , we can take  $u$  as  $u = \text{id}_Z$ . The proof is completely analogous to that of Proposition 6, so we omit it.

(3) By the same method as in the proof of Theorems 10 and 11, we can show: Under the same assumption as in Theorem 10 or 11,  $Y$  has no nonisomorphic endomorphism.

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