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Optimal decay rate for higher-order derivatives of the solution to the Lagrangian-averaged Navier–Stokes-α equation in $\mathbb{R}^3$

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Abstract. Recently, Bjorland and Schonbek [Ann. Inst. H. Poincaré Anal. Non Linéaire 25 (2008) 907–936] investigated the upper bound of the decay rate for the solution to the Lagrangian-averaged Navier–Stokes-α equation under the condition that the initial data belongs to $L^1(\mathbb{R}^n) \cap H^N_\sigma(\mathbb{R}^n)$ with $n = 2, 3, 4$. The decay rate can eventually be shown to be optimal if the average of the initial data is nonzero. Thus, the target in this paper is to study the optimal decay rate of the solution when the average of the initial data is zero. If the initial data belongs to $L^1(\mathbb{R}^3) \cap H^N_\sigma(\mathbb{R}^3)$ and some weighted Sobolev space, we show that the lower and upper bounds of decay rates for the kth-order ($k \in [0, N]$) spatial derivatives of the solution tending to zero in L^2-norm are $(1+t)^{-\frac{5+2k}{4}}$, which implies these decay rates are optimal. As a by-product, we show that the optimal decay rate (including lower and upper bounds) of the time derivative of the solution tending to zero in L^2-norm is $(1+t)^{-\frac{9}{4}}$.

1. Introduction

In this paper we are interested in establishing the upper and lower bounds of decay rate for the unfiltered velocity in the Lagrangian-averaged Navier–Stokes-α equation (LANS-α) (also known as the viscous Camassa–Holm equation (VCHE) or Navier–Stokes-α (NS-α) equation) in three-dimensional whole space. For this purpose, we consider the following LANS-α equation in $\mathbb{R}^3$:

$$\begin{cases} \partial_t v + u \cdot \nabla v + v \cdot \nabla u^T = \nu \Delta v - \nabla p, \\ \operatorname{div} u = 0, \end{cases} \tag{1.1}$$

where $u = \mathcal{O}^{-1} v = (I - \alpha^2 \Delta)^{-1} v$, and the operator $\mathcal{O} = I - \alpha^2 \Delta$ denotes the Helmholtz operator. We adopt the notation $v \cdot \nabla u^T = \sum_i v_i \nabla u_i$. Also, $t \geq 0$ and $x = (x^1, x^2, x^3) \in \mathbb{R}^3$ are the time and spatial variables respectively, the material coefficient (normal stress moduli) $\alpha > 0$ is a regularization length-scale parameter representing the width of the filter, the unknown function $u = (u_1, u_2, u_3)(t, x)$ represents the filtered fluid velocity and the unknown function $v = u - \alpha^2 \Delta u$ denotes the unfiltered velocity. The function p

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denotes the filtered pressure. The mean quantities $u(t,x)$ and p can reflect the uncertainty of accurately reproducing the initial data when the same fluid experiment is repeated many times. The constant $\nu > 0$ is the viscosity coefficient. We consider the solution of system (1.1) with the initial data

$$v(x,t)|_{t=0} = v_0. \tag{1.2}$$

Moreover, we assume the far field behavior is

$$\lim_{|x|\to\infty} u(x) = 0. \tag{1.3}$$

Chen et al. ([15]) gave the detailed derivation of the LANS-α equation. Holm et al. ([34]; see also [42]) derived the LANS-α equation by virtue of variational principles in the Lagrangian formalism. Based on modifying the Navier–Stokes system, Foais et al. ([22]) derived the LANS-α equation. The LANS-α equation is built to provide a valid value simulation of three-dimensional turbulence over a periodic domain; see [21, 33]. It should be demonstrated analytically and numerically that this model is an approximation of the Navier–Stokes equation (also called N-S), where in partial terms, the unknown velocity v of the Navier–Stokes equation is represented by the smoother velocity function u, that is, $v = u - \alpha^2 \Delta u$. Meanwhile, this model gives an approximation in the research on many problems related to turbulent flow; refer to [32, 44] and the references therein. For a discussion of the physical importance and mathematical significance, see [15–18].

Many researchers have addressed the well-posedness of the LANS-α equation. Based on a fixed point argument, Marsden and Shkoller proved global well-posedness and regularity of the LANS-α equation in a three-dimensional bounded domain with a smooth boundary in [42]. Coutand et al. ([19]) established the global well-posedness and regularity of weak solutions for the case of no-slip boundary conditions on a three-dimensional bounded domain, which generalizes the periodic-box results of [23]. Global-in-time well-posedness of weak solutions to the Cauchy problem in a three-dimensional periodic region was first shown by Foais et al. with the help of the Galerkin method and the smoothing property of solutions was also studied in [23]. Later, Bjorland and Schonbek ([4]) improved the work in [42] by the Galerkin method on an n-dimensional bounded domain or n-dimensional whole space with $n = 2, 3, 4$. Zhou and Fan ([57]) showed various regularity criteria for the strong solution in n-dimensional whole space and established the existence of a global smooth solution when $n \leq 4$. For more results on the well-posedness of the LANS-α equation, see [43] and the references therein.

It should be noted that the LANS-α equation generalizes the incompressible Navier–Stokes equation due to the fact that the LANS-α equation reduces to the incompressible Navier–Stokes equation as $\alpha = 0$. See [4,22,23] and the references therein for more results about the convergence of the LANS-α equation to the Navier–Stokes equation. Also note that when $\nu \to 0$, the solution v of the LANS-α equation will converge to u^α, which satisfies the well-known Euler-α equation (also known as the Lagrangian-averaged Euler equations),

$$\begin{cases} \partial_t(u^\alpha - \alpha^2\Delta u^\alpha) + u^\alpha\cdot\nabla(u^\alpha - \alpha^2\Delta u^\alpha) + (u^\alpha - \alpha^2\Delta u^\alpha)\cdot\nabla(u^\alpha)^T = -\nabla p^\alpha, \\ \operatorname{div} u^\alpha = 0, \\ u^\alpha(x,t)|_{t=0} = u_0^\alpha(x). \end{cases}$$

We remark here that the Euler-α equation arises as the zero-viscosity case of the incompressible non-Newtonian fluids of second grade introduced in [20]. Chen et al. ([15, 16]), took the viscosity factor into account in the Euler-α equation and introduced the LANS-α equation as a consequence. By a variable formulation, the Euler-α equation introduced in [34] is a natural mathematical generalization of the integrable invisible one-dimensional Camassa–Holm equation which was discovered in [11]. Local well-posedness of solutions to the Euler-α equation has been shown in some contexts; refer to [7, 9, 37, 38, 53] for instance. For the three-dimensional axisymmetric Euler-α equation without swirl, the global well-posedness of classical solutions was obtained by Busuioc and Ratiu in [7]; later, the global existence and uniqueness of strong solutions on a bounded domain were shown in [8], and the global existence of a weak solution was established by Jiu et al. ([35]) very recently. However, the global existence of smooth solutions to the Euler-α equation in $\mathbb{R}^3$ is still an open problem, which parallels the famous open problem of the existence of smooth solutions to the three-dimensional Euler equations. See [6, 39, 40, 55] and the references therein for more results on the convergence of the solution of the Euler-α equation to the solutions of the incompressible Euler equation.

In the past two decades, the following Camassa–Holm (CH) equation, describing the one-way propagation of shallow water waves, has attracted a great deal of attention from scholars in the study of nonlinear integrable equations:

$$\partial_t v + u\partial_x v + 2v\partial_x u = 0, \quad v = u - \partial_{xx}u,$$

where the function u denotes the height of the water at the bottom. Local well-posedness of the CH equation in the Sobolev space $H^s(\mathbb{R})$ with $s > \frac{3}{2}$ has been shown in [36]. Later, local ill-posedness was shown in [10] and [31] for the cases $s < \frac{3}{2}$ and $s = \frac{3}{2}$ respectively. For the CH equation with fractional dissipation, Gui and Liu ([28]) proved global well-posedness in $H^{\frac{3}{2}-\gamma}(\mathbb{R})$ with the dissipative operator power $\gamma \in [\frac{1}{2}, 1)$. For the case of the dissipative operator power $\gamma = 2$ and the constant $\nu > 0$, it reduces to the classical viscous CH equation; Xin and Zhang showed the global well-posedness of the weak solution to the Cauchy problem in [54]. Also, there are other types of CH equations, for instance the modified CH equations, the two-component CH equations and the μ-CH equations. Mathematical properties of the modified CH equations that could be viewed as a cubic extension of the CH equation have been studied in many works; for example, the blow-up mechanism has been investigated in [14] and results about the well-posedness of solutions have been shown in [24, 29, 30, 48]. When the quantity which is related to the free surface elevation from equilibrium equals zero, the two-component CH equations reduce to the CH equations. Many researchers have studied the two-component CH equations. The well-posedness of solutions has been studied in several contexts; refer to [26, 27] for instance.

The wave breaking phenomenon, one of the most interesting problems of water wave theory, has also attracted the attention of a large number of scholars; see [12, 26, 27, 56] for instance. The μ-CH equations, the midway equations between the CH equations and the Hunter–Saxton equations (which are a short-wave limit of the CH equations), share many remarkable properties as well. For example, well-posedness, the blow-up structure and wave breaking have been investigated extensively; see [13, 25, 46, 47, 49].

The study of the large time behavior of the solution to the LANS-α equation has attracted researchers' attention. Bjorland and Schonbek ([4]) have used the Fourier splitting method to show that the upper bound of decay rate for the Nth-order derivative of the unfiltered velocity in L^2-norm is $(1+t)^{-\frac{n+2N}{4}}$ in n-dimensional whole space with $n = 2, 3, 4$, provided the initial data belongs to $L^1 \cap H^N_\sigma$ for any positive integer N. It is worth noting that the L^1 integrating condition mentioned before is to ensure that the Fourier transform of the initial data in lower frequency has an upper bound. Therefore, the decay rates obtained in [4] are eventually optimal (i.e., the lower bound of decay rate coincides with the upper one) if we assume the Fourier transform of the initial data at the zero point (equivalent to the average of the initial data) is nonzero. *Thus, the target in this paper is to study the optimal decay rate of the solution when the average of the initial data is zero.* On this subject, Schonbek and her collaborators ([51, 52]) have completed some original celebrated results involving a zero average of the initial data for both the classical incompressible Navier–Stokes and MHD equations. However, to the authors' knowledge, there are no relevant results concerning the optimal decay rate for the LANS-α equation when the average of the initial data is zero. *Thus, the purpose of this paper is to establish some decay rate results along this direction.* For more decay rate results concerning the LANS-α equation, we also refer interested readers to [1, 3, 5].

Notation. Throughout this paper, we utilize the symbol ∇^k with integer $k \geq 0$ to stand, as usual, for any spatial derivative of order k. Denote $L^p(\mathbb{R}^3)$ as the usual Lebesgue space. Denote $H^s(\mathbb{R}^3) = W^{s,2}(\mathbb{R}^3)$ with $s \geq 0$ as the usual Sobolev space. Denote $\dot{H}^s(\mathbb{R}^3)$ with $s \geq 0$ as the usual homogeneous Sobolev space with norm $\|u\|_{\dot{H}^s} = (\int_{\mathbb{R}^3} |\xi|^{2s} |\hat{u}|^2 \, d\xi)^{\frac{1}{2}} < \infty$. We use H^s_σ to represent the completion of the set $\{\phi \in C_0^\infty(\mathbb{R}^3) \mid \operatorname{div} \phi = 0\}$ under the H^s-norm. We adopt the following simplified notation: $\int f \, dx := \int_{\mathbb{R}^3} f \, dx$. We introduce two sets as follows:

$$\mathcal{V} = \{v \in C_0^\infty(\mathbb{R}^3) : \operatorname{div} v = 0\}, \quad H = \text{closure of } \mathcal{V} \text{ in } L^2(\mathbb{R}^3).$$

Denote by W_1 and W_2 the two weighted Lebesgue spaces

$$W_1 = \{v(x) : \mathbb{R}^3 \to \mathbb{R},\ \|v\|_{W_1} = \int_{\mathbb{R}^3} |x|^2 |v(x)| \, dx < \infty\},$$
$$W_2 = \{v(x) : \mathbb{R}^3 \to \mathbb{R},\ \|v\|_{W_2} = \big(\int_{\mathbb{R}^3} |x| \, |v(x)|^2 \, dx\big)^{\frac{1}{2}} < \infty\}.$$

The set M is expressed as

$$M = \Big\{\nabla_\xi \hat{v}_0(0) = 0,\ \lim_{t\to\infty} \frac{\pi^{\frac{3}{2}}}{15\Gamma(\frac{3}{2})} \Big(\sum_{k\neq j} (\mathcal{A}_{kk}(t) - \mathcal{A}_{jj}(t))^2 + 6 \sum_{k\neq j} \mathcal{A}_{kj}(t)^2\Big) = 0\Big\},$$

where an element of the 3×3 matrix $\mathcal{A}_{kj}(t)$ is expressed by

$$\mathcal{A}_{kj}(t) = \int_0^t a_{kj}(0, s)\, ds,$$

with the function a_{kj} defined in Section 3 and v_0 the initial data of the LANS-α equation (1.1).

We first recall the result on an estimate of the solution to the LANS-α equation (1.1) obtained by Bjorland and Schonbek ([4]).

Theorem 1.1 ([4]). *Let v (the unfiltered velocity) be the solution to system (1.1) in $\mathbb{R}^n$ with $n = 2, 3, 4$. Assume that the initial data $v_0(x)$ satisfies $v_0(x) \in H^m_\sigma$ for any $m \geq 0$; then the solution satisfies the estimate*

$$\|\partial_t^p \nabla^k v(t)\|_{L^2}^2 + \nu \int_0^t \|\partial_t^p \nabla^{k+1} v(t)\|_{L^2}^2\, ds \leq C, \tag{1.4}$$

with $2p + k \leq m$ and C a positive constant dependent on the initial data but independent of time.

In this paper we establish upper and lower bounds of decay rates for the solution to the LANS-α equation (1.1) under the H^N $(N \geq 3)$ framework. Our main results are stated in the following theorems.

Theorem 1.2. *Assume that the initial data of the unfiltered velocity v satisfies that $v_0 \in H \cap L^1 \cap H^N \cap W_1 \cap W_2$, and the initial data of the filtered velocity u satisfies $u_0 \in W_2$ and $\nabla u_0 \in W_2$. Additionally, provided that the velocity $v \notin M$, then for all $t > T$ there exists a positive constant C independent of time such that the following decay rate holds:*

$$c_1(1+t)^{-\frac{5+2k}{2}} \leq \|\nabla^k v\|_{L^2}^2 \leq C_1(1+t)^{-\frac{5+2k}{2}}, \quad k \in [0, N]. \tag{1.5}$$

Here, T is a positive large time, and the two positive constants c_1 and C_1 depend on ν, α but do not depend on time.

Remark 1.1. Bjorland and Schonbek investigated the upper bound of decay rate for the solution under the condition that the initial data belongs to $L^1(\mathbb{R}^n) \cap H^N_\sigma(\mathbb{R}^n)$ with $N \geq 3$ and $n = 2, 3, 4$. The L^1 integrating condition proposed in [4] by Bjorland and Schonbek is used to guarantee that the Fourier transform of the initial data $v_0(x)$ is bounded. That is to say, when the initial data $v_0(x) \in L^1(\mathbb{R}^n)$, it holds that

$$|\hat{v}_0(\xi)| \leq C,$$

with C a positive constant that is independent of time. Then by applying the Fourier splitting method and an inductive argument, the authors obtained decay rates for the unfiltered velocity $v(x, t)$ and all of its derivatives. It should be noted that the decay rates obtained in [4] could be shown to be optimal (i.e., the lower bound of decay rate coincides with the upper bound) provided the Fourier transform of the initial data at the zero point (equivalent to the average of the initial data) is nonzero.

In our manuscript we establish the upper bound of decay rate for the solution provided the initial data of the solution $v(x,t)$ satisfies $v_0(x) \in L^1(\mathbb{R}^3) \cap H^N_\sigma(\mathbb{R}^3) \cap W_1(\mathbb{R}^3)$. We use the Borchers lemma to obtain that the initial data satisfies $\hat{v}_0(0) = 0$. The condition that the initial data $v_0(x)$ belongs to the weighted Lebesgue space W_1 is used to ensure that the Fourier transform of the initial data is twice continuously differentiable with bounded partial derivatives. That is to say, we can obtain

$$|\hat{v}_0(\xi)| \leq C|\xi|;$$

see (2.5). So, in this case, we obtain that the upper bounds of decay rates for the kth-order ($k \in [0, N]$) spatial derivatives of the solution tending to zero in L^2-norm are $(1+t)^{-\frac{5+2k}{4}}$ in the H^N ($N \geq 3$) framework. *Our interest is to establish the optimal decay rate for the solution when the average of the initial data is zero*. If the conditions given in Theorem 1.2 hold, we can derive the lower bound of decay rate for the solution, which coincides with the upper bound.

Remark 1.2. In the H^N framework, the highest derivative of the filtered velocity u is $N+2$. With the help of decay rate (1.5), it is easy to deduce that the upper and lower bounds of decay rates for the kth-order ($k \in [0, N-1]$) spatial derivatives of u tending to zero in L^2-norm are $(1+t)^{-\frac{5+2k}{4}}$. For the case $k \in [N, N+2]$, due to the fact that the faster upper bounds of decay rates for the kth-order ($k > N$) spatial derivatives of u are unavailable, we can get that the upper bounds of decay rates for the kth-order spatial derivatives of the filtered velocity u tending to zero in L^2-norm are $(1+t)^{-\frac{5+2N}{4}}$ but the lower bounds of decay rates for the kth-order spatial derivatives of the filtered velocity u tending to zero in L^2-norm remain unknown.

Remark 1.3. Based on the results in Theorem 1.2, we can apply the Sobolev interpolation inequality to obtain that the upper bounds of decay rates for the kth-order ($k \in [0, N-1]$) spatial derivatives of the filtered velocity tending to zero in $L^p (2 < p \leq 6)$-norm are $(1+t)^{-(2+\frac{k}{2}-\frac{3}{2p})}$. Additionally, the upper bounds of decay rates for the kth-order ($k = 0, \ldots, N-2$) spatial derivatives of the filtered velocity tending to zero in L^∞-norm are $(1+t)^{-\frac{1}{2}(k+4)}$.

Finally, we state the decay rate for the time derivative of the unfiltered velocity v to the LANS-α equation (1.1).

Theorem 1.3. *Assume the conditions in Theorem 1.2 hold; then for all $t > T$, the unfiltered velocity satisfies the following decay rate:*

$$c_3(1+t)^{-\frac{9}{2}} \leq \|\partial_t v\|_{L^2}^2 \leq C_3(1+t)^{-\frac{9}{2}},$$

with T a positive large time and two positive constants c_3 and C_3 independent of time.

Remark 1.4. By using the same method, the upper and lower bounds of decay rates for the higher-order time derivatives for the unfiltered velocity can be easily derived.

We now make some comments on this paper. First of all, we hope to apply the Fourier splitting method (see [50]) to establish the upper decay rate of the solution. However, it is not easy to apply the above method to obtain the decay rate for the unfiltered velocity directly because of the appearance of the second-order derivative in the convective term. Motivated by the energy differential equality (2.8), our strategy is to derive the upper bounds of decay rates for the filtered velocity itself and its first-order derivative. Furthermore, the upper bound of decay rate for the first-order derivative of the unfiltered velocity can be obtained by using the decay rate of the filtered velocity obtained before. Thus, the decay rate of the unfiltered velocity follows from the relation between the unfiltered velocity and the filtered one. Then we use mathematical induction to derive the upper bounds of decay rates for the higher-order derivatives of the unfiltered velocity.

Next we follow the technique introduced in [51, 52] to derive the lower bounds of decay rates for the solution itself and its high-order derivatives. The main objective is to compare the decay rates of solutions to the LANS-α equation to those of the homogeneous heat equations. When the initial data belongs to $L^1 \cap H^N$ and some weighted Sobolev spaces, the Fourier transform of the kth-order ($k \leq N$) derivative of solution has the form (3.8), which ensures that the lower bounds of decay rates for the solution to the homogeneous heat equation could be obtained by virtue of Proposition 3.1 when the initial data of the homogeneous heat equation equals the solution to the LANS-α equation for some time T_0. Meanwhile, this guarantees that the difference between the solution to the homogeneous heat equation and the LANS-α equation decays faster when T_0 is large enough.

Finally, we hope to establish the optimal decay rate for the time derivative of the unfiltered velocity by using equation (1.1). Thus, the main task turns into establishing the upper decay rate of the first-order derivative of pressure ∇p. Our method here is to apply the Sobolev interpolation inequality to control this difficult term ∇p by using the quantities p and Δp. The latter is a good quantity since it is not involved with the time derivative of the unfiltered velocity due to the divergence-free condition. Thus, the optimal decay rate of the time derivative of the unfiltered velocity can be obtained by using the optimal decay rate of the solution itself and its spatial derivatives.

The remainder of this paper is organized as follows. We establish the upper bounds of decay rates for the unfiltered velocity in Section 2. We propose some properties of the heat equation in Section 3.1, and Section 3.2 is devoted to establishing the lower bounds of decay rates for the higher-order derivatives of the unfiltered velocity. Section 4 is devoted to addressing the upper and lower bounds of decay rate for the time derivative of the unfiltered velocity. In Section 5 we establish the claimed estimates that are used in Section 3.

2. The upper bound of decay rate

This section is devoted to obtaining the upper bounds of decay rates for the spatial derivatives of the solution itself and its derivatives under the H^N ($N \geq 3$) framework.

In order to achieve this goal, we will bound the Fourier transform of the solution in the following proposition.

Proposition 2.1. *Assume that the initial unfiltered velocity $v_0(x)$ satisfies $v_0(x) \in L^1 \cap H \cap W_1$. Then the Fourier transform of the unfiltered velocity $v(x)$ satisfies*

$$|\hat{v}(\xi)| \leq C|\xi| + C|\xi|\left(\int_0^t \|u\|_{L^2}^2\, ds\right)^{\frac{1}{2}}\left(\int_0^t \|v\|_{L^2}^2\, ds\right)^{\frac{1}{2}},$$

where the positive constant C does not depend on t and α.

Proof. Let us denote $H := \nabla p + u \cdot \nabla v + v \cdot \nabla u^T$; then the first equation of (1.1) could be expressed as

$$\partial_t v - \nu \Delta v = -H. \tag{2.1}$$

Applying the Fourier transform to equation (2.1), it holds that

$$\hat{v}(\xi) = e^{-\nu t|\xi|^2}\hat{v}_0(\xi) - \int_0^t e^{-\nu(t-s)|\xi|^2}\widehat{H}(\xi, s)\, ds. \tag{2.2}$$

In what follows we estimate the term $\widehat{H}(\xi, s)$ first of all. Applying the operator div to equation (2.1), due to the fact that $\operatorname{div} v = 0$, it follows that $\operatorname{div} H = 0$, that is to say,

$$\Delta p = -\operatorname{div}(u \cdot \nabla v) - \operatorname{div}(v \cdot \nabla u^T). \tag{2.3}$$

Because $\operatorname{div} u = 0$, we can easily deduce that

$$\operatorname{div}(u \cdot \nabla v) = \sum_{j,k} \partial^j \partial^k (u_j v_k)$$

and

$$u \cdot \nabla v = \sum_j \partial^j (u_j v).$$

Applying the Fourier transform to equality (2.3), one arrives at

$$\hat{p} = -\sum_{j,k} \frac{\xi^j \xi^k}{|\xi|^2}\mathcal{F}(u_j v_k) + \frac{i\xi^t}{|\xi|^2}\mathcal{F}(v \cdot \nabla u^T).$$

Then we can obtain

$$\begin{aligned}
\widehat{H} &= \mathcal{F}(u \cdot \nabla v) + \mathcal{F}(v \cdot \nabla u^T) + \mathcal{F}(\nabla p) \\
&= i\sum_j \xi^j \mathcal{F}(u_j v) + \mathcal{F}(v \cdot \nabla u^T) + i\xi\hat{p} \\
&= i\sum_j \xi^j \mathcal{F}(u_j v) + \mathcal{F}(v \cdot \nabla u^T) + \left(-\sum_{j,k}\frac{\xi^j \xi^k}{|\xi|^2}\mathcal{F}(u_j v_k) + \frac{i\xi^t}{|\xi|^2}\mathcal{F}(v \cdot \nabla u^T)\right)i\xi \\
&= i\sum_j \xi^j \mathcal{F}(u_j v) - \sum_{j,k}\frac{\xi^j \xi^k}{|\xi|^2}\mathcal{F}(u_j v_k)\xi.
\end{aligned}$$

Setting

$$a_{kj} := \mathcal{F}(u_j v_k), \quad \mu_{kj} := \frac{\xi^k \xi^j}{|\xi|^2},$$

we introduce the 3×3 matrix $A = [a_{kj}]$. Then the term $\widehat{H}$ can be expressed as

$$\widehat{H} = i(I - \mu(\xi))A(\xi, t)\xi.$$

So we have

$$|A(\xi, t)| \leq C \sum_{k,j} |a_{kj}| \leq C\|u\|_{L^2}\|v\|_{L^2},$$

with C a positive constant. Then the term $|\widehat{H}|$ can be estimated by

$$|\widehat{H}| \leq C\|u\|_{L^2}\|v\|_{L^2}|\xi|. \tag{2.4}$$

Due to the fact that $v_0(x) \in L^1 \cap H$, by Lemma A.4, we find

$$\hat{v}_0(0) = \int v_0(x)\,dx = 0.$$

Since $v_0 \in W_1$, it is easy to deduce that $\hat{v}_0(\xi)$ is twice continuously differentiable with bounded partial derivatives. We expand $\hat{v}_0(\xi)$ in Taylor series around the origin up to terms of order 1 as

$$\hat{v}_0(\xi) = \hat{v}_0(0) + D_\xi \hat{v}_0(0)\xi + o(\xi);$$

then it holds that

$$|\hat{v}_0(\xi)| \leq C|\xi|, \tag{2.5}$$

with C a positive constant that is independent of time. Substituting inequalities (2.4) and (2.5) into equality (2.2), together with the Hölder inequality, we acquire

$$\begin{aligned} |\hat{v}(\xi)| &\leq Ce^{-\nu t|\xi|^2}|\xi| + C\int_0^t |\widehat{H}(\xi, s)|\,ds \\ &\leq C|\xi| + C|\xi|\int_0^t \|u\|_{L^2}\|v\|_{L^2}\,ds \\ &\leq C|\xi| + C|\xi|\left(\int_0^t \|u\|_{L^2}^2\,ds\right)^{\frac{1}{2}}\left(\int_0^t \|v\|_{L^2}^2\,ds\right)^{\frac{1}{2}}. \end{aligned} \tag{2.6}$$

Therefore, we have completed the proof of this lemma. ■

Lemma 2.2. *Let v be the solution to the LANS-α equation (1.1) which satisfies $v_0 \in L^1 \cap H \cap H^N \cap W_1$ with $N \geq 3$. Then for all $t \geq 0$, the solution has decay rate*

$$\|u\|_{L^2}^2 + \alpha^2\|\nabla u\|_{L^2}^2 \leq C(1 + t)^{-\frac{5}{2}}, \tag{2.7}$$

where the positive constant C depends on α and ν but not on time.

Proof. By multiplying both sides of the first equation of system (1.1) by u, with the help of the incompressibility condition $\operatorname{div} u = 0$, we integrate the result over $\mathbb{R}^3$ to get

$$\frac{1}{2}\frac{d}{dt}\Big(\int |u|^2\,dx + \alpha^2\int |\nabla u|^2\,dx\Big) + \nu\Big(\int |\nabla u|^2\,dx + \alpha^2\int |\nabla^2 u|^2\,dx\Big) = 0, \quad (2.8)$$

where we have used integration by parts to derive

$$\int u\cdot\nabla v\cdot u\,dx + \int v\cdot\nabla u^T\cdot u\,dx = 0.$$

With the aid of the Plancherel theorem, we rewrite equation (2.8) as

$$\frac{d}{dt}\int (1+\alpha^2|\xi|^2)|\hat{u}|^2\,d\xi + 2\nu\int |\xi|^2(1+\alpha^2|\xi|^2)|\hat{u}|^2\,d\xi = 0.$$

As was shown in [4], similarly, let $B(\rho)$ be the ball with radius ρ and $\rho^2 = \frac{f'}{2\nu f}$, where the positive increasing function f will be defined below. Then it is easy to deduce that

$$\begin{aligned}\frac{d}{dt}\int (1+\alpha^2|\xi|^2)|\hat{u}|^2\,d\xi &+ 2\nu\rho^2\int (1+\alpha^2|\xi|^2)|\hat{u}|^2\,d\xi \\ &\le 2\nu\rho^2\int_{B(\rho)} (1+\alpha^2|\xi|^2)|\hat{u}|^2\,d\xi. \end{aligned} \quad (2.9)$$

Since $v = u - \alpha^2\Delta u$, the Plancherel theorem implies

$$\hat{u} = \frac{\hat{v}}{1+\alpha^2|\xi|^2};$$

then it holds that

$$(1+\alpha^2|\xi|^2)|\hat{u}|^2 \le |\hat{v}|^2 \le C|\xi|^2 + C|\xi|^2\Big(\int_0^t \|u\|_{L^2}^2\,ds\Big)\Big(\int_0^t \|v\|_{L^2}^2\,ds\Big). \quad (2.10)$$

Inserting the equality $v = u - \alpha^2\Delta u$ into the above inequality (2.10), we have

$$\begin{aligned}(1+\alpha^2|\xi|^2)|\hat{u}|^2 &\le C|\xi|^2 + C|\xi|^2\Big(\int_0^t \|u\|_{L^2}^2\,ds\Big)\Big(\int_0^t \|u\|_{L^2}^2\,ds + \int_0^t \|\Delta u\|_{L^2}^2\,ds\Big)\\ &\le C|\xi|^2 + C|\xi|^2\Big(\int_0^t \|u\|_{L^2}^2\,ds\Big)\Big(\int_0^t \|u\|_{L^2}^2\,ds + \int_0^t \|\nabla v\|_{L^2}^2\,ds\Big)\\ &\le C|\xi|^2 + C|\xi|^2\Big(\int_0^t \|u\|_{L^2}^2\,ds\Big)^2\\ &\quad + C|\xi|^2\Big(\int_0^t \|u\|_{L^2}^2\,ds\Big)\Big(\int_0^t \|\nabla v\|_{L^2}^2\,ds\Big)\\ &\le C|\xi|^2 + C|\xi|^2 t\Big(\int_0^t \|u\|_{L^2}^2\,ds\Big) + C|\xi|^2\Big(\int_0^t \|u\|_{L^2}^2\,ds\Big),\end{aligned}$$

where we have used bound (1.4) and the facts that $\|u\|_{L^2}^2 \le \|v\|_{L^2}^2$ and $\|\Delta u\|_{L^2}^2 \le C\|\nabla v\|_{L^2}^2$. Together with this bound, we get the estimate

$$\begin{aligned}
&\frac{d}{dt}\int (1+\alpha^2|\xi|^2)|\hat{u}|^2\,d\xi + 2\nu\rho^2\int (1+\alpha^2|\xi|^2)|\hat{u}|^2\,d\xi \\
&\quad\le 2\nu\rho^2\int_{B(\rho)}(1+\alpha^2|\xi|^2)|\hat{u}|^2\,d\xi \\
&\quad\le C\rho^2\int_{B(\rho)}|\xi|^2\,d\xi + C\rho^2 t\int_{B(\rho)}|\xi|^2\Big(\int_0^t \|u\|_{L^2}^2\,ds\Big)\,d\xi \\
&\qquad + C\rho^2\int_{B(\rho)}|\xi|^2\Big(\int_0^t \|u\|_{L^2}^2\,ds\Big)\,d\xi \\
&\quad\le C\rho^7 + C\rho^7(1+t)\int_0^t \|u\|_{L^2}^2\,ds.
\end{aligned}$$

Denoting $\int (1+\alpha^2|\xi|^2)|\hat{u}|^2\,d\xi := F$, it holds that

$$\frac{d}{dt}F + 2\nu\rho^2 F \le C\rho^7 + C\rho^7(1+t)\int_0^t F\,ds.$$

We choose $f = (1+t)^{\frac{7}{2}}$. Substituting $\rho^2 = \frac{f'}{2\nu f}$ into the above inequality directly implies

$$\frac{d}{dt}[(1+t)^{\frac{7}{2}}F] \le C + C(1+t)\int_0^t F\,ds.$$

Integrating the above inequality with respect to time over $[0,t]$, we find

$$(1+t)^{\frac{7}{2}}F - F_0 \le Ct + C(1+t)^2\int_0^t F\,ds,$$

which implies

$$F \le C(1+t)^{-\frac{5}{2}} + C(1+t)^{-\frac{3}{2}}\int_0^t F\,ds.$$

We use the Grönwall inequality to arrive at

$$\|u\|_{L^2}^2 + \alpha^2\|\nabla u\|_{L^2}^2 = F \le C(1+t)^{-\frac{3}{2}}. \tag{2.11}$$

Plugging estimate (2.11) into (2.10) and then into (2.9), we can obtain

$$\frac{d}{dt}F + 2\nu\rho^2 F \le C\rho^2\int_{B(\rho)}|\xi|^2\,d\xi.$$

We still choose $f = (1+t)^{\frac{7}{2}}$. By substituting $\rho^2 = \frac{f'}{2\nu f}$ into the above inequality we get

$$\frac{d}{dt}[(1+t)^{\frac{7}{2}}F] \le C.$$

Finally, by integrating the above inequality with respect to time over $[0,t]$, one arrives at

$$\|u\|_{L^2}^2 + \alpha^2\|\nabla u\|_{L^2}^2 = F \le C(1+t)^{-\frac{5}{2}}.$$

So we have finished the proof of this lemma. ∎

Lemma 2.3. *Let v be the solution to the LANS-α equation (1.1) which satisfies $v_0 \in L^1 \cap H \cap H^N \cap W_1$ with $N \geq 3$. Then for all $t \geq T_1$ with T_1 given below, the solution has decay rate*

$$\|\nabla v\|_{L^2}^2 \leq C(1+t)^{-\frac{7}{2}}, \tag{2.12}$$

where the positive constant C depends on α and ν but not on time.

Proof. By multiplying both sides of the first equation of (1.1) by ∇v and integrating the result over $\mathbb{R}^3$, together with integration by parts, the incompressibility condition and the Hölder and Sobolev inequalities, it is easy to deduce that

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\nabla v\|_{L^2}^2 + \nu\|\Delta v\|_{L^2}^2 &\leq \left|\int u \cdot \nabla v \cdot \Delta v \, dx\right| + \left|\int v \cdot \nabla u^T \cdot \Delta v \, dx\right| \\
&\leq C \int |u|\,|\nabla v|\,|\Delta v| \, dx \\
&\leq C\|u\|_{L^3}\|\nabla v\|_{L^6}\|\Delta v\|_{L^2} \\
&\leq C\|u\|_{H^1}\|\Delta v\|_{L^2}^2 \\
&\leq C(1+t)^{-\frac{5}{4}}\|\Delta v\|_{L^2}^2,
\end{aligned}$$

where we have use the basic fact that

$$v \cdot \nabla u^T = \sum_i \nabla(v_i u_i) - u \cdot \nabla v^T,$$

to arrive at

$$\int v \cdot \nabla u^T \cdot \Delta v \, dx = -\int u \cdot \nabla v^T \cdot \Delta v \, dx.$$

We choose t to be large enough such that $C(1+t)^{-\frac{5}{4}} \leq \frac{\nu}{2}$, i.e., there exists some time T_1 such that for $t \geq T_1$, it holds that $C(1+t)^{-\frac{5}{4}} \leq \frac{\nu}{2}$, so that we have

$$\frac{d}{dt}\|\nabla v\|_{L^2}^2 + \nu\|\Delta v\|_{L^2}^2 \leq 0.$$

Similar to (2.7), let $B(\rho)$ be the ball with radius ρ, but the radius ρ here satisfies $\rho^2 = \frac{f'}{\nu f}$ with positive increasing f given below. By the Plancherel theorem, it is easy to derive

$$\frac{d}{dt}\int |\xi|^2|\hat{v}|^2 \, d\xi + \nu\rho^2 \int |\xi|^2|\hat{v}|^2 \, d\xi \leq \nu\rho^4 \int_{B(\rho)} |\hat{v}|^2 \, d\xi. \tag{2.13}$$

Substituting $\|u\|_{L^2}^2 \leq C(1+t)^{-\frac{5}{2}}$ into (2.10) directly yields

$$\begin{aligned}
|\hat{v}|^2 &\leq C|\xi|^2 + C|\xi|^2\left(\int_0^t \|u\|_{L^2}^2 \, ds\right)\left(\int_0^t \|v\|_{L^2}^2 \, ds\right) \\
&\leq C|\xi|^2 + C|\xi|^2\left(\int_0^t \|u\|_{L^2}^2 \, ds\right)^2 + C|\xi|^2\left(\int_0^t \|u\|_{L^2}^2 \, ds\right)\left(\int_0^t \|\Delta u\|_{L^2}^2 \, ds\right) \\
&\leq C|\xi|^2 + C|\xi|^2 \int_0^t \|\nabla v\|_{L^2}^2 \, ds,
\end{aligned} \tag{2.14}$$

where we have used the basic fact that

$$\|\Delta u\|_{L^2}^2 \le C\|\nabla v\|_{L^2}^2,$$

since $v = u - \alpha^2 \Delta u$. Then by plugging (2.14) into (2.13) we have

$$\begin{aligned}
&\frac{d}{dt}\int |\xi|^2|\hat{v}|^2\,d\xi + \nu\rho^2\int |\xi|^2|\hat{v}|^2\,d\xi \\
&\quad \le C\rho^4\int_{B(\rho)}|\xi|^2\,d\xi + C\rho^4\int_{B(\rho)}|\xi|^2\int_0^t\|\nabla v\|_{L^2}^2\,ds\,d\xi \\
&\quad \le C\rho^9 + C\rho^9\int_0^t\|\nabla v\|_{L^2}^2\,ds,
\end{aligned}$$

which implies

$$\frac{d}{dt}\|\nabla v\|_{L^2}^2 + \nu\rho^2\|\nabla v\|_{L^2}^2 \le C\rho^9 + C\rho^9\int_0^t\|\nabla v\|_{L^2}^2\,ds.$$

Inserting $\rho^2 = \frac{f'}{\nu f}$ into the above inequality and letting $f = (1+t)^{\frac{9}{2}}$, it holds that

$$\frac{d}{dt}[(1+t)^{\frac{9}{2}}\|\nabla v\|_{L^2}^2] \le C + C\int_0^t\|\nabla v\|_{L^2}^2\,ds.$$

By integrating the above inequality with respect to time over $[T_1, t]$, together with inequality (1.4) we arrive at

$$\|\nabla v\|_{L^2}^2 \le C(1+t)^{-\frac{7}{2}} + C(1+t)^{-\frac{7}{2}}\int_0^t\|\nabla v\|_{L^2}^2\,ds.$$

Then the Grönwall inequality directly yields

$$\|\nabla v\|_{L^2}^2 \le C(1+t)^{-\frac{7}{2}}.$$

Thus we have proved the lemma. ■

In the following we will use estimates (2.7) and (2.12) to give the upper bound of decay rate for the solution.

Lemma 2.4. *Under the conditions of Theorem 1.2, for $t \ge T_1$ with T_1 given above, the following estimate holds:*

$$\|v\|_{L^2}^2 \le C(1+t)^{-\frac{5}{2}},$$

with C a positive constant independent of time.

Proof. Due to the fact that

$$\|\nabla v\|_{L^2}^2 = \|\nabla u\|_{L^2}^2 + 2\alpha^2\|\nabla^2 u\|_{L^2}^2 + \alpha^4\|\nabla^3 u\|_{L^2}^2,$$

we can easily obtain

$$\|\nabla^2 u\|_{L^2}^2 \le C\|\nabla v\|_{L^2}^2 \le C(1+t)^{-\frac{7}{2}}. \tag{2.15}$$

By combining estimates (2.7) and (2.15), we have for $t \ge T_1$,

$$\|v\|_{L^2}^2 = \|u\|_{L^2}^2 + 2\alpha^2\|\nabla u\|_{L^2}^2 + \alpha^4\|\Delta u\|_{L^2}^2 \le C(1+t)^{-\frac{5}{2}}.$$

Therefore, we have completed the proof of this lemma. ∎

The following lemma is devoted to establishing the upper bounds of decay rates for the kth-order derivative of the solution with $k \in [0, N]$.

Lemma 2.5. *Let v be the solution to the LANS-α equation (1.1) which satisfies $v_0 \in L^1 \cap H^N \cap W_1$ with $N \ge 3$. Then for all $t \ge T_*$, with T_* defined below, the solution has the following decay rate:*

$$\|\nabla^k v\|_{L^2}^2 \le C(1+t)^{-\frac{5+2k}{2}}, \quad k \in [0, N], \tag{2.16}$$

where C is a positive constant independent of time.

Proof. Above, we have obtained for $k = 0, 1$,

$$\|\nabla^k v\|_{L^2}^2 \le C(1+t)^{-\frac{5+2k}{2}}.$$

In what follows we need to verify that for the case $k \in [2, N]$, decay estimate (2.16) is also true. We will prove it by mathematical induction. Assume that, for some time T_2, decay rate (2.16) is true for $k \in [0, N-1]$ when $t \ge T_2$. We shall prove that the decay rate is true when $k = N$. Multiplying both sides of the first equation of the LANS-α equation (1.1) by $\Delta^N v$ and integrating the result over $\mathbb{R}^3$, with the aid of integration by parts, the incompressibility condition and the Young inequality, it holds that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\nabla^N v\|_{L^2}^2 + \nu\|\nabla^{N+1} v\|_{L^2}^2 &\le C\|\nabla^{N-1}(u\cdot\nabla v)\|_{L^2}\|\nabla^{N+1} v\|_{L^2} \\ &\le \frac{\nu}{2}\|\nabla^{N+1} v\|_{L^2}^2 + C\|\nabla^{N-1}(u\cdot\nabla v)\|_{L^2}^2,\end{aligned} \tag{2.17}$$

where we used the fact that

$$\begin{aligned}\int v\cdot\nabla u^T\cdot\Delta^N v\,dx &= \sum_i \int \nabla(u_i v_i)\cdot\Delta^N v\,dx - \sum_{i,j}\int u_i\partial_j v_i\cdot\Delta^N v\,dx \\ &= -\int u\cdot\nabla v^T\cdot\Delta^N v\,dx.\end{aligned}$$

The Newton–Leibniz formula and the Hölder and Sobolev inequalities imply

$$\begin{aligned}\|\nabla^{N-1}(u\cdot\nabla v)\|_{L^2}^2 &\le C\sum_{m=0}^{N-1}\|\nabla^m u\cdot\nabla^{N-m} v\|_{L^2}^2 \\ &\le C\|\nabla^{N-1} u\cdot\nabla v\|_{L^2}^2 + C\|u\cdot\nabla^N v\|_{L^2}^2 \\ &\quad + C\sum_{m=1}^{N-2}\|\nabla^m u\cdot\nabla^{N-m} v\|_{L^2}^2\end{aligned}$$

$$
\begin{aligned}
&\le C\|\nabla^{N-1}u\|_{L^\infty}^2\|\nabla v\|_{L^2}^2 + C\|u\|_{L^3}^2\|\nabla^N v\|_{L^6}^2 \\
&\quad + C\sum_{m=1}^{N-2}\|\nabla^m u\|_{L^\infty}^2\|\nabla^{N-m}v\|_{L^2}^2 \\
&\le C\|\nabla^{N-1}v\|_{L^2}^2\|\nabla v\|_{L^2}^2 + C\|v\|_{L^2}^2\|\nabla^{N+1}v\|_{L^2}^2 \\
&\quad + C\sum_{m=1}^{N-2}\|\nabla^m v\|_{L^2}^2\|\nabla^{N-m}v\|_{L^2}^2,
\end{aligned}
$$

where we have used the interpolation inequality to deduce that

$$
\|\nabla^m u\|_{L^\infty} \le C\|\nabla^{m+1}u\|_{L^2}^{\frac{1}{2}}\|\nabla^{m+2}u\|_{L^2}^{\frac{1}{2}} \le C\|\nabla^m v\|_{L^2}.
$$

Substituting the above inequality into (2.17), based on the assumption we get

$$
\begin{aligned}
\frac{d}{dt}\|\nabla^N v\|_{L^2}^2 + \nu\|\nabla^{N+1}v\|_{L^2}^2 &\le C\|\nabla^{N-1}v\|_{L^2}^2\|\nabla v\|_{L^2}^2 + C\|v\|_{L^2}^2\|\nabla^{N+1}v\|_{L^2}^2 \\
&\quad + C\sum_{m=1}^{N-2}\|\nabla^m v\|_{L^2}^2\|\nabla^{N-m}v\|_{L^2}^2 \\
&\le C(1+t)^{-(5+N)} + C(1+t)^{-\frac{5}{2}}\|\nabla^{N+1}v\|_{L^2}^2. \qquad (2.18)
\end{aligned}
$$

By choosing t large enough such that $C(1+t)^{-\frac{5}{2}} \le \frac{\nu}{2}$, i.e., there exists some time T_3 such that for all $t \ge T_3$, it holds that $C(1+t)^{-\frac{5}{2}} \le \frac{\nu}{2}$, then we have

$$
\frac{d}{dt}\|\nabla^N v\|_{L^2}^2 + \frac{\nu}{2}\|\nabla^{N+1}v\|_{L^2}^2 \le C(1+t)^{-(N+5)}.
$$

For some positive constant R given below, denote the time sphere by

$$
S_0 := \left\{\xi \in \mathbb{R}^3 \;\middle|\; |\xi| \le \left(\tfrac{R}{1+t}\right)^{\frac{1}{2}}\right\},
$$

it follows immediately that

$$
\|\nabla^{N+1}v\|_{L^2}^2 \ge \frac{R}{1+t}\|\nabla^N v\|_{L^2}^2 - \frac{R^2}{(1+t)^2}\|\nabla^{N-1}v\|_{L^2}^2.
$$

Then we can obtain

$$
\begin{aligned}
\frac{d}{dt}\|\nabla^N v\|_{L^2}^2 + \frac{\nu R}{2(1+t)}\|\nabla^N v\|_{L^2}^2 &\le \frac{\nu R^2}{2(1+t)^2}\|\nabla^{N-1}v\|_{L^2}^2 + C(1+t)^{-(N+5)} \\
&\le C(1+t)^{-(N+\frac{7}{2})} + C(1+t)^{-(N+5)} \\
&\le C(1+t)^{-(N+\frac{7}{2})}.
\end{aligned}
$$

Let $T_* = \max\{T_2, T_3\}$. By choosing $R = \frac{2N+7}{\nu}$ and multiplying both sides of the above inequality by $(1+t)^{N+\frac{5}{2}}$ and then integrating with respect to time over $[T_*, t]$, it holds that

$$
\|\nabla^N v\|_{L^2}^2 \le C(1+t)^{-(N+\frac{5}{2})},
$$

where we have used the uniform bound of estimate (1.4). Hence, by the general step of induction, we have given the proof for estimate (2.16). ∎

3. The lower bound of decay rate

In this section we will address the lower bounds of decay rates for the spatial derivatives of the unfiltered velocity. To this end we will establish the lower bounds of decay rates for the homogeneous heat equation and the upper bounds of decay rates for the difference between the homogeneous heat equation and the LANS-α equation (1.1).

3.1. The property of the homogeneous heat equation

This subsection is devoted to establishing decay rates for the solution to the homogeneous heat equation. For this aim, we consider the following heat equation:

$$\begin{cases} \partial_t \omega - \nu \Delta \omega = 0, \\ \operatorname{div} \omega = 0, \\ \omega|_{t=0} = v_0(x). \end{cases} \tag{3.1}$$

The divergence-free condition proposed here guarantees that the difference between the solution to the LANS-α equation (1.1) and the heat equation (3.1) satisfies the divergence-free condition. The following proposition has been shown in [52] for the case $k = 0$. We generalize it to the case $k \geq 1$ in what follows.

Proposition 3.1. *Assume that the initial data $v_0(x)$ of the heat equation (3.1) belongs to $H^N(\mathbb{R}^3)$ $(N \geq 3)$ and there exists a positive constant δ such that the Fourier transform $\mathcal{F}(v_0) = \hat{v}_0$ satisfies*

$$\hat{v}_0(\xi) = P(\xi)\xi + h(\xi), \quad |\xi| \leq \delta, \tag{3.2}$$

where $P(\xi)$ is a homogeneous, 3×3 matrix-valued function of degree zero satisfying

$$\|P\| = \sup_{|\xi|=1} |P(\xi)|,$$

with $|P(\xi)|$ a matrix norm of $P(\xi)$, and there exists a positive constant M such that for all $\xi \in \mathbb{R}^3$,

$$|h(\xi)| \leq M|\xi|^2, \quad |\xi| \leq \delta.$$

Then for any nonnegative integer k, it holds that for all $t \geq 1$,

$$C(\delta)\left(\int_{S^2} |P(\omega')\omega|^2 \, d\omega'\right) t^{-\frac{5}{2}-k} \leq \|\nabla^k \omega\|_{L^2}^2 \leq C(\delta)\left(\int_{S^2} |P(\omega')\omega'|^2 \, d\omega'\right) t^{-\frac{5}{2}-k}. \tag{3.3}$$

Here the positive constant $C(\delta)$ satisfies

$$\left(\frac{\delta^2}{2}\right)^{\frac{5}{2}} e^{-2\delta^2} \leq C(\delta) \leq \left(\frac{1}{2}\right)^{\frac{5}{2}} \Gamma\left(\frac{5}{2}\right).$$

Remark 3.1. It should be noticed that for the case $k = 0$, Schonbek et al. ([52]) have given the proof in detail. Applying the same method used in [52], it is easy to prove (3.3) for the case $k \geq 1$. For the sake of simplicity, we omit the proof here.

3.2. Lower bounds of decay rate

This subsection is devoted to addressing the lower bounds of decay rates for the solution itself and its derivatives. For this purpose, we will establish the upper bound of decay rate for the difference between the solution to the LANS-α equation (1.1) and the heat equation (3.1).

Denote the difference between the solution to the LANS-α equation (1.1) and the heat equation (3.1) by D, i.e., $D := v - \omega$. Then D satisfies

$$\begin{cases} \partial_t D - \nu \Delta D = -u \cdot \nabla v - v \cdot \nabla u^T - \nabla p, \\ \operatorname{div} D = 0, \\ D|_{t=0} = 0. \end{cases} \tag{3.4}$$

Lemma 3.2. *Suppose the function $D(t,x)$ is the difference between the LANS-α equation (1.1) and the heat equation (3.1). Then D satisfies*

$$|\widehat{D}| \leq C|\xi| \int_0^t \|v\|_{L^2}^2 \, d\xi, \tag{3.5}$$

where the positive constant C does not depend on time.

Proof. Taking the Fourier transform of the first equation of system (3.4), it is easy to deduce that

$$\widehat{D}(\xi) = -\int_0^t e^{-\nu(t-s)|\xi|^2} \widehat{H}(\xi, s) \, ds,$$

where the term H is denoted by $H := \nabla p + u \cdot \nabla v + v \cdot \nabla u^T$. Since

$$|\widehat{H}| \leq C\|u\|_{L^2}\|v\|_{L^2}|\xi| \leq C\|v\|_{L^2}^2|\xi|,$$

one arrives at

$$|\widehat{D}(\xi)| \leq C \int_0^t |\widehat{H}(\xi, s)| \, ds \leq C|\xi| \int_0^t \|v\|_{L^2}^2 \, d\xi.$$

So we have given the proof for this lemma. ∎

Lemma 3.3. *Under the conditions of Theorem 1.2, there exists a time T_0 (which will be determined later) such that for all $t \geq T_0$,*

$$\|\nabla^k D\|_{L^2}^2 \leq C T_0^{-3} t^{-\frac{5}{2}-k}, \quad k \in [0, N], \tag{3.6}$$

with C a positive constant independent of time.

Proof. We use the symbol $O_t(\xi)$ to represent the quantity that depends on ξ and time. We claim (a proof will be given in Section 5) that

$$|\nabla_\xi a_{kj}| \leq Ct, \tag{3.7}$$

where the positive constant C depends on the initial data but not on time. Due to the fact that

$$\widehat{H} = i(I - \mu(\xi))A(\xi, t)\xi,$$

then we can obtain

$$\widehat{H} = i(I - \mu(\xi))A(0, t)\xi + O_t(\xi)|\xi|^2.$$

Since

$$\hat{v}(\xi) = e^{-\nu t|\xi|^2}\hat{v}_0(\xi) - \int_0^t e^{-\nu(t-s)|\xi|^2}\widehat{H}(\xi, s)\,ds,$$

by Taylor expansion around the origin to second order, it is easy to deduce that

$$\hat{v}(\xi) = D_\xi \hat{v}_0(0)\xi - i(I - \mu(\xi))\mathcal{A}(t)\xi + O_t(\xi)|\xi|^2,$$

where the integration $\mathcal{A}(t)$ is defined by

$$\mathcal{A}(t) = \int_0^t A(0, s)\,ds.$$

With the help of estimates (1.4) and (2.16) we can deduce that

$$\begin{aligned}\int_0^t A(0, s)\,ds &\le \int_0^{T_1} A(0, s)\,ds + \int_{T_1}^t A(0, s)\,ds \\ &\le C\int_0^{T_1} \|v\|_{L^2}^2\,ds + C\int_{T_1}^t \|v\|_{L^2}^2\,ds \le C.\end{aligned}$$

Denote

$$P(\xi) := D_\xi \hat{v}_0(0) - i(I - \mu(\xi))\mathcal{A}(t),$$

so that

$$\hat{v}(\xi) = P(\xi)\xi + O_t(\xi)|\xi|^2.$$

Obviously, we can also obtain for any $k \ge 0$,

$$\mathcal{F}(\nabla^k v)(\xi) = (i\xi)^k(P(\xi)\xi + O_t(\xi)|\xi|^2). \tag{3.8}$$

We note that $(I - \mu(\xi))^2 = I - \mu(\xi)$, and by Lemma A.5 the integral $\int_{S^2} |P(\omega')\omega'|^2\,d\omega'$ can be written in the form

$$\int_{S^2} |P(\omega')\omega'|^2\,d\omega' = \frac{\pi^{\frac{3}{2}}}{15\Gamma(\frac{3}{2})}\Big(\sum_{k\neq j}(\mathcal{A}_{kk}(t) - \mathcal{A}_{jj}(t))^2 + 6\sum_{k\neq j}\mathcal{A}_{kj}(t)^2\Big),$$

provided $D_\xi \hat{v}_0(0) = 0$.

For some time $T_0 \ge \max\{T_*, 1\}$ with T_0 given below, for $t \ge T_0$, let the initial data of the heat equation $\omega(0)$ be equal to $v(T_0)$ with $v(t, x)$ the solution to the LANS-α equation (1.1), i.e., $\omega(0) = v(T_0)$. Then by virtue of Proposition 3.1, we can obtain

$$\|\nabla^k \omega\|_{L^2}^2 \ge \Big(\frac{\delta^2}{2}\Big)^{\frac{5}{2}} e^{-2\delta^2}\Big(\int_{S^2} |P(\omega')\omega'|^2\,d\omega'\Big)t^{-(\frac{5}{2}+k)}. \tag{3.9}$$

As was shown in [52], by use of Lemma A.6, it is easy to deduce that there exists a positive constant m such that

$$\|\nabla^k \omega\|_{L^2}^2 \geq \left(\frac{\delta^2}{2}\right)^{\frac{5}{2}} e^{-2\delta^2} m t^{-(\frac{5}{2}+k)}. \tag{3.10}$$

We will consider the upper bounds of decay rates for $\nabla^k D(t)$ with $k \geq 0$ in the following. Let $D(t) := v(t+T_0) - \omega(t)$; then $D(t)$ satisfies the system

$$\begin{cases} \partial_t D(t) - \nu \Delta D(t) = -H(t+T_0), \\ \operatorname{div} D = 0, \end{cases} \tag{3.11}$$

where the function H is defined by $H := u \cdot \nabla v + v \cdot \nabla u^T + \nabla p$. We apply the operator ∇^k to equation (3.11) and take the scalar product in L^2 with $\nabla^k D$ to get

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|\nabla^k D\|_{L^2}^2 + \nu\|\nabla^{k+1} D\|_{L^2}^2 &= -\int \nabla^k H \cdot \nabla^k D\, dx \\ &= -\int \nabla^k (u \cdot \nabla v) \cdot \nabla^k D\, dx \\ &\quad - \int \nabla^k (v \cdot \nabla u^T) \cdot \nabla^k D\, dx \\ &=: -K_1 - K_2. \end{aligned} \tag{3.12}$$

For the term K_2, we use the Hölder inequality and decay rates (2.16) and (3.9) to obtain

$$\begin{aligned} K_2 &= \int \nabla^k (v \cdot \nabla u^T) \cdot \nabla^k D\, dx \\ &= \int \nabla^k (v \cdot \nabla u^T) \cdot \nabla^k v\, dx - \int \nabla^k (v \cdot \nabla u^T) \cdot \nabla^k \omega\, dx \\ &\leq C(\|\nabla^k v\|_{L^2} + \|\nabla^k \omega\|_{L^2})\|\nabla^k (v \cdot \nabla u^T)\|_{L^2} \\ &\leq C(1+t)^{-(\frac{5}{4}+\frac{k}{2})}(\|\nabla^k v\|_{L^2}\|\nabla u\|_{L^\infty} + \|v\|_{L^\infty}\|\nabla^{k+1} u\|_{L^2}) \\ &\leq C(1+t)^{-(\frac{5}{4}+\frac{k}{2})}(\|\nabla^k v\|_{L^2}\|\nabla^2 u\|_{L^2}^{\frac{1}{2}}\|\nabla^3 u\|_{L^2}^{\frac{1}{2}} + \|\nabla v\|_{L^2}^{\frac{1}{2}}\|\nabla^2 v\|_{L^2}^{\frac{1}{2}}\|\nabla^k v\|_{L^2}) \\ &\leq C(1+t)^{-(\frac{9}{2}+k)}, \end{aligned} \tag{3.13}$$

where we have used the fact that $\|\nabla^{k+1} u\|_{L^2} \leq C\|\nabla^k v\|_{L^2}$ for any nonnegative integer k. In the following we only need to estimate the term K_1. Integration by parts and the Young inequality imply

$$\begin{aligned} K_1 &= \int \nabla^k (u \cdot \nabla v) \cdot \nabla^k D\, dx \\ &\leq \|\nabla^{k-1}(u \cdot \nabla v)\|_{L^2}\|\nabla^{k+1} D\|_{L^2} \\ &\leq \frac{\nu}{2}\|\nabla^{k+1} D\|_{L^2}^2 + C\|\nabla^{k-1}(u \cdot \nabla v)\|_{L^2}^2. \end{aligned}$$

We use the Sobolev and interpolation inequalities to find

$$
\begin{aligned}
\|\nabla^{k-1}(u\cdot\nabla v)\|_{L^2} &\leq C\|\nabla^{k-1}u\|_{L^2}\|\nabla v\|_{L^\infty}+C\|u\|_{L^\infty}\|\nabla^k v\|_{L^2}\\
&\leq C\|\nabla^{k-1}u\|_{L^2}\|\nabla^2 v\|_{L^2}^{\frac12}\|\nabla^3 v\|_{L^2}^{\frac12}+C\|\nabla u\|_{L^2}^{\frac12}\|\nabla^2 u\|_{L^2}^{\frac12}\|\nabla^k v\|_{L^2}\\
&\leq C\|\nabla^{k-1}v\|_{L^2}\|\nabla^2 v\|_{L^2}^{\frac12}\|\nabla^3 v\|_{L^2}^{\frac12}+C\|\nabla v\|_{L^2}^{\frac12}\|\nabla^2 v\|_{L^2}^{\frac12}\|\nabla^k v\|_{L^2}\\
&\leq C(1+t)^{-\frac12(k+\frac{13}{2})},
\end{aligned}
$$

where we have used estimate (2.16). So it is easy to obtain

$$
K_1 \leq \frac{\nu}{2}\|\nabla^{k+1}D\|_{L^2}^2+C(1+t)^{-(k+\frac{13}{2})}. \tag{3.14}
$$

Inserting estimates (3.13) and (3.14) into (3.12), we have

$$
\begin{aligned}
\frac{d}{dt}\|\nabla^k D\|_{L^2}^2+\nu\|\nabla^{k+1}D\|_{L^2}^2 &\leq C(1+t)^{-(\frac92+k)}+C(1+t)^{-(\frac{13}{2}+k)}\\
&\leq C(1+t)^{-(\frac92+k)}.
\end{aligned}
$$

Multiplying both sides of the above inequality by $G(t)=e^{2\int_0^t g^2(s)\,ds}$ directly yields

$$
\begin{aligned}
\frac{d}{dt}[G(t)\|\nabla^k D\|_{L^2}^2] &\leq 2G(t)[g^2(t)\|\nabla^k D\|_{L^2}^2-\nu\|\nabla^{k+1}D\|_{L^2}^2]\\
&\quad +CG(t)(1+t)^{-(\frac92+k)}.
\end{aligned} \tag{3.15}
$$

We choose $G(t)=t^\gamma$ (i.e., $g^2(t)=\frac{\gamma}{2t}$). The Plancherel theorem implies

$$
\begin{aligned}
g^2(t)\|\nabla^k D\|_{L^2}^2-\nu\|\nabla^{k+1}D\|_{L^2}^2 &= \int |\xi|^{2k}(g^2(t)-\nu|\xi|^2)|\widehat{D}|^2\,d\xi\\
&\leq Cg^{2+2k}(t)\int_{\nu|\xi|^2\leq g^2(t)}|\widehat{D}|^2\,d\xi\\
&= Cg^{2+2k}(t)\int_{2\nu t|\xi|^2\leq\gamma}|\widehat{D}|^2\,d\xi.
\end{aligned}
$$

By virtue of estimate (3.5) and decay rate (2.16), we can get

$$
\begin{aligned}
|\widehat{D}| \leq C|\xi|\int_0^t\|v(s+T_0)\|_{L^2}^2\,ds &= C|\xi|\int_{T_0}^{t+T_0}\|v(s_1)\|_{L^2}^2\,ds_1\\
&\leq C|\xi|\int_{T_0}^{\infty}(1+s_1)^{-\frac52}\,ds_1\leq C|\xi|T_0{}^{-\frac32}.
\end{aligned}
$$

Then it is easy to derive

$$
\int_{2\nu t|\xi|^2\leq\gamma}|\widehat{D}|^2\,d\xi \leq CT_0^{-3}\int_{2\nu t|\xi|^2\leq\gamma}|\xi|^2\,d\xi\leq CT_0^{-3}t^{-\frac52}.
$$

Substituting these estimates into inequality (3.15) we have

$$\frac{d}{dt}[t^\gamma\|\nabla^k D\|_{L^2}^2] \le Ct^\gamma \cdot t^{-\frac{1}{2}\cdot(2+2k)}T_0^{-3}t^{-\frac{5}{2}} + Ct^{\gamma-\frac{9}{2}-k}.$$

We take $\gamma > \frac{9}{2} + k$ and then integrate the above inequality about time over $[T_0, t]$ to obtain for $k \in [0, N]$,

$$\|\nabla^k D\|_{L^2}^2 \le CT_0{}^{-3}t^{-\frac{5}{2}-k}, \tag{3.16}$$

where we have used the basic fact that

$$\|\nabla^k D(T_0)\|_{L^2}^2 \le C\|\nabla^k\omega(T_0)\|_{L^2}^2 + C\|\nabla^k v(T_0)\|_{L^2}^2 \le C.$$

Therefore we have completed this proof. ∎

In what follows we will combine estimates (3.10) and (3.6) to derive the lower bounds of decay rates for the solution.

Lemma 3.4. *Under the conditions of Theorem 1.2, the estimate*

$$\|\nabla^k v\|_{L^2}^2 \ge C(1+t)^{-\frac{5+2k}{2}}, \quad k \in [0, N],$$

holds for $t > T$ with T a positive large time and C a positive constant independent of time.

Proof. We choose $T_0 = \max\{T_*, 1, (4C(\frac{\delta^2}{2})^{-\frac{5}{2}}e^{2\delta^2}m^{-1})^{\frac{1}{3}}\}$. Then it holds for any $k \in [0, N]$ that

$$\begin{aligned}\|\nabla^k v(t+T)\|_{L^2}^2 &\ge \frac{1}{2}\|\nabla^k\omega\|_{L^2}^2 - \|\nabla^k D\|_{L^2}^2\\ &\ge \frac{1}{2}\Big(\frac{\delta^2}{2}\Big)^{\frac{5}{2}}e^{-2\delta^2}mt^{-\frac{5}{2}-k} - CT_0{}^{-3}t^{-\frac{5}{2}-k}\\ &\ge Ct^{-\frac{5}{2}-k}.\end{aligned} \tag{3.17}$$

Thus we have obtained the lower bounds of decay rates for the spatial derivatives of the unfiltered velocity. ∎

We have given the upper and lower bounds of decay rates for the solution to the LANS-α equation in, respectively, Lemmas 2.5 and 3.4, so we have finished the proof of Theorem 1.2. ∎

4. Optimal decay rate for the time derivative

This section is devoted to establishing the upper bound of decay rate for the time derivative of the unfiltered velocity. For this purpose, we use the upper and lower bounds of decay rates for the spatial derivatives of the unfiltered velocity obtained in Section 2 and the first equation of the LANS-α equation (1.1).

Lemma 4.1. *Provided the conditions of Theorem 1.2 are true, then the unfiltered velocity has the decay rate*

$$C(1+t)^{-\frac{9}{2}} \le \|\partial_t v\|_{L^2}^2 \le C(1+t)^{-\frac{9}{2}},$$

where $t \ge T_0$ and C is a positive constant independent of time.

Proof. We establish the upper bound of decay rate for the time derivative of the unfiltered velocity first of all. On the basis of the upper bounds of decay rates for the unfiltered velocity (1.5), with the help of the interpolation inequality, we can easily obtain

$$\begin{aligned}\|u\cdot\nabla v\|_{L^2} &\le C\|u\|_{L^\infty}\|\nabla v\|_{L^2} \le C\|\nabla u\|_{L^2}^{\frac{1}{2}}\|\nabla^2 u\|_{L^2}^{\frac{1}{2}}\|\nabla v\|_{L^2}\\ &\le C\|\nabla v\|_{L^2}^{\frac{3}{2}}\|\nabla^2 v\|_{L^2}^{\frac{1}{2}} \le C(1+t)^{-\frac{15}{4}}\end{aligned} \tag{4.1}$$

and

$$\begin{aligned}\|v\cdot\nabla u^T\|_{L^2} &\le C\|v\|_{L^\infty}\|\nabla u\|_{L^2} \le C\|\nabla v\|_{L^2}^{\frac{1}{2}}\|\nabla^2 v\|_{L^2}^{\frac{1}{2}}\|\nabla v\|_{L^2}\\ &\le C\|\nabla v\|_{L^2}^{\frac{3}{2}}\|\nabla^2 v\|_{L^2}^{\frac{1}{2}} \le C(1+t)^{-\frac{15}{4}}.\end{aligned} \tag{4.2}$$

By combining equation (1.1) and estimates (1.5), (4.1) and (4.2), we can easily get

$$\begin{aligned}\|\partial_t v\|_{L^2}^2 &\le C\|\Delta v\|_{L^2}^2 + C\|\nabla p\|_{L^2}^2 + C\|u\cdot\nabla v\|_{L^2}^2 + C\|v\cdot\nabla u^T\|_{L^2}^2\\ &\le C(1+t)^{-\frac{9}{2}} + C\|\nabla p\|_{L^2}^2\end{aligned}$$

and

$$\begin{aligned}\|\partial_t v\|_{L^2}^2 &\ge C\|\Delta v\|_{L^2}^2 - C\|\nabla p\|_{L^2}^2 - C\|u\cdot\nabla v\|_{L^2}^2 - C\|v\cdot\nabla u^T\|_{L^2}^2\\ &\ge C(1+t)^{-\frac{9}{2}} - C\|\nabla p\|_{L^2}^2.\end{aligned}$$

Thus, we only need to estimate the upper bound of decay rate for the first-order derivative of the pressure in the following. By use of the interpolation inequality we have

$$\|\nabla p\|_{L^2}^2 \le C\|p\|_{L^2}\|\Delta p\|_{L^2}.$$

With the aid of equality (2.3), the Sobolev and interpolation inequalities and estimate (1.5), we can get

$$\begin{aligned}\|\Delta p\|_{L^2} &\le C\|\operatorname{div}(u\cdot\nabla v)\|_{L^2} + \|\operatorname{div}(v\cdot\nabla u^T)\|_{L^2}\\ &\le C\|\nabla u\|_{L^2}\|\nabla v\|_{L^\infty} + \|u\|_{L^\infty}\|\nabla^2 v\|_{L^2}\\ &\quad + \|\nabla v\|_{L^2}\|\nabla u\|_{L^\infty} + \|v\|_{L^\infty}\|\nabla^2 u\|_{L^2}\\ &\le C\|\nabla v\|_{L^2}\|\nabla^2 v\|_{L^2}^{\frac{1}{2}}\|\nabla^3 v\|_{L^2}^{\frac{1}{2}} + \|\nabla v\|_{L^2}^{\frac{1}{2}}\|\nabla^2 v\|_{L^2}^{\frac{3}{2}}\\ &\le C(1+t)^{-\frac{17}{4}}.\end{aligned}$$

By use of the Plancherel theorem and estimates (1.5) and (5.1), it is easy to find that

$$\|p\|_{L^2} = \|\hat{p}\|_{L^2} \le C(1+t)^{-\frac{5}{2}}.$$

Thus we have

$$\|\nabla p\|_{L^2}^2 \le C\|p\|_{L^2}\|\Delta p\|_{L^2} \le C(1+t)^{-\frac{27}{4}}.$$

Then one arrives at

$$C(1+t)^{-\frac{9}{2}} \le \|\partial_t v\|_{L^2}^2 \le C(1+t)^{-\frac{9}{2}}.$$

So we have finished the proof of this lemma. ■

5. Proofs of several technical estimates

In this section we will prove the claimed estimate (3.7), which was used in Section 3.

Proof of inequality (3.7). Recall $a_{kj} = \mathcal{F}(u_k v_j)$. By virtue of the Cauchy inequality, we can easily deduce that

$$|\nabla_\xi a_{kj}| \le C\int |x|\,|u|\,|v|\,dx \le C\int |x|\,|u|^2\,dx + C\int |x|\,|v|^2\,dx.$$

We will prove that the following inequality holds:

$$\int |x|\,|u|^2\,dx + \int |x|\,|v|^2\,dx \le Ct.$$

To this end, we multiply both sides of equation (2.1) by $|x|u$ and integrate over the whole space to acquire

$$\begin{aligned}\int |x|u\cdot\partial_t v\,dx - \nu\int |x|u\Delta v\,dx = &-\int |x|u\cdot(u\cdot\nabla v)\,dx\\ &-\int |x|u\cdot(v\cdot\nabla u^T)\,dx\\ &-\int |x|u\cdot\nabla p\,dx.\end{aligned}$$

Then it holds that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\Big(\int |x|\,|u|^2\,dx + \alpha^2\int |x|\,|\nabla u|^2\,dx - 2\alpha^2\int \frac{|u|^2}{|x|}\,dx\Big)\\ &\quad+\nu\Big(\int |x|\,|\nabla u|^2\,dx + \alpha^2\int |x|\,|\Delta u|^2\,dx\Big)\\ &= \alpha^2\int u_t\frac{x}{|x|}\nabla u\,dx - \nu\int\frac{x}{|x|}u\nabla u\,dx - 2\alpha^2\nu\int \Delta u\cdot\frac{1}{|x|}u\,dx\\ &\quad-2\alpha^2\nu\int\frac{x}{|x|}\Delta u\nabla u\,dx + \int\frac{(u\cdot x)}{|x|}v\cdot u\,dx + \int p\cdot\frac{x}{|x|}u\,dx,\end{aligned}$$

where we have used integration by parts and the incompressibility condition $\operatorname{div} u = 0$ to obtain

$$
\begin{aligned}
-&\int (u\cdot\nabla v)\cdot|x|u\,dx - \int (v\cdot\nabla u^T)|x|u\,dx \\
&= -\sum_{i,j}\int u_i\partial^i v_j|x|u_j\,dx - \sum_{i,j}\int v_j\partial^i u_j|x|u_i\,dx \\
&= \sum_{i,j}\int \partial^i u_i v_j|x|u_j\,dx + \sum_{i,j}\int u_i v_j\partial^i|x|u_j\,dx \\
&\quad + \sum_{i,j}\int u_i v_j|x|\partial^i u_j\,dx - \sum_{i,j}\int v_j\partial^i u_j|x|u_i\,dx \\
&= \sum_{i,j}\int u_i v_j\partial^i|x|u_j\,dx \\
&= \int \frac{(u\cdot x)}{|x|}v\cdot u\,dx.
\end{aligned}
$$

Then by the Hölder inequality, one arrives at

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\Big(\int |x|\,|u|^2\,dx + \alpha^2\int |x|\,|\nabla u|^2\,dx - 2\alpha^2\int \frac{|u|^2}{|x|}\,dx\Big) \\
&\qquad + \nu\Big(\int |x|\,|\nabla u|^2\,dx + \alpha^2\int |x|\,|\Delta u|^2\,dx\Big) \\
&\quad \le C\|u_t\|_{L^2}\|\nabla u\|_{L^2} + C\|u\|_{L^2}\|\nabla u\|_{L^2} + C\Big\|\frac{u}{|x|}\Big\|_{L^2}\|\Delta u\|_{L^2} + C\|\nabla u\|_{L^2}\|\Delta u\|_{L^2} \\
&\qquad + C\|u\|_{L^4}^2\|v\|_{L^2} + C\|p\|_{L^2}\|u\|_{L^2}.
\end{aligned}
$$

With the aid of the Hardy inequality, we can obtain

$$
\Big\|\frac{u}{|x|}\Big\|_{L^2} \le C\|\nabla u\|_{L^2}.
$$

By the Plancherel theorem, we have $\|p\|_{L^2} = \|\hat{p}\|_{L^2}$. Since

$$
\begin{aligned}
v\cdot\nabla u^T &= \sum\nabla(u_i v_i) - \sum u_i\nabla v_i \\
&= \sum\nabla(u_i v_i) - \sum u_i\partial^j v_i \\
&= \sum\nabla(u_i v_i) - \sum u_i\partial^j u_i + \alpha^2\sum u_i\partial^j\partial^k\partial^k u_i \\
&= \sum\nabla(u_i v_i) - \frac{1}{2}\sum\partial^j|u_i|^2 + \alpha^2\sum\partial^k(u_i\partial^j\partial^k u_i) - \frac{\alpha^2}{2}\sum\partial^j|\partial^k u_i|^2,
\end{aligned}
$$

by virtue of the Hölder inequality and the Plancherel theorem, it is easy to deduce that

$$
\begin{aligned}
\|\hat{p}\|_{L^2} &\le C\|\mathscr{F}(u\cdot v)\|_{L^2} + \|\mathscr{F}(u\cdot u)\|_{L^2} + \|\mathscr{F}(u\cdot\nabla^2 u)\|_{L^2} + \|\mathscr{F}|\nabla u|^2\|_{L^2} \\
&= C\|u\cdot v\|_{L^2} + \|u\cdot u\|_{L^2} + \|u\cdot\nabla^2 u\|_{L^2} + \|\,|\nabla u|^2\|_{L^2} \\
&\le C\|u\|_{L^\infty}\|v\|_{L^2} + C\|u\|_{L^4}^2 + C\|u\|_{L^\infty}\|\nabla^2 u\|_{L^2} + C\|\nabla u\|_{L^3}\|\nabla u\|_{L^6} \\
&\le C\|v\|_{L^2}^2. \qquad (5.1)
\end{aligned}
$$

Hence we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\Big(\int |x|\,|u|^2\,dx + \alpha^2\int |x|\,|\nabla u|^2\,dx - 2\alpha^2\int \frac{|u|^2}{|x|}\,dx\Big)\\
&\quad + \nu\Big(\int |x|\,|\nabla u|^2\,dx + \alpha^2\int |x|\,|\Delta u|^2\,dx\Big)\\
&\le C\|u_t\|_{L^2}\|\nabla u\|_{L^2} + C\|u\|_{L^2}\|\nabla u\|_{L^2} + C\|\nabla u\|_{L^2}\|\nabla^2 u\|_{L^2}\\
&\quad + C\|u\|_{L^4}^2\|v\|_{L^2} + C\|v\|_{L^2}^2\|u\|_{L^2}\\
&\le C\|u_t\|_{L^2}\|\nabla u\|_{L^2} + C\|v\|_{L^2}^2 + C\|v\|_{L^2}^3\\
&\le C\|v_t\|_{L^2}\|\nabla u\|_{L^2} + C\|v\|_{L^2}^2 + C\|v\|_{L^2}^3,
\end{aligned}$$

where we have used the interpolation inequality and the basic facts that

$$\|v\|_{L^2}^2 = \|u\|_{L^2}^2 + 2\alpha^2\|\nabla u\|_{L^2}^2 + \alpha^4\|\nabla^2 u\|_{L^2}^2$$

and

$$\|\partial_t u\|_{L^2} \le \|\partial_t v\|_{L^2}.$$

With the aid of bound (1.4), one arrives at

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\Big(\int |x|\,|u|^2\,dx + \alpha^2\int |x|\,|\nabla u|^2\,dx - 2\alpha^2\int \frac{|u|^2}{|x|}\,dx\Big)\\
&\quad + \nu\Big(\int |x|\,|\nabla u|^2\,dx + \alpha^2\int |x|\,|\Delta u|^2\,dx\Big) \le C.
\end{aligned}$$

Finally, we integrate the above inequality with respect to time over $[0, t]$ to acquire

$$\begin{aligned}
&\int |x|\,|u|^2\,dx + \alpha^2\int |x|\,|\nabla u|^2\,dx - 2\alpha^2\int \frac{|u|^2}{|x|}\,dx\\
&\quad + 2\nu\int_0^t\Big(\int |x|\,|\nabla u|^2\,dx + \alpha^2\int |x|\,|\Delta u|^2\,dx\Big)\,ds \le Ct,
\end{aligned}$$

where we have used the Hölder inequality to obtain

$$\begin{aligned}
\int \frac{|u_0|^2}{|x|}\,dx &= \int_{|x|\le 1}\frac{|u_0|^2}{|x|}\,dx + \int_{|x|\ge 1}\frac{|u_0|^2}{|x|}\,dx\\
&\le \Big(\int_{|x|\le 1}\frac{1}{|x|^2}\,dx\Big)^{\frac{1}{2}}\|u_0\|_{L^4}^2 + \|u_0\|_{L^2}^2\\
&\le C\|u_0\|_{L^4}^2 + C\|u_0\|_{L^2}^2\\
&\le C.
\end{aligned} \tag{5.2}$$

Next, we multiply both sides of the first equation of system (1.1) by $|x|v$ and integrate the result over $\mathbb{R}^3$ to get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int |x|\,|v|^2\,dx + \nu\int |x|\,|\nabla v|^2\,dx\\
&\quad = \nu\int \frac{x}{|x|}v\cdot\nabla v\,dx - \int u\cdot\nabla v|x|v\,dx - \int v\cdot\nabla u^T|x|v\,dx + \int \frac{x}{|x|}v\cdot p\,dx.
\end{aligned}$$

We exploit integration by parts, the incompressibility condition, the Hölder and Sobolev inequalities and the bound (1.4) to find

$$\begin{aligned}\int u\cdot\nabla v|x|v\,dx &= \sum_{i,j}\int u_i\partial^i v_j|x|v_j\,dx = -\frac{1}{2}\int\frac{(u\cdot x)}{|x|}|v|^2\,dx\\ &\le C\int|u|\,|v|^2\,dx \le C\|u\|_{L^\infty}\|v\|_{L^2}^2 \le C.\end{aligned}$$

Similarly, we also have

$$\begin{aligned}\int v\cdot\nabla u^T|x|v\,dx &= \int u\cdot\nabla u^T|x|u\,dx - \alpha^2\int\Delta u\cdot\nabla u^T|x|u\,dx\\ &\quad -\alpha^2\int u\cdot\nabla u^T|x|\Delta u\,dx + \alpha^4\int\Delta u\cdot\nabla u^T|x|\Delta u\,dx\\ &\le C\int|x|\,|u|^2|\nabla u|\,dx + C\int|x|\,|u|\,|\nabla u|\,|\Delta u|\,dx\\ &\quad + C\int|x|\,|\nabla u|\,|\Delta u|^2\,dx\\ &\le C\|\,|x|^{\frac12}\nabla u\|_{L^2}\|\,|x|^{\frac12}u\|_{L^6}\|u\|_{L^3}\\ &\quad + C\|u\|_{L^\infty}\|\,|x|^{\frac12}\nabla u\|_{L^2}\|\,|x|^{\frac12}\Delta u\|_{L^2} + C\|\nabla u\|_{L^\infty}\|\,|x|^{\frac12}\Delta u\|_{L^2}^2\\ &\le C\|\,|x|^{\frac12}\nabla u\|_{L^2}^2 + C\|\,|x|^{\frac12}\Delta u\|_{L^2}^2 + C\|\,|x|^{\frac12}u\|_{L^6}^2.\end{aligned}$$

Similar to estimate (5.2), the Sobolev inequality directly implies

$$\begin{aligned}\|\,|x|^{\frac12}u\|_{L^6}^2 \le C\|\nabla(|x|^{\frac12}u)\|_{L^2}^2 &\le C\int\frac{|u|^2}{|x|}\,dx + C\|\,|x|^{\frac12}\nabla u\|_{L^2}^2\\ &\le C + C\|\,|x|^{\frac12}\nabla u\|_{L^2}^2.\end{aligned}$$

Then we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int|x|\,|v|^2\,dx + \nu\int|x|\,|\nabla v|^2\,dx\\ &\quad\le C + C\|v\|_{L^2}\|\nabla v\|_{L^2} + C\int|x|\,|\nabla u|^2\,dx + C\int|x|\,|\Delta u|^2\,dx + C\|v\|_{L^2}\|p\|_{L^2}\\ &\quad\le C + C\|v\|_{H^1}^2 + C\|v\|_{L^2}^3 + C\int|x|\,|\nabla u|^2\,dx + C\int|x|\,|\Delta u|^2\,dx\\ &\quad\le C + C\int|x|\,|\nabla u|^2\,dx + C\int|x|\,|\Delta u|^2\,dx.\end{aligned}$$

Finally, by integrating the above inequality with respect to time over $[0,t]$, it holds that

$$\int|x|\,|v|^2\,dx + 2\nu\int_0^t\int|x|\,|\nabla v|^2\,dx\,ds \le Ct.$$

Thus, we have completed the proof for the claimed estimate (3.7). ■

A. Analytic tools

In what follows we give several useful lemmas which will be applied to our proof. The following two inequalities play an important role in establishing estimates. The first one is called the Sobolev interpolation of the Gagliardo–Nirenberg inequality.

Lemma A.1 ([45]). *Let* $0 \le m, \alpha \le l$ *and the function* $f \in C_0^\infty(\mathbb{R}^n)$*; then we have*

$$\|\nabla^\alpha f\|_{L^p} \le C\|\nabla^m f\|_{L^2}^{1-\theta}\|\nabla^l f\|_{L^2}^{\theta}, \tag{A.1}$$

where θ *satisfies*

$$0 \le \theta \le 1$$

and α, m, l *satisfy*

$$\frac{1}{p} - \frac{\alpha}{n} = \Big(\frac{1}{2} - \frac{m}{n}\Big)(1-\theta) + \Big(\frac{1}{2} - \frac{l}{n}\Big)\theta.$$

Lemma A.2 ([41]). *Let* $|k| \le s$ *and assume that the functions* $f, g \in H^s \cap L^\infty$*; then there exists a constant* C *that depends on* s *such that*

$$\|\nabla^k(fg)\|_{H^s} \le C(\|\nabla^s f\|_{L^2}\|g\|_{L^\infty} + \|f\|_{L^\infty}\|\nabla^s g\|_{L^2}).$$

The following Hardy inequality is useful for singular weighted estimates.

Lemma A.3 ([2]). *For any function* $f \in \dot{H}^1(\mathbb{R}^n)$ *with* $n \ge 3$*, it holds that*

$$\Big(\int_{\mathbb{R}^n} \frac{|f(x)|^2}{|x|^2}\,dx\Big)^{\frac{1}{2}} \le \frac{2}{n-2}\|\nabla f\|_{L^2}.$$

The so-called Borchers lemma could be used to obtain some properties of the initial data of the unfiltered velocity. The proof of this lemma can be found in [52].

Lemma A.4 ([52]). *Suppose that the function* $u \in L^1(\mathbb{R}^n)^n \cap H$ *for any* $n \ge 1$*; then*

$$\int_{\mathbb{R}^n} u\,dx = 0.$$

The following two auxiliary lemmas, whose proofs are shown in [52], are used to derive the lower bound of decay rate for the heat equation.

Lemma A.5 ([52]). *Denote* $P(\xi) := I - \mu(\xi)$ *with* I *the identity matrix and*

$$\mu(\xi) = \frac{1}{|\xi|^2}(\xi_k\xi_j)_{1\le k,j\le 3},$$

where $\xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3 \setminus \{0\}$*. Suppose that* $S = (s_{kj})$ *is a symmetric matrix; then it holds that*

$$\int_{S^2} P(\omega')S\omega' \cdot S\omega'\,d\omega' = \frac{\pi^{\frac{3}{2}}}{6\Gamma(\frac{3}{2})}\Big(\sum_{k\ne j}(s_{kk} - s_{jj})^2 + 6\sum_{k\ne j} s_{kj}^2\Big).$$

Lemma A.6 ([52]). *Let V be a 3×3 complex matrix and B be a 3×3 real matrix. Suppose that for every $\omega' \in S^2$, it holds that*

$$V\omega' - i(I - \mu(\omega'))B\omega' = 0;$$

then we have that the matrix $V = 0$ and the matrix B is a scalar matrix.

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