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Filtrations on Chow Groups and Transcendence Degree†

By

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Abstract

For a smooth complex projective variety X defined over a number field, we have filtrations on the Chow groups depending on the choice of realizations. If the realization consists of mixed Hodge structure without any additional structure, we can show that the obtained filtration coincides with the filtration of Green and Griffiths, assuming the Hodge conjecture. In the case the realizations contain Hodge structure and etale cohomology, we prove that if the second graded piece of the filtration does not vanish, it contains a nonzero element which is represented by a cycle defined over a field of transcendence degree one. This may be viewed as a refinement of results of Nori, Schoen, and Green-Griffiths-Paranjape. For higher graded pieces we have a similar assertion assuming a conjecture of Beilinson and Grothendieck's generalized Hodge conjecture.

Introduction

Let $X_{\mathbb{C}}$ be a smooth complex projective variety, and $\mathrm{CH}^p(X_{\mathbb{C}})_{\mathbb{Q}}$ be the Chow group with rational coefficients. Choosing a category $\mathcal{M}$ of realizations (see [12], [13], [20]), we can define a filtration $F_{\mathcal{M}}$ on $\mathrm{CH}^p(X_{\mathbb{C}})_{\mathbb{Q}}$ by spreading cycles out, see (1.4) below (and also [1], [16], [26], [30]). By definition $F^1_{\mathcal{M}}\mathrm{CH}^p(X_{\mathbb{C}})_{\mathbb{Q}}$ consists of null homologous cycles, and $F^2_{\mathcal{M}}\mathrm{CH}^p(X_{\mathbb{C}})_{\mathbb{Q}}$ is contained in the kernel of the Abel-Jacobi map (tensored with $\mathbb{Q}$). It is conjectured that the filtration $F_{\mathcal{M}}$ does not depend on the choice of $\mathcal{M}$, and coincides with

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Murre's conjectural filtration [24]. We can verify this conjecture, assuming a conjecture of Beilinson on the injectivity of the Abel-Jacobi map for smooth projective varieties over number fields [2] together with the Hodge conjecture.

In this paper we assume that $X_{\mathbb{C}}$ is defined over a number field k. Then a similar filtration has been defined by M. Green and P. Griffiths [16], and we have

Proposition 0.1. *If $\mathcal{M}$ is the category of mixed Hodge structure without any additional structure, then the filtration $F_{\mathcal{M}}$ coincides with the filtration F_G of Green and Griffiths* [16], *assuming the Hodge conjecture.*

Let X be a smooth projective k-variety whose base change by $k \to \mathbb{C}$ is $X_{\mathbb{C}}$. Let K be a subfield of $\mathbb{C}$ containing k, and having finite transcendence degree. Let X_K be the base change of X by $k \to K$. Then $\mathrm{CH}^p(X_K)_{\mathbb{Q}}$ is identified with a subgroup of $\mathrm{CH}^p(X_{\mathbb{C}})_{\mathbb{Q}}$, and has the induced filtration $F_{\mathcal{M}}$. It has been observed by Green and Griffiths [16] that the property of this induced filtration is very much influenced by the transcendence degree d of K. For example, $\mathrm{Gr}^r_{F_{\mathcal{M}}}\mathrm{CH}^p(X_K)_{\mathbb{Q}}$ vanishes for $r > d+1$ if the realization consists of mixed Hodge structure. If $d = 0$, it is conjectured that $F^2_{\mathcal{M}}\mathrm{CH}^p(X_K)_{\mathbb{Q}} = 0$ by the above conjecture of Beilinson. However, for $d = 1$, it is shown by M. Nori and C. Schoen [32] that the kernel of the Albanese map for certain surfaces has a nontrivial cycle defined over a subfield K of transcendence degree 1. Here we can show also the nonvanishing of $\mathrm{Gr}^2_{F_{\mathcal{M}}}\mathrm{CH}^2(X_{\mathbb{C}})_{\mathbb{Q}}$ (see [26]), which implies that the above estimate is optimal. The results of Nori and Schoen are recently generalized by Green-Griffiths-Paranjape [17] to the case of surfaces having a nontrivial global 2-form. Considering these, we may have

Conjecture 0.2. *If $\mathrm{Gr}^r_{F_{\mathcal{M}}}\mathrm{CH}^p(X_{\mathbb{C}})_{\mathbb{Q}} \neq 0$ with $r \geq 1$, then it contains a nonzero element which is represented by a cycle defined over a subfield of transcendence degree $r-1$.*

In this paper we prove

Theorem 0.3. *Assume that the realizations contain mixed Hodge structure and étale cohomology with Galois action. Then Conjecture* (0.2) *is true for $r = 1, 2$. Assume further that Grothendieck's generalized Hodge conjecture holds, and the filtration $F_{\mathcal{M}}$ coincides with the filtration F_{MHS} associated to the category of realization consisting of mixed Hodge structure. Then Conjecture* (0.2) *is true also for $r \geq 3$.*

The proof uses Terasoma's argument on Hilbert's irreducibility theorem [35] as in [17]. The same argument was also indicated by A. Tamagawa when we tried to construct an l-adic theory of normal functions [29]. It is quite interesting that we cannot prove Theorem (0.3) by using only Hodge theory. The hypothesis of (0.3) for $r = 2$ is satisfied for 0-cycles if X has a nontrivial global 2-from [30] (this follows from Murre's Albanese motive [23] and Bloch's diagonal cycle [7]). So Theorem (0.3) may be viewed as a refinement of the result of Green-Griffiths-Paranjape [17]. For cycles of arbitrary codimension, we have a similar assertion if the standard conjecture of Lefschetz-type holds for X.

If we restrict to the subgroup $\mathrm{CH}^p_{\mathrm{alg}}(X_{\mathbb{C}})_{\mathbb{Q}}$ consisting of cycles algebraically equivalent to zero, $F^2_{\mathcal{M}}$ coincides with the kernel of the Abel-Jacobi map (or that of the l-adic Abel-Jacobi map). This applies to the case of 0-cycles on surfaces, and we have in general $\mathrm{Gr}^r_{F_{\mathcal{M}}}\mathrm{CH}^p(X_{\mathbb{C}})_{\mathbb{Q}} = 0$ for $r > p$ (see also [16]). However it is not yet clear whether the nonvanishing of the kernel of the Albanese map for a surface $X_{\mathbb{C}}$ implies that $\mathrm{Gr}^2_{F_{\mathcal{M}}}\mathrm{CH}^2(X_{\mathbb{C}})_{\mathbb{Q}} \neq 0$, because it is not proved that the filtration $F_{\mathcal{M}}$ is separated.

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§1. Filtrations on Chow Groups

§1.1. Realizations

We will denote by $\mathcal{M}$ a category of (systems of) realizations, see [12], [13], [20], etc. The simplest example in our case is the abelian category MHS of $\mathbb{Q}$-mixed Hodge structures whose graded pieces Gr^W_m are polarizable [11]. In this paper we choose a number field k contained in $\mathbb{C}$. Then we have the category MHS_k of mixed $\mathbb{Q}$-Hodge structures with k-structure, see [26], [30], etc. Let $\overline{k}$ be the algebraic closure of k in $\mathbb{C}$, and put $G = \mathrm{Gal}(\overline{k}/k)$. For a prime number l, we have an abelian category $\mathcal{M}_l$ whose object consists of filtered vector spaces $(H_{\mathbb{Q}}, W)$ over $\mathbb{Q}$, (H_l, W) over $\mathbb{Q}_l$ and $(H_{\mathbb{C}}, F)$ over $\mathbb{C}$ together with isomorphisms

$$\alpha_l : (H_{\mathbb{Q}}, W)\otimes_{\mathbb{Q}}\mathbb{Q}_l = (H_l, W), \quad \alpha_{\mathbb{C}} : H_{\mathbb{Q}}\otimes_{\mathbb{Q}}\mathbb{C} = H_{\mathbb{C}},$$

where (H_l, W) is endowed with a continuous action of G and $(\mathrm{Gr}^W_m H_{\mathbb{Q}}, \mathrm{Gr}^W_m (H_{\mathbb{C}}, F))$ is a polarizable $\mathbb{Q}$-Hodge structure of weight m for any m (here W denotes also the induced filtration on $H_{\mathbb{C}}$), see [12], [13], [20], etc.

A polarization of a pure object H of weight n is a compatible system of perfect pairings on the underlying vector spaces which gives a polarization of

Hodge structure, and induces a morphism $H \otimes H \to \mathbb{Q}(-n)$ (or equivalently $H \to (\mathbb{D}H)(-n)$, where $\mathbb{D}H$ is the dual of H and the last morphism is an isomorphism). In particular, a polarization is compatible with the Galois action. Note that the restriction of a polarization to a subobject H' is a polarization, and this implies the semisimplicity of pure objects, because the injection $H' \to H$ induces a splitting $H \to H'$ using the polarization. (This semisimplicity does not necessarily imply the semisimplicity of the Galois action, and conversely, even if there exists a splitting for each realization, the compatibility of the splittings is not trivial unless a polarization is used.)

For other examples, we have $\mathcal{M}_{\text{ét}}$ by considering H_l for any prime numbers l, and $\mathcal{M}_{k,l}, \mathcal{M}_{k,\text{ét}}$ by considering also the k-structure. It is also possible to consider the category of systems of realizations as in [20].

Note that the category $\mathcal{M}$ can be extended naturally to the category of mixed sheaves $\mathcal{M}(S/k)$ for any k-variety S, and there is a forgetful functor from $\mathcal{M}(S/k)$ to the category of perverse sheaves [6], see [30] for the details.

§1.2. Deligne cohomology

Let $\mathcal{M}$ be one of the categories of realizations as in (1.1). Let X be a smooth k-variety. Then the cohomology $H^i(X/k, \mathbb{Q})$ is well-defined in $\mathcal{M}$, using de Rham cohomology of X_k, étale cohomology of $X_{\overline{k}}$, and cohomology of $X_{\mathbb{C}}$ together with comparison isomorphisms, see [12], [13], [20], etc. Furthermore, there exists canonically $K_{\mathcal{H}}(X/k)$ in the bounded derived category $D^b\mathcal{M}$ whose cohomology is isomorphic to the cohomology of X (using, for example, two sets of affine open coverings associated to general hyperplane sections [5], see also [26, 1.1]).

We define Deligne cohomology by

$$H^i_{\mathcal{D}}(X/k, \mathbb{Q}(j)) = \mathrm{Hom}_{D^b\mathcal{M}}(\mathbb{Q}, K_{\mathcal{H}}(X/k)(j)[i]),$$

where (j) is the Tate twist, and $[i]$ is the shift of complexes. If $\mathcal{M} = \mathrm{MHS}$, it is called the absolute p-Hodge cohomology in [3]. If $\mathcal{M} = \mathrm{MHS}$ or MHS_k, then higher extension groups vanish in $\mathcal{M}$ as a corollary of [10] (see [30]), and we have canonical short exact sequences

(1.2.1)
$$0 \to \mathrm{Ext}^1_{\mathcal{M}}(\mathbb{Q}, H^{i-1}(X/k, \mathbb{Q})(j)) \to H^i_{\mathcal{D}}(X/k, \mathbb{Q}(j)) \\ \to \mathrm{Hom}_{\mathcal{M}}(\mathbb{Q}, H^i(X/k, \mathbb{Q})(j)) \to 0.$$

For a closed subvariety Z of X, we can define similarly the Deligne local cohomology $H^i_{\mathcal{D},Z}(X/k, \mathbb{Q}(j))$ using a complex $K_{\mathcal{H},Z}(X/k)$, which is the shifted mapping cone of $K_{\mathcal{H}}(X/k) \to K_{\mathcal{H}}((X \setminus Z)/k)$.

We have the cycle map

$$cl : \mathrm{CH}^p(X)_{\mathbb{Q}} \to H^{2p}_{\mathcal{D}}(X/k, \mathbb{Q}(p)), \tag{1.2.2}$$

which is compatible with the usual cycle class map to $H^{2p}(X_{\mathbb{C}}, \mathbb{Q})(p)$. Its restriction to the null homologous cycles coincides with Griffiths' Abel-Jacobi map [18] tensored with $\mathbb{Q}$ if $k = \mathbb{C}$, $\mathcal{M} = \mathrm{MHS}$ and X is smooth proper, see [9], [14], [15], [19], etc. We can show that (1.2.2) is compatible with the direct image by a proper morphism and the pull-back by any morphism, and hence with the action of a correspondence, cf. [28].

§1.3. Leray filtration

Let X, S be a smooth k-varieties. Then the Deligne cohomology $H^i_{\mathcal{D}}(X\times_k S/k, \mathbb{Q}(j))$ has the (decreasing) Leray filtration F_L induced by the canonical filtration τ on $K_{\mathcal{H}}(X/k)$ using the canonical isomorphism

$$K_{\mathcal{H}}(X\times_k S/k) = K_{\mathcal{H}}(X/k)\otimes K_{\mathcal{H}}(S/k).$$

Here F_L^r on $H^i_{\mathcal{D}}(X\times_k S/k, \mathbb{Q}(j))$ is induced by $\tau_{\le i-r}$ as in [11]. Assume X is smooth proper. Then the filtration F_L splits because we have a non canonical isomorphism

$$K_{\mathcal{H}}(X/k) \simeq \sum_j H^j(X/k, \mathbb{Q})[-j] \quad \text{in } D^b\mathcal{M}. \tag{1.3.1}$$

(This follows from a general property of pure complexes, see e.g. [27].) In particular, for a morphism $S' \to S$, the filtration F_L is strictly compatible with the pull-back morphism

$$H^i_{\mathcal{D}}(X\times_k S/k, \mathbb{Q}(j)) \to H^i_{\mathcal{D}}(X\times_k S'/k, \mathbb{Q}(j)).$$

By the canonical filtration on $K_{\mathcal{H}}(S/k)$, we have for each $m \in \mathbb{Z}$ the Leray spectral sequence

$$\begin{aligned} E_2^{p,q} &= \mathrm{Ext}^{p-m}_{\mathcal{M}}(\mathbb{Q}, H^m(X/k, \mathbb{Q})\otimes H^q(S/k, \mathbb{Q})(j)) \\ &\Rightarrow \mathrm{Gr}^{p+q-m}_{F_L} H^{p+q}_{\mathcal{D}}(X\times_k S/k, \mathbb{Q}(j)) \end{aligned} \tag{1.3.2}$$

It is conjectured that this degenerates at E_2, because $K_{\mathcal{H}}(S/k)$ would be defined in the (conjectural) category of motives where higher extension groups should vanish so that a decomposition similar to (1.3.1) would hold.

We will denote by F_L' the decreasing filtration on $\mathrm{Gr}^r_{F_L} H^i_{\mathcal{D}}(X\times_k S/k, \mathbb{Q}(j))$ induced by the canonical filtration τ on $K_{\mathcal{H}}(S/k)$ so that $\mathrm{Gr}^s_{F_L'} \mathrm{Gr}^r_{F_L} H^i_{\mathcal{D}}$

$(X\times_k S/k,\mathbb{Q}(j))$ is a subquotient of $\mathrm{Ext}^s_{\mathcal{M}}(\mathbb{Q},H^{i-r}(X/k,\mathbb{Q})\otimes H^{r-s}(S/k,\mathbb{Q})(j))$.

In the case $\mathcal{M}=\mathrm{MHS}$ or MHS_k, the higher extension groups really vanish so that (1.3.2) degenerates at E_2 and we get canonical short exact sequences

(1.3.3)
$$0\to \mathrm{Ext}^1_{\mathcal{M}}(\mathbb{Q},H^{i-r}(X/k,\mathbb{Q})\otimes H^{r-1}(S/k,\mathbb{Q})(j))\to \mathrm{Gr}^r_{F_L}H^i_{\mathcal{D}}(X\times_k S/k,\mathbb{Q}(j))$$
$$\to \mathrm{Hom}_{\mathcal{M}}(\mathbb{Q},H^{i-r}(X/k,\mathbb{Q})\otimes H^r(S/k,\mathbb{Q})(j))\to 0.$$

In particular, $F'^2_L\mathrm{Gr}^r_{F_L}=0$ in this case.

§1.4. Filtration on Chow groups

Let X a smooth k-variety, and K be a subfield of $\mathbb{C}$ containing k, and having finite transcendence degree over k. Put $X_K=X\otimes_k K$, and $X_{\mathbb{C}}=X\otimes_k\mathbb{C}$. Then we have natural injections

$$\mathrm{CH}^p(X_K)_{\mathbb{Q}}\to \mathrm{CH}^p(X_{\mathbb{C}})_{\mathbb{Q}}, \tag{1.4.1}$$

and $\cup_K\mathrm{CH}^p(X_K)_{\mathbb{Q}}=\mathrm{CH}^p(X_{\mathbb{C}})_{\mathbb{Q}}$.

Let $\zeta\in \mathrm{CH}^p(X_K)_{\mathbb{Q}}$. By spreading out [7], there exists an irreducible smooth affine k-variety S such that $k(S)=K$ and ζ is defined over S, i.e. there exists $\zeta_S\in \mathrm{CH}^p(X\times_k S)_{\mathbb{Q}}$ whose restriction to X_K is ζ, where X_K is identified with the generic fiber of $X\times_k S\to S$. For an open subvariety S' of S, let $\zeta_{S'}$ denote the restriction of ζ_S over S'. Then the limit of $\zeta_{S'}$ is well-defined, see [7].

Let k_S be the algebraic closure of k in $\Gamma(S,\mathcal{O}_S)$, and put $S_{\mathbb{C}}=S\otimes_{k_S}\mathbb{C}$. This is an irreducible variety, i.e. S is geometrically irreducible over k_S. (If we consider $S\otimes_k\mathbb{C}$ instead of $S\otimes_{k_S}\mathbb{C}$, then the former is a disjoint union of copies of the latter in the case k_S is a normal extension of k.) Note that $X\times_k S=X_{k_S}\times_{k_S}S$, and this allows us to replace k with k_S. Actually we can replace k with any finite extension, because we take the limit over K.

The cycle map (1.2.2) induces

$$cl:\mathrm{CH}^p(X\times_k S)_{\mathbb{Q}}\to H^{2p}_{\mathcal{D}}(X\times_k S/k_S,\mathbb{Q}(p)), \tag{1.4.2}$$

and the filtration $F_{\mathcal{M}}$ on $\mathrm{CH}^p(X\times_k S)_{\mathbb{Q}}$ is defined to be the induced filtration by the Leray filtration F_L on $H^{2p}_{\mathcal{D}}(X\times_k S/k_S,\mathbb{Q}(p))$. Then, taking the inductive limit over the non empty open subvarieties of S, we get the filtration $F_{\mathcal{M}}$ on $\mathrm{CH}^p(X_K)_{\mathbb{Q}}$.

This means that $\zeta \in F_{\mathcal{M}}^r \mathrm{CH}^p(X_K)_{\mathbb{Q}}$ if $cl(\zeta_S) \in F_L^r H_{\mathcal{D}}^{2p}(X \times_k S/k_S, \mathbb{Q}(p))$ for some S, and hence $\mathrm{Gr}_{F_{\mathcal{M}}}^r \zeta$ is nonzero in $\mathrm{Gr}_{F_{\mathcal{M}}}^r \mathrm{CH}^p(X_K)_{\mathbb{Q}}$ if the restrictions of $\mathrm{Gr}_{F_L}^r cl(\zeta_S)$ to $\mathrm{Gr}_{F_L}^r H_{\mathcal{D}}^{2p}(X \times_k S'/k_S, \mathbb{Q}(p))$ does not vanish for any non empty open subvarieties S' of S.

We can show that $F_{\mathcal{M}}$ is strictly compatible with the base change by $K \to K'$, see [30]. This implies that $\mathrm{CH}^p(X)_{\mathbb{Q}}$ has the filtration $F_{\mathcal{M}}$ which is strictly compatible with (1.4.1).

§1.5. Filtration of Green and Griffiths

In the case $\mathcal{M} = \mathrm{MHS}$, a similar filtration is constructed by M. Green and P. Griffiths [16]. They assume that the S in (1.4) are smooth *projective*, and then, roughly speaking, consider everything modulo ambiguity coming from cycles over proper closed subvarieties of S (here they also assume Grothendieck's generalized Hodge conjecture). More precisely, for a smooth projective k-variety S and a divisor Z of S defined over k_S, we have an exact sequence

$$\mathrm{CH}^{p-1}(X \times_k Z)_{\mathbb{Q}} \to \mathrm{CH}^p(X \times_k S)_{\mathbb{Q}} \to \mathrm{CH}^p(X \times_k (S \setminus Z))_{\mathbb{Q}} \to 0, \tag{1.5.1}$$

and we define the filtration F_G of Green and Griffiths on $\mathrm{CH}^p(X \times_k (S \setminus Z))_{\mathbb{Q}}$ in this paper to be the quotient filtration of $F_{\mathcal{M}}$ on $\mathrm{CH}^p(X \times_k S)_{\mathbb{Q}}$, where $\mathcal{M} = \mathrm{MHS}$. Then we take the inductive limit as before.

Proposition 1.6. *$F_G = F_{\mathcal{M}}$, assuming the Hodge conjecture.*

Proof. It is enough to show the assertion on $\mathrm{CH}^p(X \times_k (S \setminus Z))_{\mathbb{Q}}$. This is reduced to the case Z is a divisor with normal crossings by using an embedded resolution. We have a canonical morphism of (1.5.1) to

$$\begin{aligned} H_{\mathcal{D},Z}^{2p}(X \times_k S/k_S, \mathbb{Q}(p)) &\to H_{\mathcal{D}}^{2p}(X \times_k S/k_S, \mathbb{Q}(p)) \\ &\to H_{\mathcal{D}}^{2p}(X \times_k (S \setminus Z)/k_S, \mathbb{Q}(p)). \end{aligned}$$

Here we may assume $k_S = k$ (and similarly for intersections of irreducible components of Z) replacing k if necessary. Assuming the Hodge conjecture, we have to prove the following:

For $\zeta \in F_{\mathcal{M}}^r \mathrm{CH}^p(X \times_k S)_{\mathbb{Q}}$ such that $\mathrm{Gr}_{F_L}^r cl(\zeta) \in \mathrm{Gr}_{F_L}^r H_{\mathcal{D}}^{2p}(X \times_k S, \mathbb{Q}(p))$ comes from $\xi \in \mathrm{Gr}_{F_L}^r H_{\mathcal{D},Z}^{2p}(X \times_k S, \mathbb{Q}(p))$, there exists $\zeta' \in \mathrm{CH}^{p-1}(X \times_k Z)_{\mathbb{Q}}$

such that the image of $cl(\zeta')$ in $H^{2p}_{\mathcal{D}}(X\times_k S, \mathbb{Q}(p))$ belongs to F^r_L, and coincides with $\mathrm{Gr}^r_{F_L} cl(\zeta)$ modulo F^{r+1}_L.

This is verified by using correspondences $\Gamma_a \in \mathrm{CH}^{\dim S-1}(S\times_k \widetilde{Z})_{\mathbb{Q}}$ such that

$$(\Gamma_a)_*:H^j(S,\mathbb{Q}) \to H^{j-2}(\widetilde{Z},\mathbb{Q})(-1)$$

vanishes for $j \neq a$, and the restriction of $i_*(\Gamma_a)_*$ to $\mathrm{Im}\, i_* \subset H^a(S, \mathbb{Q})$ is the identity for $j = a$, where $\widetilde{Z}$ is the normalization of Z. Indeed, if we denote by

$$\xi_0 \in \mathrm{Hom}_{\mathrm{MHS}}(\mathbb{Q},H^{2p-r}(X,\mathbb{Q})\otimes H^r_Z(S,\mathbb{Q})(p))$$

the image of ξ by the canonical morphism, then the Hodge conjecture implies the existence of $\zeta' \in \mathrm{CH}^{p-1}(X\times_k \widetilde{Z})_{\mathbb{Q}}$ such that the Künneth component of the cycle class of ζ' in $\mathrm{Hom}(\mathbb{Q}, H^{2p-a}(X, \mathbb{Q})\otimes H^a_Z(S, \mathbb{Q})(p))$ coincides with ξ_0 for $a = r$, and is zero otherwise. We may assume further that the image of $cl(\zeta')$ in $H^{2p}_{\mathcal{D}}(X\times_k S, \mathbb{Q}(p))$ belongs to F^r_L by modifying ζ' using Γ_a for $a < r$ together with the decomposition (1.3.1). So the assertion is reduced to the case $\xi_0 = 0$ by modifying ζ using ζ'. Then the assertion follows by using Γ_a for $a = r$. □

§2. Proof of Theorem (0.3)

§2.1. Hilbert's irreducibility theorem

We first recall Terasoma's argument [35] on Hilbert's irreducibility theorem, which is essential for the proof of (0.3). Let U be a non empty open subvariety of $\mathbb{P}^1_k$, and

$$0 \to L \to \widetilde{L} \to \mathbb{Q}_{l,U} \to 0 \qquad (2.1.1)$$

be a short exact sequence of smooth $\mathbb{Q}_l$-sheaves on U, where $\mathbb{Q}_{l,U}$ denotes the constant sheaf of rank one on U. Put $K = k(U)$, and let $\overline{K}$ be an algebraic closure of K. There exists a k-valued point x of U such that (2.1.1) splits if and only if its restriction over x does. Indeed, choosing a geometric point over x on each Galois étale covering of U in a compatible way with natural projections, we get a morphism of $\mathrm{Gal}(\overline{k}/k)$ to $\pi_1(U, \mathrm{Spec}\, \overline{K})$, and hence to the arithmetic monodromy group of $\widetilde{L}$. Then we have infinitely many k-valued points x such that the last morphism is surjective by Hilbert's irreducibility theorem [22] together with the structure of the l-adic monodromy group [33], see [35]. Related to the l-adic theory of normal functions, the same argument was indicated by A. Tamagawa, see [29].

Here it is also possible to get infinitely many k-valued points x such that the above property holds for the monodromy groups of L and $\widetilde{L}$ simultaneously by the theory of Hilbert set. Note also that the exact sequence (2.1.1) can be replaced by a short exact sequence $0 \to L^1 \to \widetilde{L} \to L^0 \to 0$ of smooth $\mathbb{Q}_l$-sheaves, because

$$\mathrm{Ext}^1(L^0, L^1) = \mathrm{Ext}^1(\mathbb{Q}_{l,U}, \mathcal{H}om(L^0, L^1)).$$

§2.2. Restriction of extension classes

Let $f : S \to U$ be a smooth projective morphism of smooth irreducible k-varieties where U is a non empty open subvariety of $\mathbb{P}^1_k$. Let $n = \dim S - 1$, and $L = R^n f_* \mathbb{Q}_X \in \mathcal{M}(U/k)$ where $\mathcal{M}(U/k)$ denotes the category of mixed sheaves on U (shifted by $\dim U$), and L is pure of weight n, see [30]. Here we assume that there is a forgetful functor from $\mathcal{M}$ to $\mathcal{M}_l$ in (1.1). By semisimplicity we have a direct sum decomposition

$$L = L' \oplus L'' \quad \text{in } \mathcal{M}(U/k)$$

such that $H^0(U/k, L') = 0$ and L'' is constant over $\mathrm{Spec}\, k$ (i.e. the pull-back of an object on $\mathrm{Spec}\, k$ by the structure morphism).

Let H be a pure object of weight $n+1$ in $\mathcal{M}$ (e.g. a direct factor of $H^i(X/k, \mathbb{Q})(q)$ for a smooth projective k-variety X where $i - 2q = n + 1$). Let $H_U = a_U^* H$, where $a_U : U \to \mathrm{Spec}\, k$ is the structure morphism. By the adjunction for a_U, we have a natural isomorphism

$$\mathrm{Ext}^1_{\mathcal{M}(U/k)}(H_U, L) = \mathrm{Hom}_{D^b\mathcal{M}}(H, (a_U)_* L[1]).$$

This implies

$$\begin{aligned} \mathrm{Ext}^1_{\mathcal{M}(U/k)}(H_U, L') &= \mathrm{Hom}_{\mathcal{M}}(H, H^1(U/k, L')), \\ \mathrm{Ext}^1_{\mathcal{M}(U/k)}(H_U, L'') &= \mathrm{Ext}^1_{\mathcal{M}}(H, L''_x), \end{aligned} \tag{2.2.1}$$

for any k-valued point x of U, because $\mathrm{Hom}_{\mathcal{M}}(H, H^1(U/k, L'')) = 0$.

Let $\xi \in \mathrm{Hom}_{\mathcal{M}}(H, H^1(U/k, L'))$. The corresponding extension class is denoted also by ξ. If $\xi \neq 0$, there exists a k-valued point x of U such that the restriction ξ_x of ξ to x does not vanish by (2.1), because (2.2.1) holds also for l-adic sheaves. Note that the same argument still holds after replacing k by a finite extension. In the case $\dim S = 1$ and $n = 0$, we may also assume that $f^{-1}(x)$ consists of one point. Then replacing U, L with $S, \mathbb{Q}_{l,S}$, the restriction of ξ to some k-valued point of S does not vanish (replacing k if necessary).

§2.3. Restriction to open subvarieties

With the above notation and assumptions, let $S_x = f^{-1}(x)$. Then L'_x is a direct factor of $H^n(S_x/k, \mathbb{Q})$, and we get

$$\xi_x \in \mathrm{Ext}^1_{\mathcal{M}}(H, H^n(S_x/k, \mathbb{Q})).$$

We now consider to restrict ξ_x to a non empty open subvariety S'_x of S_x. We assume that the underlying Hodge structure of H does not have a nontrivial subobject with level $< n$, where the level of a Hodge structure is the difference between the maximal and minimal numbers p such that $\mathrm{Gr}^p_F \neq 0$ (and the difference between level and weight is even). Let

$$H' = H^n(S'_x/k, \mathbb{Q}).$$

It has weights $\geq n$. If S'_x is sufficiently small, we have

$$W_n H' = H^n(S_x/k, \mathbb{Q})/N^1 H^n(S_x/k, \mathbb{Q}),$$

where N is the 'coniveau' filtration.

We have $\mathrm{Hom}_{\mathcal{M}}(H, H'/W_n H') = \mathrm{Hom}_{\mathcal{M}}(H, \mathrm{Gr}^W_{n+1} H') = 0$, because $\mathrm{Gr}^W_{n+1} H'$ has level $< n$ (see [11]). This implies the injection

$$\mathrm{Ext}^1_{\mathcal{M}}(H, W_n H') \to \mathrm{Ext}^1_{\mathcal{M}}(H, H'),$$

by the long exact sequence associated to

$$0 \to W_n H' \to H' \to H'/W_n H' \to 0. \tag{2.3.1}$$

Using also the long exact sequence associated to

$$0 \to N^1 H^n(S_x/k, \mathbb{Q}) \to H^n(S_x/k, \mathbb{Q}) \to W_n H' \to 0,$$

we see that the restriction of ξ_x to S'_x does not vanish if its image in $\mathrm{Ext}^1_{\mathcal{M}}$ $(H, W_n H')$ does not vanish, i.e. if ξ_x does not come from $\mathrm{Ext}^1_{\mathcal{M}}(H, N^1 H^n$ $(S_x/k, \mathbb{Q}))$.

In the case $\dim S = 2$ and $n = 1$, the last condition is trivially satisfied because $N^1 H^1(S_x/k, \mathbb{Q}) = 0$. Furthermore, $H'/W_n H'$ is a direct sum of copies of $\mathbb{Q}$, replacing k with a finite extension (depending on S'_x) if necessary. Indeed, it is given by taking a basis of the kernel of the cycle class map $\sum_i \mathbb{Z}[D_i] \to H^2(S_x/k, \mathbb{Q})(1)$ where the D_i are the irreducible components of $S_x \setminus S'_x$, which may be assumed to be absolutely irreducible (replacing k if necessary). This fact will be used in (2.4).

In general, $L'_{1,x} := N^1H^n(S_x/k,\mathbb{Q}) \cap L'_x$ does not vanish. However, it corresponds to a $\mathbb{Q}_l$-submodule stable by the action of $\mathrm{Gal}(\overline{k}/k)$, and is hence extended to an étale subsheaf L'_1 of L' by the argument in (2.1). Let $s = \operatorname{rank} L'_1$. Then taking the pull-back to $U_{\mathbb{C}}$, it determines a subsheaf with $\mathbb{Q}$ coefficients, and the latter underlies a variation of Hodge structure. Indeed, $\wedge^s L'_1$ determines a variation of Hodge structure of rank 1 contained in $\wedge^s L'$ by the global invariant cycle theorem (using a finite covering if necessary, because the monodromy of $\wedge^s L'_1$ is defined over $\mathbb{Z}$ and is finite, see [11]). Then L'_1 is the kernel of $L' \to \wedge^{s+1}L'$ defined locally by a generator of $\wedge^s L'_1$, and hence underlies a variation of Hodge structure.

This argument implies that the restriction of ξ_x to S'_x does not vanish if $\xi \in \mathrm{Hom}_{\mathcal{M}}(H, H^1(U/k, L'))$ is nonzero. Indeed, if the restriction vanishes, the corresponding l-adic extension class comes from $L'_{1,x}\,(\subset L'_x)$ which is extended to $L'_1\,(\subset L')$. We apply some argument in (2.1) also to L'/L'_1, where we may assume $H = \mathbb{Q}$ by the last remark of (2.1) and the monodromy group of the extension for L'/L'_1 is a quotient of that for L'. Then we see that ξ comes from $\mathrm{Hom}(H, H^1(U/k, L'_1))$, where Hom is considered in $\mathcal{M}_l$. But $H^1(U/k, L'_1)$ has level $< n$, because the stalk of L'_1 has level $\le n-2$, see [36]. So ξ vanishes by the hypothesis on the level of H, and the assertion follows.

§2.4. Proof of (0.3)

By hypothesis there exists a smooth irreducible affine k-variety S together with $\zeta \in \mathrm{CH}^p(X\times_k S)_{\mathbb{Q}}$ such that its cycle class $cl(\zeta)$ in $H^{2p}_{\mathcal{D}}(X\times_k S/k, \mathbb{Q}(p))$ belongs to F^r_L, and the restriction of $\mathrm{Gr}^r_{F_L} cl(\zeta)$ to $\mathrm{Gr}^r_{F_L} H^{2p}_{\mathcal{D}}(X\times_k S'/k,\mathbb{Q}(p))$ does not vanish for any non empty open subvariety S' of S. Using the spectral sequence (1.3.2), $\mathrm{Gr}^r_{F_L} cl(\zeta)$ induces

$$\xi_0 \in \mathrm{Hom}_{\mathcal{M}}(\mathbb{Q}, H^{2p-r}(X/k,\mathbb{Q})\otimes H^r(S'/k,\mathbb{Q})(p)).$$

We first consider the case where ξ_0 does not vanish for any S'. Let $d = \dim X - p$. Since ξ_0 corresponds to the morphism

$$\xi'_0 : H^{2d+r}(X/k,\mathbb{Q})(d) \to H^r(S'/k,\mathbb{Q})),$$

this nonvanishing is equivalent to that the image of ξ'_0 has level r (assuming Grothendieck's generalized Hodge conjecture for $r > 2$). So we may assume $\dim S = r$ by the weak Lefschetz theorem. Put $n = r-1$. Let

$$H = H^{2d+r}(X/k,\mathbb{Q})(d),$$

and $H_{<n}$ be the largest subobject of H which has level $< n$. By semisimplicity there exists a subobject $H_{>n}$ with a decomposition $H = H_{<n} \oplus H_{>n}$, and the restriction of ξ'_0 to $H_{>n}$ does not vanish. So the assertion follows from (2.2-3) applied to a Lefschetz pencil.

Now we may assume $\xi_0 = 0$, i.e. $\mathrm{Gr}^r_{F_L} cl(\zeta) \in F'^1_L \mathrm{Gr}^r_{F_L}$, see (1.3). Then $\mathrm{Gr}^r_{F_L} cl(\zeta)$ induces

$$\begin{aligned}\xi_1 &\in \mathrm{Ext}^1_{\mathcal{M}}(\mathbb{Q}, H^{2p-r}(X/k,\mathbb{Q})\otimes H^{r-1}(S'/k,\mathbb{Q})(p)) \\ &= \mathrm{Ext}^1_{\mathcal{M}}(H, H^{r-1}(S'/k,\mathbb{Q})).\end{aligned}$$

Consider the case where ξ_1 does not vanish for any S'. If $r = 1$, we may replace S' with any point (replacing k if necessary), and the assertion is clear. So we may assume $r > 1$. In this case we have to show the nonvanishing of its restriction to any non empty open subvariety C' of a general hyperplane section $\overline{C}$ of a smooth projective compactification $\overline{S}$ of S.

If $r = 2$, let $\mathcal{P}_{\overline{S}/k}, \mathcal{P}_{\overline{C}/k}$ be the Picard variety of $\overline{S}, \overline{C}$. Then we have an injective morphism of k-varieties $\mathcal{P}_{\overline{S}/k} \to \mathcal{P}_{\overline{C}/k}$, and any k-valued point on the image can be lifted to a k-valued point of $\mathcal{P}_{\overline{S}/k}$. So the assertion follows using the short exact sequence (2.3.1) for S' and C' (and replacing k if necessary).

If $r > 2$, we may assume Grothendieck's generalized Hodge conjecture, and the 'coniveau' filtration N coincides with the filtration by the level of Hodge structure. If S' is a sufficiently small open affine subvarieties of $\overline{S}$, then

$$W_n H^n(S'/k,\mathbb{Q}) = H^n(\overline{S}/k,\mathbb{Q})/N^1 H^n(\overline{S}/k,\mathbb{Q}),$$

and similarly for C'. By the weak Lefschetz theorem, the restriction morphism

$$H^n(\overline{S}/k,\mathbb{Q}) \to H^n(\overline{C}/k,\mathbb{Q})$$

is injective, and splits by semisimplicity.

Assume that the pull-back of ξ_1 by $C' \to S'$ vanishes. Then, using the long exact sequence associated to (2.3.1), we see that ξ_1 factors through a direct factor of H (or equivalently, of $H^{2p-r}(X/k,\mathbb{Q})$) with level $< n$, because $\mathrm{Gr}^W_{n+1} H^n(C'/k,\mathbb{Q})$ has level $< n$. By the Hodge conjecture there exists a smooth proper k-variety Y of pure dimension $r-2$ together with a correspondence $\Gamma \in \mathrm{CH}^{p-1}(Y\times_k X)_{\mathbb{Q}}$ such that the image of

$$\Gamma_* : H^{r-2}(Y/k,\mathbb{Q}) \to H^{2p-r}(X/k,\mathbb{Q})(p-r+1)$$

coincides with $N^{p-r+1}H^{2p-r}(X/k,\mathbb{Q})(p-r+1)$ (i.e. the maximal subobject with level $\leq r-2$). We have also a correspondence $\Gamma' \in \mathrm{CH}^{\dim X-p+r-1}$

$(X\times_k Y)_{\mathbb{Q}}$ such that the restriction of $\Gamma_*\Gamma'_*$ to $\operatorname{Im}\Gamma_* \subset H^{2p-r}(X/k,\mathbb{Q})$ is the identity. So we may replace ζ with $\Gamma_*\Gamma'_*\zeta$ to show the vanishing of ξ_1. Here ζ is extended to $X\times_k\overline{S}$ by taking the closure, and the correspondences preserve the filtration τ because they induce morphisms of complexes $K_{\mathcal{H}}(X/k)$, see [28]. Since Γ' induces

$$\Gamma'_* : \mathrm{CH}^p(X\times_k\overline{S}) \to \mathrm{CH}^{r-1}(Y\times_k\overline{S}),$$

we see that $\operatorname{supp}\Gamma'_*\zeta \subset Y\times_k Z$ with Z a divisor on $\overline{S}$, because $r-1 > \dim Y$. So we get the assertion, because $\operatorname{supp}\Gamma_*\Gamma'_*\zeta \subset X\times_k Z$.

Now we may assume further $\xi_1 = 0$, i.e. $\mathrm{Gr}^r_{F_L} cl(\zeta) \in F'^2_L\mathrm{Gr}^r_{F_L}$. If $r > 2$, we have $\mathrm{Gr}^r_{F_L} cl(\zeta) = 0$ by the hypothesis on the coincidence of the two filtrations, because $F'^2_L\mathrm{Gr}^r_{F_L} = 0$ for $\mathcal{M} = \mathrm{MHS}$ and the filtrations $F_{\mathcal{M}}$ and $F'_{\mathcal{M}}$ in (1.4) are functorial for $\mathcal{M}$. So we may assume $r = 2$, since the case $r = 1$ is trivial by the vanishing of $H^{r-2}(S'/k,\mathbb{Q})$. Then $\mathrm{Gr}^r_{F_L} cl(\zeta)$ induces

$$\xi_2 \in \mathrm{Ext}^2_{\mathcal{M}}(\mathbb{Q}, H^{2p-2}(X/k,\mathbb{Q})\otimes H^0(S'/k,\mathbb{Q})(p)),$$

because $d_2 : E_2^{m,1} \to E_2^{m+2,0}$ vanishes in (1.3.2) (replacing k if necessary) where $m = 2p-2, j = p$. Indeed, $H^0(S'/k,\mathbb{Q}) = \mathbb{Q}$ is a direct factor of $K_{\mathcal{H}}(S'/k)$ by choosing a k-valued point x of S', because we have $\mathbb{Q} \to K_{\mathcal{H}}(S'/k) \to \mathbb{Q}$ by the structure morphism and x. In this case, the assertion is clear because $H^0(S'/k,\mathbb{Q}) = H^0(C'/k,\mathbb{Q}) = \mathbb{Q}$ (replacing k if necessary). Thus we have verified all the cases, because $\mathrm{Gr}^r_{F_L} cl(\zeta) = 0$ if $\xi_2 = 0$ and $r = 2$. This completes the proof of Theorem (0.3).

Remark 2.5. (i) It is conjectured that the filtration $F_{\mathcal{M}}$ is separated, and gives the conjectural "motivic" filtration of Beilinson [4] and Bloch [7]. This depends on the injectivity of the Abel-Jacobi map for smooth projective k-varieties, which is also a conjecture of Beilinson [2], see also [8], [16], [30], [31], etc. It is expected that the filtration $F_{\mathcal{M}}$ does not depend on the choice of $\mathcal{M}$, and coincides with Murre's (conjectural) filtration F_{Mur} [24]. Indeed, we have

$$(2.5.1)\qquad F_{\mathrm{Mur}} \subset F_{\mathcal{M}} \quad \text{and} \quad F_{\mathrm{Mur}} = F_{\mathcal{M}} \mod \cap_i F^i_{\mathcal{M}},$$

see [30]. The existence of F_{Mur} can be deduced from the separatedness of the filtration $F_{\mathcal{M}}$ assuming the algebraicity of the Künneth components of the diagonal, see [21]. The separatedness of $F_{\mathcal{M}}$ is reduced to the above conjecture of Beilinson on the Abel-Jacobi map for k-varieties, assuming the Hodge conjecture in the case the codimension of cycles is more than 2.

(ii) We have $\mathrm{Gr}^r_{F_{\mathcal{M}}}\mathrm{CH}^p(X_{\mathbb{C}})_{\mathbb{Q}} = 0$ for $r > p$. If $\mathcal{M} = \mathrm{MHS}$ or MHS_k, then $\mathrm{Gr}^r_{F_{\mathcal{M}}}\mathrm{CH}^p(X_K)_{\mathbb{Q}} = 0$ for $r > \mathrm{tr\ deg}\, K/k + 1$. These follow from the vanishing of $H^i(S_{\mathbb{C}}, \mathbb{Q})$ for a smooth affine variety S and $i > \dim S$, together with the compatibility of the cycle map with the pull-back by a closed embedding (and the vanishing of higher extension groups). These assertions have been shown by M. Green and P. Griffiths [16] for their filtration, assuming the above conjecture of Beilinson and Grothendieck's generalized Hodge conjecture. Note that these conjectures imply also that the filtration is separated and ends at the p-th step.

(iii) Restricted to the subgroup $\mathrm{CH}^p_{\mathrm{alg}}(X)_{\mathbb{Q}}$ consisting of cycles algebraically equivalent to 0, the kernel of the Abel-Jacobi map coincides with $F^2_{\mathcal{M}}\mathrm{CH}^p_{\mathrm{alg}}(X_{\mathbb{C}})_{\mathbb{Q}}$, see [31], 3.9. Indeed, for a curve C and a correspondence $\Gamma \in \mathrm{CH}^p(C \times X)_{\mathbb{Q}}$, we have a decomposition $H^1(C, \mathbb{Q}) = \mathrm{Im}\,\Gamma_* \oplus \mathrm{Ker}\,\Gamma_*$ induced by idempotents of $\mathrm{CH}^1(C \times C)_{\mathbb{Q}}$, where $\Gamma_* : H^1(C, \mathbb{Q}) \to H^{2p-1}(X, \mathbb{Q})(p-1)$. By a similar argument, the kernel of the usual Abel-Jacobi map coincides with that of the l-adic Abel-Jacobi map on $\mathrm{CH}^p_{\mathrm{alg}}(X)_{\mathbb{Q}}$, because we have the injectivity in the divisor case using the Kummer sequence, see also [25] for the case of 0-cycles.

(iv) It has been remarked by the referee that some arguments in this paper are related to a remark of T. Shioda [34, 3(c)] on an analogue of the result of Terasoma [35] concerning a result of Griffiths on the triviality of the image of the Abel-Jacobi map for general hypersurfaces. Indeed, his conjecture can be proved by using Terasoma's argument together with the facts that the image of the Abel-Jacobi map is trivial if a certain member of the coniveau filtration vanishes (because it is enough to consider a family of cycles parametrized by a curve) and that the subvarieties used in the definition of the coniveau filtration can be defined over the given number field. Note that the nontriviality of the coniveau filtration would induce a non trivial local subsystem by the Hilbert irreducibility theorem (i.e. by the surjectivity to the monodromy group), and it would contradict the irreducibility of the local system, which follows from the Picard-Lefschetz formula together with the irreducibility of the discriminant as well known.

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