Unbounded growth of the energy density associated to the Schrödinger map and the binormal flow

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Abstract. We consider the binormal flow equation, which is a model for the dynamics of vortex filaments in Euler equations. Geometrically, it is a flow of curves in three dimensions, explicitly connected to the one-dimensional Schrödinger map with values on the two-dimensional sphere, and to the one-dimensional cubic Schrödinger equation. Although these equations are completely integrable, we show the existence of an unbounded growth of the energy density. The density is given by the amplitude of the high frequencies of the derivative of the tangent vectors of the curves, thus giving information about oscillation at small scales. In the setting of vortex filaments, the variation of the tangent vectors is related to the derivative of the direction of the vorticity, which according to the Constantin–Fefferman–Majda criterion is relevant in the possible development of singularities for the Euler equations.

1. Introduction

In this paper we show the existence of an unbounded flux of the energy density of the solutions of two related partial differential equations which are completely integrable and have a connection with fluid mechanics. The first equation is the Schrödinger map in one dimension with values on the two-dimensional sphere, known in the physics literature as the classical continuous Heisenberg chain model in ferromagnetism:

$$T_t = T \times T_{xx}. \tag{1}$$

Here t will represent time and x the spatial variable. Equation (1) can be also written in divergence form. This is due to the fact that (1) can be obtained by simple differentiation in the spatial variable from the following second equation on curves $\chi(t, x)$ in $\mathbb{R}^3$:

$$\chi_t = \chi_x \times \chi_{xx}, \quad \chi_x = T, \ |T| = 1. \tag{2}$$

This latter equation, known as the localized induction approximation (LIA), and also as the vortex filament equation (VFE) and as the binormal flow (BF), appears naturally as a formal approximation (see [2, 8, 14, 39]), after a renormalization of time, of the location

evolution of vortex filaments that move according to the Euler equations. This model is conjectured to give the right dynamics of vortex filaments in certain situations (see [28] and the references therein). Through this model, at any given time t, the curve $\chi(t, \cdot)$ represents the location of the vortex filament, and the tangent vector to the curve $T = \chi_x$ represents the direction of the vorticity.

A simple use of the Frenet equations immediately gives that (2) can be written as

$$\chi_t = cb, \tag{3}$$

which explains the binormal flow name, and that

$$T_x = cn, \tag{4}$$

where n is the normal vector, b the binormal vector, and c the curvature. It is also easy to see that $c^2(t, x)\, dx$ is an energy density that from (3) describes the kinetic energy of the filament and from (4) the interaction energy of the chain. More precisely,

$$\int |\chi_t(t, x)|^2\, dx = \int |T_x(t, x)|^2\, dx = \int c^2(t, x)\, dx, \tag{5}$$

and for smooth solutions these quantities are conserved in time if they are finite.

Instead of the classical Frenet frame given by the tangent, normal, and the binormal vectors, for analytical reasons it is much more convenient to use the one given by parallel frames (T, e_1, e_2) constructed as solutions of

$$\begin{cases} T_x = \alpha e_1 + \beta e_2, \\ e_{1x} = -\alpha T, \\ e_{2x} = -\beta T. \end{cases} \tag{6}$$

Above, α and β are real scalars. A further simplification can be made defining the complex vector $N = e_1 + i e_2 \in \mathbb{S}^2 + i\mathbb{S}^2$ and the complex scalar

$$u = \alpha + i\beta \tag{7}$$

to obtain

$$\begin{cases} T_x = \Re(\bar{u}N), \\ N_x = -uT. \end{cases} \tag{8}$$

It was proved in [27] that in order for the constraint $T_{xt} = T_{tx}$ to hold, u has to solve the one-dimensional focusing non-linear Schrödinger equation (NLS)

$$iu_t + u_{xx} + \frac{1}{2}(|u|^2 - a(t))u = 0, \tag{9}$$

with $a(t)$ a real scalar, and the tangent and normal vector have to satisfy the linear system

$$\begin{cases} T_t = \Im(\overline{u_x}N), \\ N_t = -iu_x T + \frac{i}{2}(|u|^2 - a(t))N. \end{cases} \tag{10}$$

Finally, (2) can be written

$$\chi_t = \Im(\bar{u}N). \tag{11}$$

It is a well-known fact that equation (9) is completely integrable and belongs to the so-called AKNS-ZS hierarchy ([1, 46, 47]; for (1) see [43, 48]). The geometric meaning of u is clarified when we write (7) in polar form,

$$u(t, x) = \rho(t, x)e^{i\theta(t,x)}, \tag{12}$$

as $\rho(t)$ and $\theta_x(t)$ are the curvature and the torsion respectively of the curve $\chi(t)$ ([27]).

Conversely, given a solution u of (9), $t_0, x_0 \in \mathbb{R}$, and $\mathcal{B}$ an orthonormal basis of $\mathbb{R}^3$, one can construct a solution of (1) by imposing $\{T, \Re N, \Im N\}(t_0, x_0) = \mathcal{B}$ and solving (10) for (t, x_0) and then (6) for (t, x). Then, given a point $P \in \mathbb{R}^3$, a solution of (2) is constructed by imposing $\chi(t_0, x_0) = P$ and solving $\chi_t = T \wedge T_x$ for (t, x_0) and then $\chi_x = T$ for (t, x). We shall call this way of constructing a solution of (1)–(2) from a solution of (9) Hasimoto's method.

For x either in the real line $\mathbb{R}$ or in the torus $\mathbb{T}$, the well-posedness theory of the initial value problem associated to (9) was established in the function spaces $L^2(\mathbb{R})$ in [44] and $L^2(\mathbb{T})$ in [6]. Observe that (9) is invariant under the scaling $u_\lambda(t, x) := \lambda u(\lambda^2 t, \lambda x)$, and that according to this scaling $L^2(\mathbb{R})$ is subcritical. Moreover, among the homogeneous Sobolev spaces, $\dot{H}^{-\frac{1}{2}}$ is the one invariant with respect to the scaling, thus there is a gap of $1/2$ derivative between L^2 and the critical space $\dot{H}^{-\frac{1}{2}}$. Beginning with [45], a lot of attention has been devoted to extend the well-posedness theory to function and distribution spaces, not necessarily given by the Sobolev class, to make this gap as small as possible. As observed in [18], a good choice is to consider the so-called Fourier–Lebesgue spaces that are defined using the L^p norm of the Fourier transform of the solution. Therefore, they are invariant under translation in phase space or, equivalently, under the so-called Galilean symmetries. The Fourier–Lebesgue space of functions with Fourier transform in L^∞ is also invariant with respect to the scaling. Several results about ill-posedness, either in the sense that the map datum–solution is not uniformly continuous, or showing what is known as the norm inflation phenomena, have been proved ([9, 11, 30, 32, 40]). On the other hand, (local) well-posedness holds for data with Fourier transform in L^p spaces, for all $2 < p < +\infty$ ([10, 18, 19]). This result can be proved using perturbation techniques and a fixed point argument. Making strong use of the complete integrability, the gap to the critical space has been also reduced, even in the quite remarkable case of the non-homogeneous Sobolev class and for global well-posedness, as recently proved in [26], to all the subcritical cases; see also [31, 34, 41] for the global-in-time result in the Fourier–Lebesgue class. As a consequence, no possible unbounded flux in the size of the Fourier transform of the solution can happen in this subcritical regime. In this paper we focus our attention on the critical case.

Geometrically, critical regularity for (2) means the possibility of having either corners or logarithmic spirals. We will concentrate on the particular case of corners, which implies the existence of jumps for the corresponding tangent vectors. The case of logarithmic spirals has been considered in [22, 23, 37, 38] and is poorly understood. Nevertheless, we

think that the extent to which the results shown in this work can be extended to logarithmic spirals is a relevant question.

The simplest way of obtaining a corner is to look for self-similar solutions of (2), that is to say, solutions that can be written as $\chi(t, x) = \sqrt{t}G(x/\sqrt{t})$ for some well-chosen G. A simple computation gives that such a curve G has to solve the non-linear ODE

$$\frac{1}{2}G - \frac{y}{2}G' = G' \times G''. \tag{13}$$

In [7, 35, 36] it is proved that a solution of (13) is characterized by the property that the curvature has to be a constant $c = \alpha$, and the torsion τ has to be $\tau(y) = y/2$. Thanks to (12) this amounts to saying that

$$u_\alpha(t, x) = \frac{\alpha}{\sqrt{t}}e^{i\frac{x^2}{4t}}.$$

As a consequence, if in (9) we take $a(t) = \frac{|\alpha|^2}{t}$, we observe that u_α solves (9) with initial condition

$$u_\alpha(0, x) = \sqrt{i}\alpha\delta(x).$$

Here δ is the Dirac-δ function located at the origin.

Observe that (2) is invariant under rotations. Therefore, it is enough to give the Frenet frame of the curve given by G at say $x = 0$ to construct all the solutions of (13). Take this frame, $\{T(0), n(0), b(0)\}$, to be the canonical orthonormal basis of $\mathbb{R}^3$, and call G_α the corresponding solution. It was proved in [21] that if

$$\chi_\alpha(t, x) = \sqrt{t}G_\alpha\left(\frac{x}{\sqrt{t}}\right),$$

then χ_α solves (2) for $t > 0$ and there exists $\chi_\alpha(0, x)$ such that

$$|\chi_\alpha(t, x) - \chi_\alpha(0, x)| \leq 2\alpha\sqrt{t}, \quad t > 0.$$

Moreover, $\chi_\alpha(0, x)$ is given by two half-lines joined at a corner at the origin. Calling θ_α the corresponding interior angle, it is proved in [21] that

$$\sin\frac{\theta_\alpha}{2} = e^{-\frac{\pi}{2}|\alpha|^2}. \tag{14}$$

In our previous works we considered two different scenarios for finding a functional setting that includes this example and such that at least a local well-posedness result can be established for the corresponding initial value problem.

The first scenario is when the polygonal line given by $\chi_\alpha(0, x)$ is perturbed in such a way that the angle remains θ_α but outside the corner location the curve is smooth and tends to two, possibly different, lines at infinity. To find these solutions we study first (9) with $a(t) = \frac{|\alpha|^2}{t}$, then we use Hasimoto's method for positive times, and eventually we deal

with the limit curves at $t = 0$. Regarding (9) we use the pseudo-conformal transformation. More concretely, we look for v with $u = \mathcal{T}(v)$ where

$$\mathcal{T}(v)(t, x) = \frac{e^{i \frac{x^2}{4t}}}{\sqrt{t}} \bar{v}\left(\frac{1}{t}, \frac{x}{t}\right). \tag{15}$$

Observe that above we also make a change of variables so that the time interval $(0, 1)$ becomes $(1, \infty)$. A standard calculation gives that v has to solve

$$i v_t + v_{xx} + \frac{1}{2t}(|v|^2 - \tilde{a}(t))v = 0, \quad \tilde{a}(t) = \frac{1}{t}a\left(\frac{1}{t}\right) = |\alpha|^2. \tag{16}$$

Solutions of the above equation (16) formally preserve the L^2 norm

$$\int |v|^2 \, dx, \tag{17}$$

and

$$E(v)(t) := \frac{1}{2} \int |v_x(t)|^2 \, dx - \frac{1}{4t} \int (|v|^2 - |\alpha|^2)^2 \, dx \tag{18}$$

satisfies

$$\partial_t E(v)(t) = \frac{1}{4t^2} \int (|v|^2 - |\alpha|^2)^2 \, dx.$$

Finally,

$$\Im \int x \bar{v}_x v \, dx$$

is also a conserved quantity. We could also consider any of the infinitely many conserved quantities of (9) but observe that from the definition of (15) it would be necessary to assume regularity and decay on v for these quantities to be finite.

Notice that $v_\alpha(t, x) := \alpha$ is a particular solution of (16), and the corresponding binormal flow solution is χ_α. In a series of papers (see the introduction of [3] for a survey of the results), we prove well-posedness and small-data modified scattering results for $v - v_\alpha$, $t \geq 1$, in some appropriate function spaces such that $E(t)$ given in (18) is finite.

The second scenario was started in [4] and considers solutions of (2) that at time $t = 0$ are given by a skew polygonal line $\chi_0(x)$ that tends to two lines when $x \to \pm\infty$. The corners are all located at integers $j \in \mathbb{Z}$. We use Hasimoto's method, and at the level of (9) this problem is related to considering data

$$\sum_j \alpha_j \delta(x - j)$$

with some appropriate conditions on the size of α_j. Following [33],[1] we look for solutions of (9) with $a(t) = M/t$, where $M := \sum_j |\alpha_j|^2$, for $t > 0$, of the type

$$u(t, x) = \sum_j A_j(t) \frac{e^{i \frac{(x-j)^2}{4t}}}{\sqrt{t}}, \tag{19}$$

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with

$$A_j(t) = e^{-i(|\alpha_j|^2 - M)\log\sqrt{t}}(\alpha_j + R_j(t)), \tag{20}$$

and $R_j(t)$ satisfying decay properties as t goes to zero. The construction is performed for $\{\alpha_j\}$ such that

$$\sum_j |j|^{2s}|\alpha_j|^2 < \infty \quad \text{for } s > 1/2. \tag{21}$$

Using Hasimoto's method we construct a solution of the binormal flow for $t > 0$ such that we recover at time $t = 0$ the curve χ_0, provided that we choose α_j in a precise way determined by the curvature and torsion angles of χ_0 at $x = j$. In particular, as (14) for self-similar solutions, we choose

$$\sin\frac{\theta_j}{2} = e^{-\frac{\pi}{2}|\alpha_j|^2}. \tag{22}$$

The need of weights in (21) comes from integrating (10) because the coefficients of that system involve u_x. Notice that making the expansion of the square phases in (19) one immediately computes which is the pseudo-conformal transformation (15) of u. Indeed, we can write $u = \mathcal{T}(v)$ with v the 2π-periodic function in the x-variable:

$$v(t,x) = \sum_j A_j\left(\frac{1}{t}\right)e^{-itj^2+ijx},$$

a solution of (16) with $\tilde{a}(t) = M$. Observe that (17), the L^2 conservation law of (16), and (20) give that for $t > 0$,

$$\sum_j |A_j(t)|^2 = M. \tag{23}$$

Recall that from (6),

$$|T_x(t,x)|^2 = |u(t,x)|^2 = \frac{1}{t}\left|v\left(\frac{1}{t},\frac{x}{t}\right)\right|^2.$$

Hence, for any $t > 0$ the function $|T_x(t,x)|^2$ is a $2\pi t$-periodic function in the x-variable and by (23) the integral on each of the periods is M. Nevertheless, this "conservation law" does not give any information about $\widehat{T_x}$, the Fourier transform of T_x. We proved in [5] that (23) can be also understood as a kind of scattering energy of $\widehat{T_x}$ for the solutions of (1) and (2) that we constructed in [4]. More precisely, if

$$\Xi(T(t)) := \lim_{k\to\infty}\int_k^{k+1} |\widehat{T_x}(t,\xi)|^2\,d\xi, \tag{24}$$

then for $t > 0$ we have the conservation law

$$\Xi(T(t)) = 4\pi\sum_j |\alpha_j|^2. \tag{25}$$

It was also proved in [5] that there is a jump discontinuity of $\Xi(T(t))$ at $t = 0$. From (24) we can see $|\widehat{T_x}(t,\xi)|^2\,d\xi$ as an asymptotic energy density in phase space. The main result of this paper is to prove that this energy density can grow in time at specific Fourier modes.

The procedure used in [4] to construct the solution u with the shape given in (19) is to solve the infinite-dimensional non-homogeneous dynamical system generated by $A_j(t)$. The choice (23) kills all the resonant frequencies except those generated by the interaction of any mode j with itself. This interaction is easily absorbed by a logarithmic modification of the phase of $A_j(t)$ that has already been incorporated in (20). Hence, a fixed point argument can be performed to solve the system and to obtain the decay properties of $R_j(t)$ mentioned above. As a consequence, there is no possible growth for $A_j(t)$.

The appearance of the logarithmic correction in the phases mentioned above is analogous to the long-range modified scattering that smooth small solutions of (9) satisfy ([42]). This modified scattering is behind the growth results of high Sobolev norms proved in [25] for the scalar cubic NLS on $\mathbb{R} \times \mathbb{T}^d$ with $d \geq 2$. In that setting, which is a mixture of periodic and continuous variables, the authors prove a *loglog* growth in time of the amplitudes of the Fourier modes. The key ingredient for this growth is that, differently to what happens in one dimension, for $d \geq 2$ the corresponding infinite-dimensional system has a non-trivial resonant subsystem that generates solutions whose high Sobolev norms grow in time – see [12, 20, 24]. All of these equations are not integrable. For this purpose we recall that growth of Sobolev norms for an integrable equation was proved in the case of the cubic Szegő equation ([16]; see also [17]).

In Theorem 1.1 below we obtain a precise logarithmic growth in time for $\widehat{T_x}$, where T is a solution of (1) and the tangent vector of a curve that evolves according to (2). This curve at time $t = 0$ is a polygonal line with just two corners of the same angle that are located at $x = 1$ and $x = -1$. Recall that T_x represents the variation of the direction of the vorticity that, as proved in [13], plays a crucial role in the possible formation of singularities of the Euler equations.

Theorem 1.1. *Consider a polygonal line $\chi_0(x)$ with two corners of angle θ located at $x \in \{-1, 1\}$. Let $\chi(t, x)$ be its evolution by the binormal flow (2) as explained above and let $T(t, x)$ be its tangent vector.*

There exist $t_\theta, \tilde{t}_\theta \in (0, 1)$ and $n_\theta \in \mathbb{N}$ such that for $n \in \mathbb{N}, n \geq n_\theta$, $t \in (\frac{\tilde{t}_\theta}{n^2 \log^2 n}, \frac{t_\theta}{n^2})$, and ξ satisfying either $|\xi - \frac{1}{2\pi t}| \leq \frac{1}{n}$ or $|\xi + \frac{1}{2\pi t}| \leq \frac{1}{n}$ the following growth holds:

$$|\widehat{T_x}(t, \xi) - V \log n| \leq \frac{1}{2}|V| \log n, \tag{26}$$

where V is the non-null vector $i(-\frac{2}{\pi}) \log(\sin \frac{\theta}{2})(T^{-\infty} - 2T^0 + T^{+\infty})$, and the vectors $T^{-\infty}$, T^0, and $T^{+\infty}$ are the directions of the polygonal line $\chi_0(x)$ on $x < -1, -1 < x < 1$, and $1 < x$ respectively.

As a consequence, for $t \in (0, \frac{t_\theta}{n_\theta^2})$ there exists $C_\theta > 0$ such that

$$\sup_\xi |\widehat{T_x}(t, \xi)| \geq C_\theta \log t. \tag{27}$$

Finally, for ξ satisfying $|\xi - \frac{1}{2\pi t}| \geq \frac{3}{8\pi t}$ and $|\xi + \frac{1}{2\pi t}| \geq \frac{3}{8\pi t}$ and $t \in (0, \frac{1}{4\pi})$ we have an upper bound of $|\widehat{T_x}(t, \xi)|$ depending only on θ.

Let us first note that our result concerns the growth of $\|\widehat{T_x}\|_\infty$, the L^∞ norm of the Fourier transform of T_x, and therefore we are looking at a critical norm in the class of Fourier–Lebesgue spaces.

The proof of Theorem 1.1, which is given in Section 2, is based on the computation of $\widehat{T_x}(\xi)$ using (8) with $u = \alpha + i\beta$ satisfying (19). This generates a first sum in j with the corresponding $\bar{A}_j$ and their quadratic phases. Then, we use (8) again to integrate by parts, and a second sum appears with some new amplitudes A_r and new quadratic phases. It was observed in [5] that a resonance can happen if $j - r \in \mathbb{Z}$ is properly chosen. It is easy to obtain a logarithmic upper bound for this resonance. It involves a small set of frequencies ξ which do not prevent (24) from holding. Our purpose in this paper is to obtain a lower bound.

In Section 3 we prove the extension of Theorem 1.1 to the case of polygonal lines with many corners. The exhibited logarithmic growth is a hint that the numerical computations given in [15] about the unboundedness of $\|\widehat{T_x}\|_\infty$ are correct – see [15, equation (8) in §5]. In that case the initial condition of (2) is a planar regular polygon. The dynamics then also becomes periodic in time and exhibits a Talbot effect, in the sense that at rational p/q multiples of the time period, skew polygons emerge with typically as many sides as q; see [29]. So this (numerical) logarithmic growth also happens at these rational times. A rigorous proof of this fact is a very challenging question that we propose to address in the future.

We shall denote systematically by $C(\|A_j(t)\|_{l^1})$ constants depending only on universal constants and on a finite number of positive powers of $\|A_j(t)\|_{l^1}$.

2. Proof of Theorem 1.1

Let $n \in \mathbb{N}^*$. First we recall that for $s > \frac{1}{2}, 0 < \gamma < 1$, it was proved in [4] that

$$i u_t + u_{xx} + \frac{1}{2}\left(|u|^2 - \frac{\sum_j |\alpha_j|^2}{t}\right) u = 0 \tag{28}$$

has an unique local solution for $t \in (0, \mathcal{T})$ of type

$$u(t, x) = \sum_{j \in \mathbb{Z}} e^{-i(|\alpha_j|^2 - M)\log\sqrt{t}}(\alpha_j + R_j(t)) \frac{e^{i\frac{(x-j)^2}{4t}}}{\sqrt{t}}, \tag{29}$$

with

$$\sup_{0 < t < \mathcal{T}} t^{-\gamma}\|R_j(t)\|_{l^{2,s}} + t\|\partial_t R_j(t)\|_{l^{2,s}} < C(\gamma)\|\alpha_j\|_{l^{2,s}}^3. \tag{30}$$

The time of existence $\mathcal{T}$ is in terms of $s, \gamma, \|\alpha_j\|_{l^{2,s}}$.

As explained in the introduction, the evolution $\chi(t)$ of χ_0 on $(0, \mathcal{T})$ is constructed by Hasimoto's method from the solution (29) of (28) with, in view of (22),

$$\alpha_j = \sqrt{\left(-\frac{2}{\pi}\right)\log\left(\sin\frac{\theta}{2}\right)} =: \alpha \ \text{ for } j \in \{\pm 1\}, \quad \alpha_j = 0 \ \text{ otherwise.} \tag{31}$$

2.1. A general analysis on locating possible growth scenarios

We shall start with a lemma that highlights the part of $\widehat{T_x}(t, \xi)$ that can grow for small times, in general cases of polygonal lines.

Lemma 2.1. *Let $\{\alpha_j\} \in l^{2,\frac{1}{2}^+}$ and let $\chi(t)$ be the evolution through the binormal flow of the corresponding polygonal line. For $t \in (0, \frac{1}{4\pi n^2})$ and $\xi \in \mathbb{R}$, the tangent vector of $\chi(t)$ satisfies*

$$\left| \widehat{T_x}(t, \xi) - i \sum_{|j-r+4\pi t\xi| < 2nt} \overline{A_j(t)} A_r(t) e^{-i\frac{j^2-r^2}{4t}} \right.$$
$$\left. \times \int_{\substack{|x-j-4\pi t\xi| > \frac{1}{n}, \\ |x-r+4\pi t\xi| > \frac{1}{n}}} e^{i\frac{x(j-r+4\pi t\xi)}{2t}} \left(\frac{1}{x-j-4\pi t\xi} - \frac{1}{x-r+4\pi t\xi} \right) T(t, x) \, dx \right|$$
$$\leq \frac{C(\|A_j(t)\|_{l^1})}{n\sqrt{t}}.$$

Proof. From (8) we have $T_x(t, x) = \Re(\bar{u}N)(t, x)$ so

$$\widehat{T_x}(t, \xi) = \int_{-\infty}^{\infty} e^{i2\pi x\xi} \Re(\bar{u}N)(t, x) \, dx$$
$$= \int_{-\infty}^{\infty} e^{i2\pi x\xi} \Re\left(\sum_j \overline{A_j(t)} \frac{e^{-i\frac{(x-j)^2}{4t}}}{\sqrt{t}} N(t, x) \right) dx$$
$$= \frac{e^{i4\pi^2 t\xi^2}}{2\sqrt{t}} \sum_j e^{i2\pi j\xi} \overline{A_j(t)} \int_{-\infty}^{\infty} e^{-i\frac{(x-j-4\pi t\xi)^2}{4t}} N(t, x) \, dx$$
$$+ \frac{e^{-i4\pi^2 t\xi^2}}{2\sqrt{t}} \sum_j e^{i2\pi j\xi} A_j(t) \int_{-\infty}^{\infty} e^{i\frac{(x-j+4\pi t\xi)^2}{4t}} \overline{N(t, x)} \, dx.$$

We start by removing bounded pieces of the integral centered in $j \pm 4\pi t\xi$, by using a cutoff function ψ_n vanishing on $B(0, \frac{1}{4n})$ and valued 1 on $^cB(0, \frac{1}{2n})$. These pieces are easy to estimate by $C\frac{\|A_j(t)\|_{l^1}}{n\sqrt{t}}$, as the integrants are of constant modulus. On the remaining pieces we integrate by parts:

$$\left| \widehat{T_x}(t, \xi) + i\sqrt{t} e^{i4\pi^2 t\xi^2} \sum_j e^{i2\pi j\xi} \overline{A_j(t)} \int_{-\infty}^{\infty} e^{-i\frac{(x-j-4\pi t\xi)^2}{4t}} \left(\frac{N(t, x)\psi_n(x-j-4\pi t\xi)}{x-j-4\pi t\xi} \right)_x dx \right.$$
$$\left. - i\sqrt{t} e^{-i4\pi^2 t\xi^2} \sum_j e^{i2\pi j\xi} A_j(t) \int_{-\infty}^{\infty} e^{i\frac{(x-j+4\pi t\xi)^2}{4t}} \left(\frac{\overline{N(t, x)}\psi_n(x-j+4\pi t\xi)}{x-j+4\pi t\xi} \right)_x dx \right|$$
$$\leq C\frac{\|A_j(t)\|_{l^1}}{n\sqrt{t}}.$$

When the derivative falls on ψ_n and on the denominator we get terms bounded by $Cn\sqrt{t}\|\{A_j(t)\}\|_{l^1}$, and $n\sqrt{t} \leq \frac{1}{n\sqrt{t}}$ as $t \in (0, \frac{1}{4\pi n^2})$. We are left with the part from

$N_x = -uT$. Then we get

$$\left| \widehat{T_x}(t,\xi) - i \sum_j \overline{A_j(t)} \int_{-\infty}^{\infty} \frac{\sum_r A_r(t) e^{i \frac{x(j-r+4\pi t\xi)}{2t}} e^{-i \frac{j^2-r^2}{4t}}}{x-j-4\pi t\xi} T(t,x) \psi_n(x-j-4\pi t\xi) \, dx \right.$$

$$\left. + i \sum_j A_j(t) \int_{-\infty}^{\infty} \frac{\sum_r \overline{A_r(t)} e^{i \frac{x(r-j+4\pi t\xi)}{2t}} e^{i \frac{j^2-r^2}{4t}}}{x-j+4\pi t\xi} T(t,x) \psi_n(x-j+4\pi t\xi) \, dx \right|$$

$$\leq C \frac{\|A_j(t)\|_{l^1}}{n\sqrt{t}}.$$

For $|\frac{\pm(j-r)+4\pi t\xi}{2t}| \geq n$ we perform an integration by parts using the linear phase and get terms bounded by $C\|A_j(t)\|_{l^1}^2$ when the derivative falls on ψ_n and on the denominator. When the derivative falls on T we get $\Re(\bar{u}N)$ which yields a quadratic phase. We complete to a quadratic phase incorporating the linear one and the terms to estimate are of the type

$$\frac{1}{n\sqrt{t}} \sum_{j,r,k} |A_j(t)A_r(t)A_k(t)| \left| \int_{-\infty}^{\infty} \frac{e^{i \frac{(x-j+r-k-4\pi t\xi)^2}{4t}}}{x-j+4\pi t\xi} N(t,x) \psi_n(x-j+4\pi t\xi) \, dx \right|.$$

We first note that we can add in the integral the cutoff ψ_n centered where the quadratic phase cancels. Indeed, as the function to integrate is upper-bounded by Cn the contribution of $1-\psi_n(x-j+r-k-4\pi t\xi)$ is upper-bounded by $\frac{C}{n\sqrt{t}}\|A_j(t)\|_{l^1}^3$. Then we perform an integration by parts from the quadratic phase and get

$$\frac{2\sqrt{t}}{n} \sum_{j,r,k} |A_j(t)A_r(t)A_k(t)| \left| \int_{-\infty}^{\infty} e^{i \frac{(x-j+r-k-4\pi t\xi)^2}{4t}} \right.$$

$$\left. \times \left(\frac{N(t,x)\psi_n(x-j+4\pi t\xi)\psi_n(x-j+r-k-4\pi t\xi)}{(x-j+4\pi t\xi)(x-j+r-k-4\pi t\xi)} \right)_x dx \right|.$$

When the derivative falls on N, as $N_x = -uT$ we can perform a Cauchy–Schwarz inequality separating the factors in the denominator to get a $C\|A_j(t)\|_{l^1}^4$ upper bound. When the derivative falls on a term other than N we get directly a $Cn\sqrt{t}\|A_j(t)\|_{l^1}^3$ upper bound.

Summarizing, we have obtained for all $\xi \in \mathbb{R}$,

$$\left| \widehat{T_x}(t,\xi) - i \sum_{|j-r+4\pi t\xi|<2nt} \overline{A_j(t)} A_r(t) I^+(t,\xi,j,r) \right.$$

$$\left. + i \sum_{|-(j-r)+4\pi t\xi|<2nt} A_j(t)\overline{A_r(t)} I^-(t,\xi,j,r) \right| \leq \frac{C(\|A_j(t)\|_{l^1})}{n\sqrt{t}},$$

where

$$I^{\pm}(t,\xi,j,r) := e^{\mp i \frac{j^2-r^2}{4t}} \int_{|x-j\mp4\pi t\xi|>\frac{1}{n}} \frac{e^{i \frac{x(\pm(j-r)+4\pi t\xi)}{2t}}}{x-j\mp4\pi t\xi} T(t,x) \, dx.$$

For simplicity, in the integrals we have replaced the cutoff function ψ_n by the characteristic function of $B(0, \frac{1}{n})$. Indeed, the functions to integrate are upper-bounded by n so the error term, which comes from integrating on a domain of size $\frac{C}{n}$, is upper-bounded by $C(\|A_j(t)\|_{l^1})$. The first discrete summation holds for (j, r) if and only if the second discrete summation holds for (r, j), so

$$\left| \widehat{T_x}(t, \xi) - i \sum_{|j-r+4\pi t\xi| < 2nt} \overline{A_j(t)} A_r(t) (I^+(t, \xi, j, r) - I^-(t, \xi, r, j)) \right| \leq \frac{C(\|A_j(t)\|_{l^1})}{n\sqrt{t}}.$$

By the same argument above we can remove from the integrals in $I^+(t, \xi, j, r)$ and $I^-(t, \xi, r, j)$ the domains of integration $\{|x-r+4\pi t\xi| \leq \frac{1}{n}\}$ and $\{|x-j-4\pi t\xi| \leq \frac{1}{n}\}$ respectively to get

$$\left| \widehat{T_x}(t, \xi) - i \sum_{|j-r+4\pi t\xi| < 2nt} \overline{A_j(t)} A_r(t) e^{-i\frac{j^2-r^2}{4t}} \right.$$
$$\left. \times \int_{\substack{|x-j-4\pi t\xi|>\frac{1}{n}, \\ |x-r+4\pi t\xi|>\frac{1}{n}}} e^{i\frac{x(j-r+4\pi t\xi)}{2t}} \left(\frac{1}{x-j-4\pi t\xi} - \frac{1}{x-r+4\pi t\xi} \right) T(t, x)\, dx \right|$$
$$\leq \frac{C(\|A_j(t)\|_{l^1})}{n\sqrt{t}}. \qquad \blacksquare$$

2.2. An analysis on locating particular solutions that can exhibit growth

To get the logarithmic growth of the theorem we shall restrict to a particular class of polygonal lines and we shall look for values of ξ_n such that $|4\pi \frac{t}{n^2} \xi_n|$ is close to the distance between the corners. We note that in the case of a single corner we have $\alpha_1 \alpha_{-1} = 0$, so the following lemma ensures that $\widehat{T_x}(\frac{t}{n^2}, \xi_n)$ is bounded. Therefore the logarithmic growth comes from the interaction of several corners.

Lemma 2.2. *Let $\{\alpha_j\}$ be such that*

$$\alpha_j = 0 \quad \text{for } |j| > 1. \tag{32}$$

We have for all $t \in (0, \frac{1}{4\pi n^2})$ and $|\delta| < \frac{1}{n}$,

$$\left| \widehat{T_x}\left(t, \frac{1}{2\pi t} + \delta\right) - i\overline{\alpha_{-1}} \alpha_1 e^{i(|\alpha_{-1}|^2 - |\alpha_1|^2)\log t} \int_{\substack{|x-1|>\frac{1}{n}, \\ |x+1|>\frac{1}{n}, \\ |x|<2}} \left(\frac{1}{x-1} - \frac{1}{x+1} \right) T(t, x)\, dx \right|$$
$$\leq \frac{C(\|A_j(t)\|_{l^1})}{n\sqrt{t}}. \tag{33}$$

A similar estimate holds at $-\frac{1}{2\pi t} + \delta$.

Proof. Denoting $\xi = \frac{1}{2\pi t} + \delta$ implies $4\pi t\xi = 2 + 4\pi t\delta$, so we have

$$|j-r+4\pi t\xi| < 2nt < \frac{1}{2\pi n} \iff r = j + 2,$$

and Lemma 2.1 gives us

$$
\left| \widehat{T_x}(t,\xi) - i \sum_j \overline{A_j(t)} A_{j+2}(t) e^{i\frac{(j+2)^2-j^2}{4t}} \right.
$$

$$
\left. \times \int_{\substack{|x-j-2-4\pi t\delta|>\frac{1}{n}, \\ |x-j+4\pi t\delta|>\frac{1}{n}}} e^{ix2\pi\delta} \left(\frac{1}{x-j-2-4\pi t\delta} - \frac{1}{x-j+4\pi t\delta} \right) T(t,x)\,dx \right|
$$

$$
\leq \frac{C(\|A_j(t)\|_{l^1})}{n\sqrt{t}}. \tag{34}
$$

First we note that, as $|4\pi t\delta| < \frac{1}{n^3}$, we obtain by Cauchy–Schwarz,

$$
\int_{\substack{|x-j-2-4\pi t\delta|>\frac{1}{n}, \\ |x-j+4\pi t\delta|>\frac{1}{n}, \\ |x-j-1|>2}} \left| \frac{1}{x-j-2-4\pi t\delta} - \frac{1}{x-j+4\pi t\delta} \right| dx
$$

$$
= \int_{\substack{|y-1-4\pi t\delta|>\frac{1}{n}, \\ |y+1+4\pi t\delta|>\frac{1}{n}, \\ |y|>2}} \left| \frac{2+8\pi t\delta}{(y-1-4\pi t\delta)(y+1+4\pi t\delta)} \right| dy \leq C.
$$

Thus we can reduce the integration in (34) to $|x-j-1| < 2$. In the remaining integrals the functions to integrate are upper-bounded by Cn on the domain of integration and also $\frac{1}{n^3}$ far from it. Then, as $|4\pi t\delta| < \frac{1}{n^3}$, we can shift the domain of integration and remove the $4\pi t\delta$ factors from the denominators to obtain

$$
\left| \widehat{T_x}(t,\xi) - i \sum_j \overline{A_j(t)} A_{j+2}(t) e^{i\frac{(j+2)^2-j^2}{4t}} \right.
$$

$$
\left. \times \int_{\substack{|x-j-2|>\frac{1}{n}, \\ |x-j|>\frac{1}{n}, \\ |x-j-1|<2}} e^{ix2\pi\delta} \left(\frac{1}{x-j-2} - \frac{1}{x-j} \right) T(t,x)\,dx \right| \leq \frac{C(\|A_j(t)\|_{l^1})}{n\sqrt{t}}.
$$

Now, using the fact that $\alpha_j = 0$ for $|j| > 1$ and that the integrals are upper-bounded by $C \log n$, we have

$$
\left| \widehat{T_x}(t,\xi) - i \overline{\alpha_{-1}} \alpha_1 e^{i(|\alpha_{-1}|^2-|\alpha_1|^2)\log t} \int_{\substack{|x-1|>\frac{1}{n}, \\ |x+1|>\frac{1}{n}, \\ |x|<2}} e^{ix2\pi\delta} \left(\frac{1}{x-1} - \frac{1}{x+1} \right) T(t,x)\,dx \right|
$$

$$
\leq \frac{C(\|A_j(t)\|_{l^1})}{n\sqrt{t}} + \sum_{|j|\leq 1} |\alpha_j|\,|R_{j+2}(t)| \log n + \sum_{|j+2|\leq 1} |R_j(t)|\,|\alpha_{j+2}| \log n
$$

$$
+ \sum_j |R_j(t)|\,|R_{j+2}(t)| \log n.
$$

Since $|x2\pi\delta| \in (0, \frac{4\pi}{n})$ and the functions to integrate are upper-bounded by n and the integration domain is bounded, the factors $e^{ix2\pi\delta}$ can be neglected. By using the decay

(30) of $R_j(t)$ with $\gamma > \frac{1}{2}$, we get

$$|R_j(t)| \leq \|R_k(t)\|_{l^{2,s}} \leq C(\gamma)t^\gamma \|\alpha_k\|_{l^{2,s}}^3 \leq \frac{C}{n}\|\alpha_k\|_{l^{2,s}}^3, \tag{35}$$

thus the conclusion (33) of the lemma. For $\xi = -\frac{1}{2\pi t} + \delta$ we proceed in the same way. ∎

2.3. Proof of the logarithmic growth

Lemma 2.2 and $(31)^2$ ensure that for $\xi_n = \frac{n^2}{2\pi t} + \delta$ with $t \in (0, \frac{1}{4\pi})$ and $|\delta| < \frac{1}{n}$ we have

$$\left|\widehat{T_x}\left(\frac{t}{n^2}, \xi_n\right) - i|\alpha|^2 \int_{\substack{|x-1|>\frac{1}{n}, \\ |x+1|>\frac{1}{n}, \\ |x|<2}} \left(\frac{1}{x-1} - \frac{1}{x+1}\right)T\left(\frac{t}{n^2}, x\right) dx\right| \leq \frac{C(\|A_j(\frac{t}{n^2})\|_{l^1})}{\sqrt{t}}.$$

Now we note that we can further reduce to

$$\left|\widehat{T_x}\left(\frac{t}{n^2}, \xi_n\right) - i|\alpha|^2 \int_{\substack{\{\frac{1}{3}>|x-1|>\frac{1}{n}\} \\ \cup\{\frac{1}{3}>|x+1|>\frac{1}{n}\}}} \left(\frac{1}{x-1} - \frac{1}{x+1}\right)T\left(\frac{t}{n^2}, x\right) dx\right| \leq \frac{C(\|A_j(\frac{t}{n^2})\|_{l^1})}{\sqrt{t}}.$$

We recall also that from [4, Lemma 4.1] we have the convergence

$$\left|T\left(\frac{t}{n^2}, x\right) - T(0, x)\right| \leq C(\|\alpha_j\|_{l^{1,1}})(1 + |x|)\sqrt{\frac{t}{n^2}}\left(\frac{1}{d(x, \frac{1}{2}\mathbb{Z})} + \frac{1}{d(x, \mathbb{Z})}\right),$$

so

$$\left|\widehat{T_x}\left(\frac{t}{n^2}, \xi_n\right) - i|\alpha|^2 \int_{\substack{\{\frac{1}{3}>|x-1|>\frac{1}{n}\} \\ \cup\{\frac{1}{3}>|x+1|>\frac{1}{n}\}}} \left(\frac{1}{x-1} - \frac{1}{x+1}\right)T(0, x)\,dx\right|$$
$$\leq \frac{C(\|A_j(\frac{t}{n^2})\|_{l^1})}{\sqrt{t}} + C|\alpha|^3\sqrt{t}.$$

As χ_0 is a polygonal line with $T^{-\infty}$, T^0, and $T^{+\infty}$ the directions on $x < -1$, $-1 < x < 1$, and $1 < x$ respectively,

$$\int_{\substack{\{\frac{1}{3}>|x-1|>\frac{1}{n}\} \\ \cup\{\frac{1}{3}>|x+1|>\frac{1}{n}\}}} \left(\frac{1}{x-1} - \frac{1}{x+1}\right)T(0, x)\,dx$$
$$= T^{-\infty}\left[\log\frac{|x-1|}{|x+1|}\right]_{-\frac{4}{3}}^{-1-\frac{1}{n}} + T^0\left[\log\frac{|x-1|}{|x+1|}\right]_{-1+\frac{1}{n}}^{-\frac{2}{3}} + T^0\left[\log\frac{|x-1|}{|x+1|}\right]_{\frac{2}{3}}^{1-\frac{1}{n}}$$
$$+ T^{+\infty}\left[\log\frac{|x-1|}{|x+1|}\right]_{1+\frac{1}{n}}^{\frac{4}{3}},$$

---

^{2}Just for simplicity we imposed $\alpha_{-1} = \alpha_1 = \alpha$, i.e. the corners of the curve χ_0 have the same angles. The computations go the same in the general case with $\overline{\alpha_{-1}}\alpha_1 e^{i(|\alpha_{-1}|^2 - |\alpha_1|^2)\log t}$ instead of $|\alpha|^2$.

and logarithmic growth in n comes from the boundary terms at $-1 - \frac{1}{n}$, $-1 + \frac{1}{n}$, $1 - \frac{1}{n}$, and $1 + \frac{1}{n}$. Therefore we obtain

$$\left| \widehat{T_x} \left(\frac{t}{n^2}, \xi_n \right) - i|\alpha|^2 (T^{-\infty} - 2T^0 + T^{+\infty}) \log n \right| \leq \frac{C(\|A_j(\frac{t}{n^2})\|_{l^1})}{\sqrt{t}} + C|\alpha|^3 \sqrt{t}.$$

Now we compute

$$|T^{-\infty} - 2T^0 + T^{+\infty}| = 2(1 - \cos(\pi - \theta)). \tag{36}$$

The smallest values of this modulus appear for θ close to π, equivalent to α close to zero in view of (22):

$$1 - \frac{(\pi - \theta)^2}{8} \stackrel{\theta \to \pi}{\approx} \cos \frac{\pi - \theta}{2} = \sin \frac{\theta}{2} = e^{-\frac{\pi}{2}|\alpha|^2} \stackrel{\alpha \to 0}{\approx} 1 - \frac{\pi}{2}|\alpha|^2,$$

thus

$$|T^{-\infty} - 2T^0 + T^{+\infty}| = 2(1 - \cos(\pi - \theta)) \stackrel{\theta \to \pi}{\approx} (\pi - \theta)^2 \approx 4\pi|\alpha|^2.$$

We then choose t small enough that

$$C|\alpha|^3 \sqrt{t} < \frac{1}{8}|\alpha|^2 |T^{-\infty} - 2T^0 + T^{+\infty}| \log n,$$

which reduces to

$$\sqrt{t} < C|\alpha| \log n,$$

and large enough that

$$\frac{C(\|A_j(\frac{t}{n^2})\|_{l^1})}{\sqrt{t}} < \frac{1}{8}|\alpha|^2 |T^{-\infty} - 2T^0 + T^{+\infty}| \log n,$$

which is implied, in view of (35), by

$$\frac{C(|\alpha| + \frac{|\alpha|^3}{n})}{|\alpha|^4 \log n} < \sqrt{t}.$$

Therefore the conditions on t and n with respect to α are

$$\frac{C(|\alpha| + \frac{|\alpha|^3}{n})}{|\alpha|^4 \log n} < \sqrt{t} < \min\{C|\alpha| \log n, \tfrac{1}{4\pi}\}. \tag{37}$$

We note that the upper and lower conditions on t imply that n has to be chosen large with respect to α.

Summarizing, we have obtained the existence of $t_\theta, \tilde{t}_\theta \in (0, 1)$ and $n_\theta \in \mathbb{N}$ such that for $n \in \mathbb{N}$, $n \geq n_\theta$, and $t \in (\frac{\tilde{t}_\theta}{\log^2 n}, t_\theta)$ the following growth holds:

$$\left| \widehat{T_x} \left(\frac{t}{n^2}, \frac{n^2}{2\pi t} + \delta \right) - i|\alpha|^2 (T^\infty - 2T^0 + T^{+\infty}) \log n) \right|$$

$$\leq \frac{1}{2}|\alpha|^2 |T^{-\infty} - 2T^0 + T^{+\infty}| \log(n) \quad \forall \delta, |\delta| < \frac{1}{n}.$$

This yields (26) in Theorem 1.1. For the analysis at $-\frac{n^2}{2\pi t} + \delta$ we proceed in the same way.

As a consequence we get the existence of $C_\theta > 0$ such that for $n \geq n_\theta$ and $t \in (\frac{\tilde{t}_\theta}{n^2}, \frac{t_\theta}{n^2}]$ we have

$$\sup_\xi |\widehat{T_x}(t, \xi)| \geq C_\theta \log n.$$

We can choose n_θ large enough that for all $\tau \in (\tilde{t}_\theta, t_\theta)$,

$$|\log \tau| < \frac{1}{2} \left| \log \frac{t_\theta}{n_\theta^2} \right| < \frac{1}{2} \left| \log \frac{\tau}{n_\theta^2} \right|.$$

Then, for $t = \frac{\tau}{n^2} \in (\frac{\tilde{t}_\theta}{n^2}, \frac{t_\theta}{n^2})$,

$$\sup_\xi |\widehat{T_x}(t, \xi)| \geq \frac{C_\theta}{2} (|\log t| - |\log \tau|) \geq \frac{C_\theta}{4} \log t.$$

Moreover, by choosing n_θ large enough that

$$\frac{t_\theta}{(n+1)^2} > \frac{\tilde{t}_\theta}{n^2},$$

we have $(0, \frac{t_\theta}{n_\theta^2}) = \bigcup_{n \geq n_\theta} (\frac{\tilde{t}_\theta}{n^2}, \frac{t_\theta}{n^2})$, so (27) holds.

2.4. Bounds away from the growth zone

Finally, we consider for $n \in \mathbb{N}^*$ those ξ such that $|\xi - \frac{n^2}{2\pi t}| \geq \frac{3n^2}{2\pi t}$ and $|\xi + \frac{n^2}{2\pi t}| \geq \frac{3n^2}{2\pi t}$, which means $|4\pi \frac{t}{n^2}\xi - 2| \geq \frac{3}{4}$ and $|4\pi \frac{t}{n^2}\xi + 2| \geq \frac{3}{4}$. We denote by $m \in \frac{\mathbb{Z}}{2}$ and $d \in [-\frac{1}{2}, \frac{1}{2}[$ the numbers such that $4\pi \frac{t}{n^2}\xi = 2m + d$, which in particular implies $m \notin \{\pm 1\}$. Then

$$\left| j - r + 4\pi \frac{t}{n^2}\xi \right| < 2n\frac{t}{n^2} \iff r = j + 2m,$$

and Lemma 2.1 gives us for $t \in (0, \frac{1}{4\pi})$,

$$\left| \widehat{T_x}\left(\frac{t}{n^2}, \xi\right) \right| \leq \sum_j \left| A_j\left(\frac{t}{n^2}\right) \right| \left| A_{j+2m}\left(\frac{t}{n^2}\right) \right|$$

$$\times \int_{\substack{|x-j-2m-d|>\frac{1}{n}, \\ |x-j+d|>\frac{1}{n}}} \left| \frac{1}{x-j-2m-d} - \frac{1}{x-j+d} \right| dx + \frac{C(\|A_j(\frac{t}{n^2})\|_{l^1})}{\sqrt{t}}.$$

The piece of integration $|x-j-m| > 4m$ is bounded and the remaining part is upper-bounded by $C \log(\max\{\langle m \rangle, n\})$. Since $m \notin \{\pm 1\}$ then

$$\sum_j \left| A_j\left(\frac{t}{n^2}\right) \right| \left| A_{j+2m}\left(\frac{t}{n^2}\right) \right|$$

$$\leq \sum_{|j| \leq 1} |\alpha_j| \left| R_{j+2m}\left(\frac{t}{n^2}\right) \right| + \sum_{|j+2m| \leq 1} \left| R_j\left(\frac{t}{n^2}\right) \right| |\alpha_{j+2m}|$$

$$+ \sum_j \left| R_j\left(\frac{t}{n^2}\right) \right| \left| R_{j+2m}\left(\frac{t}{n^2}\right) \right|$$

$$= |\alpha| \left(\left| R_{-1+2m}\left(\frac{t}{n^2}\right) \right| + \left| R_{1+2m}\left(\frac{t}{n^2}\right) \right| + \left| R_{-1-2m}\left(\frac{t}{n^2}\right) \right| + \left| R_{1-2m}\left(\frac{t}{n^2}\right) \right| \right)$$
$$+ \sum_j \left| R_j\left(\frac{t}{n^2}\right) \right| \left| R_{j+2m}\left(\frac{t}{n^2}\right) \right|.$$

From (30) we have

$$\sum_j |j|^{2s} \left| R_j\left(\frac{t}{n^2}\right) \right|^2 \leq C \frac{|\alpha|^3}{n},$$

so for all $j \in \mathbb{Z}$,

$$\left| R_j\left(\frac{t}{n^2}\right) \right| \leq \frac{C|\alpha|^3}{\sqrt{n}\langle j\rangle^s}.$$

Thus

$$\left| \widehat{T_x}\left(\frac{t}{n^2}, \xi\right) \right| \leq C(\alpha) \log(\max\{\langle m\rangle, n\}) \frac{1}{\sqrt{n}\langle m\rangle^s} + \frac{C(\alpha)}{\sqrt{t}}.$$

Therefore, for ξ such that $|\xi - \frac{n^2}{2\pi t}| \geq \frac{3n^2}{2\pi t}$ and $|\xi + \frac{n^2}{2\pi t}| \geq \frac{3n^2}{2\pi t}$ and $t \in (0, \frac{1}{4\pi})$ we obtain

$$\left| \widehat{T_x}\left(\frac{t}{n^2}, \xi\right) \right| \leq \frac{C(\alpha)}{\sqrt{t}},$$

and the proof of Theorem 1.1 is complete.

3. Several corners

Instead of 2 corners, we consider a planar polygonal line with $2N$ corners located at $-2N+1, \ldots, 2N-1$, with the same angle θ. Let $m \in \{1, \ldots, N\}$. Proceeding similarly to the proof of Lemma 2.2, we obtain (with constants that can depend on N) for $|\delta| < \frac{1}{n}$,

$$\left| \widehat{T_x}\left(\frac{t}{n^2}, \frac{mn^2}{2\pi t} + \delta\right) \right.$$
$$\left. - i|\alpha|^2 \sum_{\substack{j \in \{(-2N+1),\ldots, \\ (2N-2m-1)\}}} e^{i\frac{(j+2m)^2-j^2}{4}\frac{t}{n^2}} \int_{\substack{\frac{1}{n}<|x-(j+2m)|<\frac{1}{3}, \\ \frac{1}{n}<|x-j|<\frac{1}{3}}} \left(\frac{1}{x-(j+2m)} - \frac{1}{x-j}\right) T(t,x)\, dx \right|$$
$$\leq \frac{C(\|A_j(\frac{t}{n^2})\|_{l^1})}{\sqrt{t}}.$$

Now we restrict to t and n such that $\frac{n^2}{t} \in 8\pi\mathbb{Z}$ to get rid of the phases in front of the integrals. Arguing as for the end of the proof of Theorem 1.1 in Section 2.3, we get

$$\left| \widehat{T_x}\left(\frac{t}{n^2}, \frac{mn^2}{2\pi t} + \delta\right) - V_m \log n \right| \leq \frac{C(\|A_j(t)\|_{l^1})}{\sqrt{t}} + C|\alpha|^3 \sqrt{t} \log n,$$

where

$$V_m = i\left(-\frac{2}{\pi}\right)\log\left(\sin\frac{\theta}{2}\right)$$
$$\times \sum_{\substack{j\in\{(-2N+1),\dots,\\(2N-2m-1)\}}} (T(j^-) - T(j^+) - T((j+2m)^-) + T((j+2m)^+))$$
$$= i\left(-\frac{2}{\pi}\right)\log\left(\sin\frac{\theta}{2}\right) \tag{38}$$
$$\times \big(T((-2N+1)^-) - T((-2N+1+2m)^-) - T((2N-1-2m)^+) + T((2N-1)^+)\big).$$

For $N = 1$ we recover the vector V in Theorem 1.1 as

$$V_1 = i\left(-\frac{2}{\pi}\right)\log\left(\sin\frac{\theta}{2}\right)$$
$$\times \big(T((-2N+1)^-) - T((-2N+1)^+) - T((2N-1)^-) + T((2N-1)^+)\big).$$

We note that for at least one $m \in \{1,\dots,N\}$ we have $V_m \neq 0$.

Continuing similarly to Sections 2.3–2.4 we obtain the following result.

Theorem 3.1. *Consider a polygonal line $\chi_0(x)$ with $2N$ corners of angle θ located at $x \in \{-2N + 1,\dots, 2N - 1\}$. Let $\chi(t, x)$ be its evolution by the binormal flow by the Hasimoto method and denote $T(t, x)$ the tangent vector.*

There exists $t_{\theta,N}, \tilde{t}_{\theta,N} \in (0, 1)$ and $n_{\theta,N} \in \mathbb{N}$ such that for $n \in \mathbb{N}$, $n \geq n_{\theta,N}$, for $t \in (\frac{\tilde{t}_{\theta,N}}{n^2\log^2 n}, \frac{t_{\theta,N}}{n^2})$ satisfying $\frac{1}{t} \in 8\pi\mathbb{Z}$, for $m \in \{1,\dots,N\}$, and for ξ satisfying either $|\xi - \frac{m}{2\pi t}| \leq \frac{1}{n}$ or $|\xi + \frac{m}{2\pi t}| \leq \frac{1}{n}$, the following holds:

$$|\widehat{T_x}(t,\xi) - V_m \log(n)| \leq \frac{1}{2}|V_m|\log n, \tag{39}$$

where V_m is defined in (38).

As a consequence, for $t \in (0, \frac{t_{\theta,N}}{n_{\theta,N}^2})$ there exists $C_{\theta,N} > 0$ such that

$$\sup_\xi |\widehat{T_x}(t,\xi)| \geq C_{\theta,N}\log t. \tag{40}$$

Finally, for ξ such that $|\xi - \frac{m}{2\pi t}| \geq \frac{3}{8\pi t}$ and $|\xi + \frac{m}{2\pi t}| \geq \frac{3}{8\pi t}$ for all $m \in \{1,\dots,N\}$, and $t \in (0, \frac{1}{4\pi})$ we have an upper bound of $|\widehat{T_x}(t,\xi)|$ depending only on θ and N.

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