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A Cabling Formula for the 2-Loop Polynomial of Knots†

By

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Abstract

The 2-loop polynomial is a polynomial presenting the 2-loop part of the logarithm of the Kontsevich invariant of knots. We show a cabling formula for the 2-loop polynomial of knots. In particular, we calculate the 2-loop polynomial for torus knots.

§1. Introduction

The Kontsevich invariant is a very strong invariant of knots (which dominates all quantum invariants and all Vassiliev invariants) and it is expected that the Kontsevich invariant will classify knots. A problem when we study the Kontsevich invariant is that it is difficult to calculate the Kontsevich invariant of an arbitrarily given knot concretely. It has recently been shown [20, 9, 6][1] that the infinite sum of the terms of the logarithm of the Kontsevich invariant with a fixed loop number is presented by using polynomials (after appropriate normalization by the Alexander polynomial). In particular, it is known[2] that

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[1] It was conjectured by Rozansky [20]. The existence of such rational presentations has been proved by Kricker [9] (though such a rational presentation itself is not necessarily a knot invariant in a general loop degree). Further, Garoufalidis and Kricker [6] defined a knot invariant in any loop degree, from which such a rational presentation can be deduced.

[2] This follows from the theory of [2] on the MMR conjecture. See also [9, 6] and references therein.

the 1-loop part is presented by the Alexander polynomial. The polynomial giving the 2-loop part is called the 2-loop polynomial. The values of the 2-loop polynomial has been calculated so far only for particular[3] classes of knots.

In this paper, we give a cabling formula for the 2-loop polynomial (Theorem 4.1), which presents the 2-loop polynomial of a cable knot (see Figure 1) of a knot K in terms of the 2-loop polynomial of K. In particular, we calculate a formula of the 2-loop polynomial for torus knots (Theorem 3.1). This formula and the cabling formula are also obtained independently by Marché [14, 15].

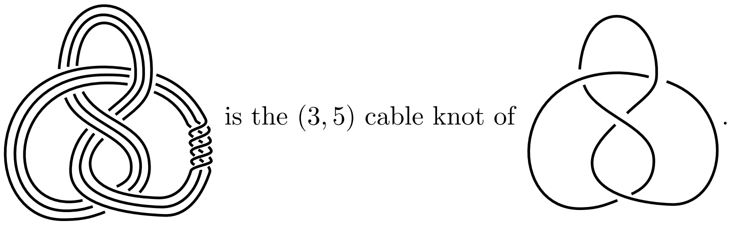

Figure 1. A cable knot of a knot

This paper is organized as follows. In Section 1 we review the definition of the 2-loop polynomial. In Section 2 we calculate the 2-loop polynomial of torus knots as the 2-loop part of the primitive part of the cabling formula of the Kontsevich invariant of the trivial knot. In Section 3 we give a cabling formula for the 2-loop polynomial. In Section 4 we show relations to some Vassiliev invariants. In Section 5 we present the sl_2 reduction of the 2-loop polynomial by a 1-variable reduction of it.

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§2. The Kontsevich Invariant and the 2-Loop Polynomial

The 2-loop polynomial is a polynomial presenting the 2-loop part of the logarithm of the Kontsevich invariant. In this section, we review its definition and a cabling formula of the Kontsevich invariant.

An *open Jacobi diagram* is a uni-trivalent graph such that a cyclic order of the three edges around each trivalent vertex of the graph is fixed. Let $\mathcal{A}(*)$ be

[3] A table of the 2-loop polynomial for knots with up to 7 crossings is given by Rozansky [21]. The 2-loop polynomial of knots with the trivial Alexander polynomial can often been calculated by surgery formulas [6, 10].

the vector space over $\mathbb{Q}$ spanned by open Jacobi diagrams subject to the AS and IHX relations; see Figure 2 for the relations.

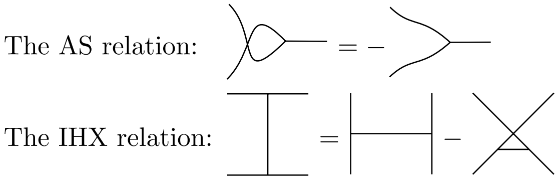


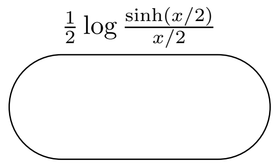
Figure 2. The AS and IHX relations

The *Kontsevich invariant* $Z^\sigma(K)$ of a framed knot K is defined in $\mathcal{A}(*)$; for a definition[4] see *e.g.* [17]. It is known [12] that the value of the Kontsevich invariant for each knot is group-like, which implies that it is presented by the exponential of some primitive element. That is, $Z^\sigma(K)$ is presented by the exponential of a primitive element, where a *primitive element* of $\mathcal{A}(*)$ is a linear sum of connected open Jacobi diagrams.

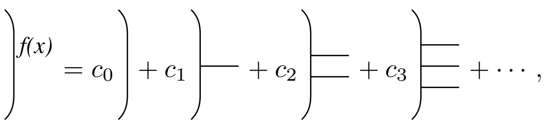
For example, it is shown [4] that the Kontsevich invariant of the trivial knot, denoted by Ω, is presented by

$$Z^\sigma(\text{the trivial knot}) = \Omega = \exp_\sqcup(\omega),$$

where $\exp_\sqcup$ denotes the exponential with respect to the disjoint-union product, and ω is defined by

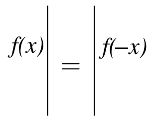
$\omega =$  .

Here, a label of a power series $f(x) = c_0 + c_1x + c_2x^2 + c_3x^3 + \cdots$ implies



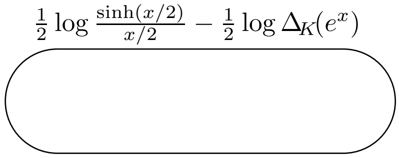
where a label is put on either of the sides of an edge, and the corresponding

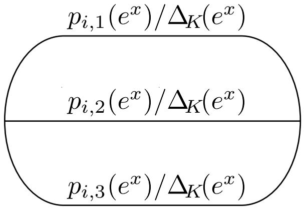
[4]In literatures, the Kontsevich invariant is often defined by $Z(K)$ in the space $\mathcal{A}(S^1)$. The version $Z^\sigma(K)$ is defined to be the image of $Z(K)$ by the inverse map σ of the Poincare-Birkhoff-Witt isomorphism $\mathcal{A}(*) \to \mathcal{A}(S^1)$.

legs are written in the same side of the edge.[5] Note that 

by the AS relation, in the notation of this paper.

Let K be a framed knot with 0 framing. (Throughout this paper, we often mean a framed knot with 0 framing also by a knot, abusing terminology.) A connected open Jacobi diagram is called an *n-loop* diagram when the first Betti number of the uni-trivalent graph of the diagram is equal to n. The *loop expansion* of the Kontsevich invariant is given by

$\log_{\sqcup} Z^{\sigma}(K) =$ 

$+ \sum_i^{\text{finite}}$ 

$+ \big(\text{terms of } (\geq 3)\text{-loop}\big),$

where $\log_{\sqcup}$ denotes the logarithm with respect to the disjoint-union product, and $\Delta_K(t)$ is the normalized[6] Alexander polynomial of K, and $p_{i,j}(e^x)$ is a polynomial in e^x. The 2-loop part is characterized by the polynomial,

$$\Theta'_K(t_1,t_2,t_3) = \sum_i p_{i,1}(t_1)p_{i,2}(t_2)p_{i,3}(t_3).$$

We call its symmetrization,[7]

$$\Theta_K(t_1,t_2,t_3) = \sum_{\substack{\varepsilon=\pm1 \\ \{i,j,k\}=\{1,2,3\}}} \Theta'_K(t_i^{\varepsilon}, t_j^{\varepsilon}, t_k^{\varepsilon}) \quad \in \mathbb{Q}[t_1^{\pm1}, t_2^{\pm1}, t_3^{\pm1}]/(t_1t_2t_3=1),$$

the *2-loop polynomial* of K, which is an invariant[8] of K. (Note that this normalization of $\Theta_K(t_1,t_2,t_3)$ is 12 times the usual normalization.) $\Theta_K(t,t^{-1},1)$

[5] Our notation is different from the notation in [6, 10] where a label of an edge is defined by setting a local orientation of the edge that determines the side in which we write the corresponding legs.
[6] We suppose that $\Delta_K(t)$ is normalized, satisfying that $\Delta_K(t)=\Delta_K(t^{-1})$ and $\Delta_K(1)=1$.
[7] With respect to the symmetry of the theta graph, of order 12.
[8] This is not trivial, since there is another 2-loop trivalent graph, what is called, a "dumbbell diagram".

is a symmetric polynomial in $t^{\pm 1}$ divisible by $t-1$ (since $\Theta_K(1,1,1)=0$) and, hence, divisible by $(t-1)^2$. We define the *reduced 2-loop polynomial* by

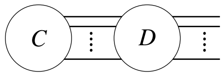
$$\hat{\Theta}_K(t) = \frac{\Theta_K(t,t^{-1},1)}{(t^{1/2}-t^{-1/2})^2} \quad \in \mathbb{Q}[t^{\pm 1}],$$

which is a symmetric polynomial in $t^{\pm 1}$. This gives the sl_2 reduction of the 2-loop polynomial; see Proposition 6.1.

Let us review the cabling formula of the Kontsevich invariant of [4]. Another version of the Kontsevich invariant, called the *wheeled Kontsevich invariant* [3], is defined by

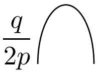
$$Z^w(K) = \partial_\Omega^{-1} Z^\sigma(K),$$

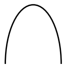
where $\partial_\Omega : \mathcal{A}(*) \to \mathcal{A}(*)$ is the *wheeling isomorphism*; see [4]. Here, for open Jacobi diagrams C and D, $\partial_C(D)$ is defined to be 0 if C has more univalent vertices than D, and the sum of all ways of gluing all univalent vertices of C to some univalent vertices of D otherwise. We graphically present it by

$$\partial_C(D) =$$  .

Let $\Psi^{(p)} : \mathcal{A}(*) \to \mathcal{A}(*)$ be the map which takes a diagram with k univalent vertices to its p^k multiple. The (p,q) *cable knot* of a knot K is the knot given by a simple closed curve on the boundary torus of a tubular neighborhood of K which winds q times in the meridian direction and p times in the longitude direction (see *e.g.* [13]); for example see Figure 1. The cabling formula of the Kontsevich invariant is given by[9]

Proposition 2.1 Le ([4], see also [22]). *Let K be a framed knot with* 0 *framing, and let $K^{(p,q)}$ be the (p,q) cable knot of K (with* 0 *framing). Then,*

$$Z^w(K^{(p,q)}) = \partial_\Omega^{-1}\Psi^{(p)}\partial_\Omega\Big(Z^w(K) \sqcup \exp_\sqcup\big(\frac{q}{2p}$$  $$- \frac{q}{48p}\theta\big)\Big)$$

$$\sqcup \exp_\sqcup\big(-\frac{pq}{2}$$  $$+ \frac{pq}{48}\theta\big).$$

§3. The 2-Loop Polynomial of a Torus Knot

In this section, we calculate the 2-loop polynomial of a torus knot, picking up the 2-loop part of the primitive part of the cabling formula of the Kontsevich

[9] Proposition 2.1 is obtained from Theorem 1 of [4] by pulling back by the isomorphism $\mathcal{A}(*) \xrightarrow{\partial_\Omega} \mathcal{A}(*) \xrightarrow{\chi} \mathcal{A}(S^1)$, and by modifying the contribution from the framing of the cable knot, noting that the (p,q) cable knot in the definition of [4] has framing $(p-1)q$.

invariant of the trivial knot. The 2-loop part of the logarithm of the Kontsevich invariant for torus knots is also calculated[10] independently by Marché [14, 15].

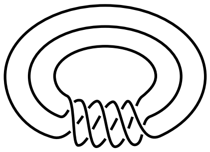


Figure 3. The $(5,3)$ torus knot

The *torus knot* $T(p,q)$ of type (p,q) is the (p,q) cable knot of the trivial knot (which is isotopic to $T(q,p)$); for example see Figure 3. It is known, see *e.g.* [13], that the Alexander polynomial of a torus knot is given by

$$\Delta_{T(p,q)}(t) = \frac{(t^{pq/2} - t^{-pq/2})(t^{1/2} - t^{-1/2})}{(t^{p/2} - t^{-p/2})(t^{q/2} - t^{-q/2})}.$$

Theorem 3.1. *The 2-loop polynomial of the torus knot $T(p,q)$ of type (p,q) is given by*[11]

$$\Theta_{T(p,q)}(t_1,t_2,t_3) = -\frac{1}{4} \sum_{\{i,j,k\}=\{1,2,3\}} \psi_{p,q}(t_i)\psi_{q,p}(t_j)\Delta_{T(p,q)}(t_k) \in \mathbb{Z}[t_1^{\pm 1}, t_2^{\pm 1}, t_3^{\pm 1}]/(t_1 t_2 t_3 = 1),$$

where $\psi_{p,q}$ is defined by

$$\begin{aligned}\psi_{p,q}(t) &= \Delta_{T(p,q)}(t) \cdot \Big(\frac{t^{p/2} + t^{-p/2}}{t^{p/2} - t^{-p/2}} - q \cdot \frac{t^{pq/2} + t^{-pq/2}}{t^{pq/2} - t^{-pq/2}} \Big) \\ &= \frac{t^{1/2} - t^{-1/2}}{(t^{p/2} - t^{-p/2})(t^{q/2} - t^{-q/2})} \\ &\qquad \times \Big((t^{p/2} + t^{-p/2}) \cdot \frac{t^{pq/2} - t^{-pq/2}}{t^{p/2} - t^{-p/2}} - q(t^{pq/2} + t^{-pq/2}) \Big).\end{aligned}$$

In particular, $\Theta_{T(p,q)}(t_1,t_2,t_3)$ is a polynomial in $t_1^{\pm 1}$, $t_2^{\pm 1}$, $t_3^{\pm 1}$ with integer coefficients of $\mathrm{degree}_{t_1}\big(\Theta_{T(p,q)}(t_1,t_2,t_1^{-1}t_2^{-1})\big) = (p-1)(q-1)$.

[10] Bar-Natan has also obtained some presentation of the wheeled Kontsevich invariant for torus knots (private communication).

[11] This value coincides with the value in [14, 15]. However, the values of the 2-loop polynomial for some torus knots in Table 2 of [21] have opposite signs to our values. The signs of some values in Table 2 of [21] might not be correct.

Remark. $\psi_{p,q}(t)$ is not a polynomial, but a rational function, while $\Theta_{T(p,q)}(t_1,t_2,t_3)$ is a polynomial. Rozansky [21] suggests that the 2-loop polynomial is a polynomial with integer coefficients; this holds for torus knots by the theorem. He also suggests a conjectural inequality

$$\mathrm{degree}_{t_1}\big(\Theta_K(t_1,t_2,t_1^{-1}t_2^{-1})\big) \le 2g(K),$$

where $g(K)$ denotes the genus of K. Since the genus of $T(p,q)$ equals $(p-1)(q-1)/2$ (see *e.g.* [13]), torus knots give the equality of the above formula.

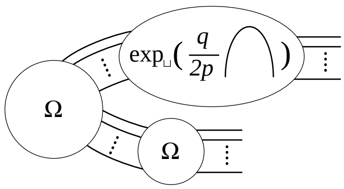
Remark. The sl_2 reduction of the n-loop part of the primitive part of the Kontsevich invariant is equal to the nth line in the expansion of the colored Jones polynomial; see Section 6. Rozansky [19] has calculated it for torus knots.

For group-like elements $\alpha,\beta\in\mathcal{A}(*)$ we write $\alpha\equiv\beta$ if $\log_\sqcup\alpha-\log_\sqcup\beta$ is equal to a linear sum of Jacobi diagrams, either, of (≥ 3)-loop, or, having a component of a trivalent graph (*i.e.*, a component with no univalent vertices).

Proof of Theorem 3.1. Since the torus knot $T(p,q)$ is obtained from the trivial knot by cabling, we have that

$$Z^w\big(T(p,q)\big)\equiv\partial_\Omega^{-1}\Psi^{(p)}\partial_\Omega\Big(\Omega\sqcup\exp_\sqcup\big(\frac{q}{2p}\frown\big)\Big)\sqcup\exp_\sqcup\big(-\frac{pq}{2}\frown\big)$$

by Proposition 2.1. The first term of the right hand side is calculated as follows. From the definition of ∂_Ω,

$$(3.1)\qquad \partial_\Omega\Big(\exp_\sqcup\big(\frac{q}{2p}\frown\big)\sqcup\Omega\Big)=$$  .

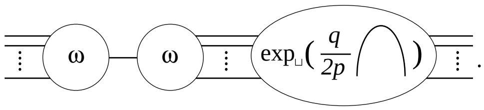
Since any component of Ω has a loop, the (≤ 1)-loop part of the primitive part of the right hand side has no edges between the two Ω's, and, hence, the exponential of this part is presented by

$$\partial_\Omega\exp_\sqcup\big(\frac{q}{2p}\frown\big)\sqcup\Omega.$$

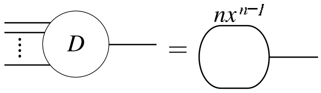
Further, its first term is given by

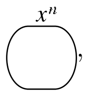
$$\partial_\Omega\exp_\sqcup\big(\frac{q}{2p}\frown\big)\equiv\exp_\sqcup\big(\frac{q}{2p}\frown\big)\sqcup\Omega_{\frac{q}{p}x},$$

where the equivalence is obtained in the same way as Lemma 6.3 of [4], and, as in [4], $\Omega_{\frac{q}{p}x}$ denotes the element obtained from Ω by replacing open Jacobi diagrams with l legs by their $(q/p)^l$ multiples. The 2-loop part of the primitive part of the right hand side of (3.1) is equal to a linear sum of diagrams, each of which has precisely one edge between the two Ω's. Hence, it is presented by

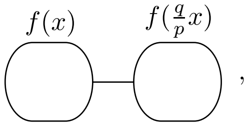


Since



for $D =$


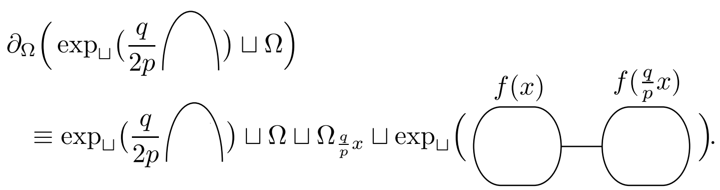
the previous diagram is equivalent to



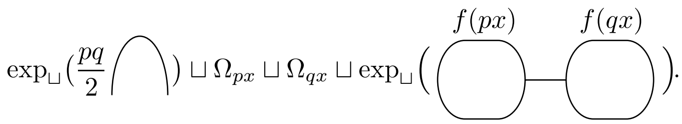
where $f(x)$ is given by

$$f(x) = \frac{d}{dx}\Big(\frac{1}{2}\log\frac{\sinh x/2}{x/2}\Big) = \frac{1}{4}\cdot\frac{e^{x/2}+e^{-x/2}}{e^{x/2}-e^{-x/2}} - \frac{1}{2x}.$$

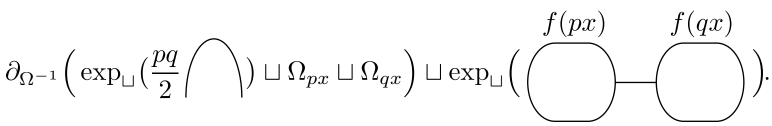
Hence, the (≤ 2)-loop part of the primitive part of (3.1) is presented by

(3.2)


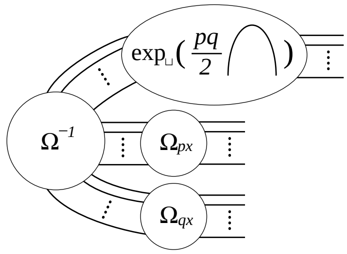
The map $\Psi^{(p)}$ sends this to



Further, ∂_Ω^{-1} sends this (modulo the equivalence) to



Its first term is graphically shown as

(3.3) 

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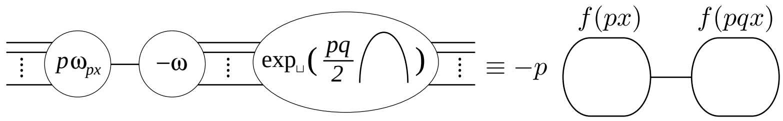
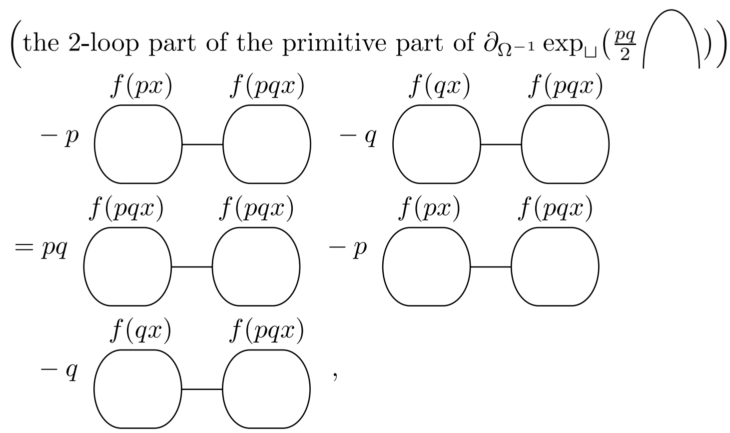
The 2-loop part of the primitive part of this diagram is calculated similarly as before; for example, when there is precisely one edge between Ω^{-1} and Ω_{px}, we have the following component,



Thus, the 2-loop part of the primitive part of (3.3) is equal to



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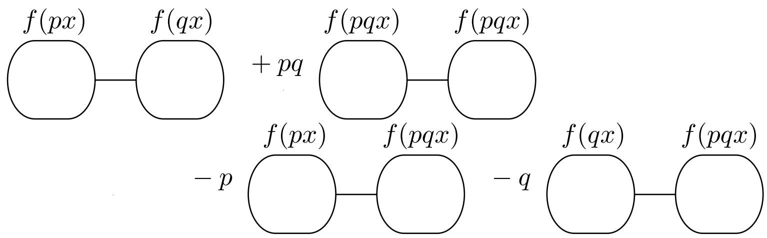
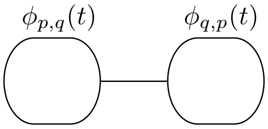
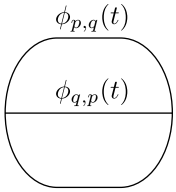
where the equality is obtained from Lemma 3.1 below. Hence, the 2-loop part of the primitive part of $Z^w\big(T(p,q)\big)$ is given by

(3.4) 

$$= \frac{1}{16}$$ 

$$= -\frac{1}{8}$$ 

,

where we put $t = e^x$ and $\phi_{p,q}$ is defined by $\phi_{p,q}(e^x) = 4\big(f(px) - qf(pqx)\big)$, that is,

$$\phi_{p,q}(t) = \frac{t^{p/2} + t^{-p/2}}{t^{p/2} - t^{-p/2}} - q \cdot \frac{t^{pq/2} + t^{-pq/2}}{t^{pq/2} - t^{-pq/2}}.$$

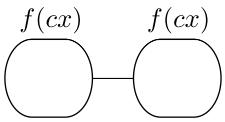
Therefore, from the definition of the 2-loop polynomial, we obtain the required formula.

By Corollary 3.1 below, the degree of $\hat{\Theta}_{T(p,q)}(t)$ equals $(p-1)(q-1)-1$. Since $(t^{1/2} - t^{-1/2})^2 \hat{\Theta}_{T(p,q)}(t) = \Theta_{T(p,q)}(t, 1, t^{-1})$ by definition, t_1-degree of $\Theta_{T(p,q)}(t_1, t_2, t_1^{-1} t_2^{-1})$ is at least $(p-1)(q-1)$. We can show that it is exactly $(p-1)(q-1)$ in the same way as the proof of Example 1. □

Corollary 3.1. *The reduced 2-loop polynomial of the torus knot $T(p,q)$ is given by*

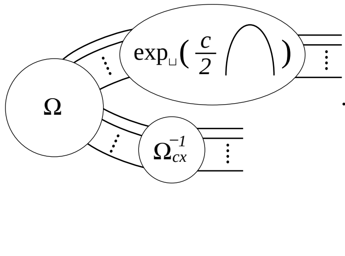
$$\begin{aligned}\hat{\Theta}_{T(p,q)}(t) &= \frac{1}{2(t^{1/2} - t^{-1/2})^2} \psi_{p,q}(t) \psi_{q,p}(t) \\ &= \frac{1}{2} \cdot \frac{1}{(t^{p/2} - t^{-p/2})^2} \cdot \Big((t^{p/2} + t^{-p/2}) \cdot \frac{t^{pq/2} - t^{-pq/2}}{t^{p/2} - t^{-p/2}} - q(t^{pq/2} + t^{-pq/2}) \Big) \\ &\quad \times \frac{1}{(t^{q/2} - t^{-q/2})^2} \cdot \Big((t^{q/2} + t^{-q/2}) \cdot \frac{t^{pq/2} - t^{-pq/2}}{t^{q/2} - t^{-q/2}} - p(t^{pq/2} + t^{-pq/2}) \Big).\end{aligned}$$

Lemma 3.1. *For a scalar c,*

$$\partial_\Omega^{-1} \exp_\sqcup\Big(\frac{c}{2} \bigcap\Big) \equiv \exp_\sqcup\Big(\frac{c}{2} \bigcap\Big) \sqcup \Omega_{cx}^{-1} \sqcup \exp_\sqcup\Big(c$$ 

$$\Big).$$

Proof. From the definition of ∂_Ω,

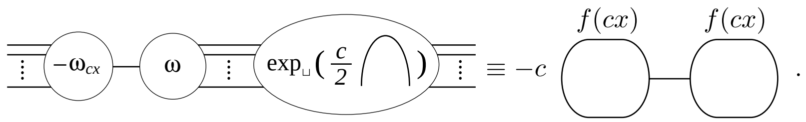
$$(3.5) \qquad \partial_\Omega\Big(\exp_\sqcup\Big(\frac{c}{2} \bigcap\Big) \sqcup \Omega_{cx}^{-1} \Big) =$$ 

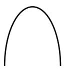
.

Similarly as in the proof of Theorem 3.1, the (≤ 1)-loop part of the primitive part of the right hand side is presented by

$$\partial_\Omega \exp_\sqcup\Big(\frac{c}{2}$$  $$\Big) \sqcup \Omega_{cx}^{-1} \equiv \exp_\sqcup\Big(\frac{c}{2}$$  $$\Big).$$

Further, the 2-loop part of the primitive part of the right hand side of (3.5) is presented by



This implies that ∂_Ω takes the right hand side of the formula of the lemma to $\exp_\sqcup\big(\frac{c}{2}$  $\big)$. □

Example 1. For the $(p,2)$ torus knot, Theorem 3.1 implies that

$$\begin{aligned}\Theta_{T(p,2)}(t_1,t_2,t_3) = {}& \frac{1}{(t_1+1)(t_2+1)(t_3+1)} \\ & \times \Bigg(\frac{p-1}{2}\big(t_1^p + t_1^{-p} + t_2^p + t_2^{-p} + t_3^p + t_3^{-p}\big) \\ & - \frac{t_1^{p-1} - t_1^{-(p-1)}}{t_1 - t_1^{-1}} - \frac{t_2^{p-1} - t_2^{-(p-1)}}{t_2 - t_2^{-1}} - \frac{t_3^{p-1} - t_3^{-(p-1)}}{t_3 - t_3^{-1}} \Bigg).\end{aligned}$$

For example, the coefficients of $\Theta_{T(7,2)}(t_1,t_2,t_3)$ are as shown in Table 1. Further,

$$\begin{aligned}\hat{\Theta}_{T(p,2)}(t) &= \frac{t^2}{(t^2-1)^2}\Big(\frac{p-1}{2}(t^p + t^{-p}) - \frac{t^{p-1} - t^{-(p-1)}}{t - t^{-1}}\Big) \\ &= \frac{t^3}{(t^2-1)^3}\Big(\frac{p-1}{2}(t^{p+1} - t^{-p-1}) - \frac{p+1}{2}(t^{p-1} - t^{-p+1})\Big).\end{aligned}$$

Proof. By definition,

$$\Delta_{T(p,2)}(t) = \frac{t^{p/2} + t^{-p/2}}{t^{1/2} + t^{-1/2}}, \quad \psi_{p,2}(t) = -\frac{t^{p/2} - t^{-p/2}}{t^{1/2} + t^{-1/2}},$$

$$\psi_{2,p}(t) = \frac{1}{(t^{1/2} + t^{-1/2})(t^{p/2} - t^{-p/2})} \cdot \Big((t + t^{-1}) \cdot \frac{t^p - t^{-p}}{t - t^{-1}} - p(t^p + t^{-p})\Big).$$

n	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6
$m=6$	·	·	·	·	·	·	3	-3	3	-3	3	-3	3
$m=5$	·	·	·	·	·	-3	·	·	·	·	·	·	-3
$m=4$	·	·	·	·	3	·	2	-2	2	-2	2	·	3
$m=3$	·	·	·	-3	·	-2	·	·	·	·	-2	·	-3
$m=2$	·	·	3	·	2	·	1	-1	1	·	2	·	3
$m=1$	·	-3	·	-2	·	-1	·	·	-1	·	-2	·	-3
$m=0$	3	·	2	·	1	·	·	·	1	·	2	·	3
$m=-1$	-3	·	-2	·	-1	·	·	-1	·	-2	·	-3	·
$m=-2$	3	·	2	·	1	-1	1	·	2	·	3	·	·
$m=-3$	-3	·	-2	·	·	·	·	-2	·	-3	·	·	·
$m=-4$	3	·	2	-2	2	-2	2	·	3	·	·	·	·
$m=-5$	-3	·	·	·	·	·	·	-3	·	·	·	·	·
$m=-6$	3	-3	3	-3	3	-3	3	·	·	·	·	·	·

Table 1. The non-zero coefficients of $t_1^n t_2^m$ in $\Theta_{T(7,2)}(t_1, t_2, t_1^{-1}t_2^{-1})$

Hence, when $\{i, j, k\} = \{1, 2, 3\}$, we have that

$$\frac{1}{2}\Big(\psi_{p,2}(t_i)\Delta_{T(p,2)}(t_k) + \psi_{p,2}(t_k)\Delta_{T(p,2)}(t_i)\Big) = \frac{t_j^{p/2} - t_j^{-p/2}}{(t_i^{1/2} + t_i^{-1/2})(t_k^{1/2} + t_k^{-1/2})}.$$

Therefore,

$$\begin{aligned}
&-\frac{1}{4}\psi_{2,p}(t_j)\cdot\Big(\psi_{p,2}(t_i)\Delta_{T(p,2)}(t_k) + \psi_{p,2}(t_k)\Delta_{T(p,2)}(t_i)\Big)\\
&= \frac{1}{(t_i^{1/2}+t_i^{-1/2})(t_j^{1/2}+t_j^{-1/2})(t_k^{1/2}+t_k^{-1/2})}\\
&\quad \times \frac{1}{2}\cdot\Big(p(t_j^p+t_j^{-p}) - (t_j+t_j^{-1})\cdot\frac{t_j^p-t_j^{-p}}{t_j-t_j^{-1}}\Big)\\
&= \frac{1}{(t_i^{1/2}+t_i^{-1/2})(t_j^{1/2}+t_j^{-1/2})(t_k^{1/2}+t_k^{-1/2})}\\
&\quad \times \Big(\frac{p-1}{2}\big(t_j^p+t_j^{-p}\big) - \frac{t_j^{p-1}-t_j^{-(p-1)}}{t_j-t_j^{-1}}\Big).
\end{aligned}$$

By Theorem 3.1, we obtain $\Theta_{T(p,2)}(t_1, t_2, t_3)$ as the sum of the above formula over $(i, j, k) = (1, 2, 3), (2, 3, 1), (3, 1, 2)$, which gives the required formula. □

Example 2. In a similar way as the previous example, we have that

$$\begin{aligned}\Theta_{T(p,3)}(t_1,t_2,t_3) &= \frac{(t_1-1)(t_2-1)(t_3-1)}{(t_1^3-1)(t_2^3-1)(t_3^3-1)}\\ &\times \Big((p-1)\big(t_1^p+t_1^{-p}+t_2^p+t_2^{-p}+t_3^p+t_3^{-p}\\ &\qquad +t_1^{2p}+t_1^{-2p}+t_2^{2p}+t_2^{-2p}+t_3^{2p}+t_3^{-2p}\\ &\qquad +t_1^{2p}t_2^p+t_1^{-2p}t_2^{-p}+t_1^pt_2^{2p}+t_1^{-p}t_2^{-2p}+t_1^pt_2^{-p}+t_1^{-p}t_2^p\big)\\ &- \frac{t_1^{3(p-1)/2}-t_1^{-3(p-1)/2}}{t_1^{3/2}-t_1^{-3/2}}\cdot\big(2t_1^{p/2}+2t_1^{-p/2}+t_2^{p/2}t_3^{-p/2}+t_2^{-p/2}t_3^{p/2}\big)\\ &- \frac{t_2^{3(p-1)/2}-t_2^{-3(p-1)/2}}{t_2^{3/2}-t_2^{-3/2}}\cdot\big(2t_2^{p/2}+2t_2^{-p/2}+t_1^{p/2}t_3^{-p/2}+t_1^{-p/2}t_3^{p/2}\big)\\ &- \frac{t_3^{3(p-1)/2}-t_3^{-3(p-1)/2}}{t_3^{3/2}-t_3^{-3/2}}\cdot\big(2t_3^{p/2}+2t_3^{-p/2}+t_1^{p/2}t_2^{-p/2}+t_1^{-p/2}t_2^{p/2}\big)\Big),\end{aligned}$$

and

$$\begin{aligned}\hat{\Theta}_{T(p,3)}(t) &= \frac{t^3(t^{p/2}+t^{-p/2})}{(t^3-1)^2}\\ &\quad\times\Big((p-1)(t^{3p/2}+t^{-3p/2})-2\cdot\frac{t^{3(p-1)/2}-t^{-3(p-1)/2}}{t^{3/2}-t^{-3/2}}\Big)\\ &= \frac{t^{p/2}+t^{-p/2}}{(t^{3/2}-t^{-3/2})^3}\\ &\quad\times\Big((p-1)(t^{3(p+1)/2}-t^{-3(p+1)/2})-(p+1)(t^{3(p-1)/2}-t^{-3(p-1)/2})\Big).\end{aligned}$$

See also Tables 2 and 3 for the values of $\Theta_{T(p,q)}$ and $\hat{\Theta}_{T(p,q)}$ for some (p,q).

§4. A Cabling Formula for the 2-Loop Polynomial

In this section, we give a cabling formula for the 2-loop polynomial. We show the formula by picking up the 2-loop part of the primitive part of the cabling formula of the Kontsevich invariant, modifying the proof of Theorem 3.1. This cabling formula is also obtained independently by Marché [15].

It is known, see *e.g.* [13], that a cabling formula for the Alexander polynomial is given by

$$\Delta_{K^{(p,q)}}(t)=\Delta_{T(p,q)}(t)\Delta_K(t^p).$$

A cabling formula for the 2-loop polynomial is given by

(p,q) : The non-zero coefficients of $t_1^n t_2^m$ in $\Theta_{T(p,q)}(t_1, t_2, t_1^{-1} t_2^{-1})$ in the fundamental domain

$(3,2)$:			-1
	$\cdot$	$\cdot$	1

					2
$(5,2)$:			-1	$\cdot$	-2
	$\cdot$	$\cdot$	1	$\cdot$	2

							-3
$(7,2)$:					2	$\cdot$	3
			-1	$\cdot$	-2	$\cdot$	-3
	$\cdot$	$\cdot$	1	$\cdot$	2	$\cdot$	3

							3
$(4,3)$:					3	-3	$\cdot$
			1	-2	$\cdot$	3	-3
	$\cdot$	$\cdot$	-1	4	-3	$\cdot$	3

									-4
							-4	$\cdot$	4
$(5,3)$:					-6	3	4	-4	$\cdot$
			-2	1	3	-6	$\cdot$	4	-4
	$\cdot$	$\cdot$	2	-2	$\cdot$	6	-4	$\cdot$	4

													6
											6	-6	$\cdot$
									10	-5	$\cdot$	6	-6
$(7,3)$:							12	-5	-5	10	-6	$\cdot$	6
					6	-5	-4	10	-10	$\cdot$	6	-6	$\cdot$
			2	-3	-1	6	-8	$\cdot$	10	-10	$\cdot$	6	-6
	$\cdot$	$\cdot$	-2	6	-4	-2	12	-10	$\cdot$	10	-6	$\cdot$	6

													-6
											-6	6	$\cdot$
									9	$\cdot$	$\cdot$	-6	6
$(5,4)$:							1	-5	$\cdot$	$\cdot$	6	-6	$\cdot$
					-5	4	-4	5	-5	$\cdot$	$\cdot$	$\cdot$	$\cdot$
			1	1	-2	$\cdot$	3	1	-4	$\cdot$	$\cdot$	6	-6
	$\cdot$	$\cdot$	-1	-2	9	-8	1	-2	9	$\cdot$	-6	$\cdot$	6

																			-9
																	-9	9	$\cdot$
															15	$\cdot$	$\cdot$	-9	9
													15	-15	$\cdot$	$\cdot$	9	-9	$\cdot$
$(7,4)$:											-6	-6	$\cdot$	15	-15	$\cdot$	$\cdot$	9	-9
									-18	7	-1	12	-15	8	7	$\cdot$	-9	$\cdot$	9
							-8	10	5	-6	1	$\cdot$	7	-15	8	$\cdot$	9	-9	$\cdot$
					5	-5	4	-11	13	-13	7	-7	8	-8	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$
			2	-4	2	4	2	-9	$\cdot$	11	-6	-5	$\cdot$	15	-15	$\cdot$	$\cdot$	9	-9
	$\cdot$	$\cdot$	-2	8	-9	2	-4	20	-18	2	4	12	-15	$\cdot$	15	$\cdot$	-9	$\cdot$	9

Table 2. The non-zero coefficients of $t_1^n t_2^m$ in $\Theta_{T(p,q)}(t_1, t_2, t_1^{-1} t_2^{-1})$ in a fundamental domain $\{0 \le 2m \le n\}$ (see [21]) for (p,q) with $p \le 7$, $q \le 4$. The array for each (p,q) is a subset of the full array such as shown in Table 1 and the most left dot is at $(n,m) = (0,0)$. We can recover the other coefficients for each (p,q) from the presented coefficients by the symmetry of $\Theta_K(t_1, t_2, t_1^{-1} t_2^{-1})$.

(p,q) :	The part of non-negative powers in $\hat{\Theta}_{T(p,q)}(t)$
$(3,2)$:	t
$(5,2)$:	$3t+2t^3$
$(7,2)$:	$6t+5t^3+3t^5$
$(9,2)$:	$10t+9t^3+7t^5+4t^7$
$(4,3)$:	$3t+4t^2+3t^5$
$(5,3)$:	$6t+4t^2+6t^4+4t^7$
$(7,3)$:	$10t+12t^2+6t^4+12t^5+10t^8+6t^{11}$
$(8,3)$:	$15t+12t^2+16t^4+7t^5+15t^7+12t^{10}+7t^{13}$
$(10,3)$:	$21t+24t^2+16t^4+25t^5+9t^7+24t^8+21t^{11}+16t^{14}+9t^{17}$
$(5,4)$:	$6t+12t^2+9t^3+8t^6+9t^7+6t^{11}$
$(7,4)$:	$15t+24t^2+9t^3+18t^5+20t^6+18t^9+12t^{10}+15t^{13}+9t^{17}$
$(9,4)$:	$21t+40t^2+27t^3+12t^5+36t^6+30t^7+28t^{10}+30t^{11}+16t^{14}+27t^{15}$ $+21t^{19}+12t^{23}$
$(6,5)$:	$10t+24t^2+27t^3+16t^4+15t^7+24t^8+18t^9+15t^{13}+16t^{14}+10t^{19}$
$(7,5)$:	$36t+12t^2+20t^3+30t^4+36t^6+24t^8+18t^9+30t^{11}+24t^{13}+18t^{16}$ $+20t^{18}+12t^{23}$
$(8,5)$:	$45t+24t^2+14t^3+48t^4+36t^6+30t^7+45t^9+21t^{11}+32t^{12}+36t^{14}$ $+30t^{17}+21t^{19}+24t^{22}+14t^{27}$
$(9,5)$:	$28t+60t^2+54t^3+16t^4+36t^6+60t^7+42t^8+40t^{11}+54t^{12}+24t^{13}$ $+40t^{16}+42t^{17}+36t^{21}+24t^{22}+28t^{26}+16t^{31}$

Table 3. The parts of non-negative powers in $\hat{\Theta}_{T(p,q)}(t)$ for (p,q) with $p \le 10$, $q \le 5$. The remaining part for each (p,q) can recover from the presented part by replacing t with t^{-1}.

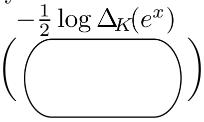
Theorem 4.1. *Let K be a knot, and let $K^{(p,q)}$ be the (p,q) cable knot of K. Then,*

$$\begin{aligned}\Theta_{K^{(p,q)}}(t_1,t_2,t_3) &= \Theta_{T(p,q)}(t_1,t_2,t_3) + \Theta_K(t_1^p,t_2^p,t_3^p)\\ &\quad + \frac{1}{2}\Delta_{T(p,q)}(t_1)\Delta_{T(p,q)}(t_2)\Delta_{T(p,q)}(t_3)\\ &\quad \times \sum_{\{i,j,k\}=\{1,2,3\}} \Delta'_K(t_i^p)\cdot t_i^p \cdot \phi_{q,p}(t_j)\Delta_K(t_j^p)\Delta_K(t_k^p).\end{aligned}$$

Proof. We show the theorem, modifying the proof of Theorem 3.1. By Proposition 2.1, we have that

$$Z^w\big(K^{(p,q)}\big) \equiv \partial_\Omega^{-1}\Psi^{(p)}\partial_\Omega\Big(Z^w(K)\sqcup \exp_\sqcup\big(\frac{q}{2p}\frown\big)\Big)\sqcup \exp_\sqcup\big(-\frac{pq}{2}\frown\big),$$

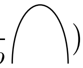
where $Z^w(K)$ is presented by

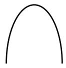
$$Z^w(K) = \Omega \sqcup \exp_{\sqcup}\Big($$ 

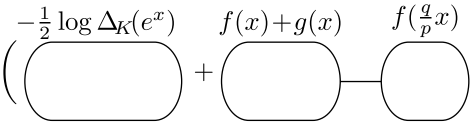
$$\Big) + (\text{terms of } (\geq 2)\text{-loop}).$$

The 2-loop part of $\log_{\sqcup} Z^w(K)$ contributes to the required formula by $\Theta_K(t_1^p, t_2^p, t_3^p)$. We calculate the contribution from the 1-loop part in the following of this proof.

In a similar way as (3.2), we have that

$$\partial_\Omega\Big(Z^w(K) \sqcup \exp_{\sqcup}\big(\frac{q}{2p}$$  $$\big)\Big)$$

$$\equiv \exp_{\sqcup}\big(\frac{q}{2p}$$  $$\big) \sqcup \Omega \sqcup \Omega_{\frac{q}{p}x}$$

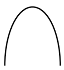
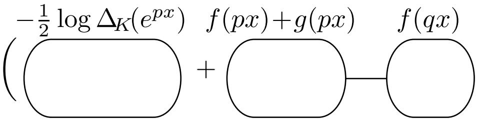
$$\sqcup \exp_{\sqcup}\Big($$ 

$$\Big),$$

where $g(x)$ is given by

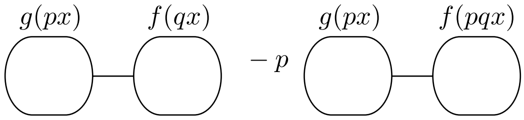
$$g(x) = \frac{d}{dx}\Big(-\frac{1}{2}\log\Delta_K(e^x)\Big) = -\frac{\Delta_K'(e^x)\cdot e^x}{2\Delta_K(e^x)}.$$

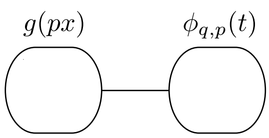
The map $\Psi^{(p)}$ sends this to

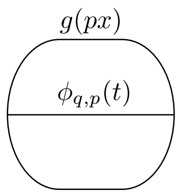
$$\exp_{\sqcup}\big(\frac{pq}{2}$$  $$\big) \sqcup \Omega_{px} \sqcup \Omega_{qx} \sqcup \exp_{\sqcup}\Big($$ 

$$\Big).$$

Calculating its image by ∂_Ω^{-1} in a similar way as in the proof of Theorem 3.1, the error term corresponding to the formula (3.4) is as follows,



$$= \frac{1}{4}$$ 

$$= -\frac{1}{2}$$ 

.

This contributes to the required formula by

$$\sum_{\{i,j,k\}=\{1,2,3\}} \frac{\Delta_K'(t_i^p)\cdot t_i^p}{2\Delta_K(t_i^p)} \cdot \Delta_{K^{(p,q)}}(t_i)\phi_{q,p}(t_j)\Delta_{K^{(p,q)}}(t_j)\Delta_{K^{(p,q)}}(t_k).$$

Noting that $\Delta_{K(p,q)}(t) = \Delta_{T(p,q)}(t)\Delta_K(t^p)$, we obtain the required formula. $\square$

A cabling formula for the reduced 2-loop polynomial is given by

Corollary 4.1. *For the notation in* Theorem 4.1,

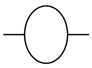
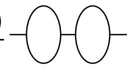
$$\hat{\Theta}_{K(p,q)}(t) = \hat{\Theta}_{T(p,q)}(t) + \frac{(t^{p/2} - t^{-p/2})^2}{(t^{1/2} - t^{-1/2})^2} \cdot \hat{\Theta}_K(t^p) - \frac{t^p}{(t^{1/2} - t^{-1/2})^2} \cdot \Delta_{T(p,q)}(t)\Delta_K(t^p)\Delta'_K(t^p)\psi_{q,p}(t).$$

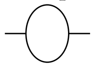
Proof. The required formula is obtained from the formula of Theorem 4.1 by putting $t_1 = t$, $t_2 = 1/t$, and $t_3 = 1$. $\square$

§5. Relations to Vassiliev Invariants

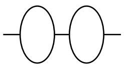
In this section we show some relations to Vassiliev invariants of degree 2, 3.

A leading part of the Kontsevich invariant is presented by

$\log_{\sqcup} Z^{\sigma}(K) - \omega = \dfrac{v_2(K)}{2}$  $+ \dfrac{v_3(K)}{4}$  $+$(terms of degree ≥ 4),

where the *degree* of a Jacobi diagram is half the number of univalent and trivalent vertices of the diagram, and v_2, v_3 are $\mathbb{Z}$-valued primitive Vassiliev invariants of degree 2, 3 respectively (see [17]). Since  has 1-loop, $v_2(K)$ can be presented by the Alexander polynomial; in fact, from the formula of the loop expansion,

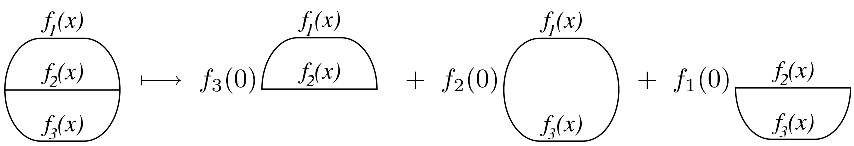
$$\begin{aligned} v_2(K) &= -\big(\text{the coefficient of } x^2 \text{ in the expansion of } \Delta_K(e^x)\big) \\ &= -\frac{1}{2}\Delta''_K(1). \end{aligned}$$

Further, since  has 2-loop, $v_3(K)$ can be presented by the 2-loop polynomial; in fact, we have

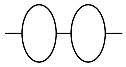
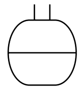
Proposition 5.1.

$$v_3(K) = \frac{1}{2}\hat{\Theta}_K(1).$$

Proof. Let us consider the map



$$\longmapsto \frac{1}{6}\sum_{\{i,j,k\}=\{1,2,3\}} f_i(x)f_j(-x)f_k(0).$$

This map takes the 2-loop part of $\log_{\sqcup} Z^{\sigma}(K)$ to $\frac{1}{12}(e^{x/2}-e^{-x/2})^2\hat{\Theta}_K(e^x)/(\Delta_K(e^x))^2$, whose coefficient of x^2 equals $\frac{1}{12}\hat{\Theta}_K(1)$. Since  =  by the AS and IHX relations, the above maps takes this diagram to $\frac{2}{3}x^2$. Hence, $\frac{1}{6}v_3(K)=\frac{1}{12}\hat{\Theta}_K(1)$, which implies the required formula. □

Example 3. A cabling formula for v_3 is given by

$$v_3(K^{(p,q)}) = p^2\cdot v_3(K) + \frac{1}{12}p(p^2-1)q\cdot\Delta_K''(1) + \frac{1}{144}p(p^2-1)q(q^2-1).$$

Proof. From Proposition 5.1 and Corollary 4.1 putting $t=1$, we have that

$$v_3(K^{(p,q)}) = v_3\big(T(p,q)\big) + p^2\cdot v_3(K) - \frac{p}{2}\Delta_K''(1)\phi_{q,p}'(1).$$

The required formula follows from it, by using

$$v_3\big(T(p,q)\big) = \frac{1}{2}\hat{\Theta}_{T(p,q)}(1) = \frac{1}{144}p(p^2-1)q(q^2-1),$$
$$\phi_{q,p}'(1) = \frac{1}{6}q(1-p^2).$$

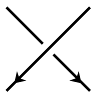
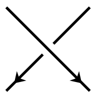
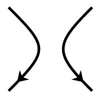
For the value of the first formula, see also [22]. □

§6. The sl_2 Reduction of the 2-Loop Polynomial

The aim of this section is to show Proposition 6.1, which implies that the sl_2 reduction of the 2-loop part of the logarithm of the Kontsevich invariant is presented by the reduced 2-loop polynomial.

The loop expansion of the colored Jones polynomial

Let us denote by $J(L;t)$ the *Jones polynomial* [8] of a link L defined by

$t^{-1}V\big($  $;t\big) - tV\big($  $;t\big) = (t^{1/2}-t^{-1/2})V\big($  $;t\big)$

and by the normalization[12] $J(\text{the trivial knot};t) = t^{1/2}+t^{-1/2}$, where the three pictures in the above formula denote three oriented links, which are identical

[12] This normalization is the normalization of the quantum sl_2 invariant (see *e.g.* [17]), which differs from the usual normalization where the value of the trivial knot is 1.

except for a ball, where they differ as shown in the pictures. The *colored Jones polynomial* [16], which we denote by $J_k(K;t)$, of a knot K is defined by

$$J(K^{(n)};t) = \sum_{0 \le k \le n/2} c_{n,k} J_{n+1-2k}(K;t)$$

where $K^{(n)}$ denotes the disconnected n cable of K with 0 framing, and $c_{n,k}$'s are scalars characterized[13] by $V_2^{\otimes n} = \bigoplus_{0 \le k \le n/2} c_{n,k} V_{n+1-2k}$; in particular $J_1(K;t) = 1$ and $J_2(K;t) = J(K;t)$. The colored Jones polynomial in another normalization, which we denote by $V_n(K;t)$, is defined by

$$V_n(K;t) = \frac{J_n(K;t)}{J_n(\text{the trivial knot};t)} = \frac{t^{1/2} - t^{-1/2}}{t^{n/2} - t^{-n/2}} \cdot J_n(K;t).$$

As in [19], based on the expansion

$$V_n(K;e^h) = \sum_{l \ge 0} h^l \sum_{k \ge 0} d_{l,k} (nh)^k,$$

the 1-loop and 2-loop parts of the colored Jones polynomial are given by

$$V^{(\text{1-loop})}(K;e^{nh}) = \sum_{k \ge 0} d_{0,k}(nh)^k,$$
$$V^{(\text{2-loop})}(K;e^{nh}) = \sum_{k \ge 0} d_{1,k}(nh)^k,$$

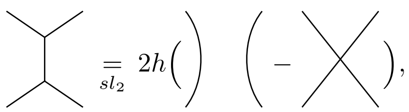
where the right hand sides are rational functions of e^{nh}, as discussed in [19]. The aim of this section is to present $V^{(\text{2-loop})}(K;t)$ by the reduced 2-loop polynomial of K.

The colored Jones polynomial is obtained from the Kontsevich invariant by[14]

$$J_n(K;e^{-h}) = W_{sl_2,V_n}\big(Z(K)\big),$$

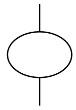
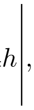
where W_{sl_2,V_n} denotes the weight system derived from the Lie algebra sl_2 and its n-dimensional irreducible representation V_n, which can be calculated recursively (see [5, 17]) by

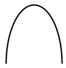
(6.1)



[13]This characterization is based on the disconnected cabling formula of quantum invariants (see *e.g.* [17]). There scalars are concretely presented by $c_{n,k} = \binom{n-1}{k} - \binom{n-1}{n+1-k}$.

[14]In the left hand side, we put, not $t = e^h$, but $t = e^{-h}$. This difference is derived from the difference of normalization between the colored Jones polynomial and the quantum sl_2 invariants.

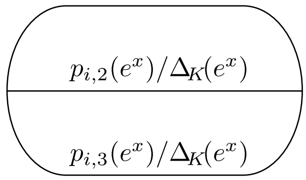
(6.2)  $\underset{sl_2}{=} 4h$  ,

(6.3)  $\sqcup\, \alpha \underset{sl_2}{=} hC \cdot \alpha,$

where we write $\alpha \underset{sl_2}{=} \beta$ if $W_{sl_2,V_n}(\alpha) = W_{sl_2,V_n}(\beta)$, and C denotes the Casimir element of sl_2, whose eigenvalue on V_n is equal to $\frac{n^2-1}{2}$. We apply these recursive relations to

$$\frac{Z^w(K)}{Z^w(O)} = \exp_{\sqcup}\Bigg($$

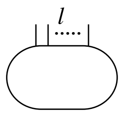
$$+\sum_i^{\text{finite}}$$



$$+ \big((\geq 3)\text{-loop part}\big)\Bigg).$$

The 1-loop part

Lemma 6.1. *For a positive integer l,*

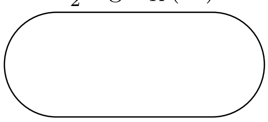


$$\underset{sl_2}{=} (2C)^{l/2} h^l \big(1 + (-1)^l\big).$$

Proof. If l is odd, the diagram is equal to 0 by the AS relation, and, hence, the lemma holds. If l is even, the lemma is proved by induction on l using (6.1) and (6.3). ☐

Putting $-\frac{1}{2}\log\Delta_K(e^x) = \sum_{k\geq 0} a_k x^{2k}$, we have that

$$\exp_{\sqcup}\Bigg($$



$$\Bigg) \underset{sl_2}{=} \sum_{k\geq 0} \exp\big(2a_k (2C)^k h^{2k}\big)$$

$$\equiv \exp\Big(2\sum_{k\geq 0} a_k (nh)^{2k}\Big) = \frac{1}{\Delta_K(e^{nh})},$$

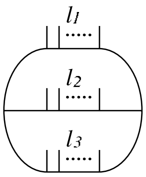
where we write $\alpha \equiv \beta$ if $\log\alpha - \log\beta$ is equal to a linear sum of contributions from (≥ 3)-loop diagrams. Hence,

$$V^{(\text{1-loop})}(K;t) = \frac{1}{\Delta_K(t)}.$$

This is nothing but the Melvin-Morton-Rozansky conjecture proved in [2].

The 2-loop part

Lemma 6.2. *Let l_1, l_2, l_3 be non-negative integers such that at least one of them is positive. Then,*

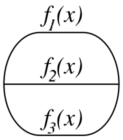


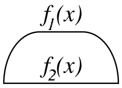
$$\underset{sl_2}{=} \begin{cases} 0 & \text{if } l_1 l_2 l_3 \neq 0, \\ 2h(2C)^{(l_i+l_j)/2} h^{l_i+l_j}\big((-1)^{l_i}+(-1)^{l_j}\big) & \text{if } l_i l_j \neq 0 \text{ and } l_k = 0, \\ 4h(2C)^{l_i/2} h^{l_i}\big(1+(-1)^{l_i}\big) & \text{if } l_i \neq 0 \text{ and } l_j = l_k = 0, \end{cases}$$

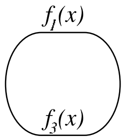
where $\{i,j,k\} = \{1,2,3\}$.

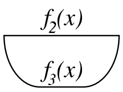
Proof. We assume that $l_1 \geq l_2 \geq l_3$ without loss of generality. If $l_1 > l_2 = l_3 = 0$, then the lemma is obtained from (6.2) and Lemma 6.1. If $l_2 > l_3 = 0$, then the lemma is obtained from (6.1) and Lemma 6.1. If $l_3 > 0$, then we obtain the lemma by induction on l_3; we can decrease l_3 by moving one of l_3 legs to upper edges by the IHX relation. □

By Lemma 6.2,



$\underset{sl_2}{=} 2h\Big(f_3(0)$ 

$+ f_2(0)$ 

$+ f_1(0)$  $\Big)$

$$\equiv 2h \sum_{\{i,j,k\}=\{1,2,3\}} f_i(nh) f_j(-nh) f_k(0).$$

Hence, similarly as in the proof of Proposition 5.1, the sl_2 reduction of the 2-loop part of $\log_{\sqcup}(Z^w(K)/Z^w(O))$ is equal to $h(e^{nh/2} - e^{-nh/2})^2 \hat{\Theta}_K(e^{nh})/(\Delta_K(e^{nh}))^2$. Therefore, we obtain

Proposition 6.1.

$$V^{(\textit{2-loop})}(K;t) = -\frac{(t^{1/2}-t^{-1/2})^2}{\big(\Delta_K(t)\big)^3}\hat{\Theta}_K(t).$$

This gives a concrete presentation of the formula of [19, Conjecture 2] in terms of the reduced 2-loop polynomial.

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