

# Solved and unsolved problems

Michael Th. Rassias

*Probability theory is nothing but common sense reduced to calculation.*

Pierre-Simon Laplace (1749–1827)

The present column is devoted to Probability Theory and related topics.

## I Six new problems – solutions solicited

Solutions will appear in a subsequent issue.

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We take for our probability space  $(X, m)$ : the unit interval  $X = [0, 1]$  equipped with the Lebesgue measure  $m$  defined on  $\mathcal{B}(X)$ , the Borel subsets of  $X$  and let  $(X, m, T)$  be an invertible measure preserving transformation, that is  $T : X_0 \rightarrow X_0$  is a bimeasurable bijection of some Borel set  $X_0 \in \mathcal{B}(X)$  of full measure so that and  $m(TA) = m(T^{-1}A) = m(A)$  for every  $A \in \mathcal{B}(X)$ .

Suppose also that  $T$  is ergodic in the sense that the only  $T$ -invariant Borel sets have either zero- or full measure ( $A \in \mathcal{B}(X)$ ,  $TA = A \Rightarrow m(A) = 0, 1$ ).

Birkhoff's ergodic theorem says that for every integrable function  $f : X \rightarrow \mathbb{R}$ ,

$$\frac{1}{n} \sum_{k=0}^{n-1} f \circ T^k \xrightarrow{n \rightarrow \infty} \mathbb{E}(f) := \int_X f dm \text{ a.s.}$$

The present exercise is concerned with the possibility of generalizing this. Throughout,  $(X, m, T)$  is an arbitrary ergodic, measure-preserving transformation as above.

#### Warm-up 1

Show that if  $f : X \rightarrow \mathbb{R}$  is measurable, and

$$m \left( \left[ \overline{\lim}_{n \rightarrow \infty} \left| \sum_{k=0}^{n-1} f \circ T^k \right| < \infty \right] \right) > 0,$$

then  $\frac{1}{n} \sum_{k=0}^{n-1} f \circ T^k$  converges in  $\mathbb{R}$  a.s.

Warm-up 1 is [1, Lemma 1]. For a multidimensional version, see [1, Conjecture 3].

#### Warm-up 2

Show that if  $f : X \rightarrow \mathbb{R}$  is as in Warm-up 1, then  $\exists g, h : X \rightarrow \mathbb{R}$  measurable with  $h$  bounded so that  $f = h + g - g \circ T^n$ .

Warm-up 2 is established by adapting the proof of [3, Theorem A].

#### Problem

Show that there is a measurable function  $f : X \rightarrow \mathbb{R}$  satisfying  $\mathbb{E}(|f|) = \infty$  so that

$$\frac{1}{n} \sum_{k=0}^{n-1} f \circ T^k$$

converges in  $\mathbb{R}$  a.s.

The existence of such  $f$  for a specially constructed ergodic measure-preserving transformation is shown in [2, example b]. The point here is to prove it for an arbitrary ergodic measure preserving transformation of  $(X, m)$ .

#### References

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- [2] D. Tanny, A zero-one law for stationary sequences. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete* **30**, 139–148 (1974)
- [3] D. Volný and B. Weiss, Coboundaries in  $L_0^\infty$ . *Ann. Inst. H. Poincaré Probab. Statist.* **40**, 771–778 (2004)

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Let  $(\Omega, \mathcal{F}, \mathbb{P})$  be a probability space and  $\{X_n : n \geq 1\}$  a sequence of independent and identically distributed (i.i.d.) random variables on  $\Omega$ . Assume that there exists a sequence of positive numbers  $\{b_n : n \geq 1\}$  such that  $\frac{b_n}{n} \leq \frac{b_{n+1}}{n+1}$  for every  $n \geq 1$ ,  $\lim_{n \rightarrow \infty} \frac{b_n}{n} = \infty$ , and  $\sum_{n=1}^{\infty} \mathbb{P}(|X_n| \geq b_n) < \infty$ . Prove that if  $S_n := \sum_{j=1}^n X_j$  for each  $n \geq 1$ , then

$$\lim_{n \rightarrow \infty} \frac{S_n}{b_n} = 0 \text{ almost surely.}$$

*Comment.* The desired statement says that if such a sequence  $\{b_n : n \geq 1\}$  exists, then  $\{X_n : n \geq 1\}$  satisfies the (generalized) Strong Law of Large Numbers (SLLN) when averaged by  $\{b_n : n \geq 1\}$ . If  $X_n \in L^1(\mathbb{P})$  for every  $n \geq 1$ , then the desired statement follows trivially from Kolmogorov's SLLN, since in that case, with probability one, we have

$$\lim_{n \rightarrow \infty} \frac{S_n}{n} = \mathbb{E}[X_1]$$

and hence

$$\frac{S_n}{b_n} = \frac{S_n}{n} \cdot \frac{n}{b_n}$$

must converge to 0 under the assumptions on  $\{b_n : n \geq 1\}$ . Therefore, the desired statement can be viewed as an alternative to Kolmogorov's SLLN for i.i.d. random variables that are not integrable.

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In Beetown, the bees have a strict rule: all clubs must have exactly  $k$  members. Clubs are not necessarily disjoint. Let  $b(k)$  be the smallest number of clubs that it is possible for the  $n \geq k^2$  bees to form, such that the set of clubs has the property that no matter how the bees divide themselves into two teams to play beeball, there will always be a club all of whose members are on the same team. Prove that

$$2^{k-1} \leq b(k) \leq Ck^2 \cdot 2^k$$

for some constant  $C > 0$ .

*Rob Morris (IMPA, Rio de Janeiro, Brasil)*

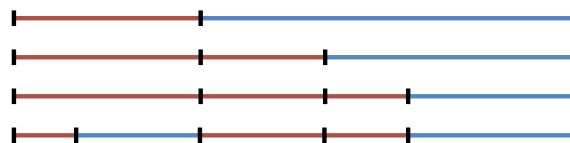
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$N$  agents are in a room with a server, and each agent is looking to get served, at which point the agent leaves the room. At any discrete time-step, each agent may choose to either shout or stay quiet, and an agent gets served in that round if (and only if) that agent is the only one to have shouted. The agents are indistinguishable to each other at the start, but at each subsequent step, every agent gets to see who has shouted and who has not. If all the agents are required to use the same randomised strategy, show that the minimum time expectation to clear the room is  $N + (2 + o(1)) \log_2 N$ .

*Bhargav Narayanan (Rutgers University, Piscataway, USA)*

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Consider the following sequence of partitions of the unit interval  $I$ : First, define  $\pi_1$  to be the partition of  $I$  into two intervals, a red interval of length  $1/3$  and a blue one of length  $2/3$ . Next, for any  $m > 1$ , define  $\pi_{m+1}$  to be the partition derived from  $\pi_m$  by splitting all intervals of maximal length in  $\pi_m$ , each into two intervals, a red one of ratio  $1/3$  and a blue one of ratio  $2/3$ , just as in the first step. For example  $\pi_2$  consists of three intervals of lengths  $1/3$  (red),  $2/9$  (red) and  $4/9$  (blue), the last two are the result of splitting the blue interval in  $\pi_1$ . The figure below illustrates  $\pi_1, \dots, \pi_4$ , from top to bottom.



Let  $m \in \mathbb{N}$  and consider the  $m$ -th partition  $\pi_m$ .

1. Choose an interval in  $\pi_m$  uniformly at random. Let  $R_m$  be the probability you chose a red interval, does the sequence  $(R_m)_{m \in \mathbb{N}}$  converge? If so, what is the limit?
2. Choose a point in  $I$  uniformly at random. Let  $A_m$  be the probability that the point is colored red, does the sequence  $(A_m)_{m \in \mathbb{N}}$  converge? If so, what is the limit?

*Yotam Smilansky (Rutgers University, NJ, USA)*

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Prove that there exist  $c < 1$  and  $\epsilon > 0$  such that if  $A_1, \dots, A_k$  are increasing events of independent boolean random variables with  $\Pr(A_i) < \epsilon$  for all  $i$ , then

$$\Pr(\text{exactly one of } A_1, \dots, A_k \text{ occurs}) \leq c.$$

(What is the smallest  $c$  that you can prove?)

Here  $A \subset \{0, 1\}^n$  is an "increasing event" if whenever  $x \in A$ , then the vector obtained by changing any coordinates of  $x$  from 0 to 1 still lies in  $A$ .

A useful fact is Harris inequality, which says that for increasing events  $A$  and  $B$  of boolean random variables,  $\Pr(A \cap B) \geq \Pr(A) \Pr(B)$ .

I learned of this problem from Jeff Kahn.

*Yufei Zhao (MIT, Cambridge, USA)*

## II Open problems

### *Equidistributed orbits in 2-adic integers*

by Hillel Furstenberg (Einstein Institute of Mathematics,  
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The Collatz problem, also known as the  $3x + 1$  problem is very well known. We will formulate a general conjecture motivated by the Collatz problem, as well as another related conjecture. Both have to do with ordinary integer sequences which should be regarded as subsequences of the 2-adics. First some notation. Every positive integer can be written as  $x = g(x)h(x)$ , with  $g(x)$  a power of 2, and  $h(x)$  odd. We define a transformation on  $\mathbb{N} = \{1, 2, 3, \dots\}$  by

$$T(x) = h(3x + 1).$$

The mysterious phenomenon is that every orbit seems to reach the fixed point 1. The Collatz problem is to prove this. Probability enters as a heuristic explanation of why the orbit is finite, always reaching some cycle. A more “robust” phenomenon is that for any odd  $b$ , if we were to set

$$T(x) = h(3x + b),$$

the orbits are finite.

We point out that it is not hard to show that  $b$  can be chosen so that the corresponding transformation has as many distinct cycles as we like. Nevertheless it seems that all orbits are finite. We propose a still more general conjecture which implies this. For this we need the notion of “equidistribution in the odd 2-adics”.

If we fix a natural number  $k$ , every odd integer lies in one of the  $2^{(k-1)}$  arithmetic progressions

$$P_{j,k} = j + 2^k \mathbb{Z}, \quad j = 1, 3, 5, \dots, 2^k - 1.$$

**Definition.** A sequence  $S$  of odd integers is equidistributed in the odd 2-adics if for every  $j, k$ , the proportion of  $S$  in  $P_{j,k}$  is  $(1/2)^{(k-1)}$ . “Proportion” means the relative density.

Now define

$$T_{a,b}(x) = h(ax + b),$$

where  $a$  and  $b$  are odd integers,  $a > 2, b > 0$ .

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**Conjecture A.** For any natural odd numbers  $a, b, x$ , the orbit of  $x$  under  $T_{a,b}$  is either finite or equidistributed in the odd 2-adics.

Consequence: All the orbits of  $T_{3,b}$  are finite.

To prove this one shows that the assumption of an infinite equidistributed orbit leads to a contradiction. This follows from the fact that on account of equidistribution, the expectation of  $\log g(x)$  is  $\log 4$ . The idea now is to observe that for large  $x$ , the effect of applying  $T_{3,b}$  is – on the average – multiplying by 3 and dividing

by 4. This can be made precise to show that an equidistributed, infinite orbit is an impossibility.

The next conjecture involves equidistribution in the full compact group of 2-adics. Define  $R(x) = 3[x/2]$ . Here as usual  $[z]$  denotes the largest integer less than  $z$ . We have

$$R(0) = R(1) = 0, \quad R(2) = 3, \quad R(3) = 3.$$

But for  $x > 3$ ,

$$R(x) > x,$$

so that for  $x > 3$ , the orbit of  $x$  will be infinite.

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**Conjecture B.** For any  $x > 3$ , the orbit under  $R$  is equidistributed in the 2-adics.

In particular, this would imply that in every such orbit, even and odd integers appear with the same frequency. However, we are unable to verify this for any orbit.

Here too we can consider  $R_a$  defined by

$$R_a(x) = a[x/2], \quad \text{for odd } a.$$

We expect the same phenomenon: orbits that are either finite or equidistributed. Little numerical work has been done on this conjecture.

## III Solutions

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Let  $\mathbb{C}^n$  stand for the space of complex column  $n$ -vectors, and let  $\mathbb{M}_n$  stand for the space of complex  $n \times n$  matrices.

The inner product  $\langle x|y \rangle$  of  $x, y \in \mathbb{C}^n$  is defined as

$$\langle x|y \rangle = x^*y \quad (\text{matrix product}).$$

Therefore  $\langle x|y \rangle$  is linear in  $y$  and conjugate linear in  $x$ .

Let  $A, B$  be  $n \times m$  complex matrices. Write them as

$$A = [a_1, \dots, a_m] \quad \text{and} \quad B = [b_1, \dots, b_m]$$

with  $a_j, b_j \in \mathbb{C}^n$  ( $j = 1, 2, \dots, m$ ).

Then it is immediate from the definition of matrix multiplication that

$$A^*B = [\langle a_j|b_k \rangle]_{j,k=1}^m \in \mathbb{M}_m.$$

Show the following relation:

$$AB^* = \sum_{j=1}^m a_j b_j^* \in \mathbb{M}_n$$

where each product  $a_j b_j^*$  ( $j = 1, \dots, m$ ) is a rank-one matrix in  $\mathbb{M}_n$ .

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*Solution by the proposer*

Consider the canonical orthonormal basis in  $\mathbb{C}^n$ ;

$$e_j := [\delta_{j,k}]_{k=1}^n, \quad (j = 1, 2, \dots, n).$$

Then it suffices to prove the identity in the assertion for the case

$$A = [0, \dots, 0, \overset{(p)}{e_j}, 0, \dots, 0], \quad \exists 1 \leq j \leq n, \exists 1 \leq p \leq m$$

and

$$B = [0, \dots, 0, \overset{(q)}{e_k}, 0, \dots, 0], \quad \exists 1 \leq k \leq n, \exists 1 \leq q \leq m$$

In this case

$$AB^* = \delta_{p,q} e_j e_k^*$$

while

$$\sum_{j=1}^m a_j b_j^* = \delta_{p,q} e_j e_k^*$$

as expected.

*Also solved by John N. Daras (Athens, Greece),  
Muhammad Thoriq (Yogyakarta, Indonesia),  
and Socratis Varelogiannis (France)*

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Let  $p$  and  $q$  be two distinct primes with  $q > p$  and  $G$  a group of exponent  $q$  for which the map  $f_p : G \rightarrow G$  defined by  $f_p(x) = x^p$ , for all  $x \in G$ , is an endomorphism. Show that  $G$  is an abelian group.

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*Solution by the proposer*

We first prove the following auxiliary result.

**Claim.** The map  $f_k : G \rightarrow G$  defined by  $f_k(x) = x^k$  is an endomorphism for all  $k \in \mathbb{Z}$  with  $k \equiv 1 \pmod{p(p-1)}$ .

First, it is not hard to see that  $(xy)^p = x^p y^p$  implies that  $(yx)^{p-1} = x^{p-1} y^{p-1}$ , for all  $x, y \in G$ . Then we observe that

$$(xy)^{(p-1)^2} = (y^{p-1} x^{p-1})^{p-1} = x^{(p-1)^2} y^{(p-1)^2}, \quad \forall x, y \in G.$$

It is also easy to show that

$$x^p y^{p-1} = y^{p-1} x^p, \quad \forall x, y \in G. \quad (\star)$$

Using the latter, we see that

$$(xy)^{p(p-1)} = (x^p y^p)^{p-1} = y^{p(p-1)} x^{p(p-1)} = x^{p(p-1)} y^{p(p-1)}$$

for all  $x, y \in G$ . We just showed that  $f_{p(p-1)}$  is an endomorphism of  $G$ . We proceed by showing that  $f_{p^2-p+1}$  is also an endomorphism.

For every  $x, y \in G$ , we have

$$(xy)^{p^2-p+1} = (xy)^p \cdot (xy)^{(p-1)^2} = x^p \cdot y^p \cdot x^{(p-1)^2} \cdot y^{(p-1)^2}.$$

In the above chain of equalities we just used that  $f_p$  and  $f_{p(p-1)}$  are endomorphisms. We previously mentioned  $(\star)$  that the middle terms commute, which shows that  $f_{p^2-p+1}$  is an endomorphism of  $G$ .

Observe that for all  $x, y \in G$  we have

$$x^{p^2-p+1} y^{p^2-p+1} = (xy)^{p^2-p+1} = (xy) x^{p^2-p} y^{p^2-p},$$

which implies that  $x^{p^2-p} y = y x^{p^2-p}$ . It follows that  $x^{p^2-p}$  belongs to the center of  $G$ , for any  $x \in G$ .

We are now ready to prove our claim. Let  $k = 1 + mp(p-1)$  for some  $m \in \mathbb{Z}$ . Then

$$(xy)^k = (xy)^{mp(p-1)} \cdot (xy) = x^{p(p-1)m} y^{p(p-1)m} \cdot (xy) = x^k y^k.$$

The claim is proved.

Since  $\gcd(q, p(p-1)) = 1$ , by the Chinese Remainder Theorem we know that there is an integer  $n$  such that

$$\begin{cases} n \equiv 1 \pmod{p(p-1)}, \\ n \equiv 2 \pmod{q}. \end{cases}$$

We therefore have that the map  $f_n : G \rightarrow G$  defined by  $f_n(x) = x^n$  is an endomorphism. However,  $x^n = x^2$  for all  $x \in G$ , since  $q$  is the exponent of  $G$ . Now, since  $f_2(x) = x^2$  is an endomorphism of  $G$ , it follows that for all  $x, y \in G$  we have  $(yx)^2 = y^2 x^2 \Leftrightarrow yxyx = y^2 x^2 \Leftrightarrow xy = yx$ .

**Remark.** The conclusion also holds if  $p$  and  $q$  are not prime. One just needs  $G$  to have a finite exponent  $q$  such that  $q$  and  $p^2 - p$  are coprime.

*Also solved by Mihaly Bencze (Romania),  
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Let  $(G, \cdot)$  be a group with the property that there is an integer  $n \geq 1$  such that the map  $f_n : G \rightarrow G, f_n(x) = x^n$  is injective and the map

$$f_{n+1} : G \rightarrow G, f_{n+1}(x) = x^{n+1}$$

is a surjective endomorphism. Prove that  $G$  is an abelian group.

*Dorin Andrica and George Cătălin Țurcaș  
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*Solution by the proposer*

From the second hypothesis we have

$$(xy)^{n+1} = x^{n+1}y^{n+1},$$

which implies that

$$(yx)^n = x^n y^n \quad \text{for all } x, y \in G.$$

Using the above, we see that  $x(yx)^n = x^{n+1}y^n$  for all  $x, y \in G$ . On the other hand,

$$x(yx)^n = (xy)^n x = y^n x^{n+1} \quad \text{for all } x, y \in G.$$

We just showed that

$$x^{n+1}y^n = y^n x^{n+1} \quad \text{for all } x, y \in G. \quad (\star)$$

Now, using the surjectivity of  $f_{n+1}$  we obtain that for every  $z \in G$  there is  $x \in G$  such that  $f_{n+1}(x) = z$ , that is  $x^{n+1} = z$ . The relation  $(\star)$  can be written as  $zy^n = y^n z$ , for all  $y, z \in G$ . From this relation we get

$$y(xy)^n = (xy)^n y \quad \text{for all } x, y \in G,$$

that is

$$y(xy)(xy)\cdots(xy) = (xy)^n y.$$

This is equivalent to

$$(yx)(yx)\cdots(yx)y = (xy)^n y,$$

hence  $(yx)^n = (xy)^n$  and the conclusion follows from the injectivity of the map  $f_n$ .

*Also solved by John N. Daras (Athens, Greece),  
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and Socratis Varelogiannis (France)*

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Let  $A$  and  $B \in \text{Mat}_k(K)$  be two matrices over a field  $K$ . We say that  $A$  and  $B$  are *similar* if there exists an invertible matrix  $C \in \text{GL}_k(K)$  such that  $B = C^{-1}AC$ .

Let  $A$  and  $B \in \text{GL}_k(\mathbb{Q})$  be two similar invertible matrices over the field of rational numbers  $\mathbb{Q}$ . Assume that for some integer  $l$ , we have  $A^{l+1}B = BA^l$ . Then  $A$  and  $B$  are the identity matrices.

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Matemáticas, CSIC-UAM-UC3M-UCM, Spain)  
and Dmitri Piontkovski (Faculty of Economic Sciences,  
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*Solution by the proposer*

Consider first a finite group  $G$  having three elements  $x, y, z$  satisfying

$$y = z^{-1}xz \quad \text{and} \quad x^{l+1}y = yx^l.$$

Let us show that  $x = y = 1$ .

By way of contradiction we assume that the order of  $x$  is  $n > 1$ . Since  $x$  and  $y$  are conjugate, the order of  $y$  is also  $n$ . The cases  $l = 0$  and  $l = -1$  are trivial, so we assume that  $l \neq 0, -1$ .

Let  $\text{GCD}(a, b)$  denote the greatest common divisor of two integers  $a$  and  $b$ . The order of  $x^l$  is  $n/\text{GCD}(n, l)$  and the order of  $x^{l+1}$  is  $n/\text{GCD}(n, l+1)$ . Since  $x^l$  and  $x^{l+1}$  are conjugate, their orders coincide. Therefore,  $\text{GCD}(n, l) = \text{GCD}(n, l+1)$ . Thus, since  $l$  and  $l+1$  are coprime, we obtain that

$$\text{GCD}(n, l) = \text{GCD}(n, l+1) = 1.$$

This implies that there exists a natural number  $q > 1$ , which is coprime with  $n$  and such that  $ql \equiv l+1 \pmod{n}$ . Note that  $yx^l y^{-1} = x^{lq}$ . Therefore,

$$x^l = y^{-n} x^l y^n = y^{-n+1} x^{lq} y^{n-1} = \dots = x^{lq^n}.$$

Since the order of  $x^l$  is  $n$ ,  $n$  divides  $q^n - 1$ . Let  $p$  be the smallest prime divisor of  $n$ . Then the integers  $n$  and  $p-1$  are coprime. Hence there exist  $a, b \in \mathbb{Z}$  such that  $an + b(p-1) = 1$ . Observe also that

$$q^n \equiv q^{p-1} \equiv 1 \pmod{p}.$$

Therefore,  $q^{an+b(p-1)} \equiv 1 \pmod{p}$ , and so

$$l \equiv l \cdot q^{an+b(p-1)} \equiv l \cdot q \equiv l+1 \pmod{p}.$$

We have arrived at a contradiction, which proves that  $x = y = 1$ .

Now let us come back to the original problem. Since  $A$  and  $B$  are similar, there exists an invertible matrix  $C \in \text{GL}_k(\mathbb{Q})$  such that  $B = C^{-1}AC$ . Let  $F$  be the subgroup of  $\text{GL}_k(\mathbb{Q})$  generated by  $A$  and  $C$ . Let  $m$  be such that  $A, A^{-1}, C, C^{-1} \in \text{GL}_k(\mathbb{Z}[\frac{1}{m}])$ . Then  $F \leq \text{GL}_k(\mathbb{Z}[\frac{1}{m}])$ .

Recall that a group  $H$  is called *residually finite* if for every non-trivial element  $h \in H$  there exists a finite quotient  $G$  of  $H$  such that the image of  $h$  in  $G$  is non-trivial. Observe that the group  $\text{GL}_k(\mathbb{Z}[\frac{1}{m}])$  is residually finite (consider the natural maps from  $\text{GL}_k(\mathbb{Z}[\frac{1}{m}])$  to  $\text{GL}_k(\mathbb{F}_p)$ , where  $p$  are prime numbers coprime with  $m$ ). Therefore we conclude that the group  $F$  is also residually finite.

If  $A$  is not the identity matrix, then there exists a finite quotient  $G$  of  $F$  such that the image of  $A$  in  $G$  is not trivial. But this contradicts what we proved at the beginning. Thus  $A$ , so also  $B$ , are the identity matrices.

**Remark.** The problem is inspired by a result of Baumslag [1] where he constructed a two-generator one-relator group having only cyclic finite quotients. Instead of  $\mathbb{Q}$  we could assume that the matrices  $A$  and  $B$  in the problem are considered over an arbitrary field  $K$ . In this case the problem can be solved using a theorem of Malcev [2] where he proved that any finitely generated group linear over a field is residually finite.

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*Also solved by John N. Daras (Athens, Greece),  
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We are trying to hang a picture on a wall. The picture has a piece of string attached to it forming a loop, and there are 3 nails in the wall that we can wrap the string around. We want to hang the picture so that it does not fall down, but it will upon the removal of any of the 3 nails.

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#### *Solution by the proposer*

Let us start with 2 nails. Wrapping a loop around two nails in some way is equivalent to choosing an element of  $\pi_1(\mathbb{C} \setminus \{0, 1\})$ . Of course, this fundamental group is the free group  $F_2 = F(a, b)$  of rank 2, with generators corresponding to loops going around one of the nails.

If the picture is not to fall down, we need a non-trivial element  $x$  of  $F_2$ . The condition that the picture is supposed to fall when any of the nails is removed means that the image of  $x$  in  $F_2/\langle\langle a \rangle\rangle \cong \mathbb{Z}$  and in  $F_2/\langle\langle b \rangle\rangle \cong \mathbb{Z}$  has to be trivial. We immediately recognise that  $x = [a, b]$  does the trick.

For three nails, we can take

$$[[a, b], c] \in F(a, b, c) \cong \pi_1(\mathbb{C} \setminus \{0, 1, 2\}).$$

The solution easily generalises to  $n$  nails.

*Also solved by Mihaly Bencze (Romania),  
and George Miliakos (Sparta, Greece)*

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Given a natural number  $n$  and a field  $k$ , let  $M_n(k)$  be the full  $n \times n$  matrix algebra over  $k$ . A matrix  $(a_{ij}) \in M_n(k)$  is said to be centrosymmetric if

$$a_{ij} = a_{n+1-i, n+1-j}$$

for  $1 \leq i, j \leq n$ . Let  $C_n(k)$  denote the set of all centrosymmetric matrices in  $M_n(k)$ . Then  $C_n(k)$  is a subalgebra of  $M_n(k)$ , called centrosymmetric matrix algebra over  $k$  of degree  $n$ . Centrosymmetric matrices have a long history (see [1, 5]) and applications in many areas, such as in Markov processes, engineering problems and quantum physics (see [2, 3, 4, 6]). In the representation theory of algebras, a fundamental problem for a finite-dimensional algebra is to know if it has finitely many nonisomorphic indecomposable modules (or in other terminology, representations). In our case, the concrete problem on  $C_n(k)$  reads as follows.

Does  $C_n(k)$  have finitely many nonisomorphic indecomposable modules? If yes, what is the number?

*Changchang Xi (School of Mathematical Sciences, Capital Normal University, Beijing, and College of Mathematics and Information Science, Henan Normal University, Xinxiang, China)*

#### *Solution by the proposer*

**Strategy.** To solve the problem, we use the fact that two algebras have the same number of nonisomorphic indecomposable modules if their module categories are equivalent. In our case of  $C_n(k)$ , a practical way is to look for a decomposition of  $C_n(k)$  as a direct sum of indecomposable left ideals, and then take the direct sum of representatives for each isomorphism classes of indecomposable left ideals. The endomorphism algebra of this direct sum of representatives is called the basic algebra of  $C_n(k)$ , denoted  $B_0(n, k)$ . Then  $C_n(k)$  and  $B_0(n, k)$  have equivalent module categories, and therefore they have the same number of nonisomorphic indecomposable modules.

**Technique.** Let  $I_n$  be the identity matrix in  $M_n(k)$ ,  $e_{ij}$  be the  $n \times n$  matrix with  $(i, j)$ -entry 1 and other entries 0, and

$$f_i = e_{ii} + e_{n+1-i, n+1-i} \text{ for } 1 \leq i \leq n.$$

Then  $f_i \in C_n(k)$ . Calculations show that, for  $n = 2m$ ,

1.  $I_n = f_1 + \cdots + f_m$ ,  $f_i f_j = \delta_{ij} f_i$ , where  $\delta_{ij}$  is the Kronecker symbol,

2.  $C_n(k)f_1 \simeq C_n(k)f_2 \simeq \dots \simeq C_n(k)f_m$  as left ideals, and
3.  $\dim_k(C_{2m}(k)) = 2m^2$ ;

and, for  $n = 2m + 1$ ,

1.  $I_n = f_1 + \dots + f_m + e_{m+1, m+1}$ ,  $f_j f_i = \delta_{ij} f_i$ , where  $\delta_{ij}$  is the Kronecker symbol, and
2.  $C_n(k)f_1 \simeq \dots \simeq C_n(k)f_m$  as left ideals, and
3.  $\dim_k(C_{2m+1}(k)) = 2m^2 + 2m + 1$ .

Thus

$$B_0(2m, k) = f_1 C_{2m}(k) f_1$$

and

$$B_0(2m + 1, k) = (f_1 + e_{m+1, m+1}) C_{2m+1}(k) (f_1 + e_{m+1, m+1}).$$

Further calculations lead to

$$B_0(2m, k) \simeq C_2(k), \quad B_0(2m + 1, k) \simeq C_3(k) \quad \text{for all } m \geq 1.$$

*Answer.* We give quiver presentations of  $B_0(n, k)$  (see [7]).

Clearly,  $B_0(1, k) = k \bullet$ . If  $\text{char}(k) \neq 2$ , then  $B_0(2, k) = k(\bullet \rightarrow \bullet)$  and  $B_0(3, k) \simeq k(\bullet \rightarrow \bullet)$ . If  $\text{char}(k) = 2$ , then

$$C_2(k) = k(\bullet \xrightarrow{\alpha} \bullet) / (\alpha^2), \quad C_3(k) = k(\bullet \xrightarrow[\beta]{\alpha} \bullet) / (\alpha\beta).$$

From these quiver presentations of the basic algebras, we can draw their Auslander–Reiten quivers and gain a complete answer to the above problem.

- (a)  $C_1(k)$  has exactly 1 nonisomorphic indecomposable module.
- (b) If  $\text{ch}(k) \neq 2$ , then  $C_n(k)$  has exactly 2 nonisomorphic indecomposable modules for all  $n \geq 2$ .
- (c) If  $\text{ch}(k) = 2$ , then  $C_{2m}(k)$  has exactly 2 nonisomorphic indecomposable modules for all  $m \geq 1$ , and  $C_{2m+1}(k)$  has exactly 5 nonisomorphic indecomposable modules for all  $m \geq 1$ .

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#### An additional interesting problem (not intimately connected to Algebra)

Intervals of monotonic changes in a polynomial are located between the roots of its derivative. A derivative of a polynomial is also a polynomial, although of a lesser degree. Using these considerations, construct an algorithm for calculating the real roots of the quadratic equation. Improve it to calculate the real roots of a polynomial of the third, fourth and generally arbitrary degree.

Igor Kostin (Moscow, Russian Federation)

#### Solution by the proposer

We assume that the coefficient at  $x^2$  of the square polynomial is greater than zero. We need to find the derivative of this polynomial.

This is a linear function and we can easily find its root. Let us denote it by  $x_{\min}$ .

It is clear that if the value of the initial square polynomial at  $x_{\min}$  is greater than zero, then such a polynomial has no real roots. Otherwise, we will look for an argument  $x_{\text{ref}}$  such that this polynomial is greater than zero. An easy way to find such an argument is to step back from  $x_{\min}$  by some step and check the value of the polynomial in such a step.

If the calculated value of the polynomial is greater than zero, then the search is completed, otherwise we will continue to retreat, each time doubling the value of the step of the retreat. Having obtained  $x_{\text{ref}}$ , we find the root by the standard dichotomy method, dividing in half the segment between  $x_{\min}$  and  $x_{\text{ref}}$ .

It is convenient to end the dichotomy process when, in the machine representation, the point dividing the segment in half coincides with one of the ends of the original segment. Due to the finite precision of real numbers, this will happen sooner or later on any computer.

We repeat the above method for calculating the root twice, departing from  $x_{\min}$  in different directions.

It is now clear that in order to calculate the roots of a polynomial of third degree, one must first calculate its derivative polynomial and find the roots of this square polynomial.

These roots will determine the boundaries of the intervals of monotonic changes in the initial polynomial of third degree. The roots of this initial polynomial will be found by the dichotomy method on the segments of monotonic variation calculated in this way.

It is clear that the ladder of the described possibilities rises, in principle, to a polynomial of an arbitrarily large degree. Of course, the complete solution to this problem should be a computer program that implements the verbal instructions listed here. In practice, the capabilities of such an algorithm are limited in that the counting time increases with the degree of the original polynomial.

Finally, we note that the  $x_{\text{ref}}$  points replacing the infinity value for the boundaries of the segments of the monotonicity of the polynomial can be calculated in a less primitive way than the step-

by-step search with doubling of the step considered above. We normalize the polynomial so that the coefficient of the highest power of the argument is equal to one.

Let  $M$  be the largest modulo value among all its coefficients. If the argument  $x$  of the polynomial is greater than  $M + 2$ , then the value of the polynomial is greater than 1. To prove this, consider the calculation of the polynomial

$$p(x) = x^n + k[n-1]x^{(n-1)} + \dots + k[1]x + k[0]$$

by the Horner scheme.

At the first step we calculate

$$p[1] = k[n-1] + x,$$

and it is obvious that  $p[1] > 1$ . In fact even if  $k[n-1] < 0$ , it does not exceed  $M$  in absolute value.

At the second step we calculate

$$p[2] = k[n-2] + xp[1],$$

and again it is obvious that  $p[2] > 1$ .

The same holds in the following steps.

At the last step we compute

$$p(x) = k[0] + xp[n-1]$$

and finally obtain  $p(x) > 1$ . Thus, if one needs to set a representative value of the polynomial with an infinite value of the argument, one should take the argument equal to  $M + 2$ .

**Solution to part (a) of Open problem 137\***  
*proposed by Ovidiu Furdui (Romania),*  
*September 2014 Issue of the EMS Newsletter*

*Solution by Seán M. Stewart (Bomaderry, NSW, Australia)*

Denote the integral to be found in (a) by  $I$ . In terms of known constants its value is:

$$I = 8 - \pi - \frac{\pi^2}{8} + \frac{\pi}{2} \log 2 + 2(\log 2)^2 - 6 \log 2 - 2G.$$

Here  $G$  is *Catalan's constant* defined by  $\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2}$ .

We first establish a number of preliminary results before evaluating the integral. The  $n$ th *harmonic number* is defined by

$$H_n = \sum_{k=1}^n \frac{1}{k}, \quad n \in \mathbb{N}.$$

By convention,  $H_0 \equiv 0$ . The harmonic numbers satisfy the following recurrence relation

$$H_{n+1} = H_n + \frac{1}{n+1}.$$

In terms of the harmonic numbers the following finite sum can be written as

$$\begin{aligned} \sum_{k=1}^n \frac{1}{2k-1} &= 1 + \frac{1}{3} + \frac{1}{5} + \dots + \frac{1}{2n-1} \\ &= \left(1 + \frac{1}{2} + \dots + \frac{1}{2n}\right) - \frac{1}{2} \left(1 + \frac{1}{2} + \dots + \frac{1}{n}\right) \\ &= H_{2n} - \frac{1}{2}H_n. \end{aligned} \quad (1)$$

The harmonic numbers are related to the *digamma function* by (see entry 5.4.14 in [1])

$$\psi(x+1) = -\gamma + H_x. \quad (2)$$

Here  $\gamma$  is the Euler–Mascheroni constant. This allows the harmonic numbers to be analytically continued to all  $x \in \mathbb{R}, x \neq -1, -2, -3, \dots$ . The functional relation for the digamma function is

$$\psi(x+1) = \psi(x) + \frac{1}{x}. \quad (3)$$

For half-integer arguments the digamma function takes the values (see entry 5.4.15 in [1])

$$\psi\left(n + \frac{1}{2}\right) = -\gamma - 2 \log 2 + 2 \sum_{k=1}^n \frac{1}{2k-1}. \quad (4)$$

Replacing  $n$  with  $n + \frac{1}{2}$  in (2) we see that

$$H_{n+\frac{1}{2}} = \psi\left(n + \frac{3}{2}\right) + \gamma = \psi\left(n + \frac{1}{2}\right) + \frac{2}{2n+1} + \gamma,$$

where we have made use of the functional for the digamma function. In view of (4) we may rewrite this as

$$H_{n+\frac{1}{2}} = -2 \log 2 + \frac{2}{2n+1} + 2 \sum_{k=1}^n \frac{1}{2k-1} = -2 \log 2 + 2 \sum_{k=1}^{n+1} \frac{1}{2k-1}.$$

By applying (1), in terms of harmonic numbers this can be expressed as

$$H_{n+\frac{1}{2}} = 2H_{2n+2} - H_{n+1} - 2 \log 2,$$

which reduces to

$$H_{n+\frac{1}{2}} = 2H_{2n+1} - H_n - 2 \log 2, \quad (5)$$

after the recurrence relation for the harmonic numbers is applied to each of the harmonic number terms that appear.

Two results from Cauchy products for power series will prove useful. The first is

$$\arctan^2(x) = - \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \left(H_{2n} - \frac{1}{2}H_n\right) x^{2n}, \quad |x| \leq 1. \quad (6)$$

Showing this we have

$$\begin{aligned} \arctan^2(x) &= \left(\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}\right) \cdot \left(\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}\right) \\ &= x^2 \sum_{n=0}^{\infty} \sum_{k=0}^n \frac{(-1)^n}{(2k+1)(2n-2k+1)} x^{2n} \\ &= \frac{x^2}{2} \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} \left[\sum_{k=0}^n \frac{1}{2k+1} - \sum_{k=0}^n \frac{1}{2k-2n-1}\right] x^{2n}, \end{aligned}$$



where we have made use of the partial fraction decomposition of

$$\frac{1}{(2k+1)(2n-2k+1)} = \frac{1}{2(n+1)(2k+1)} - \frac{1}{2(n+1)(2k-2n-1)}.$$

Reindexing the second sum  $k \mapsto n-k$  we have

$$\arctan^2(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} \left( \sum_{k=0}^n \frac{1}{2k+1} \right) x^{2n+2}.$$

Reindexing the infinite sum  $n \mapsto n-1$  and the finite sum  $k \mapsto k-1$  yields

$$\arctan^2(x) = - \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \left( \sum_{k=1}^n \frac{1}{2k-1} \right) x^{2n},$$

with the desired result then following on application of (1).

The second Cauchy product for power series is

$$\frac{x \arctan x}{1+x^2} = \sum_{n=1}^{\infty} (-1)^{n+1} \left( H_{2n} - \frac{1}{2} H_n \right) x^{2n}, \quad |x| < 1. \quad (7)$$

Showing this we have

$$\begin{aligned} \frac{x \arctan x}{1+x^2} &= x^2 \left( \sum_{n=0}^{\infty} (-1)^n x^{2n} \right) \cdot \left( \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2n+1} \right) \\ &= \sum_{n=0}^{\infty} (-1)^n \left( \sum_{k=0}^n \frac{1}{2k+1} \right) x^{2n+2} \\ &= \sum_{n=0}^{\infty} (-1)^n \left( \sum_{k=1}^{n+1} \frac{1}{2k-1} \right) x^{2n+2} \\ &= \sum_{n=0}^{\infty} (-1)^n \left( H_{2n+2} - \frac{1}{2} H_{n+1} \right) x^{2n+2}, \end{aligned}$$

where (1) has been used. The desired result then follows after a reindexing of  $n \mapsto n-1$ .

Turning our attention to the integral  $I$  we have

$$\begin{aligned} I &= - \int_0^1 \log(1-x^2) \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{n} dx \\ &= - \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \int_0^1 x^{2n} \log(1-x^2) dx. \end{aligned}$$

Here the interchange made between the summation and the integration is permissible due to Fubini's theorem. Using the variable change  $x \mapsto \sqrt{x}$  leads to

$$I = -\frac{1}{2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \int_0^1 x^{n-\frac{1}{2}} \log(1-x) dx. \quad (8)$$

From the well-known result of (see, for example, [2, p. 2, Eq. (1.4)])

$$\int_0^1 x^{n-1} \log(1-x) dx = -\frac{H_n}{n},$$

replacing  $n$  with  $n + \frac{1}{2}$  we see that

$$\int_0^1 x^{n-\frac{1}{2}} \log(1-x) dx = -\frac{2H_{n+\frac{1}{2}}}{2n+1},$$

allowing us to rewrite the integral appearing in (8) as

$$I = \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \frac{H_{n+\frac{1}{2}}}{2n+1}. \quad (9)$$

In view of (5) this can be rewritten as

$$I = \sum_{n=1}^{\infty} \frac{(-1)^n}{n(2n+1)} (2H_{2n+1} - H_n - 2 \log(2)). \quad (10)$$

From the partial fraction decomposition of

$$\frac{1}{n(2n+1)} = \frac{1}{n} - \frac{2}{2n+1},$$

and the recurrence relation for the harmonic numbers, (10) may be expressed as

$$\begin{aligned} I &= 2 \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \left( H_{2n} - \frac{1}{2} H_n \right) - 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{2n+1} \left( H_{2n} - \frac{1}{2} H_n \right) \\ &\quad + 2(1 - \log 2) \sum_{n=1}^{\infty} \frac{(-1)^n}{n} + 4(\log 2 - 1) \sum_{n=1}^{\infty} \frac{(-1)^n}{2n+1} \\ &\quad - 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n+1)^2} \\ &= 2S_1 - 4S_2 + 2(1 - \log 2)S_3 + 4(\log 2 - 1)S_4 - 4S_5. \quad (11) \end{aligned}$$

We now find each of these five sums. For the first of these, setting  $x = 1$  in (6) we see that

$$S_1 = -\arctan^2(1) = -\frac{\pi^2}{16}.$$

For the second of the sums,  $S_2$ , integrating (7) with respect to  $x$  from 0 to 1 we see that

$$S_2 = - \int_0^1 \frac{x \arctan x}{1+x^2} dx.$$

Using the variable change  $x \mapsto \tan x$  before integrating by parts produces

$$S_2 = -\frac{\pi}{8} \log 2 - \int_0^{\frac{\pi}{4}} \log(\cos x) dx.$$

The integral that has now appeared can be evaluated as follows. Recalling Euler's famous log-sine integral

$$\int_0^{\frac{\pi}{2}} \log(\sin x) dx = -\frac{\pi}{2} \log 2,$$

and let

$$J_S = \int_0^{\frac{\pi}{4}} \log(\sin x) dx \quad \text{and} \quad J_C = \int_0^{\frac{\pi}{4}} \log(\cos x) dx.$$

Using the variable change  $x \mapsto \frac{\pi}{2} - x$  in  $J_C$  gives

$$J_C = \int_{\pi/4}^{\pi/2} \log(\sin x) dx.$$

Thus

$$J_S + J_C = \int_0^{\frac{\pi}{2}} \log(\sin x) dx = -\frac{\pi}{2} \log 2.$$

Also

$$J_5 - J_C = \int_0^{\frac{\pi}{4}} \log(\tan x) dx.$$

Substituting  $y = \tan x$  we obtain

$$J_5 - J_C = \int_0^1 \frac{\log(y)}{1+y^2} dy = \sum_{n=0}^{\infty} (-1)^n \int_0^1 y^{2n} \log(y) dy.$$

Integrating by parts gives

$$J_5 - J_C = - \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \int_0^1 y^{2n} dy = - \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} = -\mathbf{G},$$

where we used the definition of Catalan's constant. Solving for  $J_C$  yields

$$J_C = \frac{\mathbf{G}}{2} - \frac{\pi}{4} \log 2.$$

Thus

$$S_2 = -\frac{\mathbf{G}}{2} + \frac{\pi}{8} \log 2.$$

The third sum comes directly from the Maclaurin series expansion for  $\log(1-x)$  evaluated at  $x = -1$ . Here

$$S_3 = -\log 2.$$

The four sum is related to the Maclaurin series expansion for  $\arctan x$ . Here

$$S_4 = -1 + \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = -1 + \arctan(1) = -1 + \frac{\pi}{4}.$$

For the fifth and final sum

$$S_5 = -1 + \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} = -1 + \mathbf{G},$$

where again the definition for Catalan's constant has been used. Combining the values found for all five sums into (11) we find

$$I = 8 - \pi - \frac{\pi^2}{8} + \frac{\pi}{2} \log 2 + 2(\log 2)^2 - 6 \log 2 - 2\mathbf{G},$$

as announced.

*Some comments on the general case (part (b) of the question)*

Denoting the integral in (b) by  $I_n$  where  $n \geq 2$  is an integer, following the identical procedure that led to (9) we find

$$I_n = \sum_{k=1}^{\infty} \frac{(-1)^k}{k} \frac{H_{k+\frac{1}{n}}}{nk+1}.$$

Using such an approach, the general problem therefore boils down to expressing  $H_{k+\frac{1}{n}}$  in terms of harmonic numbers whose indices are of integer rather than fractional order before evaluating each of the resulting series. Unfortunately, how this can be achieved for cases when  $n > 2$  is currently not clear to me, suggesting that to find  $I_n$ , we may need a different approach from the one we used to find  $I_2$ .

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*We would like to invite you to submit solutions to the proposed problems and/or ideas on the open problems. Send your solutions by email to Michael Th. Rassias, Institute of Mathematics, University of Zürich, Switzerland, michail.rassias@math.uzh.ch.*

*We also solicit your new problems, together with their solutions, for the next "Solved and Unsolved Problems" column, which will be devoted to the topic of Game Theory.*