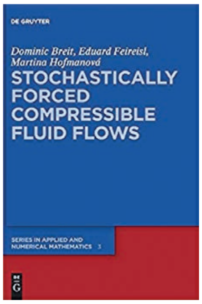
Book review

Stochastically Forced Compressible Fluid Flows by Dominic Breit, Eduard Feireisl and Martina Hofmanová

Reviewed by Donatella Donatelli



The book is focused on systematically developing a consistent mathematical theory of compressible fluids driven by random initial data and stochastic external forces in the context of classical continuum fluid mechanics.

The theory of continuum fluid mechanics is derived from basic physical principles under the assumption that all quantities – fields – are smooth, and the Navier–Stokes system became a well-established model working as a reliable basis of investigation for both theoretical and applied aspects. Built on the foundation of conservation laws, fluid mechanics helps to describe the flow and interactions of gases, liquids and/or plasmas, as well as the forces acting on them. Until fairly recently, these forces have largely been considered to be deterministic. This means that they are functions of microscopic space and time parameters, so that at any given instant of time the fluid position in space is expected to be known. There are still many important open problems, but the literature concerning the deterministic case is very well-established and extensive; see for example the monographs [E. Feireisl, *Dynamics of Viscous Compressible Fluids*, Oxford Lecture Series in Mathematics and its Applications, vol. 26, Oxford University Press, Oxford, 2004] or [P. L. Lions, *Mathematical Topics in Fluid Mechanics*, Vol. 2: *Compressible Models*, Oxford Lecture Series in Mathematics and its Applications, vol. 10, The Clarendon Press, Oxford Science Publications, Oxford University Press, New York, 1998].

However, this description is a fairly weak idealisation, which is obvious already from the fact that we are still unable to model extreme fluid mechanic events like turbulence to a sufficient level of accuracy. In fact, the modelling of turbulence can be considered as the prime motivation for the introduction of stochasticity in the study of fluids. Turbulence is frequently associated with an intrinsic element of randomness, and furthermore, experimental studies of turbulence lead more to a statistical approach than to a deterministic one. Moreover, the addition of stochastic terms to the basic governing equations is often used to account for other numerical, empirical or physical uncertainties. Therefore it becomes important, in the framework of partial differential equations, to set up a stochastic PDE theory for fluid flow.

Nowadays there exists a large amount of literature concerning the dynamics of incompressible fluids driven by stochastic forcing. The first results can be found in the pioneering work by Bensoussan–Temam (1973). See also the lecture notes [A. Debussche, Ergodicity results for the stochastic Navier–Stokes equations: An introduction, In *Topics in Mathematical Fluid Mechanics*, volume 2073 of Lecture Notes in Math., pages 23–108, Springer, Heidelberg, 2013], [Flandoli, An introduction to 3D stochastic fluid dynamics, In *SPDE in Hydrodynamic: Recent Progress and Prospects*, volume 1942 of Lecture Notes in Math., pages 51–150, Springer, Berlin, 2008]. Nevertheless, far less is known in the case of compressible fluids. Important questions of well-posedness and even mere existence of solutions to problems dealing with stochastic perturbations of compressible fluids are largely open, with only a few rigorous results available. This monograph is an exhaustive and up-to-date overview of the most recent results by different authors on stochastic compressible fluids.

The book contains eight chapters and is divided into three parts. It starts with Part I, a very didactic introduction providing the necessary background. In a very clear manner, Part I provides the non-expert readers in the field with all the basic results of the theory and, at the same time, a description of more advanced tools in the theory of stochastic PDEs. Part II is the core of the book, containing all that is really new and original compared to the existing literature. The most recent existence results on compressible stochastic fluids are described. This part consists of five chapters, which guide the reader step by step towards the proof of the existence of solutions. Each chapter is devoted to one of the main aspects of the existence theory: the setup of the model, approximation schemes and their convergence, energy inequalities, relative energy inequality, and weak strong uniqueness. In particular, it starts with the existence of local strong solutions defined on a maximal time interval bounded above by a positive stopping time that may depend on the size of the initial data; then, because all real world problems require

EMS MAGAZINE 120 (2021) — DOI 10.4171/MAG-33

solutions defined globally in time, one has to switch to the notion of weak solutions. This approach is based on the idea of including some form of the energy/entropy balance as an integral part of a weak formulation, and goes back to Dafermos (1979) concerning conservation laws and to Germain (2011) who introduced a similar concept in the context of the deterministic compressible Navier–Stokes system. Therefore, the solutions constructed in this part of the book are the so-called dissipative martingale solutions, which are weak martingale solutions also satisfying a variant of the energy balance.

Finally, Part III of the book is focused on applications such as singular limits. Indeed, by scaling the equations by means of appropriately chosen reference units, the parameters determining the behaviour of the system become evident. Asymptotic analysis and/or singular limits provide a useful tool in situations where these parameters vanish or become infinite. In this part, the authors describe a rigorous mathematical approach to asymptotic analysis in the case of incompressible and inviscid–incompressible limits for the compressible Navier–Stokes system with stochastic perturbations.

To conclude, this is the first book in which one can find a complete description of the available theory on compressible stochastic fluid equations. Compared to the previous literature, this is a new point of view that makes the book original and of very high quality. It is a really valuable and much-needed contribution to the literature in the domain. This monograph is built in a masterly manner, in such a way as to provide not only a complete and up-to-date overview of the problems under consideration, but also a detailed introduction to the topic for the uninitiated reader. The book is very well and rigorously structured, having the excellent attribute of being valuable to both experienced researchers in the domain and to graduate students who wish to explore the different topics in this challenging area of research. Overall, it constitutes an ideal book for researchers (in the broadest sense) who want to enlarge their mathematical knowledge of fluid mechanics.

Dominic Breit, Eduard Feireisl and Martina Hofmanová, *Stochastically Forced Compressible Fluid Flows*. De Gruyter Series in Applied and Numerical Mathematics 3. De Gruyter, 2018, 330 pages
ISBN 978-3-11-049050-3. eBook ISBN 978-3-11-049255-2.

——

donatella.donatelli@univaq.it

New editor appointed

Donatella Donatelli is full professor in mathematical analysis at the Department of Information Engineering, Computer Science and Mathematics of the University of L'Aquila, Italy. Her research interests cover partial differential equations of hyperbolic type, relaxation limits for nonlinear hyperbolic systems, incompressible and compressible Navier–Stokes equations, and mathematical models of fluid dynamical type. In particular, her main achievements are on scale limit analysis for fluid dynamic equations with relevant results in the related acoustic waves analysis. In 2011, Donatella was awarded the Marisa Bellisario Prize in the section "Special recognition of young talents in research" in Mathematical and Computer Sciences (this is a national prize under the auspices of the Italian President, lnx.fondazionebellisario.org/online/2011-2/).

She has been the Coordinator of the LLP-EU Intensive Programs Fluid2Bio 2012–2011, contract N.2012-1-IT2-ERA10-38827, 2011, contract N.2011-1-IT2-ERA10-27088 and the Scientific Coordinator of the L'Aquila node of the European Project Marie Curie Actions-MSCA-ITN-2014-ETN, Horizon 2020 "ModCompShock-Modelling and Computation of Shocks and Interfaces", 2016–2020.