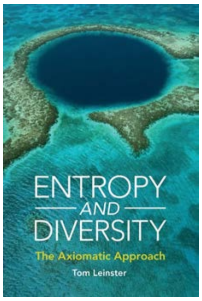
Book reviews

Entropy and Diversity: The Axiomatic Approach by Tom Leinster

Reviewed by Stefan Forcey



At least since the recording of the Noahic directive "two of every kind", humans have instinctively felt that extinction is loss. Because the viable forms of DNA-based life are developed by such arduous processes, it seems like a profound waste of hard-won information to let species become extinct. Furthermore, we know that there are many dependencies between relatively unrelated taxa: for instance birds may depend on berries, or insects, or fish, and humans depend on all the above. Due to habitat loss, pollution and climate change, the potential for a catastrophic domino effect appears to be increasingly imminent. It could be wished that there was not such a clear need for a book like Tom Leinster's new book *Entropy and Diversity: The Axiomatic Approach*.

However, in times of natural or artificial disaster, we may be forced into triage situations. With finite resources at our disposal, we may be compelled to decide which biome or which species to save, and measurements of biological diversity help make that decision. Greater diversity means less chance of cascading loss, at least as a first approximation. Further justifications are found in the study of drug resistant bacteria, as well as the study of human gut health, as Dr. Leinster points out in the introduction. Recent experience demonstrates that sometimes we may even desire to minimize diversity, as in the example of virus mutations that have the potential to be worse than their common ancestor.

The thesis of *Entropy and Diversity* is that by beginning with the list of properties we wish our diversity measurement to obey, we can often completely describe the function or family of functions that will fit those requirements. This is what is called the axiomatic approach. Showing a tight relationship between structure and properties allows the author to review a long list of measurements of diversity (and inversely, entropy) to demonstrate how they are specializations of general principles. At another level, we see that both the contribution of a single individual and the diversity of a community are special cases of a concept of *value* that is axiomatically determined. Furthermore, many mathematical invariants measuring size (such as cardinality, volume, surface area, fractional dimension, and Euler characteristic) arise from a single concept, of a general invariant called the *magnitude of an enriched category*. It is shown that this magnitude is closely related to maximum diversity: indeed in some cases they are precisely the same.

Perhaps the most distinctive new contribution here is Leinster's work (with C. A. Cobbold) on defining a family of diversity measures that depends on both the relative abundances of the species in a population and the pairwise differences between them. The similarity (or dissimilarity) between two species can be measured in many different ways: genetic, phylogenetic, or functional. By defining the value of a species to be its expected similarity (on average) to a randomly chosen individual from the population, it is shown that Leinster and Cobbold's diversity measures are special cases of an aggregate value function, which also captures the Hill numbers and the phylogenetic diversity of Chao, Chiu and Jost. Not only does the new family of diversity measures respect the similarity matrix (finite metric), and obey the desired properties, it also has the surprising feature of being simultaneously maximizable. Given a similarity matrix, it is shown that the entire family of diversity measures is maximized for a single probability distribution of the species in that ecosystem. The common maximum is yet another invariant measurement, but one which measures (the magnitude of) the metric itself.

In the early chapters, Dr. Leinster motivates and explains the basic problem of deciding how to measure biological diversity, and covers the steps of solving an equation to find a missing function. Then he begins answering those questions with an exposition of Shannon entropy from information theory. Deformations and relative versions of entropy are also covered, each with its corresponding inverse concept of diversity. The central chapters introduce the concepts of mathematical size, value, means and magnitude, and relate them back to the special cases of diversity measures. Along the way, there is a chapter on using probabilistic methods to solve functional equations. Finally there is a nice axiomatic characterization of information loss, discussion of entropy modulo a prime number, and the promised deep dive into the category-theoretical foundations.

Entropy and Diversity is a thorough presentation of the mathematics of measuring diversity, including many new results of uniqueness, unification and utility. Beyond the practical value, it is also a display of mathematical art. Beautiful patterns, at deeper and deeper levels of abstraction, are exhibited to clarify the simplicity of what at first might appear to be abstruse formulas. Leinster approaches the subject like a craftsman, paying attention to every detail. The book is over 450 pages long, but it is so nicely organized and readable that I felt immediately drawn in rather than intimidated. The book is directly accessible to a general audience comfortable with mathematical reasoning. It will be a valuable reference for both mathematicians and mathematical ecologists. The new material has already engendered a lot of discussion on future directions, as can be seen in some recent online conversations:

- johncarlosbaez.wordpress.com/2011/11/07/measuring-biodiversity
- golem.ph.utexas.edu/category/2020/12/entropy_and_diversity_the_axio.html

Tom Leinster, *Entropy and Diversity: The Axiomatic Approach.* Cambridge University Press, 2021, 458 pages, Paperback ISBN 978-1-108-96557-6, eBook ISBN 978-1-108-96217-9.

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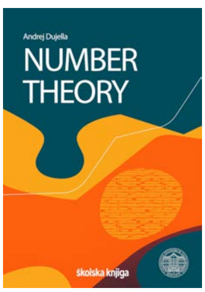
Number Theory by Andrej Dujella

Reviewed by Jean-Paul Allouche



As a student of Number Theory, I really appreciated the famous book of G. H. Hardy and E. M. Wright, while some of my friends frequently mentioned the book of Z. I. Borevich and I. R. Shafarevich. Of course, even these two books did not cover the whole (huge) field of Number Theory, and several other excellent books could be cited as well. More recently, many books have been devoted to (parts of) this vast field whose characteristic is to be both primary and not primary (pun intended). The meaning of the expression "Number Theory" itself has changed over time – partly because the domain has exploded – to the point that some contemporary authors now refer to "Modern Number Theory" …

A very recent book, entitled *Number Theory* and based on teaching materials, has been written by A. Dujella. Devoted to several subfields of this domain, this book is both extremely nice to read and to work from. It starts from primary results given in the first three chapters, ranging from the Peano axioms to the principle of induction, from the Fibonacci numbers to Euclid's algorithm, from prime numbers to congruences, and so on. Chapter 3 ends with primitive roots, decimal representation of rationals, and pseudoprimes. Chapters 4 and 5 then deal with quadratic residues (including the computation of square roots modulo a prime number) and quadratic forms (including the representation of integers as sums of 2, 4, or 3 squares). Chapters 6 and 7 are devoted to arithmetic functions (in particular, multiplicative functions, asymptotic behaviour of the summatory function of classical arithmetic functions, and the Dirichlet product), and to the distribution of primes (elementary estimates for the number of primes less than a given number, the Riemann function, Dirichlet characters, and a proof that an infinite number of primes are congruent to ℓ modulo k when $\gcd(\ell, k) = 1$). Chapter 8 deals with first results on Diophantine approximation, from continued fractions to Newton approximations and the LLL algorithm, while Chapter 9 studies applications of Diophantine approximation to cryptography (RSA, attacks on RSA, etc.). Actually two more chapters are devoted to Diophantine approximation, Chapter 10 (linear Diophantine approximation, Pythagorean triangles, Pellian equations, the Local-global principle, …) and Chapter 14 (Thue equations, the method of Tzanakis, linear forms in logarithms, Baker–Davenport reduction, …). Chapters 11, 12, and 13 deal with polynomials, algebraic numbers, and approximation of algebraic numbers. The book ends with Chapters 15 and 16 which cover elliptic curves and Diophantine problems.

This quick and largely incomplete description clearly shows that this book addresses many jewels of number theory. This is done in a particularly appealing way, mostly elementary when possible, with many well-chosen examples and attractive exercises. I arbitrarily choose two delightful examples, the kind of "elementary" statements that a beginner could attack, but whose proofs require some ingenuity, namely the unexpected statements 4.6 and 4.7:

Example 4.6. Let $p > 5$ be a prime number. Prove that there are two consecutive positive integers that are both quadratic residues and two consecutive positive integers that are both quadratic nonresidues modulo p.

Example 4.7. Let n be an integer of the form $16k + 12$ and let $\{b_1, b_2, b_3, b_4\}$ be a set of integers such that $b_i \cdot b_j + n$ is a perfect square for all i, j such that $i \neq j$. Prove that all numbers b_i are even.