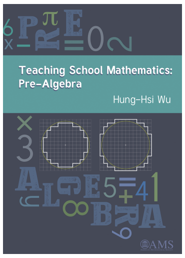
Book reviews

Teaching School Mathematics: Pre-Algebra by Hung-Hsi Wu

Reviewed by António de Bivar Weinholtz



This is the second book of a series of six covering the entire K-12 curriculum, as an instrument for the mathematical education of school teachers. It follows a first volume entitled "Understanding Numbers in Elementary School Mathematics", which covers a substantial part of the mathematical curriculum of grades K-6 (namely, numbers and operations), but also some topics more likely to be part of the curriculum of grades 7 and 8, the reason being that clearly the mathematical knowledge of elementary school teachers should go beyond what they actually have to teach.

I strongly recommend readers of this review to start by reading the review [1] of that first volume.[1] There, can one find the reasons why I deem this set of books a milestone in the struggle for a sound mathematical education of schoolchildren. I shall not repeat here all the historical and scientific arguments that support this claim, but regarding the second volume, I have to state that once again, although it is written for school teachers as a tool for their mathematical education (both during pre-service years and for their professional development while in activity), and also as a resource for textbook authors, the set of its potential readers should not be restricted to those for which it was primarily intended. Indeed, exactly as I wrote regarding the first volume, I claim that its readers may and should include anyone with the basic ability to appreciate the beauty of the use of human reasoning in our quest to understand the world around us and the capacity and the will to make the necessary effort required here, as it is for any enterprise that is really worthwhile.

This volume and the one the author explicitly refers to as its companion, namely the third volume in the series "Teaching School Mathematics: Algebra", both address the mathematics generally taught in grades 6–8 (as explained in the third volume), but also including topics that can be part of the curriculum of grades 5 and 9 (as explained in this volume). The first three chapters deal with numbers and operations, starting with fractions and including rational (relative) numbers, and also some other specific topics such as finite probability, the Euclidean algorithm, and some of its applications. The final two chapters give an introduction to geometry and geometric measurement.

As this volume together with its companion are meant to be self-contained as far as the mathematics middle school teachers have to teach is concerned, the chapters dealing with numbers and operations needed to have some overlap with the corresponding chapters in the first volume of this series, with the sole exception of the section on finite probability. What is given here, however, is a new presentation of the different subjects, although along the same lines. Therefore, much that was written in the review of the first volume still applies to the corresponding chapters in this second volume. However, we can find some new ideas in the details, and the topics are presented in a somewhat more synthetic form. It is a pleasure to revisit these fundamental ideas that can be considered, in a certain sense, the main core of school mathematics, and even someone who has thoroughly read the first volume can benefit from this new synthesis. Of the new details, let me refer for instance to the motivation of the concept of the product of fractions. Not only is the priority in this volume no longer given, as it was in the first volume, to the formula for the area of a rectangle, but rather to the concept of "a fraction of a fraction", but also, in this volume, when presenting the latter concept as the basic way to introduce the definition of multiplication (which was only presented as a second possible equivalent definition in the first volume), the author shows that the resulting product is the only possible one if one wants to have an operation that is associative and at the same time extends directly to fractions the meaning of "multiplying by a whole number". It is also understood, of course, that the definition of division by a whole number is exactly the same as the one already adopted for whole numbers, once we

[1] In the printed version of the review [1], there is an unfortunate misspelling of the name of the author of the reviewed book, both in the title and in the text; the correct spelling of the name of the author is, of course, Hung-Hsi Wu.

have the concept of multiplication of fractions by such numbers (defined, as expected, as an iterated addition). In this way, a new independent argument is added for the way one should extend multiplication to fractions, leading to the simple rule that, sadly, is so often presented as something students just have to learn by rote. One can, of course, devise similar alternative ways of motivating this definition; for instance, one can make use of the requirement of commutativity to obtain the rule to multiply a whole number by a fraction, having first figured out what it should naturally mean to multiply any fraction by a whole number, and consequently what it should mean to divide any fraction by a whole number. Then, once one notices that this forces us to define multiplication of a whole number by an fraction exactly as division of the whole number by the denominator of the fraction followed by multiplication of the result by the numerator, one can naturally extend this rule, as a definition, from whole numbers to fractions in general. While one can argue whether this slightly less synthetic (and in a certain sense less "logically compelling") argument, or some similar one, would be more or less appealing to students of a specific grade, what is essential is the requirement that, in general, operations with fractions should be understood and presented as natural extensions of the corresponding operations with whole numbers, and that a clear distinction should be made between "definitions" and "rules that can be justified using definitions". With this set of books at hand, there is no excuse for school teachers, textbook authors and government officials to persist in the unfortunate practice of trying to serve to students this fundamental part of school mathematics in a way that is in fact unlearnable …

Another example of a new formulation is a somewhat more explicit wording of the "Fundamental Assumption of School Mathematics" regarding irrational numbers.

The two final chapters, on geometry and geometrical measurement, are completely new with respect to the first volume. This part of the book will thus give rise to a somewhat more extensive commentary here. Before engaging with the subject itself, the author explains the main goals to be attained, and describes the problems one faces when trying to attain these goals, including an analysis of the situation of school geometry in the last few decades. The core of the geometrical topics to be treated in middle school is defined as a working knowledge of similar triangles, as a tool to set up the intuitive foundation for a more precise discussion of the concepts of congruence and similarity in high school geometry; the role of similar triangles in the study of linear equations of two variables is emphasized, and we are reminded of the continuing crisis in the teaching of school geometry over the last four decades or more, and of the essential discontinuity between the middle school and high school curricula all along that extensive period. While this historical analysis is mainly valid for the USA, it can certainly apply, at least partially, to many other countries around the world. The tension between two extremes, i.e., on one side a purely axiomatic approach and on the other side an exclusively "intuitive" consideration of geometric entities with scarcely any concern for even minimal organization, strongly relying on "manipulatives" and computer software, has done extensive damage to the teaching of geometry in schools, and the author points out the urgent need to get past this unfortunate situation. With these considerations in mind, he sets up a progression starting with some advice on the use of free hand drawing as a way to gain some initial necessary geometric intuition, followed by a rich set of geometric constructions using plastic triangles, ruler and compass; finally, he addresses the problem of reaching the main results on congruence and similarity in a way that introduces students to sound deductions in geometry without the undue burden of a strict axiomatic construction. The chosen main ingredients for such a path are the basic isometries in a plane, namely, translations along a vector, reflections across a line and rotations around a point, complemented by dilations, for the definition of similarity; a carefully weighed equilibrium between mathematical precision and geometric intuition in order to attain the prescribed goals is aimed at and successfully reached. The author advocates the use of transparencies to supply students with the needed geometrical intuition on the assumptions that have to be made at this stage regarding the basic isometries, and on the use of these assumptions for proper justification of the basic geometric theorems to be taught. The manner in which this can be successfully accomplished is described and illustrated in complete detail.

Unlike the case of the topics on numbers and operations, in the case of geometry there may be a wider choice of different ways to attain the same goals. Other paths respecting the same general basic principles have been proposed, even in recent curricular reforms. Just to give a hint of some of the other ways of dealing with the difficult equilibrium between mathematical precision and intuition, let me point out a few items that can be used as a basis for an alternative path. One could use a property that, in a particular case, justifies the construction used in this book to draw an angle with amplitude equal to a given angle, on a given half-line (i.e., having that half-line as one of its sides) as a definition of "equality of amplitude for angles". Admitting the effectivity and consistency of a slightly generalized form of this construction is, in a certain sense, equivalent to the usual axiom that in some axiomatic constructions of elementary (Euclidean or even absolute) geometry replaces the classical SAS criterion of congruence of triangles. This could be a basis for a justification of the other congruence criteria for triangles. Parallelism can be dealt with using, as a criterion for two lines in a plane to be parallel, the equality of the corresponding angles determined in the pair of lines by an intersecting third line. This same property can also be used to justify the construction of a line parallel to a given line and passing through a given point, like the one illustrated in this volume. These two stated criteria, admitted without proof (part of what is formulated in the second one is in fact equivalent, in a certain sense, to the parallel axiom), can be made "intuitive" by the fact that they provide practical ways of obtaining the "transport" of an angle and the construction of a parallel to a given line through a given point respectively, in each case using only a finite number of points and the transport of distance between two points. As for the concepts of congruence and similarity, one could directly use the manner in which the author defines the scale drawing of a figure as a definition of similarity, and define congruence to be the case where the scaling factor is equal to 1. The basic isometries in the plane, and vectors, can be introduced in a second stage, and their properties similarly studied with a good equilibrium between precision and intuition.

We are perhaps in a situation where we still lack the practical experience that allows us to decide if among the various different proposals for "paths of presentation" of school geometry, we should clearly prefer one or another, considering only those that satisfy the basic principles stated in this book. We can try to examine what has been done in the more or less ancient past in schools around the world, and what the evolution has been over recent decades and in some cases even over the last few years; there is still perhaps some room for a confrontation of different hypotheses. I am nevertheless convinced that the general principles stated in this book, and the diagnosis of the disastrous evolution of school geometry in many countries in the last half-century should serve as a compelling guide in this matter. At any rate, this set of volumes is pioneering in that it gives a full presentation of a path that can be followed from middle school to high school, in a manner that respects all of these sound principles for a renewed teaching of school geometry. It is to be hoped that this or perhaps other equivalent ways of meeting these requirements will have the opportunity to prove their effectiveness in schools, thus reversing the sad diagnosis of the present situation made in this volume.

As expected, the final chapter on geometrical measurement is also an excellent example of balance between mathematical precision and controlled intuition, leading to the usual formulas for length, area and volume of basic geometric entities, with a unified vision of geometrical measure.

As in the first volume of this series, the author provides the reader with numerous illuminating activities for every topic, as well as an excellent choice of a wide range of exercises.

Hung-Hsi Wu, *Teaching School Mathematics: Pre-Algebra.* American Mathematical Society, 2016, 383 pages, Hardback ISBN 978-1-4704-2720-7, eBook ISBN 978-1-4704-3009-2.

References

[1] A. de Bivar Weinholtz, Book review: Understanding numbers in elementary school mathematics. *Eur. Math. Soc. Mag.* **122**, 66–67 (2021)

António de Bivar Weinholtz is a retired associate professor of mathematics of the University of Lisbon Faculty of Science, where he taught from 1975 to 2009. He was a member of the scientific coordination committee of the new curricula of mathematics for all the Portuguese pre-university grades (2012–2014).

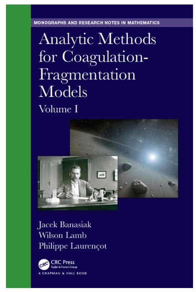
abivar@sapo.pt

DOI 10.4171/MAG-76

Analytic Methods for Coagulation-Fragmentation Models, Volumes I+II

by Jacek Banasiak, Wilson Lamb and Philippe Laurençot

Reviewed by Barbara Niethammer



Coagulation and fragmentation are of fundamental importance in a vast variety of applications such as aerosol physics, polymerization, blood agglomeration or grouping of certain species, to name just a few. The first approach to describe such processes goes back to Marian von Smoluchowski, who in 1916 developed the first deterministic model for coagulation in a colloidal gold solution. The main assumption in his approach was that only two clusters combine at a time and that the rate at which this happens depends only on the product of the number-density of the respective cluster sizes, a hypothesis similar to the molecular chaos assumption in the Boltzmann equation. This leads to an infinite system of coupled differential equations for the density of cluster sizes and involves a so-called rate kernel which depends on the microscopic details of the specific coagulation process. Later the model was extended in various ways; in particular, different forms of fragmentation were added, allowing a cluster to fragment into two or more smaller ones. While the original model is discrete, allowing only for integer cluster sizes, the continuous version of the model is also of interest, in particular if one is interested in the behaviour of the system over large times.

It should be emphasized that coagulation-fragmentation equations are not only relevant from the point of view of applications; they are also fascinating due to the rich structure that solutions can display. Indeed, depending on the rate kernels, coagulation-fragmentation equations feature the possibility of loss of mass which can happen in finite or infinite time. In pure coagulation, for example, if the kernel grows faster than linearly, infinitely large particles are created in finite time, a phenomenon known as gelation. This can be directly linked to gelation in polymers where polymer chains can become so long such that suddenly in