of obtaining the "transport" of an angle and the construction of a parallel to a given line through a given point respectively, in each case using only a finite number of points and the transport of distance between two points. As for the concepts of congruence and similarity, one could directly use the manner in which the author defines the scale drawing of a figure as a definition of similarity, and define congruence to be the case where the scaling factor is equal to 1. The basic isometries in the plane, and vectors, can be introduced in a second stage, and their properties similarly studied with a good equilibrium between precision and intuition.

We are perhaps in a situation where we still lack the practical experience that allows us to decide if among the various different proposals for "paths of presentation" of school geometry, we should clearly prefer one or another, considering only those that satisfy the basic principles stated in this book. We can try to examine what has been done in the more or less ancient past in schools around the world, and what the evolution has been over recent decades and in some cases even over the last few years; there is still perhaps some room for a confrontation of different hypotheses. I am nevertheless convinced that the general principles stated in this book, and the diagnosis of the disastrous evolution of school geometry in many countries in the last half-century should serve as a compelling guide in this matter. At any rate, this set of volumes is pioneering in that it gives a full presentation of a path that can be followed from middle school to high school, in a manner that respects all of these sound principles for a renewed teaching of school geometry. It is to be hoped that this or perhaps other equivalent ways of meeting these requirements will have the opportunity to prove their effectiveness in schools, thus reversing the sad diagnosis of the present situation made in this volume.

As expected, the final chapter on geometrical measurement is also an excellent example of balance between mathematical precision and controlled intuition, leading to the usual formulas for length, area and volume of basic geometric entities, with a unified vision of geometrical measure.

As in the first volume of this series, the author provides the reader with numerous illuminating activities for every topic, as well as an excellent choice of a wide range of exercises.

Hung-Hsi Wu, *Teaching School Mathematics: Pre-Algebra.*
American Mathematical Society, 2016, 383 pages, Hardback ISBN 978-1-4704-2720-7, eBook ISBN 978-1-4704-3009-2.

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Analytic Methods for Coagulation-Fragmentation Models, Volumes I+II

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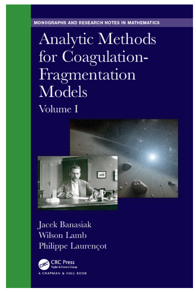

Coagulation and fragmentation are of fundamental importance in a vast variety of applications such as aerosol physics, polymerization, blood agglomeration or grouping of certain species, to name just a few. The first approach to describe such processes goes back to Marian von Smoluchowski, who in 1916 developed the first deterministic model for coagulation in a colloidal gold solution. The main assumption in his approach was that only two clusters combine at a time and that the rate at which this happens depends only on the product of the number-density of the respective cluster sizes, a hypothesis similar to the molecular chaos assumption in the Boltzmann equation. This leads to an infinite system of coupled differential equations for the density of cluster sizes and involves a so-called rate kernel which depends on the microscopic details of the specific coagulation process. Later the model was extended in various ways; in particular, different forms of fragmentation were added, allowing a cluster to fragment into two or more smaller ones. While the original model is discrete, allowing only for integer cluster sizes, the continuous version of the model is also of interest, in particular if one is interested in the behaviour of the system over large times.

It should be emphasized that coagulation-fragmentation equations are not only relevant from the point of view of applications; they are also fascinating due to the rich structure that solutions can display. Indeed, depending on the rate kernels, coagulation-fragmentation equations feature the possibility of loss of mass which can happen in finite or infinite time. In pure coagulation, for example, if the kernel grows faster than linearly, infinitely large particles are created in finite time, a phenomenon known as gelation. This can be directly linked to gelation in polymers where polymer chains can become so long such that suddenly in time the viscosity increases significantly and the polymer displays completely different properties. Similarly, in pure fragmentation mass can be lost due to the formation of zero sized particles, also called dust, a phase transition called scattering. In some combined coagulation-fragmentation equations, the loss of mass might also occur in infinite time, in the sense that the solution converges to an equilibrium that has lower mass than the solution for any finite time. This feature, typically termed nucleation, is prominent for example in the case of the well-known Becker–Döring system, that can describe the formation of liquid particles in a supersaturated gas.

Even though Smoluchowski's theory was developed more than 100 years ago, a thorough mathematical theory of coagulation-fragmentation equations going beyond explicitly solvable models started only in the 1980s. The first and until recently only book on the mathematical analysis of deterministic models appeared in 1994, written by Pavel Dubovski. However, in particular since the turn of the century the analysis of these equations has experienced another boost and a corresponding progress in new methods.

Thus the monograph under review is a very welcome, useful and state-of-the-art addition to the literature, written by three experts who have made fundamental contributions to the analysis of coagulation-fragmentation equations. The book takes up new analytical developments and provides an exhaustive treatment in particular of the two main methods that have been further developed in the last twenty years, namely semi-group theory and weak compactness methods. The work consists of two volumes, and starts after a short introduction with a description of coagulation and fragmentation processes in important applications, including numerous references, different modeling approaches and some classical results in particular for so-called solvable models. It also addresses many additional aspects, such as gelation and scattering, or the approach to universal self-similar long-time behaviour. Chapter 3 introduces notation and conventions that are used throughout the book, including in particular relevant function spaces, such as weighted L^1-spaces, as well as linear operators.

The remainder of Volume I is entirely devoted to the semi-group approach for linear fragmentation models. Chapter 4 starts with motivating why semi-groups are useful in this context, and gives an introduction to possible difficulties, such as loss of mass due to shattering or the possible non-uniqueness of solutions. The remaining subsections of Chapter 4 provide a comprehensive summary of semi-group theory. It includes the relevant definitions, the major theorems and further aspects such as inhomogeneous problems, semi-linear equations and perturbation methods. This informative section is not only the basis for Chapter 5 but also of interest in itself, and can be recommended to any reader who wants a brief introduction to semi-group theory.

In Chapter 5 the results from semi-group theory are applied to fragmentation equations. There are two main parts: in Chapter 5.1 pure fragmentation is considered and a rather complete theory is presented for the case where the fragmentation coefficients are separable. Chapter 5.2 deals with the case where transport in size space is added to the fragmentation term, an extension that is particularly relevant from the point of view of applications. The semi-group approach leads to a satisfactory theory of the well-posedness of fragmentation equations and their extensions, and gives information on the analyticity and so-called honesty of the corresponding fragmentation semi-group. In addition, topics such as the approximation of solutions by some cut-off are discussed, as well as the long-time behaviour of solutions.

Volume II addresses nonlinear models which in particular include coagulation. In the case where fragmentation dominates coagulation, semi-group theory can be applied to establish existence and uniqueness of solutions. If coagulation is dominant, which includes the case of pure coagulation, weak compactness methods have been particularly successful. In this case, approximate solutions are constructed, typically by a cut-off of the respective kernels. For these approximate solutions, moment and uniform integrability estimates need to be established to ensure weak compactness of the approximating sequences. This flexible method provides existence results, both for solutions to the original equations and for self-similar solutions. Uniqueness and regularity of solutions must be proved separately, however, and typically require further assumptions on the kernels.

The mathematical tools for this approach are summarized in Chapter 7, while Chapter 8 deals with the well-posedness of coagulation-fragmentation equations. Of particular relevance is the case of mass conserving solutions which are obtained if the rate kernels do not grow too quickly for small and large cluster sizes. The results of this chapter are up-to date and contain, for example, well-posedness results for singular coefficients that have been established only rather recently. Chapter 9 discusses the phenomenon of gelation, both instantaneous or for later times, and scattering. In the not explicitly solvable cases, these results are established via integral inequalities.

Chapter 10 is devoted to the important issue of long-time behaviour of solutions. For homogeneous kernels, one expects for pure fragmentation or coagulation respectively that this is universal and solutions converge to a self-similar solution. This issue is quite well understood for the fragmentation equation and for the coagulation equations with solvable kernel, but it is still mostly open for the pure coagulation equation with non-solvable kernels. Self-similar solutions can be constructed using approximation schemes and weak compactness methods, but apart from a few special cases, their dynamic stability has not been established yet, and for particular kernels one may even expect instability. The authors give a complete account of current results on these questions.

Finally, Chapter 11 provides a short introduction into material that cannot be covered in detail in the two volumes, such as the Becker–Döring equations and coagulation-fragmentation equations with diffusion. Numerous references are provided for readers interested in these topics.

To study the book, a basic background in functional analysis is needed, but all further tools that are used are introduced in the two *Mathematical Toolbox* chapters. The theory presented there is rather detailed and complete in the sense that the assumptions on the initial data and the rate coefficients are very general and the proofs are presented in full detail. These two volumes provide an informative, extensive and inspiring introduction to the subject accessible to all researchers from graduate students to experienced scientists.

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