Solved and unsolved problems

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The present column is devoted to Geometry/Topology.

I Six new problems – solutions solicited

Solutions will appear in a subsequent issue.

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Prove that the space of unordered couples of distinct points of a circle is the (open) Möbius band. More formally, consider

$$(S^1 \times S^1) \setminus \{(x,x) \mid x \in S^1\}$$

and the equivalence relation on this space $(x,y) \equiv (y,x)$; prove that the quotient topological space is the (open) Möbius band.

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In the Euclidean plane, let γ_1 and γ_2 be two concentric circles of radius respectively r_1 and r_2, with $r_1 < r_2$. Show that the locus γ of points P such that the polar line of P with respect to γ_2 is tangent to γ_1 is a circle of radius r_2^2/r_1.

Acknowledgement. I want to thank the professors who guided me in the first part of my career for giving me the ideas for these problems.

Paola Bonacini (Mathematics and Computer Science Department, University of Catania, Italy)

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Let $A \subseteq \mathbb{R}^3$ be a connected open subset of Euclidean space, and suppose that the following conditions hold:

(1) Every smooth irrotational vector field on A admits a potential (i.e., it is the gradient of a smooth function).

(2) The closure $\overline{A}$ of A is a smooth compact submanifold of $\mathbb{R}^3$ (of course, with non-empty boundary).

Show that A is simply connected. Does this conclusion hold even if we drop condition (2) on A?

Roberto Frigerio (Dipartimento di Matematica, Università di Pisa, Italy)

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A *regulus* is a surface in $\mathbb{R}^3$ that is formed as follows: We consider pairwise skew lines $\ell_1, \ell_2, \ell_3 \subset \mathbb{R}^3$ and take the union of all lines that intersect each of ℓ_1, ℓ_2, and ℓ_3. Prove that, for every regulus U, there exists an irreducible polynomial $f \in \mathbb{R}[x,y,z]$ of degree two that vanishes on U.

Adam Sheffer (Department of Mathematics, Baruch College, City University of New York, NY, USA)

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(Enumerative Geometry). How many lines pass through 4 generic lines in a 3-dimensional complex projective space $\mathbb{CP}^3$?

Mohammad F. Tehrani (Department of Mathematics, University of Iowa, USA)

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I learned about the following problem from Shmuel Weinberger. It can be viewed as a topological analogue of Arrow's Impossibility Theorem.

(a) A group of n friends have decided to spend their summer cottaging together on an undeveloped island, which happens to be a perfect copy of the closed 2-disk D^2. Their first task is to decide where on this island to build their cabin. Being democratically-minded, the friends decide to vote on the question. Each friend chooses his or her favourite point on D^2. The friends want a function that will take as input their n votes, and give as output a suitable point on D^2 to build. They believe, to be reasonable and fair, their "choice" function should have the following properties:

- (*Continuity*) It should be continuous as a function $(D^2)^n \to D^2$. This means, if one friend changes their vote by a small amount, the output will change only a small amount.
- (*Symmetry*) The n friends should be indistinguishable from each other. If two friends swap votes, the final choice should be unaffected.
- (*Unanimity*) If all n friends chose the same point x, then x should be the final choice.

For which values of n does such a choice function exist?

(b) The friends' second task is to decide where along the shoreline of the island they will build their dock. The shoreline happens to be a perfect copy of the circle S^1. Again, they decide to take the problem to a vote. For which values of n does a continuous, symmetric, and unanimous choice function $(S^1)^n \to S^1$ exist?

These are special cases of the following general problem in topological social choice theory: given a topological space X, for what values of n does X admit a social choice function that is continuous, symmetric, and unanimous? In other words, when is there a function $A\colon X^n \to X$ satisfying

- A is continuous,
- $A(x_1, \dots, x_n)$ is independent of the ordering of $x_1, \dots, x_n$, and
- $A(x, x, x, \dots, x) = x$ for all $x \in X$?

Jenny Wilson (Department of Mathematics, University of Michigan, USA)

II Open problem

Embeddings of contact domains

by Yakov Eliashberg (Department of Mathematics, Stanford University, USA)

One of the cornerstones of symplectic topology, *Gromov's non-squeezing theorem*, see [6], asserts that for $n > 1$ the ball of radius $R > 1$ in the standard symplectic space $(\mathbb{R}^{2n}, \omega = \sum_1^n dx_j \wedge dy_j)$ does not admit a symplectic embedding into the domain

$$\{x_1^2 + y_1^2 < 1\} \subset \mathbb{R}^{2n},$$

while there is no volume constraints to do that. Since that time, the theory of symplectic embedding has made a lot of progress (see F. Schlenk's survey [8] for recent results).

The (non-)embedding results in contact geometry, which is an odd-dimensional analogue of symplectic geometry, are rarer; below, we discuss a few open problems.

Recall that a contact structure ξ on an $(2n+1)$-dimensional manifold M is a completely non-integrable hyperplane field. If ξ is defined by a Pfaffian equation $\alpha = 0$ for a differential 1-form α (and such a form can always be found locally, and if ξ is co-orientable even globally) then the complete non-integrability can be expressed by the condition that $\alpha \wedge d\alpha^n$ is a non-vanishing $(2n+1)$-form on M.

In this set of problems, we will restrict our attention to domains in the contact manifold $X \coloneqq \mathbb{R}^{2n} \times S^1$, $S^1 = \mathbb{R}/\mathbb{Z}$, endowed with the contact structure

$$\xi \coloneqq \Big\{dz + \frac{1}{2}\sum_1^n x_j\,dy_j - y_j\,dx_j = 0\Big\}.$$

Given a *bounded* domain $U \subset \mathbb{R}^{2n}$, we set

$$\hat{U} \coloneqq U \times S^1 \subset \mathbb{R}^{2n} \times S^1 = X,$$

and refer to $\hat{U}$ as a *quantized* domain U. We say that a domain $\hat{U}_1$ admits a *contact embedding* into a domain $\hat{U}_2$ if *there is a contact isotopy $f_t\colon \hat{U}_1 \to X$, starting with the inclusion $f_0\colon \hat{U}_1 \hookrightarrow X$ such that $f_1(\hat{U}_1) \subset \hat{U}_2$*. Note that any Hamiltonian isotopy which moves U_1 into U_2 lifts to a contact isotopy moving $\hat{U}_1$ into $\hat{U}_2$. Hence we will refer to the problem of contact embeddings between the domains $\hat{U}_1$ and $\hat{U}_2$ as a quantized version of the corresponding symplectic embedding problem of U_1 to U_2.

Denote by $B^{2n}(R)$ the $2n$-dimensional open ball of radius R and by $P(r_1, \dots, r_k)$ the polydisk $B^2(r_1) \times \dots \times B^2(r_n) \subset \mathbb{R}^{2n}$, where $0 < r_1 \le r_2 \le \dots \le r_n$. It was shown in [3] that *if $\pi r_1^2 < k < \pi r_2^2$ for any integer $k \ge 1$, then $\hat{B}^{2n}(r_2)$ does not admit a contact embedding into $\hat{B}^{2n}(r_1)$*. Another theorem from [3] states that *if $\pi R^2 < 1$, then $\hat{B}^{2n}(R)$ admits a contact embedding into $\hat{B}^{2n}(r)$ for any $r > 0$*. The former result was improved in [2, 5] to show that for any $r_1 < r_2$ with $\pi r_2^2 > 1$ there is no contact embedding of $\hat{B}^{2n}(r_2)$ into $\hat{B}^{2n}(r_1)$. Recall that Gromov's *symplectic width* $\mathrm{Width}_{\mathrm{Gr}}(U)$ of a domain $U \subset \mathbb{R}^{2n}$ can be defined as the supremum of $\pi\rho^2$ such that $B^{2n}(\rho)$ can be symplectically embedded into the domain U. The above results can be slightly generalized to the following statement, see [3].

If, for two domains $U_1, U_2 \subset \mathbb{R}^{2n}$, we have

$$\mathrm{Width}_{\mathrm{Gr}}(U_2) > \mathrm{Width}_{\mathrm{Gr}}(U_1) \quad \text{and} \quad \mathrm{Width}_{\mathrm{Gr}}(U_2) > 1,$$

then the quantized domain $\hat{U}_2$ does not admit a contact embedding into $\hat{U}_1$.

Very little is known about embeddings of contact domains beyond the above results. Let us formulate a couple of concrete problems concerning quantized versions of some relatively old embedding results in symplectic topology. As we already mentioned above, many new obstructions to symplectic embeddings were found in recent years. It is unknown whether any of them hold in the quantized versions.

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Contact packing problem. *Suppose that $\pi R^2 > \pi r^2 > 1$. Is there a maximal number of quantized balls $\hat{B}^{2n}(r)$ which admit a contact packing into $\hat{B}^{2n}(R)$? And if the answer is "yes", then what is this number?*

Here we say that $\hat{B}^{2n}(R)$ admits a *contact packing* by k quantized balls $\hat{B}^{2n}(r)$ if the disjoint union

$$\underbrace{\hat{B}^{2n}(r) \sqcup \cdots \sqcup \hat{B}^{2n}(r)}_{k} \subset X$$

admits a contact embedding into $\hat{B}^{2n}(R)$. Note that the corresponding *symplectic packing* problem was intensively studied beginning with the seminal paper by Gromov [6], where he proved that the packing of the ball $B^{2n}(R)$ with 2 disjoint balls $B^{2n}(r)$ is possible if and only if $R^2 > 2r^2$. For $n = 2$, the problem was significantly advanced by D. McDuff and L. Polterovich in [7], and then completely solved by P. Biran in [1]. In the contact case, Problem 258* is completely open.

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Rotating quantized polydisks. *For $r < R$, consider the standard contact inclusion $(x, y) \mapsto (-y, x)$. Let $\hat{j} = j \times \mathrm{Id}$, $\hat{\psi} = \psi \times \mathrm{Id}$ be the corresponding contact inclusion $\hat{P}(r, r) \to \hat{P}(R, R)$, and consider the contactomorphism $\psi \times \mathrm{Id} \colon X = \mathbb{R}^{2n} \times S^1 \to \mathbb{R}^{2n} \times S^1 = X$. When are the embeddings $\hat{j}, \hat{\psi} \circ \hat{j} \colon \hat{P}(r, r) \hookrightarrow \hat{P}(R, R)$ contact isotopic?*

Note that a theorem of Floer–Hofer–Wysocki, see [4], states that when $2r^2 > R^2$ the symplectic embeddings $j, \psi \colon P(r, r) \to \mathbb{P}(R, R)$ are not symplectically isotopic.

References

[1] P. Biran, Symplectic packing in dimension 4. *Geom. Funct. Anal.* **7**, 420–437 (1997)

[2] S.-F. Chiu, Nonsqueezing property of contact balls. *Duke Math. J.* **166**, 605–655 (2017)

[3] Y. Eliashberg, S. S. Kim and L. Polterovich, Geometry of contact transformations and domains: orderability versus squeezing. *Geom. Topol.* **10**, 1635–1747 (2006)

[4] A. Floer, H. Hofer and K. Wysocki, Applications of symplectic homology. I. *Math. Z.* **217**, 577–606 (1994)

[5] M. Fraser, Contact non-squeezing at large scale in $\mathbb{R}^{2n} \times S^1$. *Internat. J. Math.* **27**, 1650107 (2016)

[6] M. Gromov, Pseudo holomorphic curves in symplectic manifolds. *Invent. Math.* **82**, 307–347 (1985)

[7] D. McDuff and L. Polterovich, Symplectic packings and algebraic geometry. *Invent. Math.* **115**, 405–434 (1994)

[8] F. Schlenk, Symplectic embedding problems, old and new. *Bull. Amer. Math. Soc. (N.S.)* **55**, 139–182 (2018)

III Solutions

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We consider a setting where there is a set of m candidates

$$C = \{c_1, \dots, c_m\}, \quad m \geq 2,$$

and a set of n voters $[n] = \{1, \dots, n\}$. Each voter ranks all candidates from the most preferred one to the least preferred one; we write $a \succ_i b$ if voter i prefers candidate a to candidate b. A collection of all voters' rankings is called a *preference profile*. We say that a preference profile is *single-peaked* if there is a total order $\lhd$ on the candidates (called the *axis*) such that for each voter i the following holds: if i's most preferred candidate is c and $a \lhd b \lhd c$ or $c \lhd b \lhd a$, then $b \succ_i a$. That is, each ranking has a single "peak", and then "declines" in either direction from that peak.

(i) In general, if we aggregate voters' preferences over candidates, the resulting majority relation may have cycles: e.g., if $a \succ_1 b \succ_1 c$, $b \succ_2 c \succ_2 a$ and $c \succ_3 a \succ_3 b$, then a strict majority (2 out of 3) voters prefer a to b, a strict majority prefer b to c, yet a strict majority prefer c to a. Argue that this cannot happen if the preference profile is single-peaked. That is, prove that if a profile is single-peaked, a strict majority of voters prefer a to b, and a strict majority of voters prefer b to c, then a strict majority of voters prefer a to c.

(ii) Suppose that n is odd and voters' preferences are known to be single-peaked with respect to an axis $\lhd$. Consider the following voting rule: we ask each voter i to report their top candidate $t(i)$, find a median voter i^*, i.e.,

$$|\{i : t(i) \lhd t(i^*)\}| < \frac{n}{2} \quad \text{and} \quad |\{i : t(i^*) \lhd t(i)\}| < \frac{n}{2},$$

and output $t(i^*)$. Argue that under this voting rule no voter can benefit from voting dishonestly, if a voter i reports some candidate $a \neq t(i)$ instead of $t(i)$, this either does not change the outcome or results in an outcome that i likes less than the outcome of the truthful voting.

(iii) We say that a preference profile is *1D-Euclidean* if each candidate c_j and each voter i can be associated with a point in $\mathbb{R}$ so that the preferences are determined by distances, i.e., there is an embedding $x \colon C \cup [n] \to \mathbb{R}$ such that for all $a, b \in C$ and $i \in [n]$, we have $a \succ_i b$ if and only if $|x(i) - x(a)| < |x(i) - x(b)|$. Argue that a 1D-Euclidean profile is necessarily single-peaked. Show that the converse is not true, i.e., there exists a single-peaked profile that is not 1D-Euclidean.

(iv) Let P be a single-peaked profile, and let L be the set of candidates ranked last by at least one voter. Prove that $|L| \leq 2$.

(v) Consider an axis $c_1 \lhd \cdots \lhd c_m$. Prove that there are exactly 2^{m-1} distinct votes that are single-peaked with respect to this axis. Explain how to sample from the uniform distribution over these votes.

These problems are based on references [4] (parts (i) and (ii)), [2] (part (iii)) and [1, 5] (part (v)); part (iv) is folklore. See also the survey [3].

Edith Elkind (University of Oxford, UK)

Solution by the proposer

(i) We can restrict the voters' preferences to the set $\{a, b, c\}$; the reader can check that a restriction of a single-peaked profile to a subset of candidates remains single-peaked. We consider three cases depending on how a, b and c are ordered by the axis $\lhd$.

Case 1: $a \lhd b \lhd c$ or $c \lhd b \lhd a$. Then all voters who prefer a over b have a as their top choice and hence prefer a to c.

Case 2: $b \lhd a \lhd c$ or $c \lhd a \lhd b$. All voters who prefer c over a have c as their top choice and hence prefer a to b; therefore, these voters are in minority.

Case 3: $a \lhd c \lhd b$ or $b \lhd c \lhd a$. This is impossible: all voters who prefer b to c have b as their top choice, so we have a strict majority preferring b over a, a contradiction.

(ii) Suppose the winner under truthful voting is a. Consider a voter i. If $t(i) = a$, then i cannot improve the outcome by lying. So suppose $t(i) \lhd a$ (the case $a \lhd t(i)$ is symmetric). If i reports a or some candidate b with $b \lhd a$, this does not change what the top choice of the median voter is, and hence does not change the outcome. If i reports a candidate c with $t(i) \lhd c$, then the median voter may shift to the right, i.e., further away from i's true top choice; as i's preferences are single-peaked, this does not improve the outcome from her perspective.

(iii) Ordering the candidates by their position, i.e., placing a before b on the axis $\lhd$ if $x(a) < x(b)$ results in an axis witnessing that the input profile is single-peaked. To show that the converse is not true, consider the following four votes:

$$b \succ_1 c \succ_1 a \succ_1 d, \quad c \succ_2 b \succ_2 a \succ_2 d,$$
$$b \succ_3 c \succ_3 d \succ_3 a, \quad c \succ_4 b \succ_4 d \succ_4 a.$$

This profile is single-peaked on $a \lhd b \lhd c \lhd d$. Now, suppose for contradiction that it is 1D-Euclidean, i.e., it admits an embedding x. Consider the positions of the four voters $x(1), x(2), x(3), x(4)$. Assume without loss of generality that $x(b) < x(c)$. Then we have

$$x(1), x(3) < \frac{1}{2}(x(b) + x(c))$$

(as voters 1 and 3 prefer b to c) and

$$x(2), x(4) > \frac{1}{2}(x(b) + x(c))$$

(as voters 2 and 4 prefer b to c). But now consider the point $\frac{1}{2}(x(a) + x(d))$. Voters 1 and 2 have to be on one side of this point and voters 3 and 4 have to be on the other side of this point, because of their preferences over a vs. d. But this is clearly impossible!

(iv) Assume without loss of generality that P is single-peaked with respect to the axis $c_1 \lhd \cdots \lhd c_m$. Clearly, P may contain a vote that ranks c_1 last or a vote that ranks c_m last. But it cannot contain a vote that ranks some c_i with $1 < i < m$ last: if the top candidate in that vote is a c_j with $j < i$, then this voter prefers c_i to c_m, and if the top candidate in that vote is a c_k with $k > i$, then this voter prefers c_i to c_1.

(v) Induction. For $m = 2$, we have $2 = 2^{2-1}$ orders, i.e., $c_1 \succ c_2$ and $c_2 \succ c_1$. Now suppose the claim has been proved for all $m' < m$. A vote that is single-peaked on $c_1 \lhd \cdots \lhd c_m$ may have c_1 in the last position, with candidates in top $m - 1$ positions forming a single-peaked vote with respect to $c_2 \lhd \cdots \lhd c_m$ (2^{m-2} options) or it may have c_m in the last position, with candidates in top $m - 1$ positions forming a single-peaked vote with respect to $c_1 \lhd \cdots \lhd c_{m-1}$ (2^{m-2} options). For sampling, we can build the vote bottom-up. At the first step, we fill the last position with c_1 or c_m, with probability $\frac{1}{2}$ each. Once k positions have been filled, $1 \le k \le m - 1$, the not-yet-ranked candidates form a contiguous segment $c_1 \lhd \cdots \lhd c_j$ of the axis. We then fill position $k + 1$ from the bottom with c_i or c_j, with equal probability. This sampling is uniform, because the probability of generating a specific ranking is exactly 2^{-m+1}: we make $m - 1$ choice, and with probability $\frac{1}{2}$ each choice is consistent with the target ranking.

References

[1] V. Conitzer, Eliciting single-peaked preferences using comparison queries. *J. Artificial Intelligence Res.* **35**, 161–191 (2009)

[2] C. H. Coombs. Psychological scaling without a unit of measurement. *Psychological Review* **57**, 145 (1950)

[3] E. Elkind, M. Lackner and D. Peters. Structured preferences. In *Trends in Computational Social Choice*, edited by U. Endriss, Chapter 10, AI Access, 187–207 (2017)

[4] H. Moulin, *Axioms of Cooperative Decision Making*. Cambridge University Press, Cambridge (1991)

[5] T. Walsh, Generating single peaked votes. arXiv:1503.02766 (2015)

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Consider a standard prisoners' dilemma game described by the following strategic form, with $\delta > \beta > 0 > \gamma$:

	C	D
C	β, β	γ, δ
D	δ, γ	0, 0

Assume that any given agent either plays C or D and that agents reproduce at a rate determined by their payoff from the strategic form of the game plus a constant f. Suppose that members of an infinite population are assorted into finite groups of size n. Let q denote the proportion of agents playing strategy C ("altruists") in the population as a whole and q_i denote the proportion of agents playing C in group i. We assume that currently $q \in (0,1)$.

The process of assortment is abstract, but we assume that it has finite expectation $E[q_i] = q$ and variance $\mathrm{Var}[q_i] = \sigma^2$. Members within each group are then randomly paired off to play one iteration of the prisoners' dilemma against another member of their group. All agents then return to the overall population.

(a) Find a condition relating q, σ^2, β, γ, δ and n under which the proportion of altruists in the overall population rises after a round of play.

(b) Now interpret this game as one where each player can confer a benefit b upon the other player by individually incurring a cost c, with $b > c > 0$, so that $\beta = b - c$, $\delta = b$ and $\gamma = -c$. Prove that, as long as (i) there is some positive assortment in group formation and (ii) the ratio $\frac{c}{b}$ is low enough, then the proportion of altruists in the overall population will rise after a round of play.

Richard Povey (Hertford College and St Hilda's College, University of Oxford, UK)

Solution by the proposer

(a) We can firstly see that the number of altruists (type C) and non-altruists (type D) in group i after a round of play will be given by

$$n_i' q_i' = \left(\beta\left(\frac{nq_i - 1}{n-1}\right) + \gamma\left(\frac{n(1-q_i)}{n-1}\right) + f\right) n q_i,$$
$$n_i'(1-q_i') = \left(\delta\left(\frac{nq_i}{n-1}\right) + f\right) n(1-q_i).$$

Summing these two equalities yields

$$n_i' = \beta q_i\left(\frac{n(nq_i-1)}{n-1}\right) + \gamma q_i\left(\frac{n^2(1-q_i)}{n-1}\right) + \delta(1-q_i)\left(\frac{n^2 q_i}{n-1}\right) + fn.$$

Given an infinite population and hence an infinite number of groups, the new proportion of altruists in the overall population will be

$$\begin{aligned} q' &= \frac{E[n_i' q_i']}{E[n_i']} \\ &= \left(\beta\left(\frac{nE[q_i^2] - E[q_i]}{n-1}\right) + n\gamma\left(\frac{E[q_i] - E[q_i^2]}{n-1}\right) + fE[q_i]\right) \\ &\quad \cdot \left(\beta q_i\left(\frac{nE[q_i^2] - E[q_i]}{n-1}\right) + n\gamma\left(\frac{E[q_i] - E[q_i^2]}{n-1}\right) \right. \\ &\qquad \left. + n\delta\left(\frac{E[q_i] - E[q_i^2]}{n-1}\right) + f\right)^{-1}. \end{aligned}$$

Substituting in $E[q_i^2] = \sigma^2 + E[q_i]^2$ and $E[q_i] = q$ gives us

$$\begin{aligned} q' = &\left(\beta\left(\frac{n\sigma^2 + nq^2 - q}{n-1}\right) + n\gamma\left(\frac{q(1-q) - \sigma^2}{n-1}\right) + fq\right) \\ &\cdot \left(\beta\left(\frac{n\sigma^2 + nq^2 - q}{n-1}\right) + n\gamma\left(\frac{q(1-q) - \sigma^2}{n-1}\right) \right. \\ &\qquad \left. + n\delta\left(\frac{q(1-q) - \sigma^2}{n-1}\right) + f\right)^{-1}. \end{aligned}$$

Assuming f is high enough to make the denominator positive, it then follows that

$$\begin{aligned} &q' - q > 0 \\ &\iff (1-q)\left(\beta\left(\frac{n\sigma^2 + nq^2 - q}{n-1}\right) + n\gamma\left(\frac{q(1-q) - \sigma^2}{n-1}\right)\right) \\ &\qquad - nq\delta\left(\frac{q(1-q) - \sigma^2}{n-1}\right) > 0. \end{aligned}$$

After some further rearrangement, we can derive the following:

$$\begin{aligned} &q' - q > 0 \\ &\iff \frac{\sigma^2}{q(1-q)} > 1 - \left(\frac{n-1}{n}\right)\left(\frac{\beta}{(1-q)(\beta-\gamma) + q\delta}\right). \quad (1) \end{aligned}$$

Since the right-hand side of (1) must be strictly between 0 and 1, this has the intuitive interpretation that the *inter-group* variance σ^2 must be sufficiently high relative to the *intra-group* variance[1] so that, although altruists do less well relative to non-altruists *within each group*, the concentration of altruists together within *particular* groups is sufficiently strong to confer enough of an evolutionary advantage to offset this and to enable altruists to do better evolutionarily than non-altruists in the overall population.[2]

(b) In the case where $\beta = b - c$, $\delta = b$ and $\gamma = -c$, condition (1) can be rearranged to give

$$\frac{c}{b} < \left(\frac{\sigma^2}{q(1-q)}\right)\left(\frac{n}{n-1}\right) - \frac{1}{n-1}. \quad (2)$$

With random assortment, q_i would be equal to $\frac{X_i}{n}$ where X_i, the number of altruists in group i would have a binomial distribution: $X_i \sim B(n,q)$. Therefore

$$\sigma^2 = \mathrm{Var}\left[\frac{X_i}{n}\right] = \frac{q(1-q)n}{n^2} = \frac{q(1-q)}{n}.$$

With perfect positive correlation between group members, we would get

$$\sigma^2 = \mathrm{Var}\left[\frac{X_i}{n}\right] = \frac{q(1-q)n^2}{n^2} = q(1-q).$$

[1] This is the variance of the Bernoulli variable $B(q)$ which takes a value of 1 of a single individual drawn from the population is an altruist and 0 otherwise, which is $q(1-q)$.

[2] This result was first proved in a general evolutionary context by George R. Price [3, 4].

With some positive assortment generating positive covariance between group members, we may therefore without loss of generality suppose that

$$\sigma^2 = \frac{q(1-q)n + q(1-q)\varepsilon(n^2-n)}{n^2} = \frac{q(1-q)}{n} + \frac{\varepsilon q(1-q)(n-1)}{n},$$

where $\varepsilon \in (0,1)$. Plugging this into (2) and simplifying, we get $\frac{c}{b} < \varepsilon$. So we can see that for any $\varepsilon > 0$ there always exists a value of $\frac{c}{b}$ low enough for altruists to expand as a proportion of the overall population.[3]

References

[1] B. Cooper and C. Wallace, Group selection and the evolution of altruism. *Oxford Economic Papers* **56**, 307–330 (2004)

[2] R. Povey, Punishment and the potency of group selection, *J. Evolutionary Economics* **24** (2014)

[3] G. R. Price, Selection and covariance, *Nature* **227**, 520–521 (1970)

[4] G. R. Price, Extension of covariance selection mathematics, *Ann. Human Genetics* **35**, 485–490 (1972)

[5] E. Sober and D. S. Wilson, *Unto Others*, Harvard University Press, Cambridge, Massachusetts (1999)

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Consider a village consisting of n farmers who live along a circle of length n. The farmers live at positions $1, 2, \dots, n$. Each of them is friends with the person to the left and right of them, and each friendship has capacity m where m is a non-negative integer. At the end of the year, each farmer does either well (her wealth is $+1$ dollars) or not well (her wealth is -1 dollars) with equal probability. Farmers' wealth realizations are independent of each other. Hence, for a large circle the share of farmers in each state is on average 1.

The farmers share risk by transferring money to their direct neighbors. The goal of risk-sharing is to create as many farmers with OK wealth (0 dollars) as possible. Transfers have to be in integer dollars and cannot exceed the capacity of each link (which is m).

A few examples with a village of size $n = 4$ serve to illustrate risk-sharing.

- Consider the case where farmers 1 to 4 have wealth

$$(+1, -1, +1, -1).$$

In that case, we can share risk completely with farmer 1 sending a dollar to agent 2 and farmer 3 sending a dollar to farmer 4. This works for any $m \geq 1$.

- Consider the case where farmers 1 to 4 have wealth

$$(+1, +1, -1, -1).$$

In that case, we can share risk completely with farmer 1 sending a dollar to farmer 2, farmer 2 sending two dollars to farmer 3 and farmer 3 sending one dollar to farmer 4. In this case, we need $m \geq 2$. If $m = 1$, we can only share risk among half the people in the village.

Show that for any wealth realization an optimal risk-sharing arrangement can be found as the solution to a maximum flow problem.

Tanya Rosenblat (School of Information and Department of Economics, University of Michigan, USA)

Solution by the proposer

We augment the village graph by adding two auxiliary nodes. The *source node s* is connected to all the farmers with positive wealth $(+1)$ and each of these links has capacity 1. The *sink node t* is connected to all the farmers with negative wealth (-1) and each of these links also capacity 1. We now look for the *maximum flow* from s to t: this is equal to the number of luck/unlucky farmer pairs who can be matched under the best risk-sharing arrangement.

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This exercise is a continuation of Problem 247 where we studied risk-sharing among farmers who live on a circle village and are friends with their direct neighbors to the left and right with friendships of a certain capacity. Assume that for any realization of wealth levels the best possible risk-sharing arrangement is implemented and denote the expected share of unmatched farmers with $U(n,m)$. Show that $U(n,m) \to \frac{1}{2m+1}$ as $n \to \infty$.

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Solution by the proposer

The solution proceeds in two stages. In Problem 247, we already established that the problem can be understood as a maximum flow problem. We first formulate a particular algorithm that implements this flow. We then use this algorithm to express $U(n,m)$ in closed form.

Risk-sharing as a Maximum Flow Problem. We next describe a matching algorithm which is to run for m rounds. The claim is that this algorithm implements the maximum flow in the augmented graph for any m. For the purpose of this algorithm, we

[3] The literature on this model has further established the results that *multiple* periods of isolation in finite groups acts to *amplify* inter-group variance, so that even with random assortment into groups altruism can evolve [1]. It has also been found that use of punishment strategies in dynamic interactions can act to weaken this group selection mechanism [2]. For an accessible book-length treatment of the topic of group selection in the biological and social sciences, see [5].

call a black agent an agent with positive shock and a white agent an agent with a negative shock.

Step I: Index all agents from 1 to n clockwise (1 is the neighbor of n on a circle). Set the counter i to 1.

Step II: If agents i and $i+1$ are of different colors, label them "matched" and move the counter clockwise to $i+2$. If agent $i+1$ has the same color as agent i, then declare agent i "unmatched" and move the counter clockwise to agent $i+1$. Repeat this step until the counter has reached the first agent again.

Step III: Define a new circle by ordering all the unmatched agents on a circle without disturbing the order of the agents. Essentially, this implies that all the matched agents are simply removed from the circle and any gaps are plugged by connected the closest unmatched agents with each other. Repeat steps I and II for this new circle. Repeat this algorithm m times.

Lemma 1. *The above matching algorithm implements the maximum flow.*

Proof. The Ford–Fulkerson algorithm computes the maximum flow by looking for open paths which can carry positive flow and then constructing a graph with augmented capacities in which the next path is found, etc. Once no more open path exists the max flow has been implemented. The above algorithm implements Ford–Fulkerson using a particular order of selecting open paths. Therefore, it implements the max flow.

Closed form solutions. We next prove the following lemma.

Lemma 2. *Assume we have a circle of size n where the probability that an agent has a neighbor of the same color is α. Then the share of unmatched agents after one round of the above algorithm converges to $\frac{\alpha}{2-\alpha}$ as $n \to \infty$.*

Proof. In each instance of step I of the algorithm, it produces an unmatched agent with probability α and a pair of matched agents with probability $1-\alpha$. The sum of unmatched and matched agents has to be n. Therefore the share of unmatched agents converges to

$$\frac{\alpha}{\alpha+2(1-\alpha)} = \frac{\alpha}{2-\alpha}.$$

The final step in the proof of the result is to derive the probability α_m that an agent is followed by a same-color agent in round m. We know that in round 1 shocks are i.i.d.; therefore $\alpha_1 = \frac{1}{2}$. We start by proving a recursive formula for calculating α_m.

Lemma 3. *If the sequential probability is α_m in round m, then the sequential probability in round $m+1$ satisfies*

$$\alpha_{m+1} = \frac{2-\alpha_m}{3-2\alpha_m}. \tag{1}$$

Proof. Consider an agent i on whom the counter rested at some point in the algorithm and who stays unmatched in the current round. This must be because he has a neighbor $i+1$ of the same color (without loss of generality assume both are black). With probability α_m agent $i+2$ is also black and therefore agent $i+1$ will survive into round $m+1$ as well and be of the same color as agent i (black). With probability $1-\alpha_m$ agent $i+2$ is white. In this case agents $i+1$ and $i+2$ can be matched. Matching can continue for the subsequent pairs of agents $(i+3, i+4)$, $(i+5, i+6)$, etc.; it will only stop if for any of these pairs agents have the same color. This will happen with probability α_m. To figure out it if this process will stop at a "white pair" or a "black pair" (let's call it the "blocking pair") it is crucial to know whether agent $i+2$, $i+4$, $i+6$, etc. (i.e., the agent just prior to the blocking pair) is white or black.

We know that agent $i+2$ is white. What is the probability that agent $i+4$ is the same color (provided $(i+3, i+4)$ is not a blocking pair)? This can only happen if the pair $(i+3, i+4)$ is a black agent followed by a white agent. If it is a white agent followed by a black agent then $i+4$ has a different color from $i+2$. So the probability of a color change is

$$\frac{\alpha_m(1-\alpha_m)}{\alpha_m(1-\alpha_m)+(1-\alpha_m)^2} = \alpha_m.$$

Assume that the probability that the agent prior to the blocking pair is of the same color as $i+2$ is q. With probability α_m the pair $(i+3, i+4)$ is a blocking pair and with probability $1-\alpha_m$ the pair is not blocking. In that case agent $i+4$ has a different color from agent $i+2$ with probability α_m. Because of the recursive nature of the problem, the probability that the agent prior to the blocking pair has the same color as $i+4$ is q. Therefore we know that

$$q = \alpha_m + (1-\alpha_m)[(1-\alpha_m)q + (1-q)\alpha_m].$$

This allows us to calculate q as

$$q = \frac{\alpha_m(2-\alpha_m)}{1-(1-\alpha_m)(1-2\alpha_m)} = \frac{2-\alpha_m}{3-2\alpha_m}.$$

So we know that the agent prior to the blocking pair is white with probability q. The blocking pair is therefore a black blocking pair with probability $q(1-\alpha_m)+(1-q)\alpha_m$. Therefore the total probability that the next unmatched agent after agent i is of the same color (black in this case) is

$$\begin{aligned}\alpha_{m+1} &= \alpha_m + (1-\alpha_m)[q(1-\alpha_m)+(1-q)\alpha_m] \\ &= q = \frac{2-\alpha_m}{3-2\alpha_m}.\end{aligned}$$

We can check that $\alpha_m = 1 - \frac{1}{2m}$ satisfies both the initial condition and the recursive equation 1. This implies that

$$\frac{\alpha_m}{2-\alpha_m} = \frac{2m-1}{2m+1}.$$

Finally, note that the share of unmatched agents (for $n \to \infty$) can be calculated by taking the product of the share of unmatched agents in each round:

$$\lim_{n\to\infty} U(n,m) = \prod_{i=1}^{m} \frac{a_i}{2-a_i} = \frac{1}{2m+1}.$$

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In a *combinatorial auction* there are m items for sale to n buyers. Each buyer i has some valuation function $v_i(\cdot)$ which takes as input a set S of items and outputs that bidder's value for that set. These functions will always be monotone ($v_i(S \cup T) \geq v_i(S)$ for all S, T), and satisfy $v_i(\emptyset) = 0$.

Definition 1 (Walrasian equilibrium). A price vector $\vec{p} \in \mathbb{R}^m_{\geq 0}$ and a list $B_1, \dots, B_n$ of subsets of $[m]$ form a *Walrasian equilbrium* for $v_1, \dots, v_n$ if the following two properties hold:

- Each $B_i \in \arg\max_S \{v_i(S) - \sum_{j\in S} p_j\}$.
- The sets B_i are disjoint, and $\bigcup_i B_i = [m]$.

Prove that a Walrasian equilibrium exists for $v_1, \dots, v_n$ if and only if there exists an integral[4] optimum to the following linear program:

$$\begin{aligned} &\text{maximize } \sum_i \sum_S v_i(S)\cdot x_{i,S} \\ &\text{such that, for all } i, \quad \sum_S x_{i,S} = 1, \\ &\qquad \text{for all } j, \quad \sum_{S\ni j}\sum_i x_{i,S} \leq 1, \\ &\qquad \text{for all } i,S, \quad x_{i,S} \geq 0. \end{aligned}$$

Hint. Take the dual, and start from there.

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Solution by the proposer

First, we take the dual of the given LP. We use the dual variable p_j for the constraints involving items, and the dual variable u_i for the constraints involving bidders. Then the dual problem is

$$\begin{aligned} &\text{minimize } \sum_i u_i + \sum_j p_j \\ &\text{such that, for all } i,S, \quad u_i + \sum_{j\in S} p_j \geq v_i(S), \\ &\qquad \text{for all } j, \quad p_j \geq 0. \end{aligned}$$

Walrasian equilibrium implies integral optimum. Now, assume that a Walrasian equilibrium exists, and let it be $p_1, \dots, p_m$ and $B_1, \dots, B_n$. Then consider the integral solution to the LP that sets $x_{i,B_i} = 1$ for all i, and all other variables to 0. This solution is clearly feasible for the LP, and has objective value equal to $\sum_i v_i(B_i)$.

Consider also the dual solution $u_i \coloneqq v_i(B_i) - \sum_{j\in B_i} p_j$, with p_j as given in the Walrasian equilibrium. We claim this is a feasible solution to the dual. To see this, observe first that each $p_j \geq 0$. Also, because $B_i \in \arg\max_S \{v_i(S) - \sum_{j\in S} p_j\}$ by definition of Walrasian equilibrium, we have that

$$u_i \coloneqq v_i(B_i) - \sum_{j\in B_i} p_j \geq v_i(S) - \sum_{j\in S} p_j \quad \text{for all } S.$$

Therefore, all dual constraints are satisfied, and this is a feasible dual. Moreover, observe that the value of the dual objective is

$$\sum_i u_i + \sum_j p_j = \sum_i \Big(v_i(B_i) - \sum_{j\in B_i} p_j\Big) + \sum_j p_j = \sum_i v_i(B_i).$$

The last equality follows because each item is in exactly one bundle B_i. So we have proved that if $(\vec{p}, \vec{B})$ is a Walrasian equilibrium, then there is an integral feasible point for the LP with objective value $\sum_i v_i(B_i)$, and also a feasible dual solution with value $\sum_i v_i(B_i)$. By LP duality, both feasible solutions are in fact optimal. Therefore, there is an integral optimum for the LP.

Integral optimum implies Walrasian equilibrium. Now, assume that the LP has an integral optimum. Observe that for this integral solution, there must exist disjoint sets $B_1, \dots, B_n$ such that for each i, $x_{i,B_i} = 1$ and all other variables are 0. The LP value for this solution is $\sum_i v_i(B_i)$. Moreover, observe that if any item $j \notin \cup_i B_i$, we can add j to an arbitrary B_i without decreasing $\sum_i v_i(B_i)$ (because each $v_i(\cdot)$ is monotone). Therefore, if there is an integral optimum to the LP, there exist disjoint $B_1, \dots, B_n$ such that $\cup_i B_i = [m]$ and $\sum_i v_i(B_i)$ is the optimal solution to the LP.

By Strong LP Duality, there also exists a feasible dual solution $p_1, \dots, p_m, u_1, \dots, u_n$ such that $\sum_i u_i + \sum_j p_j = \sum_i v_i(B_i)$. We will claim that $(\vec{p}, \vec{B})$ form a Walrasian equilibrium.

For a proof by contradiction, assume that this is not the case. Then there must be some bidder i such that $B_i \notin \arg\max_S \{v_i(S) - \sum_{j\in S} p_j\}$. In particular, this means that there exists some B_i' such that $v_i(B_i') - \sum_{j\in B_i'} > v_i(B_i) - \sum_{j\in B_i}$. Because u_i is a feasible solution for the dual, we then conclude that

$$u_i \geq v_i(B_i') - \sum_{j\in B_i'} > v_i(B_i) - \sum_{j\in B_i} p_j.$$

We claim that this contradicts the fact that $\sum_i u_i + \sum_j p_j = \sum_i v_i(B_i)$, since

$$\sum_i u_i > \Big(\sum_i v_i(B_i) - \sum_{j\in B_i} p_j\Big) > \sum_i v_i(B_i) - \sum_j p_j.$$

The first inequality holds because $u_i \geq v_i(B_i) - \sum_{j\in B_i} p_j$ for all i, and the inequality is strict for at least one i. Therefore, $(\vec{p}, \vec{B})$ must be a Walrasian equilibrium.

[4] That is, a point such that each $x_{i,S} \in \{0,1\}$.

Wrapup. This concludes the proof. We have shown that there is an integral optimum to the LP if and only if a Walrasian equilibrium exists. The solution to this problem is given by Nisan et al. in [3, Corollary 11.16]; they cite [1, 2].

References

[1] S. Bikhchandani and J. W. Mamer, Competitive equilibrium in an exchange economy with indivisibilities. *J. Econom. Theory* **74**, 385–413 (1997)

[2] S. Bikhchandani and J. M. Ostroy, The package assignment model. *J. Econom. Theory* **107**, 377–406 (2002)

[3] N. Nisan, T. Roughgarden, É. Tardos and V. V. Vazirani, *Algorithmic game theory*. Cambridge University Press (2007)

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Consider a game played on a network and a finite set of players $\mathcal{N} = \{1, 2, \dots, n\}$.[5] Each node in the network represents a player and edges capture their relationships. We use $\mathbf{G} = (g_{ij})_{1 \le i,j \le n}$ to represent the adjacency matrix of a undirected graph/network, i.e., $g_{ij} = g_{ji} \in \{0, 1\}$. We assume $g_{ii} = 0$. Thus, $\mathbf{G}$ is a zero-diagonal, squared and symmetric matrix. Each player, indexed by i, chooses an action $x_i \in \mathbb{R}$ and obtains the following payoff:

$$\pi_i(x_1, x_2, \dots, x_n) = x_i - \frac{1}{2}x_i^2 + \delta \sum_{j \in \mathcal{N}} g_{ij} x_i x_j.$$

The parameter $\delta > 0$ captures the strength of the direct links between different players. For simplicity, we assume $0 < \delta < \frac{1}{n-1}$.

A Nash equilibrium is a profile $\mathbf{x}^* = (x_1^*, \dots, x_n^*)$ such that, for any $i = 1, \dots, n$,

$$\pi_i(x_1^*, \dots, x_n^*) \ge \pi_i(x_1^*, \dots, x_{i-1}^*, x_i, x_{i+1}^*, \dots, x_n^*) \quad \text{for any } x_i \in \mathbb{R}.$$

In other words, at a Nash equilibrium, there is no profitable deviation for any player i choosing x_i^*.

Let $\mathbf{w} = (w_1, w_2, \dots, w_n)'$, $w_i > 0$ for all i (the transpose of a vector $\mathbf{w}$ is denoted by $\mathbf{w}'$), and $\mathbf{I}_n$ the $n \times n$ identity matrix. Define the *weighted* Katz–Bonacich centrality vector as

$$\mathbf{b}(\mathbf{G}, \mathbf{w}) = [\mathbf{I}_n - \delta\mathbf{G}]^{-1}\mathbf{w}.$$

Here $\mathbf{M} \coloneqq [\mathbf{I} - \delta\mathbf{G}]^{-1}$ denote the inverse Leontief matrix associated with network $\mathbf{G}$, while m_{ij} denote its ij entry, which is equal to the discounted number of walks from i to j with decay factor δ. Let $\mathbf{1}_n = (1, 1, \dots, 1)'$ be a vector of 1s. Then the *unweighted* Katz–Bonacich centrality vector can be defined as

$$\mathbf{b}(\mathbf{G}, \mathbf{1}) = [\mathbf{I} - \delta\mathbf{G}]^{-1}\mathbf{1}_n.$$

(1) Show that this network game has a unique Nash equilibrium $\mathbf{x}^*(\mathbf{G})$. Can you link this equilibrium to the Katz–Bonacich centrality vector defined above?

(2) Let $x^*(\mathbf{G}) = \sum_{i=1}^n x_i^*(\mathbf{G})$ denote the sum of actions (total activity) at the unique Nash equilibrium in part 1. Now suppose that you can remove a single node, say i, from the network. Which node do you want to remove such that the sum of effort at the new Nash equilibrium is reduced the most? (Note that, after the deletion of node i, we remove all the links of node i, and the remaining network, denoted by $\mathbf{G}_{-i}$, can be obtained by deleting the i-th row and i-th column of $\mathbf{G}$.)
Mathematically, you need to solve the *key player problem*[6]

$$\max_{i \in \mathcal{N}} \big(x^*(\mathbf{G}) - x^*(\mathbf{G}_{-i})\big).$$

In other words, you want to find a player who, once removed, leads to the highest reduction in total action in the remaining network.
Hint. You may come up with an index c_i for each i such that the key player is the one with the highest c_i. This c_i should be expressed using the Katz–Bonacich centrality vector defined above.

(3) Now instead of deleting a single node, we can delete any pair of nodes from the network. Can you identify the key pair, that is, the pair of nodes that, once removed, reduces total activity the most?[7]

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Solution by the proposer

(1) Suppose that $\mathbf{x}^*(\mathbf{G})$ is a Nash equilibrium. Then we obtain the following optimality equation for player i:

$$x_i^*(\mathbf{G}) = 1 + \delta \sum_{j \in \mathcal{N}} g_{ij} x_j^*(\mathbf{G}).$$

In matrix form,

$$\mathbf{x}^*(\mathbf{G}) = \mathbf{1}_n + \delta\mathbf{G}\mathbf{x}^*(\mathbf{G})$$

or

$$\mathbf{x}^*(\mathbf{G}) = [\mathbf{I}_n - \delta\mathbf{G}]^{-1}\mathbf{1}_n \coloneqq \mathbf{b}(\mathbf{G}, \mathbf{1}).$$

In other words, the Nash equilibrium effort is exactly equal to the unweighted Katz–Bonacich vector.

Note that, under the assumption that $0 < \delta < 1/(n-1)$, the matrix $[\mathbf{I}_n - \delta\mathbf{G}]$ is invertible, and the inverse matrix has the following infinite sum representation:

$$[\mathbf{I}_n - \delta\mathbf{G}]^{-1} = \mathbf{I}_n + \delta\mathbf{G} + \delta^2\mathbf{G}^2 + \delta^3\mathbf{G}^3 + \cdots.$$

For uniqueness, it is obvious.

[5] For an overview of the literature on network games, see [2].

[6] The key player problem has been introduced in [1].

[7] For the analysis of group players, see [1, 3].

(2) Before solving it, we first enrich the baseline model by taking into account heterogeneous individual weights $w_i > 0$,

$$\pi_i(x_1, x_2, \dots, x_n) = w_i x_i - \frac{1}{2}x_i^2 + \delta \sum_{j \in \mathcal{N}} g_{ij} x_i x_j. \qquad (1)$$

The unique equilibrium of this extended model corresponds to the weighted Katz–Bonacich centrality

$$\mathbf{x}^*(\mathbf{G}, \mathbf{w}) = [\mathbf{I}_n - \delta \mathbf{G}]^{-1}\mathbf{w} \coloneqq \mathbf{b}(\mathbf{G}, \mathbf{w}),$$

or, equivalently, for each $i = 1, \dots, n$:

$$x_i^*(\mathbf{G}, \mathbf{w}) = \sum_{j=1}^{n} m_{ij} w_j = \sum_{j=1}^{n} \sum_{k=0}^{\infty} \delta^k g_{ij}^{[k]} w_j = b_i(\mathbf{G}, \mathbf{w}),$$

where $g_{ij}^{[k]} \geq 0$ gives the number of walks of length $k \geq 1$ from i to j in the network and $b_i(\mathbf{G}, \mathbf{w})$ is the weighted Katz–Bonacich centrality of player i.

The aggregate equilibrium action is then equal to

$$x^*(\mathbf{G}, \mathbf{w}) = \mathbf{1}_n'[\mathbf{I} - \delta \mathbf{G}]^{-1}\mathbf{w} = \mathbf{b}'(\mathbf{G}, \mathbf{1})\mathbf{w}.$$

Intuitively, when w_i increases by 1 unit, $x_j^*(\mathbf{G}, \mathbf{w})$, each player j's equilibrium effort increases by $m_{ji} = m_{ij}$, and the total equilibrium action increases by $b_i(\mathbf{G}, \mathbf{1}) = \sum_j m_{ij} = \sum_j m_{ji}$ (note that $\mathbf{M}$ is a symmetric matrix). Mathematically,

$$\frac{\partial x_j^*(\mathbf{G}, \mathbf{w})}{\partial w_i} = m_{ji} = m_{ij} \quad \text{for all } i, j, \qquad (2)$$

$$\frac{\partial x^*(\mathbf{G}, \mathbf{w})}{\partial w_i} = b_i(\mathbf{G}, \mathbf{1}). \qquad (3)$$

To solve the key player problem, it suffices to prove that, for $i \in \mathcal{N}$,

$$c_i(\mathbf{G}) \coloneqq \big(x^*(\mathbf{G}) - x^*(\mathbf{G}_{-i})\big) = \frac{[b_i(\mathbf{G}, \mathbf{1})]^2}{m_{ii}}.$$

And the key player is the player i that maximizes $c_i(\mathbf{G})$.

To prove this, we take the following approach. Instead of removing node i (and all its links with others), we reduce the weight of player from $w_i = 1$ to $\hat{w}_i = 1 - \frac{b_i(\mathbf{G},\mathbf{1})}{m_{ii}}$, while keeping the weights of other players at 1 as in the baseline model, i.e., $w_j = 1$ for all $j \neq i$. (It will be clear why we pick this particular $\hat{w}_i$.)

We claim that, after this reduction in weight, the resulting equilibrium is the same as the one when i is removed from the network.

To see this, we first ask: what is the new equilibrium after this change in w_i? We claim that player i would choose exactly zero action. This is because, by (2), the change in equilibrium action by player i,

$$\Delta x_i^*(\mathbf{G}) \coloneqq x_i^*(\mathbf{G}, w_i = \hat{w}_i) - x_i^*(\mathbf{G}, w_i = 1),$$

is given by

$$\Delta x_i^*(\mathbf{G}) = m_{ii} \times (\hat{w}_i - w_i) = -m_{ii}\frac{b_i(\mathbf{G}, \mathbf{1})}{m_{ii}} = -b_i(\mathbf{G}, \mathbf{1})$$

from the construction of $\hat{w}_i$. Since initially player i chooses $b_i(\mathbf{G}, \mathbf{1})$, we have $x_i^*(\mathbf{G}, w_i = \hat{w}_i) = 0$ and the claim follows. What happens to other nodes? By (2), player j's equilibrium action would change by $m_{ij} \times (\hat{w}_i - w_i)$, and the aggregate equilibrium action, by (3), would change by

$$\begin{aligned} x^*(\mathbf{G}_{-i}) - x^*(\mathbf{G}) &= b_i(\mathbf{G}, \mathbf{1}) \times (\hat{w}_i - w_i) \\ &= -\frac{b_i(\mathbf{G}, \mathbf{1})^2}{m_{ii}} = -c_i(\mathbf{G}). \end{aligned}$$

This completes the proof of our claim. Thus, the key player in a network is the player i who has the highest $c_i(\mathbf{G})$.

(3) For any group $S \subset \mathcal{N}$, we can define the inter-centrality measure

$$d_S(\mathbf{G}) = \mathbf{b}_S'(\mathbf{G}, \mathbf{1})\mathbf{M}_{SS}^{-1}\mathbf{b}_S(\mathbf{G}, \mathbf{1}),$$

where $\mathbf{M}_{SS} = (m_{kl})$, $k, l \in S$, is the submatrix of $\mathbf{M}$, that is, the $|S| \times |S|$ $\mathbf{M}$ matrix of the subnetwork formed by players in S. Similarly, $\mathbf{b}_S(\mathbf{G}, \mathbf{1})$ is a subvector of the unweighted Katz–Bonacich centrality vector $\mathbf{b}(\mathbf{G}, \mathbf{1})$ for indices in the set S. It can be shown that

$$d_S(\mathbf{G}) = x^*(\mathbf{G}) - x^*(\mathbf{G}_{-S}),$$

where $\mathbf{G}_{-S}$ is the network obtained after removing all nodes and their links in S. The proof is similar to the one in part (2), since removing S from the network has the same effect on the equilibrium as changing the weight vector from $\mathbf{w}_S = \mathbf{1}_S$ to $\hat{\mathbf{w}}_S = \mathbf{1}_S - \mathbf{M}_{SS}^{-1}\mathbf{b}_S(\mathbf{G}, \mathbf{1})$, while the weights of the nodes in the complement of S remain fixed at 1 as in the baseline model.

When S is a pair (i, j) with $i \neq j$, we can explicitly express the index as follows:

$$d_{\{i,j\}}(\mathbf{G}) = [b_i(\mathbf{G}, \mathbf{1})\ b_j(\mathbf{G}, \mathbf{1})] \begin{bmatrix} m_{ii} & m_{ij} \\ m_{ji} & m_{jj} \end{bmatrix}^{-1} \begin{bmatrix} b_i(\mathbf{G}, \mathbf{1}) \\ b_j(\mathbf{G}, \mathbf{1}) \end{bmatrix}$$

The *key pair* (i, j) is the pair (i, j) with the largest $d_{\{i,j\}}(\mathbf{G})$.

References

[1] C. Ballester, A. Calvó-Armengol and Y. Zenou, Who's who in networks. Wanted: the key player. *Econometrica* **74**, 1403–1417 (2006)

[2] M. O. Jackson and Y. Zenou, Games on networks. In *Handbook of Game Theory. Vol. 4*, edited by H. P. Young and S. Zamir, Handbooks in Economics, 91–157, Elsevier/North-Holland, Amsterdam (2015)

[3] Y. Sun, W. Zhao and J. Zhou, Structural interventions in networks. arXiv:2101.12420 (2021)

We hope to receive your solutions to the proposed problems and your ideas on the open problems. Send your solutions to Michael Th. Rassias by email to mthrassias@yahoo.com.

We also solicit your new problems with their solutions for the next "Solved and unsolved problems" column, which will be devoted to Differential Equations.